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Part 6. Heavy tails

GENERAL INVERSE PROBLEMS FOR REGULAR VARIATION

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APPLIED PROBABILITY TRUST
DECEMBER 2014

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Abstract

Regular variation of distributional tails is known to be preserved by various linear transformations of some random structures. An inverse problem for regular variation aims at understanding whether the regular variation of a transformed random object is caused by regular variation of components of the original random structure. In this paper we build on previous work, and derive results in the multivariate case and in situations where regular variation is not restricted to one particular direction or quadrant.

Keywords: Regular variation; inverse problem; linear process; Breiman's result; random matrix

1. Introduction

The four authors of this paper are very much honored to contribute to this special issue of one of the oldest journals in applied probability. We wish this excellent journal a happy jubilee: long may it continue. Two of us, Thomas Mikosch and Gennady Samorodnitsky, were invited to contribute short papers to this special issue. With the editors' permission, we have combined our efforts leading to this longer and more substantial paper.

In this paper we study the regular variation of the tails of measures on $\mathbb{R}^d$, most importantly of probability measures. Stated somewhat vaguely, it is well known that regular variation tends to be preserved by various linear operations (such as linear transformations of the space, convolutions, integrals, etc.). We should like to understand to what extent the inverse statements are valid, or in other words, if the result of a linear operation on a measure is regularly varying in the appropriate space, was the original measure necessarily regularly varying as well?

Questions of this type are often referred to as *inverse problems for regular variation,* and in an earlier paper, Jacobsen *et al.* (2009), a fairly complete answer to this problem for certain nonnegative linear transformations of one-dimensional measures was given. Our aims in this paper are to treat the inverse problem in the multivariate case and to dispense with the non-negativity assumption on the linear transformations. We are fairly successful in our latter goal, but only partially so in the former.

Now let us be more precise. Let $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{\mathbf{0}\}$ and $\overline{\mathbb{R}}_0^d = \overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Recall that a random vector X with values in $\mathbb{R}^d$ is said to be *regularly varying* if on the Borel σ-field of $\overline{\mathbb{R}}_0^d$ there exists a nonnull Radon measure μ_X that does not charge the set of points at ∞ and is such that

$$\frac{\mathbb{P}\{s^{-1}X \in \cdot\}}{\mathbb{P}\{\|X\| > s\}} \xrightarrow{\text{v}} \mu_X \quad \text{as } s \to \infty,$$

where '$\overset{v}{\to}$' denotes vague convergence on the Borel σ-field of $\overline{\mathbb{R}}_0^d$; see, e.g. Kallenberg (1983) or Resnick (1987). Recall that, in this context, a Borel measure is Radon if it is finite outside any ball of positive radius centered at the origin. The measure μ_X necessarily satisfies the relation $\mu_X(tA) = t^{-\alpha}\mu_X(A)$, $t > 0$, for all Borel sets A and some $\alpha > 0$. We refer to α as the *index of regular variation* and to μ_X as the *tail measure.* The notion of regular variation applies equally well to σ-finite Borel measures on $\mathbb{R}^d$ that are finite outside any ball of positive radius centered at the origin. Specifically, any such measure ν is said to be *regularly varying* if, as above, there is a nonnull Radon measure μ on $\overline{\mathbb{R}}_0^d$ that does not charge the set of infinite points and is such that

$$\frac{\nu(s \cdot)}{\nu(\{\boldsymbol{y} \colon \|\boldsymbol{y}\| > s\})} \overset{v}{\to} \mu \quad \text{as } s \to \infty.$$

As in the case of probability measures, the limiting measure μ scales with index $\alpha > 0$. We will write $\nu \in \mathrm{RV}(\alpha, \mu)$. Of course, this language allows the measure ν to be the law of a random vector $\boldsymbol{X}$, but in the case of random vectors it is even more common to simply write $\boldsymbol{X} \in \mathrm{RV}(\alpha, \mu_X)$.

To give a taste of the linear operations on regularly varying measures such as we have in mind, we give two examples. The reader will notice that these examples are more general versions of the corresponding examples in Jacobsen *et al.* (2009).

Example 1. (*Weighted sums.*) Let Ψ_j, $j = 1, 2, \ldots$, be (nonrandom) $d \times m$ matrices, and let $\{\boldsymbol{Z}^{(j)}\}$ be an independent and identically distributed (i.i.d.) sequence of regularly varying $\mathbb{R}^m$-valued random (column) vectors with a generic element $\boldsymbol{Z} \in \mathrm{RV}(\alpha, \mu_{\boldsymbol{Z}})$. Then, under appropriate size conditions on the matrices $\{\Psi_j\}$, the series $\boldsymbol{X} = \sum_{j=1}^{\infty} \Psi_j \boldsymbol{Z}^{(j)}$ converges with probability 1, and $\boldsymbol{X}$ is regularly varying with index α and

$$\frac{\mathbb{P}\{s^{-1}\boldsymbol{X} \in \cdot\}}{\mathbb{P}\{\|\boldsymbol{Z}\| > s\}} \overset{v}{\to} \sum_{j=1}^{\infty} \mu_{\boldsymbol{Z}} \circ \Psi_j^{-1} \quad \text{as } s \to \infty, \tag{1}$$

assuming that the right-hand side does not vanish; see Hult and Samorodnitsky (2008, 2010). This statement is always true if the sum is finite; see Resnick and Willekens (1991) and Basrak *et al.* (2002b).

Is the converse statement true? That is, if $\boldsymbol{X}$ is regularly varying, does it follow that the i.i.d. vectors $\boldsymbol{Z}_i$ are regularly varying as well? In Jacobsen *et al.* (2009) this problem was solved for i.i.d. *real-valued* Z_i and *nonnegative* scalars $\Psi_j = \psi_j$. (Here and in what follows, we use the symbol ψ_j for scalars instead of genuine matrices Ψ_j.) It was shown that, under appropriate size conditions in the case of an infinite sum, Z_i inherits regular variation with index α from X if the condition

$$\sum_{j=1}^{\infty} \psi_j^{\alpha+\mathrm{i}\theta} \neq 0 \quad \text{for all } \theta \in \mathbb{R} \tag{2}$$

holds. Moreover, if (2) fails then one can find i.i.d. $\{Z_i\}$ which are not regularly varying but X is regularly varying. In this paper we aim to extend the result to the multivariate case and/or drop the assumption of nonnegative coefficients.

Example 2. (*Products.*) Let $\boldsymbol{Z} \in \mathrm{RV}(\alpha, \mu_{\boldsymbol{Z}})$ be a random (column) vector in $\mathbb{R}^m$, and let $\boldsymbol{A}$ be a random $d \times m$ matrix, independent of $\boldsymbol{Z}$ and such that its matrix norm satisfies $\mathbb{E}\|\boldsymbol{A}\|^{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$. Then $\boldsymbol{X} = \boldsymbol{A}\boldsymbol{Z}$ is regularly varying with index α in $\mathbb{R}^d$, and

$$\frac{\mathbb{P}\{s^{-1}\boldsymbol{X} \in \cdot\}}{\mathbb{P}\{\|\boldsymbol{Z}\| > s\}} \overset{v}{\to} \mathbb{E}[\mu_{\boldsymbol{Z}} \circ \boldsymbol{A}^{-1}] \quad \text{as } s \to \infty, \tag{3}$$

provided the measure on the right-hand side does not vanish; see Basrak *et al.* (2002a). Once again, is the converse statement true? That is, if X is regularly varying, does it follow that the vector Z is regularly varying (assuming that the random matrix A is suitably small)? In Jacobsen *et al.* (2009) it was shown that, if A and Z are real valued and $A > 0$, then Z inherits regular variation with index α from X if and only if

$$\mathbb{E}[A^{\alpha+i\theta}] \neq 0 \quad \text{for all } \theta \in \mathbb{R}.$$

We would like to remove the restriction to one dimension and the assumption of nonnegativity.

As in the one-dimensional nonnegative case, these questions turn out to be related to a certain cancellation property of measures; we address this issue in Section 2. Proof of the cancellation property requires some abstract Fourier analytic arguments. The reader interested in applications of these results in the spirit of Examples 1 and 2 is referred to Sections 3–5. In Section 3 we study the inverse problem for weighted sums of a multivariate i.i.d. sequence. In Section 4 we consider the corresponding problem for matrix products, where the random matrix has diagonal structure. Some examples in the case of nondiagonal deterministic matrices are given in Section 5. While the results in Section 3 yield a rather complete picture for weighted sums, the results in the other two sections are by way of examples, leaving room for further investigations.

2. The generalized cancellation property

Let ρ and ν be σ-finite measures on $\mathbb{R}^d$. We define the *multiplicative convolution* of ν and ρ as a (not necessarily σ-finite) measure on $\mathbb{R}^d$ given by

$$(\nu \circledast \rho)(B) = \int_{\mathbb{R}^d} \nu(T_x^{-1}(B)) \, \rho(\mathrm{d}x) \quad \text{for any Borel set } B \subset \mathbb{R}^d,$$

where $T_x = \mathrm{diag}(x)$ for $x \in \mathbb{R}^d$.

Our first result motivates the cancellation property discussed in this section.

Theorem 1. *Assume that* $\alpha > 0$, *and let* ρ *and* ν *be* σ-*finite measures such that* ρ *is not concentrated on any proper coordinate subspace of* $\mathbb{R}^d$, *that is,*

$$\inf_{j=1,\ldots,d} \rho(\{x : x_j \neq 0\}) > 0, \tag{4}$$

$\nu \circledast \rho \in \mathrm{RV}(\mu, \alpha)$, *and*

$$\int_{\mathbb{R}^d} (\|y\|^{\alpha-\delta'} \vee \|y\|^{\alpha+\delta'}) \, \rho(\mathrm{d}y) < \infty \tag{5}$$

for some $0 < \delta' < \alpha$ *and*

$$\lim_{b \downarrow 0} \limsup_{s \to \infty} \frac{\int_{0 < |y_j| \leq b} \rho(\{x : |x_j| > s/|y_j|\}) \, \nu(\mathrm{d}y)}{(\nu \circledast \rho)(\{x : \|x\| > s\})} = 0 \tag{6}$$

for each $j = 1, \ldots, d$. *Then the family of measures on* $\mathbb{R}_0^d$ *given by*

$$\mu_s(\cdot) = \frac{\nu(s \cdot)}{(\nu \circledast \rho)(\{x : \|x\| > s\})}, \qquad s \geq 1, \tag{7}$$

is relatively compact in the vague topology on $\overline{\mathbb{R}}_0^d$. *Furthermore, any limit measure* μ_* *(limit as* $s \to \infty$*) does not charge the set of infinite points and satisfies the equation*

$$\mu_* \circledast \rho = \mu. \tag{8}$$

Proof. By (4), we can choose $\theta > 0$ such that $\rho(\{\boldsymbol{x}: |x_j| \geq \theta\}) \geq \delta > 0$ for every $j = 1, \ldots, d$ and sufficiently small δ. For every j and $s > 0$,

$$\rho(\{\boldsymbol{x}: |x_j| \geq \theta\}) \, \nu\left(\left\{\boldsymbol{x}: |x_j| > \frac{s}{\theta}\right\}\right) \leq (\nu \circledast \rho)(\{\boldsymbol{x}: \|\boldsymbol{x}\| > s\}),$$

so

$$\nu(\{\boldsymbol{x}: \|\boldsymbol{x}\| > s\}) \leq \sum_{j=1}^{d} \nu\left(\left\{\boldsymbol{x}: |x_j| > \frac{s}{d}\right\}\right)$$

$$\leq (\nu \circledast \rho)\left(\left\{\boldsymbol{x}: \|\boldsymbol{x}\| > \frac{\theta s}{d}\right\}\right) \sum_{j=1}^{d} \frac{1}{\rho(\{\boldsymbol{x}: |x_j| \geq \theta\})}$$

$$\leq \frac{d}{\delta} (\nu \circledast \rho)\left(\left\{\boldsymbol{x}: \|\boldsymbol{x}\| > \frac{\theta s}{d}\right\}\right).$$

Let B_τ denote the closed ball of radius $\tau > 0$ centered at the origin. Then, as $s \to \infty$,

$$\mu_s(B_\tau^c) \leq \frac{d}{\delta} \frac{(\nu \circledast \rho)(\{\boldsymbol{x}: \|\boldsymbol{x}\| > \theta \tau s/d\})}{(\nu \circledast \rho)(\{\boldsymbol{x}: \|\boldsymbol{x}\| > s\})} \to \frac{d}{\delta}\left(\frac{\theta \tau}{d}\right)^{-\alpha} \mu(B_1) < \infty.$$

Hence, $\{\mu_s\}$ is relatively compact (see Proposition 3.16 of Resnick (1987)). Let $\{s_k\}$ be a sequence for which $s_k \to \infty$ as $k \to \infty$ and such that $\mu_{s_k} \overset{\mathrm{v}}{\to} \mu_*$ in $\overline{\mathbb{R}}_0^d$ for some limit measure μ_*. Then μ_* does not charge the set of infinite points.

For $\boldsymbol{a} \in \mathbb{R}^d$, define

$$D_{\boldsymbol{a}} = \left\{\boldsymbol{y} \in \overline{\mathbb{R}}_0^d : \mu_*\left(\left\{\boldsymbol{z}: z_j = \frac{a_j}{y_j} \text{ for some } j = 1, \ldots, d\right\}\right) = 0\right\}.$$

The argument after Equation (2.22) of Jacobsen *et al.* (2009) shows that there are sets $A_1, \ldots, A_d$ of real numbers, each set at most countable, such that

$$\rho(D_{\boldsymbol{a}}) = 0 \quad \text{for } \boldsymbol{a} \in \prod_{j=1}^{d} A_j^c. \tag{9}$$

Consider $\boldsymbol{a}$ such that

$$a_1 > 0, \qquad a_j \geq 0, \quad j = 2, \ldots, d, \qquad a_j \notin A_j, \quad j = 1, \ldots, d. \tag{10}$$

The set $C_d(\boldsymbol{a}) = \prod_{j=1}^{d}[a_j, \infty)$ is bounded away from the origin and is a continuity set for the tail measure μ of $\nu \circledast \rho$. Therefore,

$$\mu(C_d(\boldsymbol{a})) = \lim_{k \to \infty} \frac{(\nu \circledast \rho)(s_k C_d(\boldsymbol{a}))}{(\nu \circledast \rho)(\{\boldsymbol{x}: \|\boldsymbol{x}\| > s_k\})} = \lim_{k \to \infty} \sum_{I_+ \subseteq \{1, \ldots, d\}} \int_{D(I_+)} f_{k, I_+}(\boldsymbol{z}) \, \rho(\mathrm{d}\boldsymbol{z}), \tag{11}$$

where, for any nonempty $I_+ \subseteq \{1, \ldots, d\}$,

$$D(I_+) = \{\boldsymbol{z}: z_j \geq 0 \text{ for } j \in I_+ \text{ and } z_j < 0 \text{ for } j \notin I_+\},$$

interpreting $[0/0, \infty) = \mathbb{R}$ and writing, for $k \geq 1$ and $\boldsymbol{v}$ such that $v_j \geq 0$ for $j \in I_+$,

$$f_{k,I_+}(\boldsymbol{v}) = \mu_{s_k}\left(\prod_{j \in I_+}\left[\frac{a_j}{v_j}, \infty\right) \times \prod_{j \notin I_+}\left(-\infty, -\frac{a_j}{|v_j|}\right]\right).$$

Choose $\varepsilon > 0$ so small that $c := \rho(\{\boldsymbol{z}\colon |z_1| \geq \varepsilon\}) > 0$, and proceed similarly to the beginning of the proof. Then, for $I_+ \subseteq \{1, \ldots, d\}$ and $\boldsymbol{z} = \boldsymbol{1} = (1, \ldots, 1)$,

$$f_{k,I_+}(\boldsymbol{1}) \leq \mu_{s_k}(\{\boldsymbol{y}\colon |y_1| > a_1\}) \leq c^{-1}\frac{(\nu \circledast \rho)(\{\boldsymbol{x}\colon |x_1| \geq a_1\varepsilon s_k\})}{(\nu \circledast \rho)(\{\boldsymbol{x}\colon \|\boldsymbol{x}\| > s_k\})}.$$

Therefore, on $D(I_+) \cap \{\boldsymbol{z}\colon |z_1| \leq Ms_k\}$ for finite $M > 0$,

$$f_{k,I_+}(\boldsymbol{z}) \leq c^{-1}\frac{(\nu \circledast \rho)(\{\boldsymbol{x}\colon |x_1| \geq a_1\varepsilon s_k/z_1\})}{(\nu \circledast \rho)(\{\boldsymbol{x}\colon \|\boldsymbol{x}\| > s_k\})} \leq C(a_1, \varepsilon, M)(|z_1|^{\alpha-\delta'} \vee |z_1|^{\alpha+\delta'}),$$

where $C(a_1, \varepsilon, M)$ is a finite positive constant and in the last step we used the Potter bounds; see Proposition 0.8 of Resnick (1987). Recalling that (9) holds for our choice of $\boldsymbol{a}$, using (5) and the dominated convergence theorem, we conclude that, for every $M > 0$, as $k \to \infty$,

$$\int_{D(I_+)\cap\{\boldsymbol{z}\colon |z_1|\leq Ms_k\}} f_{k,I_+}(\boldsymbol{z})\rho(\mathrm{d}\boldsymbol{z}) \to \int_{D(I_+)} \mu_*\left(\prod_{j \in I_+}\left[\frac{a_j}{z_j}, \infty\right) \times \prod_{j \notin I_+}\left(-\infty, -\frac{a_j}{|z_j|}\right]\right)\rho(\mathrm{d}\boldsymbol{z}).$$

Furthermore,

$$\int_{D(I_+)\cap\{\boldsymbol{z}\colon |z_1|>Ms_k\}} f_{k,I_+}(\boldsymbol{z})\,\rho(\mathrm{d}\boldsymbol{z}) \leq \frac{\rho(\{\boldsymbol{z}\colon |z_1| > Ms_k\})\,\nu(\{\boldsymbol{y}\colon |y_1| > a_1/M\})}{(\nu \circledast \rho)(\{\boldsymbol{x}\colon \|\boldsymbol{x}\| > s_k\})}$$
$$+ \frac{\int_{0<|y_1|\leq a_1/M} \rho(\{\boldsymbol{z}\colon |z_1| > s_k a_1/|y_1|\})\,\nu(\mathrm{d}\boldsymbol{y})}{(\nu \circledast \rho)(\{\boldsymbol{x}\colon \|\boldsymbol{x}\| > s_k\})}$$
$$:= A_k + B_k.$$

Since

$$\rho(\{\boldsymbol{z}\colon |z_1| > Ms_k\}) \leq \frac{1}{(Ms_k)^{\alpha+\delta}}\int_{\mathbb{R}^d} |z_1|^{\alpha+\delta}\,\rho(\mathrm{d}\boldsymbol{z}),$$

it follows from (5) that $A_k \to 0$ as $k \to \infty$ for every finite $M > 0$, while, by (6),

$$\lim_{M\to\infty}\limsup_{k\to\infty} B_k = 0.$$

Thus, we have proved that, for any $\boldsymbol{a}$ satisfying (10) and $I_+ \subseteq \{1, \ldots, d\}$, as $k \to \infty$,

$$\int_{D(I_+)} f_{k,I_+}(\boldsymbol{z})\,\rho(\mathrm{d}\boldsymbol{z}) \to \int_{D(I_+)} \mu_*\left(\prod_{j \in I_+}\left[\frac{a_j}{z_j}, \infty\right) \times \prod_{j \notin I_+}\left(-\infty, -\frac{a_j}{|z_j|}\right]\right)\rho(\mathrm{d}\boldsymbol{z}).$$

Then, in view of (11),

$$\mu(C_d(\boldsymbol{a})) = \sum_{I_+\subseteq\{1,\ldots,d\}}\int_{D(I_+)} \mu_*\left(\prod_{j \in I_+}\left[\frac{a_j}{z_j}, \infty\right) \times \prod_{j \notin I_+}\left(-\infty, -\frac{a_j}{|z_j|}\right]\right)\rho(\mathrm{d}\boldsymbol{z})$$
$$= (\mu_* \circledast \rho)(C_d(\boldsymbol{a})). \tag{12}$$

Using the continuity of measures from above, we can now extend (12) to any $\boldsymbol{a}$ satisfying $a_1 > 0$, $a_j \geq 0$, $j = 2, \ldots, d$. This means that the measures μ and $\mu_* \circledast \rho$ coincide on the set $\{\boldsymbol{x} : x_1 > 0, \ x_j \geq 0, \ j = 2, \ldots, d\}$.

Now we can repeat this argument for each distinct coordinate $k = 1, \ldots, d$, implying that the measures μ and $\mu_* \circledast \rho$ coincide on each of the d sets $\{\boldsymbol{x} : x_k > 0, \ x_j \geq 0, \ j = 1, \ldots, d\}$, $k = 1, \ldots, d$. Since the union of these sets is the first quadrant $[0, \infty)^d \setminus \{\boldsymbol{0}\}$, we conclude that these two measures coincide on this set. An identical argument can be used for all other quadrants of $\overline{\mathbb{R}}_0^d$. Thus, (8) holds and the proof of the theorem is complete.

Seemingly, only a small step remains between the conclusion of Theorem 1 and the statement that ν is regularly varying with index α. This step consists of showing that, for fixed μ and ρ, (8) has a unique solution μ_*. Indeed, if this could be established then all limits as $s \to \infty$ of subsequences of the family $\{\mu_s\}$ in (7) would be equal. In turn, $\{\mu_s\}$ would converge vaguely and ν would be regularly varying.

Unfortunately, this step is not so small and it turns out that, in some cases, (8) has multiple solutions; see the following discussion and, in particular, Remark 2. Therefore, our next step is to establish conditions under which the solution to (8) is, indeed, unique. We start by reducing the problem to a slightly different form. Uniqueness of the solution to (8) would follow if the measure ρ had the following property: for ν_1 and ν_2 within a relevant class of σ-finite measures,

$$\text{if } \nu_1 \circledast \rho = \nu_2 \circledast \rho \text{ then } \nu_1 = \nu_2. \tag{13}$$

This property can be viewed as the *cancellation property of the measure ρ with respect to the operation '$\circledast$'*.

A similar situation was considered in Jacobsen *et al.* (2009) where the case $d = 1$ was treated. It was assumed there that all measures are supported on the positive half-line $(0, \infty)$. In particular, all regularly varying measures supported on $(0, \infty)$ have tail measures proportional to one another. It is natural in this situation to study the cancellation property if one of the measures ν_1 and ν_2 is such a canonical measure. Accordingly, we define a measure ν^α on $(0, \infty)$, $\alpha \in \mathbb{R}$, with a power density given by

$$\nu^\alpha(\mathrm{d}x) = \alpha \, x^{-(\alpha+1)} \, \mathrm{d}x. \tag{14}$$

In fact, at this point Jacobsen *et al.* (2009) allow any real value of α. In the present paper, we allow only positive α, though the statement of Theorem 2 below can be extended to the more general case.

Jacobsen *et al.* (2009) investigated what measures ρ have the cancellation property

$$\nu \circledast \rho = \nu^\alpha \circledast \rho \quad \Longrightarrow \quad \nu = \nu^\alpha.$$

They showed that a measure ρ satisfying

$$\int_0^\infty y^{\alpha-\delta} \vee y^{\alpha+\delta} \, \rho(\mathrm{d}y) < \infty \quad \text{for some } \delta > 0$$

has this cancellation property if and only if

$$\int_0^\infty y^{\alpha+\mathrm{i}\theta} \, \rho(\mathrm{d}y) \neq 0 \quad \text{for all } \theta \in \mathbb{R}.$$

In order to understand the more general cancellation property (13), we start by replacing the single equation by a system of linear equations that include only measures concentrated on the positive quadrant of $\mathbb{R}^d$.

For $d \geq 1$, consider the set $Q_d = \{-1, 1\}^d$ equipped with coordinatewise (binary) multiplication. Let $\alpha_1, \ldots, \alpha_d$ be positive numbers, let $\{\rho_v, \ v \in Q_d\}$ be σ-finite measures on $(0, \infty)^d$, and let $\{v_v^{(i)}, \ v \in Q_d\}$, $i = 1, 2$, be two collections of σ-finite measures on $[0, \infty)^d$. We assume that, for a certain nonempty subset K of $\{1, \ldots, d\}$,

$$\int_{(0,\infty)^d} x_j^{\alpha_j} \, \rho_v(\mathrm{d}\boldsymbol{x}) < \infty \quad \text{for each } v \in Q_d \text{ and } j \in K, \tag{15}$$

and, for $i = 1, 2$,

$$\sup_{s>0} s^{\alpha_j} v_v^{(i)}(\{\boldsymbol{x} : x_j > s\}) < \infty \quad \text{for each } v \in Q_d \text{ and } j \in K. \tag{16}$$

Assume now that these measures satisfy the following system of 2^d linear equations:

$$\sum_{w \in Q_d} v_w^{(1)} \circledast \rho_{vw} = \sum_{w \in Q_d} v_w^{(2)} \circledast \rho_{vw} \quad \text{for each } v \in Q_d. \tag{17}$$

The following result characterizes those measures $\{\rho_v, \ v \in Q_d\}$ which can be 'canceled' in this system of equations.

Theorem 2. *Let $\{\rho_v, \ v \in Q_d\}$ be σ-finite measures on $(0, \infty)^d$, and let $\{v_v^{(i)}, \ v \in Q_d\}$, $i = 1, 2$, be σ-finite measures on $[0, \infty)^d$. Assume that, for some nonempty set $K \subseteq \{1, \ldots, d\}$,*

$$v_v^{(i)}(\{\boldsymbol{x} : x_k = 0 \text{ for each } k \in K\}) = 0, \qquad i = 1, 2, \ v \in Q_d, \tag{18}$$

and that (15) and (16) hold for this set K. Suppose that, for each $j \in K$, $m_1, \ldots, m_d \in \{0, 1\}$, and $\theta_1, \ldots, \theta_d \in \mathbb{R}$,

$$\sum_{v \in Q_d} \prod_{k=1}^{d} v_k^{m_k} \int_{(0,\infty)^d} x_j^{\alpha_j} \prod_{k=1}^{d} x_k^{\mathrm{i}\theta_k} \, \rho_v(\mathrm{d}\boldsymbol{x}) \neq 0 \tag{19}$$

with the usual notation $v = (v_1, \ldots, v_d) \in Q_d$ and $\boldsymbol{x} = (x_1, \ldots, x_d) \in [0, \infty)^d$. If these measures satisfy the system of equations (17) then

$$v_v^{(1)} = v_v^{(2)} \quad \text{for each } v \in Q_d. \tag{20}$$

Remark 1. In applications to regular variation the measures $\{v_v^{(i)}, \ v \in Q_d\}$, $i = 1, 2$, will appear as (restrictions to the different quadrants of) certain vague limits v in $\overline{\mathbb{R}}_0^d$; hence, they automatically put no mass at the origin, so the set $K = \{1, \ldots, d\}$ always satisfies (18). This is the maximal possible choice of K which requires the largest possible set of conditions in (19). The smaller the set K can be chosen, the fewer conditions one needs to check. If, for example, v is absolutely continuous then $K = \{1\}$ and (18) gives 2^d conditions.

Before proving Theorem 2, we discuss some special cases. Consider first the scalar case, $d = 1$, for which the system of equations (17) becomes

$$\begin{aligned}
v_1^{(1)} \circledast \rho_1 + v_{-1}^{(1)} \circledast \rho_{-1} &= v_1^{(2)} \circledast \rho_1 + v_{-1}^{(2)} \circledast \rho_{-1}, \\
v_1^{(1)} \circledast \rho_{-1} + v_{-1}^{(1)} \circledast \rho_1 &= v_1^{(2)} \circledast \rho_{-1} + v_{-1}^{(2)} \circledast \rho_1.
\end{aligned} \tag{21}$$

The only choice is $K = \{1\}$, and conditions (19) for the cancellation property are equivalent to requiring that, for all $\theta \in \mathbb{R}$,

$$\int_0^\infty x^{\alpha_1 + i\theta} \rho_1(\mathrm{d}x) + \int_0^\infty x^{\alpha_1 + i\theta} \rho_{-1}(\mathrm{d}x) \neq 0,$$

$$\int_0^\infty x^{\alpha_1 + i\theta} \rho_1(\mathrm{d}x) - \int_0^\infty x^{\alpha_1 + i\theta} \rho_{-1}(\mathrm{d}x) \neq 0. \tag{22}$$

In dimension one the measure ν^α, $\alpha > 0$, given in (14), is particularly important when studying regular variation. Suppose that $\nu_i^{(2)} = c_i \nu^\alpha$, $i = \pm 1$, for nonnegative constants c_1 and c_{-1}. If we choose $\alpha_1 = \alpha$ then assumption (16) automatically holds for the measures $\nu_1^{(2)}$ and $\nu_{-1}^{(2)}$. Assuming that the measures ρ_1 and ρ_{-1} satisfy (15), and writing $\|\rho_i\|_\alpha^\alpha = \int_0^\infty x^\alpha \rho_i(\mathrm{d}x)$, $i = \pm 1$, system (21) takes the form

$$\nu_1^{(1)} \circledast \rho_i + \nu_{-1}^{(1)} \circledast \rho_{-i} = (c_1 \|\rho_i\|_\alpha^\alpha + c_{-1} \|\rho_{-i}\|_\alpha^\alpha) \, \nu^\alpha, \qquad i = \pm 1. \tag{23}$$

Note that (23) already implies that (16) holds for the measures $\nu_1^{(1)}$ and $\nu_{-1}^{(1)}$ as well. We therefore obtain the following corollary of Theorem 2.

Corollary 1. *Let $\alpha_1 = \alpha > 0$, and let ρ_1 and ρ_{-1} be σ-finite measures on $(0, \infty)$ satisfying (15). If, for $i = \pm 1$, the σ-finite measures $\nu_i^{(1)}$ on $[0, \infty)$ satisfy the system of equations (23), and if the cancellation conditions (22) are satisfied, then $\nu_i^{(1)} = c_i \nu^\alpha$, $i = \pm 1$.*

Remark 2. Assume that all the conditions of Corollary 1 except (22) are satisfied. For example, if the first condition in (22) is not satisfied for some $\theta = \theta_0 \in \mathbb{R}$ then the measures

$$\nu_i^{(1)}(\mathrm{d}x) = [c_i + a \, \cos(\theta_0 \log x) + b \sin(\theta_0 \log x)] \, \nu^\alpha(\mathrm{d}x), \qquad i = \pm 1,$$

for a and b such that $0 \leq a^2 + b^2 \leq 1$ satisfy the system of equations (23). Similarly, if the second condition in (22) fails for some $\theta = \theta_0 \in \mathbb{R}$ then the measures

$$\nu_i^{(1)}(\mathrm{d}x) = [c_i + (-1)^i (a \, \cos(\theta_0 \log x) + b \, \sin(\theta_0 \log x))] \, \nu^\alpha(\mathrm{d}x), \qquad i = \pm 1,$$

with the same choice of a and b as above satisfy (23).

Another useful special case of Theorem 2 corresponds to the situation in which only one of the measures $\{\rho_v, \, v \in Q_d\}$ is nonnull (we will see in the sequel that this case arises naturally in inverse problems for regular variation). Assume without loss of generality that the nonnull measure corresponds to the unit $v = (1, \dots, 1)$ in Q_d; for simplicity, denote this measure by ρ. Then the system of equations (17) decouples, and becomes

$$\nu_v^{(1)} \circledast \rho = \nu_v^{(2)} \circledast \rho \quad \text{for each } v \in Q_d.$$

However, the decoupled system of equations does not provide us with any additional insight over a single equation, so we drop the subscript and consider the equation

$$\nu^{(1)} \circledast \rho = \nu^{(2)} \circledast \rho \tag{24}$$

for two σ-finite measures $\nu^{(1)}$ and $\nu^{(2)}$. If we interpret (15), (16), and (18) by disregarding the subscripts, we obtain another corollary of Theorem 2.

Corollary 2. *Let $\alpha_1, \ldots, \alpha_d$ be positive numbers, let ρ be a σ-finite measure on $(0, \infty)^d$, and let $\nu^{(1)}$ and $\nu^{(2)}$ be σ-finite measures on $[0, \infty)^d$. Suppose that the nonempty set $K \subseteq \{1, \ldots, d\}$ satisfies (18), and that (15) and (16) hold. If (24) is satisfied and*

$$\int_{(0,\infty)^d} x_j^{\alpha_j} \prod_{k=1}^{d} x_k^{i\theta_k} \rho(\mathrm{d}\boldsymbol{x}) \neq 0 \qquad (25)$$

for each $j \in K$ and all $\theta_1, \ldots, \theta_d \in \mathbb{R}$, then $\nu^{(1)} = \nu^{(2)}$.

The conclusion of Corollary 2 in the case $d = 1$ is the same as the direct part of Theorem 2.1 of Jacobsen *et al.* (2009).

Proof of Theorem 2. The general idea of the proof is similar to the proof of Theorem 2.1 of Jacobsen *et al.* (2009). Fix $j \in K$, and define

$$h_j^{(v,i)}(\boldsymbol{y}) = y_j^{\alpha_j} \nu_v^{(i)}(\{\boldsymbol{z} \colon 0 \leq z_k \leq y_k,\ k \neq j,\ z_j > y_j\}), \qquad v \in Q_d,\ i = 1, 2,$$

for $\boldsymbol{y} = (y_1, \ldots, y_d)$ with all $y_k > 0$. It follows from (16) that all these functions are bounded on their domain. The system of equations (17) then tells us that

$$\sum_{w \in Q_d} \int_{[0,\infty)^d} h_j^{(w,1)}\left(\frac{x_1}{z_1}, \ldots, \frac{x_d}{z_d}\right) \rho_{vw}(\mathrm{d}\boldsymbol{z}) = \sum_{w \in Q_d} \int_{[0,\infty)^d} h_j^{(w,2)}\left(\frac{x_1}{z_1}, \ldots, \frac{x_d}{z_d}\right) \rho_{vw}(\mathrm{d}\boldsymbol{z})$$

for each $v \in Q_d$ and $x_k > 0$, $k = 1, \ldots, d$. Next, define functions for $\boldsymbol{y} \in \mathbb{R}^d$ by

$$g_j^{(v,i)}(\boldsymbol{y}) = h_j^{(v,i)}(\mathrm{e}^{y_1}, \ldots, \mathrm{e}^{y_d}), \qquad v \in Q_d,\ i = 1, 2,$$

and define finite measures on $\mathbb{R}^d$ by

$$\mu_j^{(v)}(\mathrm{d}\boldsymbol{x}) = (\mathrm{e}^{\alpha_j x_j} \rho_v) \circ T_{\log}^{-1}(\mathrm{d}\boldsymbol{x}),$$

where $T_{\log}(\boldsymbol{y}) = (\log y_1, \ldots, \log y_d)$ for $\boldsymbol{y} \in (0, \infty)^d$. We can now write

$$\sum_{w \in Q_d} \int_{\mathbb{R}^d} g_j^{(w,1)}(\boldsymbol{z} - \boldsymbol{y}) \mu_j^{(vw)}(\mathrm{d}\boldsymbol{y}) = \sum_{w \in Q_d} \int_{\mathbb{R}^d} g_j^{(w,2)}(\boldsymbol{z} - \boldsymbol{y}) \mu_j^{(vw)}(\mathrm{d}\boldsymbol{y})$$

for each $v \in Q_d$ and $\boldsymbol{z} \in \mathbb{R}^d$. Therefore, the bounded functions

$$g_j^{(v)}(\boldsymbol{y}) := g_j^{(v,1)}(\boldsymbol{y}) - g_j^{(v,2)}(\boldsymbol{y}), \qquad \boldsymbol{y} \in \mathbb{R}^d,\ v \in Q_d,$$

satisfy

$$\sum_{w \in Q_d} \int_{\mathbb{R}^d} g_j^{(w)}(\boldsymbol{z} - \boldsymbol{y}) \mu_j^{(vw)}(\mathrm{d}\boldsymbol{y}) = 0 \qquad (26)$$

for each $v \in Q_d$ and $\boldsymbol{z} \in \mathbb{R}^d$. For fixed $m_k \in \{0, 1\}$, $k = 1, \ldots, d$, and $j \in K$, define a signed bounded measure on $\mathbb{R}^d$ by

$$\mu_j = \sum_{v \in Q_d} \prod_{k=1}^{d} v_k^{m_k} \mu_j^{(v)}$$

and a bounded function on $\mathbb{R}^d$ by

$$g_j = \sum_{v \in Q_d} \prod_{k=1}^{d} v_k^{m_k} g_j^{(v)}.$$

Then the system of equations (26) implies that

$$\int_{\mathbb{R}^d} g_j(z - y) \, \mu_j(\mathrm{d}y) = 0, \qquad z \in \mathbb{R}^d, \tag{27}$$

and we want to show that $g_j = 0$ everywhere.

Note now that the right-hand side of (19) is exactly the Fourier transform of μ_j at the point $s = (\theta_1, \ldots, \theta_d)$. Let φ be the standard normal density in $\mathbb{R}^d$. Then in standard notation for additive convolution we have $\varphi * \mu_j \in L^1(\mathbb{R}^d)$, and (27) tells us that $g_j * (\varphi * \mu_j) \equiv 0$. Let the 'hat' symbol (caret) over a function or signed measure denote the distributional Fourier transform of that function or signed measure. Then Theorem 9.3 of Rudin (1973) implies that, in the distributional sense,

$$\operatorname{supp}(\widehat{g_j}) \subseteq \{s \in \mathbb{R}^d : \widehat{\varphi}(s)\widehat{\mu}_j(s) = 0\} = \{s \in \mathbb{R}^d : \widehat{\mu}_j(s) = 0\} = \varnothing,$$

where the last equation is just condition (19). Therefore, we conclude that the support of the Fourier transform $\widehat{g_j}$ is empty; hence, $g_j = 0$ almost everywhere. Since the function g_j is coordinatewise right continuous, we see that $g_j = 0$ everywhere.

The $2^d \times 2^d$ matrix A with the entries

$$a_{m_1, \ldots, m_d, v_1, \ldots, v_d} = \prod_{k=1}^{d} v_k^{m_k}, \qquad m_j \in \{0, 1\}, \; v_j \in \{-1, 1\}, \; j = 1, \ldots, d,$$

is nondegenerate (in fact, $|\det A| = 2^{d2^{d-1}}$). Therefore, it follows from the definition of the function g_j that, for each $v \in Q_d$, $g_j^{(v)} \equiv 0$; hence, $g_j^{(v,1)} \equiv g_j^{(v,2)}$. We conclude that, for each $v \in Q_d$, $y \in (0, \infty)^d$, and $j \in K$,

$$\nu_v^{(1)}(\{z : 0 \leq z_k \leq y_k, \; k \neq j, \; z_j > y_j\}) = \nu_v^{(2)}(\{z : 0 \leq z_k \leq y_k, \; k \neq j, \; z_j > y_j\}).$$

This means that, for each $v \in Q_d$, the measures $\nu_v^{(1)}$ and $\nu_v^{(2)}$ coincide on the set $\{y_j > 0\}$ for each $j \in K$. By definition of the set K we obtain (20), completing the proof.

The conditions for the cancellation property in (19) and its special cases above are somewhat implicit. On the other hand, in the case of one dimension and a single equation, the presence of a sufficiently large atom in the measure ρ already guarantees the cancellation property (see Corollary 2.2 of Jacobsen *et al.* (2009)). A similar phenomenon, described in the following statement, occurs in general.

Corollary 3. *Let $\{\rho_v, \; v \in Q_d\}$ be σ-finite measures on $(0, \infty)^d$, and, for $i = 1, 2$, let $\{\nu_v^{(i)}, \; v \in Q_d\}$ be σ-finite measures on $[0, \infty)^d$. Suppose that K is a nonempty set satisfying (18). Assume further that (15) and (16) hold for this set K.*

Suppose that these measures satisfy the system of equations (17). If, for every $j \in K$, there is $v^{(j)} \in Q_d$ and an atom $x^{(j)} = (x_1^{(j)}, \ldots, x_d^{(j)})$ of $\rho_{v^{(j)}}$ with mass $w^{(j)}$ so large that

$$w^{(j)}(x_j^{(j)})^{\alpha_j} > \int_{x \neq x^{(j)}} x_j^{\alpha_j} \, \rho_{v^{(j)}}(\mathrm{d}x) + \sum_{v \neq v^{(j)}} \int_{(0, \infty)^d} x_j^{\alpha_j} \, \rho_v(\mathrm{d}x),$$

then conclusion (20) holds.

Proof. An application of the triangle inequality shows that in fact the assumptions of the corollary imply (19). Indeed, let $j \in K$. Then, from the assumptions, for any $m_1, \ldots,$ $m_d \in \{0, 1\}$ and $\theta_1, \ldots, \theta_d \in \mathbb{R}$,

$$\left| \sum_{v \in Q_d} \prod_{k=1}^{d} v_k^{m_k} \int_{(0,\infty)^d} x_j^{\alpha_j} \prod_{k=1}^{d} x_k^{\mathrm{i}\theta_k} \rho_v(\mathrm{d}\boldsymbol{x}) \right|$$

$$\geq w^{(j)}(x_j^{(j)})^{\alpha_j} - \int_{\boldsymbol{x} \neq \boldsymbol{x}^{(j)}} x_j^{\alpha_j} \rho_{v^{(j)}}(\mathrm{d}\boldsymbol{x}) - \sum_{v \neq v^{(j)}} \int_{(0,\infty)^d} x_j^{\alpha_j} \rho_v(\mathrm{d}\boldsymbol{x})$$

$$> 0,$$

so none of the expressions in (19) can vanish.

We now combine Theorems 1 and 2, and obtain an inverse regular variation result for multiplicative convolutions. It is a multivariate extension of Theorem 2.3 of Jacobsen *et al.* (2009).

Theorem 3. *Let ρ and v be σ-finite measures on $\mathbb{R}^d$ such that $(v \circledast \rho) \in \mathrm{RV}(\alpha, \mu)$ for some $\alpha > 0$, and*

$$\rho(\{\boldsymbol{x} \colon x_i = 0\}) = 0 \quad \text{for every } i = 1, \ldots, d.$$

Assume that (5) and (6) hold, and that, for each $j = 1, \ldots, d, m_1, \ldots, m_d \in \{0, 1\}$, and $\theta_1, \ldots, \theta_d \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} |x_j|^\alpha \prod_{k=1}^{d} |x_k|^{\mathrm{i}\theta_k} (\mathrm{sgn}(x_k))^{m_k} \rho(\mathrm{d}\boldsymbol{x}) \neq 0. \tag{28}$$

Then the measure v is regularly varying with index α. Moreover, the family of measures $\{\mu_s\}$ in (7) converges vaguely in $\overline{\mathbb{R}}_0^d$, as $s \to \infty$, to a measure μ_* satisfying (8).

Proof. Recalling Theorem 1, we only need to prove that the vague limits $v^{(1)}$ and $v^{(2)}$ of any two subsequences in that theorem coincide. Note that $v^{(1)}$ and $v^{(2)}$ are two solutions to (8), so in order to prove that $v^{(1)} = v^{(2)}$ we translate our problem to the cancellation property situation of Theorem 2. For $v \in Q_d$, define

$$\mathcal{Q}_v = \{\boldsymbol{x} \colon x_j v_j \geq 0 \text{ for each } j = 1, \ldots, d\}$$

and

$$\rho_v(\cdot) = \rho(\{\boldsymbol{x} \in \mathcal{Q}_v \colon (|x_1|, \ldots, |x_d|) \in \cdot\}).$$

Similarly, define two collections of σ-finite measures on $[0, \infty)^d$, $\{v_v^{(i)}, v \in Q_d\}, i = 1, 2$, by restricting the measures $v^{(1)}$ and $v^{(2)}$ to the appropriate quadrants. By assumption, $v^{(1)} \circledast \rho = v^{(2)} \circledast \rho$. By writing out this equality of measures on $\mathbb{R}^d$ for each quadrant of $\mathbb{R}^d$, we immediately see that the sets of measures $\{\rho_v, v \in Q_d\}$ and $\{v_v^{(i)}, v \in Q_d\}, i = 1, 2$, satisfy the system of equations (17).

Let $\alpha_j = \alpha$ for $j = 1, \ldots, d$ and $K = \{1, \ldots, d\}$. Then (18) holds because the measures $v^{(1)}$ and $v^{(2)}$ are vague limits in $\overline{\mathbb{R}}_0^d$, and, hence, place no mass at the origin in $\mathbb{R}^d$. Assumption (15) follows from (5). Assumption (16) follows from the fact that both $v^{(1)}$ and $v^{(2)}$ satisfy (8) and the scaling property of the tail measure μ. Finally, condition (19) follows from (28), and elementary manipulation of the sums and integrals. Therefore, Theorem 2 applies, and $v_v^{(1)} = v_v^{(2)}$ for each $v \in Q_d$. This means that $v^{(1)} = v^{(2)}$.

Remark 3. If the tail measure μ of $\nu \circledast \rho$ satisfies

$$\mu(\{\boldsymbol{x} : x_k = 0 \text{ for each } k \in K\}) = 0 \tag{29}$$

for some nonempty set $K \subseteq \{1, \ldots, d\}$, then every measure μ_* satisfying (8) has the same property:

$$\mu_*(\{\boldsymbol{x} : x_k = 0 \text{ for each } k \in K\}) = 0.$$

Therefore, the measures $\{\nu_v^{(i)}, \ v \in Q_d\}$, $i = 1, 2$, defined in the proof of Theorem 3 satisfy (18), and we can apply Theorem 2 with this smaller set K. In other words, if (29) holds then condition (28) has to be checked only for $j \in K$.

We can extend Theorem 3 to the situation where the measure ρ puts positive mass on the axes. The next corollary follows from the theorem by splitting the space $\mathbb{R}^d$ into subspaces of different dimensions, by setting some of the coordinates equal to 0. We omit the details.

Corollary 4. *Let ρ and ν be σ-finite measures on $\mathbb{R}^d$ such that $(\nu \circledast \rho) \in \mathrm{RV}(\alpha, \mu)$ for some $\alpha > 0$ and (5) and (6) hold. Assume that, for every $I_0 \subset \{1, \ldots, d\}$ such that*

$$\rho(\{\boldsymbol{x} \in \mathbb{R}^d : x_i = 0 \text{ for all } i \in I_0\}) > 0,$$

it holds that, for every I such that $I_0 \cup I = \{1, \ldots, d\}$,

$$\int_{\mathbb{R}^d} |x_j|^\alpha \prod_{k \in I} |x_k|^{\mathrm{i}\theta_k} (\mathrm{sgn}(x_k))^{m_k} \, \rho(\mathrm{d}\boldsymbol{x}) \neq 0$$

for each $j \in I, m_k \in \{0, 1\}$, and $\theta_k \in \mathbb{R}, k \in I$. Then the conclusions of Theorem 3 hold.

3. The inverse problem for weighted sums

In this section we revisit the weighted sums of i.i.d. random vectors introduced in Example 1. We consider the special case of diagonal coefficient matrices. Our goal is to apply the generalized cancellation theory of the previous section to investigate under what conditions on the coefficient matrices regular variation of the weighted sum implies regular variation of the underlying i.i.d. random vectors.

Let $\{\boldsymbol{Z}^{(i)}\}$ be an i.i.d. sequence of $\mathbb{R}^d$-valued random column vectors with a generic element $\boldsymbol{Z}$, and let the $\{\boldsymbol{d}^{(i)}\}$ be deterministic vectors in $\mathbb{R}^d$. The ith coefficient matrix Ψ_i is a diagonal matrix with $\boldsymbol{d}^{(i)}$ on the main diagonal: $\Psi_i = \mathrm{diag}(\{\boldsymbol{d}^{(i)}\})$. The following theorem is the main result of this section. The corresponding result for $d = 1$ and positive weights ψ_i was proved in Jacobsen *et al.* (2009, Theorem 3.3).

Theorem 4. *Assume that the series $\boldsymbol{X} := \sum_{i=1}^{\infty} \Psi_i \boldsymbol{Z}^{(i)}$ converges almost surely (a.s.) and that $\boldsymbol{X} \in \mathrm{RV}(\alpha, \mu_X)$ for some $\alpha > 0$. Suppose that all nonzero vectors $\{\boldsymbol{d}^{(i)}\}$ have nonvanishing coordinates, and that, for some $0 < \delta' < \alpha$,*

$$\sum_{i=1}^{\infty} \|\boldsymbol{d}^{(i)}\|^{\alpha - \delta'} < \infty. \tag{30}$$

If, for all $j = 1, \ldots, d, m_1, \ldots, m_d \in \{0, 1\}$, and $\theta_1, \ldots, \theta_d \in \mathbb{R}$,

$$\sum_{l=1}^{\infty} \left(|d_j^{(l)}|^\alpha \prod_{k=1}^{d} |d_k^{(l)}|^{\mathrm{i}\theta_k} (\mathrm{sgn}(d_k^{(l)}))^{m_k} \right) \neq 0, \tag{31}$$

then $\boldsymbol{Z}$ is regularly varying with index α and (1) holds.

Remark 4. Of course, if some of the nonzero vectors $\{d^{(i)}\}$ have vanishing coordinates, we can use Corollary 4 instead of Theorem 3, and obtain regular variation of the vector Z under a more extensive set of conditions than (31).

Before presenting the proof of Theorem 4 we prove the following lemma.

Lemma 1. *Assume that the conditions of Theorem 4 hold, except that here the vectors d_i, $i = 1, 2, \ldots$, may contain zero components. Then, for any Borel set $A \subset \mathbb{R}^d$ bounded away from the origin and such that A is a μ_X-continuity set,*

$$\mathbb{P}\{s^{-1}X \in A\} \sim \sum_{i=1}^{\infty} \mathbb{P}\{s^{-1}\Psi_i Z \in A\} \quad as\ s \to \infty. \tag{32}$$

Proof. For every $j = 1, \ldots, d$, we may assume that there is $i(j) = 1, 2, \ldots$ such that $d_j^{(i(j))} \neq 0$ for, if this is not the case, we can simply delete the jth coordinate. Define

$$Y^{(j)} = X - \Psi_{i(j)} Z^{(i(j))},$$

and choose $M_j > 0$ such that $\mathbb{P}\{\|Y^{(j)}\| \leq M_j\} > 0$, $j = 1, \ldots, d$. For $s > 0$ and $j = 1, \ldots, d$,

$$\mathbb{P}\{\|X\| > s\} \geq \mathbb{P}\{\|Y^{(j)}\| \leq M_j\}\,\mathbb{P}\{|d_j^{(i(j))}|\,|Z_j| > s + M_j\},$$

and the regular variation of X implies that there exists $C_j > 0$ such that

$$\mathbb{P}\{|Z_j| > s\} \leq C_j\,\mathbb{P}\{\|X\| > s\}, \qquad s > 0,$$

and, therefore, there exists $C > 0$ such that

$$\mathbb{P}\{\|Z\| > s\} \leq C\,\mathbb{P}\{\|X\| > s\} \quad \text{for all } s > 0. \tag{33}$$

Write $X_q = \sum_{i=1}^{q} \Psi_i Z^{(i)}$ and $X^q = X - X_q$ for $q \geq 1$. In the usual notation,

$$A^\epsilon = \{y \in \mathbb{R}_0^d : d(y, A) \leq \epsilon\}, \qquad A_\epsilon = \{y \in A : d(y, A^c) > \epsilon\},$$

we have

$$\mathbb{P}\{s^{-1}X_q \in A_\epsilon\}\,\mathbb{P}\{\|X^q\| \leq \epsilon s\} \leq \mathbb{P}\{s^{-1}X \in A\}$$
$$\leq \mathbb{P}\{s^{-1}X_q \in A^\epsilon\} + \mathbb{P}\{\|X^q\| > \epsilon s\}. \tag{34}$$

Proceeding as in the appendix of Mikosch and Samorodnitsky (2000) and using (33), we obtain

$$\lim_{q \to \infty} \limsup_{s \to \infty} \frac{\mathbb{P}\{\|X^q\| > s\}}{\mathbb{P}\{\|X\| > s\}} = 0.$$

By virtue of (34), it therefore suffices to prove the lemma for X_q instead of X. In what follows, we assume that $q < \infty$ and suppress the dependence of X on q in the notation.

Let $M = \max_{i=1,\ldots,q,\ j=1,\ldots,d} |d_j^{(i)}|$. For $\epsilon > 0$, we have

$$\mathbb{P}\{s^{-1}X \in A_\epsilon\} \leq \sum_{j=1}^{q} \mathbb{P}\{s^{-1}\Psi_j Z \in A\} + \frac{q\,(q-1)}{2}\left(\mathbb{P}\left\{\|Z\| > \frac{s\,\epsilon}{(q-1)M}\right\}\right)^2.$$

Hence, by (33) and regular variation of X,

$$\mu_X(A_\epsilon) \leq \liminf_{s \to \infty} \frac{\mathbb{P}\{s^{-1}X \in A_\epsilon\}}{\mathbb{P}\{\|X\| > s\}} \leq \liminf_{s \to \infty} \frac{\sum_{j=1}^{q} \mathbb{P}\{s^{-1}\Psi_j Z \in A\}}{\mathbb{P}\{\|X\| > s\}}.$$

Letting $\epsilon \downarrow 0$ and using the fact that A is a μ_X-continuity set, we have

$$\mu_X(A) \leq \liminf_{s \to \infty} \frac{\sum_{j=1}^q \mathbb{P}\{s^{-1}\Psi_j \mathbf{Z} \in A\}}{\mathbb{P}\{\|\mathbf{X}\| > s\}},$$

and (32) follows once we show that

$$\mu_X(A) \geq \limsup_{s \to \infty} \frac{\sum_{j=1}^q \mathbb{P}\{s^{-1}\Psi_j \mathbf{Z} \in A\}}{\mathbb{P}\{\|\mathbf{X}\| > s\}}. \tag{35}$$

Let $\delta := \inf\{\|\mathbf{x}\| : \mathbf{x} \in A\} > 0$. For $0 < \epsilon < \delta$, $\mathbb{P}\{s^{-1}\mathbf{X} \in A^\epsilon\}$ is bounded below by

$$\mathbb{P}\left(\bigcup_{i=1}^q \left\{s^{-1}\Psi_i \mathbf{Z}^{(i)} \in A, \left\|\sum_{1 \leq l \neq i \leq q} \Psi_l \mathbf{Z}^{(l)}\right\| \leq s\epsilon\right\}\right)$$

$$\geq \sum_{i=1}^q \mathbb{P}\left\{s^{-1}\Psi_i \mathbf{Z}^{(i)} \in A, \left\|\sum_{1 \leq l \neq i \leq q} \Psi_l \mathbf{Z}^{(l)}\right\| \leq s\epsilon\right\} - \frac{q(q-1)}{2}\left(\mathbb{P}\left\{\|\mathbf{Z}\| \geq \frac{s\delta}{M}\right\}\right)^2$$

$$\geq \sum_{i=1}^q \mathbb{P}\{s^{-1}\Psi_i \mathbf{Z}^{(i)} \in A\} - \frac{q(q-1)}{2}\left(\mathbb{P}\left\{\|\mathbf{Z}\| \geq \frac{s\delta}{M}\right\}\right)^2$$

$$- q(q-1)\mathbb{P}\left\{\|\mathbf{Z}\| \geq \frac{s\delta}{M}\right\}\mathbb{P}\left\{\|\mathbf{Z}\| \geq \frac{s\epsilon}{(q-1)M}\right\}.$$

Thus, by regular variation of $\mathbf{X}$ and (33),

$$\mu_X(A^\epsilon) \geq \limsup_{s \to \infty} \frac{\mathbb{P}\{s^{-1}\mathbf{X} \in A^\epsilon\}}{\mathbb{P}\{\|\mathbf{X}\| > s\}} \geq \limsup_{s \to \infty} \frac{\sum_{j=1}^q \mathbb{P}\{s^{-1}\Psi_j \mathbf{Z} \in A\}}{\mathbb{P}\{\|\mathbf{X}\| > s\}}.$$

Letting $\epsilon \downarrow 0$ and using the μ_X-continuity of A, we obtain the desired relation (35).

Proof of Theorem 4. It follows from Lemma 1 that the measure

$$\mu(\cdot) = \sum_{i=1}^\infty \mathbb{P}\{\Psi_i \mathbf{Z} \in \cdot\} \quad \text{on } \mathbb{R}^d$$

is regularly varying with index α. Note that $\mu = \nu \circledast \rho$, where ν is the law of $\mathbf{Z}$ (a probability measure), and $\rho = \sum_{i=1}^\infty \delta_{\mathbf{d}^{(i)}}$, with the usual notation $\delta_{\mathbf{a}}$ denoting the measure with unit mass at the point $\mathbf{a} \in \mathbb{R}^d$. Note that the conditions of Theorem 3 are satisfied; in particular, (6) holds because the measure ρ has bounded support. Therefore, the conclusion of Theorem 4 follows.

Example 3. Consider the vector AR(1) difference equation $\mathbf{X}_t = \Psi \mathbf{X}_{t-1} + \mathbf{Z}_t$, $i \in \mathbb{Z}$, for an i.i.d. $\mathbb{R}^d$-valued sequence $\{\mathbf{Z}_t\}$ and a matrix $\Psi = \text{diag}(\mathbf{d})$ for some deterministic vector $\mathbf{d} \in \mathbb{D}^d$ with nonvanishing coordinates. A unique stationary causal solution to the AR(1) equation exists if and only if $\max_{i=1,\ldots,d} |d_i| < 1$ and $\mathbf{Z}_1$ has some finite logarithmic moment. A generic element $\mathbf{X}$ of the solution satisfies the relation $\mathbf{X} \stackrel{\mathrm{D}}{=} \sum_{j=0}^\infty \Psi^j \mathbf{Z}_j$. Assume that $\mathbf{X}$ is regularly varying with index $\alpha > 0$. Then (30) is trivially satisfied and (31) reads as follows: for every $j = 1, \ldots, d$, any $m_i \in \{0, 1\}$, and $\theta_i \in \mathbb{R}$, $i = 1, \ldots, d$,

$$|d_j|^\alpha \prod_{k=1}^d |d_k|^{i\theta_k}(\text{sgn}(d_k))^{m_k}\left(1 - |d_j|^\alpha \prod_{k=1}^d |d_k|^{i\theta_k}(\text{sgn}(d_k))^{m_k}\right)^{-1} \neq 0.$$

This condition is always satisfied. Hence, any $\mathbf{Z}_t$ is regularly varying with index $\alpha > 0$.

A special case of the setup of this section is a sum with scalar weights, of the type $X = \sum_{i=1}^{\infty} \psi_i \mathbf{Z}^{(i)}$, where $\{\psi_i\}$ is a sequence of scalars. Applying Theorem 4 with $d_j^{(i)} = \psi_i$, $j = 1, \ldots, d$, for $i = 1, 2, \ldots$, we obtain the following corollary.

Corollary 5. *Let $\alpha > 0$, and suppose that, for some $0 < \delta < \alpha$,*

$$\sum_{i=1}^{\infty} |\psi_i|^{\alpha - \delta} < \infty. \tag{36}$$

Assume that the series $X = \sum_{i=1}^{\infty} \psi_i \mathbf{Z}^{(i)}$ converges a.s. and that X is regularly varying with index α. If, for all $\theta \in \mathbb{R}$,

$$\sum_{j=1}^{\infty} |\psi_j|^{\alpha + \mathrm{i}\theta} \neq 0 \tag{37}$$

$$and \quad \sum_{\{j\,:\,\psi_j > 0\}} \psi_j^{\alpha + \mathrm{i}\theta} \neq \sum_{\{j\,:\,\psi_j < 0\}} |\psi_j|^{\alpha + \mathrm{i}\theta}, \tag{38}$$

then $\mathbf{Z} \in \mathrm{RV}(\alpha, \mu_{\mathbf{Z}})$ and the tail measure $\mu_{\mathbf{Z}}$ satisfies

$$\frac{\mathbb{P}\{s^{-1} X \in \cdot\}}{\mathbb{P}\{|\mathbf{Z}| > s\}} \xrightarrow{\mathrm{v}} \psi_+ \mu_{\mathbf{Z}}(\cdot) + \psi_- \mu_{\mathbf{Z}}(-\cdot) \quad as\ s \to \infty,$$

where

$$\psi_+ = \sum_{\{j\,:\,\psi_j > 0\}} \psi_j^{\alpha} \quad and \quad \psi_- = \sum_{\{j\,:\,\psi_j < 0\}} |\psi_j|^{\alpha}.$$

Remark 5. Corollary 5 has a natural converse statement. Specifically, if either (37) or (38) fails to hold for some real θ, then there is a random vector $\mathbf{Z}$ that is not regularly varying but $X = \sum_{i=1}^{\infty} \psi_i \mathbf{Z}^{(i)}$ is regularly varying. Indeed, suppose, for example, that (37) fails for some real θ_0. We use a construction similar to that in Jacobsen *et al.* (2009). Choose real numbers a and b satisfying $0 < a^2 + b^2 \leq 1$, and define a measure on $(0, \infty)$ by

$$\nu_0(\mathrm{d}x) = [1 + a \cos(\theta_0 \log x) + b \sin(\theta_0 \log x)] \nu^{\alpha}(\mathrm{d}x), \tag{39}$$

where ν^{α} is given in (14). Choose $r > 0$ large enough so that $\nu_0(r, \infty) \leq 1$, and respectively define probability laws on $(0, \infty)$ and $\mathbb{R}$ by

$$\mu_0(B) = \nu_0(B \cap (r, \infty)) + [1 - \nu_0(r, \infty)]\mathbf{1}_B(1) \quad \text{for any Borel set } B$$

and

$$\mu_*(\cdot) = \tfrac{1}{2}\mu_0(\cdot) + \tfrac{1}{2}\mu_0(-\cdot).$$

Obviously, μ_* is not a regularly varying probability measure. Therefore, neither is the random vector $\mathbf{Z} = (Z, 0, \ldots, 0)$ regularly varying, where Z has distribution μ_*.

Since the vector $\mathbf{Z}$ is symmetric, the series $X = \sum_{i=1}^{\infty} \psi_i \mathbf{Z}^{(i)}$ converges a.s. under assumption (36); see Lemma A.3 of Mikosch and Samorodnitsky (2000), and the argument in Jacobsen *et al.* (2009) shows that X is regularly varying with index α.

On the other hand, suppose that (38) fails for some real θ_0. Define ν_0 as in (39), and define another measure on $(0, \infty)$ by

$$\nu_1(\mathrm{d}x) = [1 - a \cos(\theta_0 \log x) - b \sin(\theta_0 \log x)] \nu^{\alpha}(\mathrm{d}x).$$

Convert ν_0 into a probability measure μ_0 as above, and similarly convert ν_1 into a probability measure μ_1. Define a probability measure on $\mathbb{R}$ by

$$\mu_*(\cdot) = \tfrac{1}{2}\mu_0(\cdot) + \tfrac{1}{2}\mu_1(-\cdot).$$

Once again, let $\mathbf{Z} = (Z_1, 0, \ldots, 0)$, where $Z_1 \sim \mu_*$. Then $\mathbf{Z}$ is not regularly varying, nor is the vector

$$\tilde{\mathbf{Z}} = \begin{cases} \mathbf{Z} & \text{if } 0 < \alpha \leq 1, \\ \mathbf{Z} - \mathbb{E}(\mathbf{Z}) & \text{if } \alpha > 1. \end{cases}$$

As before, the series $X = \sum_{i=1}^{\infty} \psi_i \tilde{\mathbf{Z}}^{(i)}$ converges a.s. under assumption (36), and X is regularly varying with index α.

We now give two examples of the situation described in Corollary 5. We say that the coefficients $\psi_1, \psi_2, \ldots$ are α-*regular variation determining* if regular variation of $X = \sum_{i=1}^{\infty} \psi_i \mathbf{Z}^{(i)}$ implies regular variation of $\mathbf{Z}$. In other words, both conditions (37) and (38) must be satisfied.

Example 4. Let $q < \infty$, and assume that $\psi_i = 1$ for $i = 1, \ldots, q$ and $\psi_i = 0$ for $i > q$. By Corollary 5, these coefficients are α-regular variation determining and $\mathbb{P}\{s^{-1}X \in \cdot\} \sim q\mathbb{P}\{s^{-1}\mathbf{Z} \in \cdot\}$ as $s \to \infty$. For $d = 1$ (only in this case is the notion of subexponentiality properly defined), this property is in agreement with the *convolution root property* of subexponential distributions; see Embrechts *et al.* (1979) or Proposition A3.18 of Embrechts *et al.* (1997). Indeed, if X is a positive random variable then regular variation of Z implies subexponentiality.

Example 5. Again, let $q < \infty$ and $\psi_j = 0$ for $j > q$. If, say, $|\psi_1|^{\alpha} > \sum_{j=2}^{q} |\psi_j|^{\alpha}$, then both conditions (37) and (38) are satisfied, and, therefore, the coefficients are α-regular variation determining. This is, of course, the same phenomenon as in Corollary 3. In the special case $q = 2$, if $\psi_1 \neq -\psi_2$ then the coefficients are α-regular variation determining. On the other hand, if $\psi_1 = -\psi_2$ then condition (38) fails, and the coefficients are not α-regular variation determining. This means that regular variation of $X = \mathbf{Z}_1 - \mathbf{Z}_2$ does not necessarily imply regular variation of $\mathbf{Z}$.

4. The inverse problem for products

We now apply the generalized cancellation theory to Example 2, concentrating on the case of multiplication by a random diagonal matrix. Specifically, let $\mathbf{A} = \text{diag}(A_1, \ldots, A_d)$ for some random variables $\{A_i, \; i = 1, \ldots, d\}$. The following theorem is an easy application of Theorem 3.

Theorem 5. *Assume that* $\mathbb{P}\{A_j = 0\} = 0$ *for* $j = 1, \ldots, d$. *Let* $\mathbf{A} = \text{diag}(A_1, \ldots, A_d)$. *Let* $\mathbf{Z}$ *be a d-dimensional random vector independent of $\mathbf{A}$ such that* $X = \mathbf{A}\mathbf{Z}$ *is regularly varying with index $\alpha > 0$. If* $\mathbb{E}\|\mathbf{A}\|^{\alpha+\delta} < \infty$ *for some* $\delta > 0$, *and, for each* $j = 1, \ldots, d$, $m_1, \ldots, m_d \in \{0, 1\}$, *and* $\theta_1, \ldots, \theta_d \in \mathbb{R}$,

$$\mathbb{E}\left[|A_j|^{\alpha} \prod_{k=1}^{d} (|A_k|^{i\theta_k} (\text{sgn}(A_k))^{m_k}) \right] \neq 0,$$

then $\mathbf{Z}$ is regularly varying with index $\alpha > 0$. Also, (3) holds.

A special case is multiplication of a random vector by an independent scalar random variable, corresponding to $A_1 = \cdots = A_d = A$ for some random variable A. The following corollary restates Theorem 5 in this special case.

Corollary 6. *Let A be a random variable independent of some d-dimensional random vector* $\mathbf{Z}$ *such that* $\mathbf{X} = A\mathbf{Z}$ *is regularly varying with index* $\alpha > 0$. *If* $\mathbb{E}|A|^{\alpha+\delta} < \infty$ *for some* $\delta > 0$ *and, for* $\theta \in \mathbb{R}$,

$$\mathbb{E}|A|^{\alpha+\mathrm{i}\theta} \neq 0 \tag{40}$$

$$\text{and} \quad \mathbb{E}A_+^{\alpha+\mathrm{i}\theta} \neq \mathbb{E}A_-^{\alpha+\mathrm{i}\theta}, \tag{41}$$

then $\mathbf{Z}$ *is regularly varying with index* $\alpha > 0$. *Also, the tail measure* $\mu_{\mathbf{Z}}$ *of* $\mathbf{Z}$ *satisfies*

$$\frac{\mathbb{P}\{s^{-1}\mathbf{X} \in \cdot\}}{\mathbb{P}\{|\mathbf{Z}| > s\}} \xrightarrow{\text{v}} \mathbb{E}A_+^{\alpha}\,\mu_{\mathbf{Z}}(\cdot) + \mathbb{E}A_-^{\alpha}\,\mu_{\mathbf{Z}}(-\cdot) \quad as\ s \to \infty,$$

where $A_+ = \max\{A, 0\}$ *and* $A_- = \max\{-A, 0\}$.

Using terminology similar to that of Section 3, we say that a random variable A is α-*regular variation determining* if regular variation with index α of $\mathbf{X} = A\mathbf{Z}$ implies regular variation of $\mathbf{Z}$. Corollary 6 shows that if A satisfies both conditions (40) and (41), then A is α-regular variation determining. On the other hand, a construction similar to that in Remark 5 shows that, if one of the conditions (40) and (41) fails, then one can construct an example of a random vector $\mathbf{Z}$ that is not regularly varying but $\mathbf{X} = A\mathbf{Z}$ is regularly varying with index α. Therefore, conditions (40) and (41) are necessary and sufficient for A to be α-regular variation determining.

Jacobsen *et al.* (2009) proved this result for positive A in their Theorem 4.2. They gave various examples of distributions on $(0, \infty)$ which are α-regular variation determining, including the gamma, log-normal, and Pareto distributions, and the distribution of the powers of the absolute value of a symmetric normal random variable, of the absolute values of a Cauchy random variable (for $\alpha < 1$), and any positive random variable whose log-transform is infinitely divisible. The condition in (41) rules out a whole class of important distributions: no member of the class of symmetric distributions is α-regular variation determining. Even nonsymmetric distributions with $\mathbb{E}A_+^{\alpha} = \mathbb{E}A_-^{\alpha}$ are not α-regular variation determining.

As a further example, consider a uniform random variable $A \sim U(a, b)$ for $a < b$. If $a = -b$ then A cannot be α-regular variation determining since it has a symmetric distribution. On the other hand, an elementary calculation shows that in all other cases both conditions (40) and (41) hold. Therefore, the only non-α-regular variation determining uniform random variables are the symmetric ones.

In financial time series analysis, models for returns are often of the form $X_t = A_t Z_t$, where $\{A_t\}$ is some volatility sequence, $\{Z_t\}$ is an i.i.d. multiplicative noise sequence such that A_t and Z_t are independent for every t, and $\{X_t\}$ constitutes a strictly stationary sequence. In most parts of the literature it is assumed that the volatility A_t is nonnegative. It is often assumed that X_t is heavy tailed, e.g. regularly varying with some index $\alpha > 0$; see Davis and Mikosch (2009a, 2009b). Note that A_t and Z_t are not observable; it depends on the model as to which of the variables A_t or Z_t one assigns regular variation. For example, in the case of a GARCH process $\{X_t\}$, $\{A_t\}$ is regularly varying with index $\alpha > 0$ and the i.i.d. noise $\{Z_t\}$ has lighter tails and is symmetric. On the other hand, if one only assumes that X_t is regularly varying with index α and $\mathbb{E}|Z|^{\alpha+\delta} < \infty$ for some $\delta > 0$ and Z is symmetric, one cannot conclude that A_t is regularly varying.

5. Nondiagonal matrices

The (direct) statements of Examples 1 and 2 of Section 1 deal with transformations of regularly varying random vectors involving matrices that do not have to be diagonal matrices.

On the other hand, all the converse statements of Sections 3 and 4 deal only with diagonal matrices. Generally, we do not know how to solve inverse problems involving nondiagonal matrices. In this section we describe one of the very few 'nondiagonal' situations where we can prove a converse statement. We restrict attention to the case of finite weighted sums and square matrices.

Theorem 6. *Let* $X = \sum_{j=1}^{q} \Psi_j Z_j$, *where* Z_j, $j = 1, \ldots, q$, *are i.i.d.* $\mathbb{R}^d$-*valued random vectors and* Ψ_j, $j = 1, \ldots, q$, *deterministic* $d \times d$ *matrices. Assume that* $X \in \mathrm{RV}(\alpha, \mu_X)$ *for some* $\alpha > 0$. *If all the matrices* Ψ_j, $j = 1, \ldots, q$, *are invertible, and*

$$(\gamma(\Psi_1))^{\alpha} > \sum_{j=2}^{q} \|\Psi_j\|^{\alpha}, \tag{42}$$

where $\gamma(\Psi_1) = \min_{z \in \mathbb{S}^{d-1}} |\Psi_1 z|$ *and* $\|\Psi_j\|$ *is the operator norm of* Ψ_j, $j = 1, \ldots, q$, *then* $Z \in \mathrm{RV}(\alpha, \mu_Z)$ *and* μ_Z *satisfies (1).*

Proof. An argument similar to that used in the proof of Lemma 1 shows that, under the assumptions of the theorem, a finite version of (32) holds: for any Borel set $A \subset \mathbb{R}^d$ bounded away from the origin such that A is a continuity set with respect to the tail measure μ_X,

$$\mathbb{P}\{s^{-1} X \in A\} \sim \sum_{i=1}^{q} \mathbb{P}\{s^{-1} \Psi_i Z \in A\} \quad \text{as } s \to \infty.$$

This allows us to proceed as in Theorem 1 to see that the family of measures

$$\left\{ \frac{\mathbb{P}\{s^{-1} Z \in \cdot\}}{\mathbb{P}\{|X| > s\}} \right\}_{s \geq 1} \tag{43}$$

is vaguely tight in $\overline{\mathbb{R}}_0^m$, and any vague limit μ_* of a subsequence of this family satisfies

$$\mu_X = \sum_{j=1}^{q} \mu_* \circ \Psi_j^{-1}. \tag{44}$$

Let $T_j = \Psi_j^{-1} \Psi_1$, $j = 2, \ldots, q$. Then, by (44), for any measurable set $B \subset \mathbb{R}^d$ bounded away from 0,

$$\mu_*(B) = \mu_X(\Psi_1 B) - \sum_{j=2}^{q} \mu_*(T_j B). \tag{45}$$

Replacing B by $T_j B$ for $j = 2, \ldots, q$ and iterating (45), we obtain, for $n = 1, 2, \ldots$,

$$
\begin{aligned}
\mu_*(B) &= \mu_X(\Psi_1 B) - \sum_{j=2}^{q} \mu_X(\Psi_1 T_j B) + \sum_{j_1=2}^{q} \sum_{j_2=2}^{q} \mu_*(T_{j_2} T_{j_1} B) \\
&= \sum_{k=0}^{n} (-1)^k \sum_{j_1=2}^{q} \cdots \sum_{j_k=2}^{q} \mu_X(\Psi_1 T_{j_k} \cdots T_{j_1} B) \\
&\quad + (-1)^{n+1} \sum_{j_1=2}^{q} \cdots \sum_{j_{n+1}=2}^{q} \mu_*(T_{j_{n+1}} \cdots T_{j_1} B).
\end{aligned}
\tag{46}
$$

Clearly, for every $n \geq 1$ and $j_1, \ldots, j_{n+1} = 2, \ldots, q$,

$$\inf_{z \in T_{j_{n+1}} \cdots T_{j_1} B} |z| \geq \inf_{z \in B} |z| \, (\gamma(\Psi_1))^{n+1} \prod_{k=1}^{n+1} \|\Psi_{j_k}\|^{-1}. \tag{47}$$

Furthermore, it follows from (44) that, for some $c > 0$,

$$\mu_*(\{z \in \mathbb{R}^d : |z| > s\}) \leq c\, s^{-\alpha}, \qquad s > 0.$$

Therefore, we conclude from (47) and (42) that, as $n \to \infty$,

$$\sum_{j_1=2}^{q} \cdots \sum_{j_{n+1}=2}^{q} \mu_*(T_{j_{n+1}} \cdots T_{j_1} B)$$

$$\leq c \left(\inf_{z \in B} |z| \right)^{-\alpha} \sum_{j_1=2}^{q} \cdots \sum_{j_{n+1}=2}^{q} \left((\gamma(\Psi_1))^{n+1} \prod_{k=1}^{n+1} \|\Psi_{j_k}\|^{-1} \right)^{-\alpha}$$

$$= c \left(\inf_{z \in B} |z| \right)^{-\alpha} (\gamma(\Psi_1))^{-\alpha(n+1)} \left(\sum_{j=2}^{q} \|\Psi_j\|^{\alpha} \right)^{n+1}$$

$$\to 0.$$

Thus, by virtue of (46),

$$\mu_*(B) = \lim_{n \to \infty} \sum_{k=0}^{n} (-1)^k \sum_{j_1=2}^{q} \cdots \sum_{j_k=2}^{q} \mu_X(T_{j_k} \cdots T_{j_1} B).$$

This means that μ_* is uniquely determined by the measure μ_X. Hence, all the vague limits of subsequences of (43) coincide. Therefore, $\mathbf{Z} \in \mathrm{RV}(\alpha, \mu_{\mathbf{Z}})$ and (1) holds.

Remark 6. In the special case of diagonal matrices $\{\Psi_j\}$ with identical elements on the diagonals, the conditions in Theorem 6 coincide with those in Example 5.

Remark 7. The conditions in Theorem 6 are weakened slightly by assuming, instead of (42), that

$$(\gamma(A\Psi_1))^{\alpha} > \sum_{j=2}^{q} \|A\Psi_j\|^{\alpha} \tag{48}$$

for some invertible matrix A. Indeed, regular variation of X implies regular variation of the vector AX, and regular variation of $A\mathbf{Z}$ implies regular variation of $\mathbf{Z}$. It is not difficult to construct examples where (48) holds but (42) fails.

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