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VALUATIONS OF NEAR POLYGONS

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Abstract. We introduce the notion of valuation of a dense near polygon. The valuations of a dense near polygon F describe the possible relations between a point of a dense near polygon $\mathcal{S}$ and any geodetically closed sub near polygon of $\mathcal{S}$ isomorphic to F. Several nice properties of valuations are given and several classes of these objects are defined. Valuations are an important tool for classifying dense near polygons.

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1. Introduction. A simple undirected connected graph Γ without loops is called a *near* $2d$*-gon* ([**9**]) if it has diameter d and if for every vertex x and every maximal clique M, there exists a unique vertex x' in M nearest to x. If Γ is a near polygon, then the point-line incidence structure $\mathcal{S}$ with points the vertices of Γ, with lines the maximal cliques of Γ and with natural incidence is a partial linear space; that is, every two points of $\mathcal{S}$ are incident with at most one line. The partial linear space $\mathcal{S}$ is also called a near polygon. The graph Γ can easily be retrieved from $\mathcal{S}$: it is the point graph or collinearity graph of $\mathcal{S}$. In the sequel we shall always adopt the geometrical point of view and interpret distances $\mathrm{d}(\cdot,\cdot)$ in $\mathcal{S}$ as if they were measured in Γ. From the geometrical point of view a near 0-gon is a point and a near 2-gon is a line.

If X_1 and X_2 are two sets of points, then $\mathrm{d}(X_1, X_2)$ denotes the minimal distance between a point of X_1 and a point of X_2. If $X_1 = \{x\}$, then we also write $\mathrm{d}(x, X_2)$ instead of $\mathrm{d}(\{x\}, X_2)$. For every $i \in \mathbb{N}$, $\Gamma_i(X_1)$ denotes the set of all points y for which $\mathrm{d}(y, X_1) = i$. If $X_1 = \{x\}$, we also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

A near $2d$-gon, $d \geq 2$, is called a *generalized* $2d$*-gon* ([**11**]) if $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = 1$ for every $i \in \{1, \dots, d-1\}$ and every two points x and y at distance i from each other. A generalized $2d$-gon is called *degenerate* if it does not contain ordinary $2d$-gons as subgeometries, or equivalently, if it contains a point which has distance at most $d-1$ from any other point. The near quadrangles are precisely the generalized quadrangles (GQ's, [**7**]). A degenerate generalized quadrangle consists of a number of lines through a point.

A nonempty set X of points in $\mathcal{S}$ is called a *subspace* if every line meeting X in at least two points is completely contained in X. A subspace X is called *geodetically closed* if every point on a shortest path between two points of X is also contained in X. Given a subspace X, we can define a subgeometry $\mathcal{S}_X$ of $\mathcal{S}$ by considering only those points and lines of $\mathcal{S}$ that are completely contained in X. If X is geodetically closed, then $\mathcal{S}_X$ clearly is a sub near polygon of $\mathcal{S}$. If $\mathcal{S}_X$ is a nondegenerate generalized quadrangle, then X and often also $\mathcal{S}_X$ will be called a *quad*. If $X_1, \dots, X_k$ are nonempty sets of points, then $\mathcal{C}(X_1, \dots, X_k)$ denotes the minimal geodetically closed sub near polygon through

$X_1 \cup \cdots \cup X_k$; that is the intersection of all geodetically closed sub near polygons through $X_1 \cup \cdots \cup X_k$. If x and y are two different points of $\mathcal{S}$, then $\mathcal{C}(\{x, y\})$ is also denoted by $\mathcal{C}(x, y)$.

A near polygon is said to have *order* (s, t) if every line is incident with exactly $s + 1$ points and if every point is incident with exactly $t + 1$ lines. A near $2d$-gon, $d \geq 2$, is called *regular* if it has an order (s, t) and if there exist constants t_i, $i \in \{0, \ldots, d\}$, such that for any two points x and y at distance i there are precisely $t_i + 1$ neighbours of y at distance $i - 1$ from x. Then $t_0 = -1$, $t_1 = 0$ and $t_d = t$.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. Dense near polygons satisfy several nice properties; see [**2**] for an overview. We mention some properties that are needed later.

Proposition 1.1. (i) (Lemma 19 of [**2**]). *Every point of a dense near polygon $\mathcal{S}$ is incident with the same number of lines.*

(ii) (Theorem 4 of [**2**]). *If x and y are two points of a dense near polygon, then $\mathcal{C}(x, y)$ is the unique geodetically closed sub near $[2 \cdot d(x, y)]$-gon through x and y. Hence, if x and y are two points at distance 2 in a dense near polygon, then these points are contained in a unique quad.*

(iii) ([**2**]) *Let $\mathcal{S}$ be a dense near $2d$-gon, $d \geq 1$, let F be a geodetically closed sub near $2i$-gon, $i \in \{0, \ldots, d - 1\}$, of $\mathcal{S}$.*

• *If L is a line which intersects F in a point, then $\mathcal{C}(F, L)$ is a geodetically closed sub near $2(i + 1)$-gon.*

• *If x is a point at distance 1 from F, then x is collinear with a unique point x' of F and $d(x, y) = 1 + d(x', y)$ for every point y of F.*

(iv) (Corollary, [**2**, p. 156]) *If x is a point of a dense near $2d$-gon, then the subgraph of Γ induced by $\Gamma_d(x)$ is connected.*

Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two near polygons. A new near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ can be derived from $\mathcal{S}_1$ and $\mathcal{S}_2$:

(1) $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$;

(2) $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$;

(3) the point (x, y) of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(z, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = z$ and $y \; \mathrm{I}_2 \; L$, the point (x, y) of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(M, u) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $x \; \mathrm{I}_1 \; M$ and $y = u$.

The near polygon $\mathcal{S}$ is called the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$, and is denoted by $\mathcal{S}_1 \times \mathcal{S}_2$. If $\mathcal{S}_i$, $i \in \{1, 2\}$, is a near $2n_i$-gon, then the direct product $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ is a near $2(n_1 + n_2)$-gon. Since $\mathcal{S}_1 \times \mathcal{S}_2 \cong \mathcal{S}_2 \times \mathcal{S}_1$ and $(\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{S}_3 \cong \mathcal{S}_1 \times (\mathcal{S}_2 \times \mathcal{S}_3)$, also the direct product of $k \geq 3$ near polygons $\mathcal{S}_1, \ldots, \mathcal{S}_k$ is well defined.

Proposition 1.2. (Theorem 1 of [**2**]) *Suppose $\mathcal{S}$ is a near polygon with the property that every two points at distance 2 have at least two common neighbours. If $k \geq 2$ different line sizes occur in $\mathcal{S}$, then $\mathcal{S}$ is isomorphic to a direct product of k near polygons, each of which has constant line size.*

Corollary 1.3. *If a dense near polygon $\mathcal{S}$ has lines of size $s + 1$, then $\mathcal{S}$ has a partition in isomorphic geodetically closed sub near polygons of order (s, t') for some $t' \geq 0$.*

2. Valuations.

2.1. Motivation. Let F_1 and F_2 denote two geodetically closed sub near polygons of a dense near polygon $\mathcal{S}$ and put $d_i := \mathrm{diam}(F_i)$, $i \in \{1, 2\}$. Depending on how the distances $\mathrm{d}(x_1, x_2)$ behave when x_1 and x_2 range over all elements of F_1 and F_2, respectively, we shall be able to say that F_1 has a "certain position" with respect to F_2. For instance, in the case $(d_1, d_2) = (1, 1)$, we can distinguish two possible line-line relations; see Proposition 2.1; in the case $(d_1, d_2) = (0, 2)$, we can distinguish two possible point-quad relations; see Proposition 2.2; in the case $(d_1, d_2) = (1, 2)$ we can distinguish five possible line-quad relations; see Proposition 2.3.

Definitions. Let Q be a generalized quadrangle. An *ovoid* of Q is a set of points of Q meeting each line of Q in exactly one point. More generally, an ovoid of a partial linear space is a set of points meeting each line in a unique point. A *fan of ovoids* of Q is a set of ovoids of Q partitioning the point set of Q. A *rosette of ovoids* of Q is a set of ovoids of Q through a common point x which partitions the set of points at distance 2 from x.

Proposition 2.1. (The line-line relations, Lemma 1 of [**2**]) *Let K and L denote two lines of a near polygon $\mathcal{S}$. Then precisely one of the following cases occurs.*

(i) *There exist unique points $k_0 \in K$ and $l_0 \in L$ such that $d(k, l) = d(k, k_0) + d(k_0, l_0) + d(l_0, l)$, for all points $k \in K$ and $l \in L$.*

(ii) *For every point $k \in K$ there exists a unique point $l \in L$ such that $d(k, l) = d(K, L)$.* In this case K and L are called *parallel.*

Proposition 2.2. (The point-quad relations, Proposition 2.6 of [**9**]) *Let x be a point and Q a quad of a dense near polygon $\mathcal{S}$. Then precisely one of the following cases occurs.*

(i) *Q contains a unique point $\pi_Q(x)$ nearest to x and for every point y of Q, $d(x, y) = d(x, \pi_Q(x)) + d(\pi_Q(x), y)$.* In this case, x is called *classical with respect to Q.*

(ii) *The set of points in Q nearest to x forms an ovoid O_x of Q.* In this case, x is called *ovoidal with respect to Q.*

For every quad Q of a dense near polygon and every $i \in \mathbb{N}$, let $X_i(Q)$ denote the set of points x at distance i from Q, $X_{i,C}(Q)$ the set of points of $X_i(Q)$ that are classical with respect to Q and $X_{i,O}(Q)$ the set of points $X_i(Q)$ that are ovoidal with respect to Q. If no confusion is possible, we also write X_i, $X_{i,C}$ and $X_{i,O}$ instead of $X_i(Q)$, $X_{i,C}(Q)$ and $X_{i,O}(Q)$.

Proposition 2.3. (The line-quad relations, Lemma (3)–(10) of [**2**]) *Let (L, Q) be a line-quad pair of a dense near polygon $\mathcal{S}$ and put $i := d(L, Q)$. Then one of the following cases occurs.*

(i) *$L \subseteq X_{i,C}$. In this case, $\pi_Q(L) := \{\pi_Q(x) \mid x \in L\}$ is a line of Q parallel with L.*

(ii) *$L \subseteq X_{i,O}$. In this case, the ovoids O_x, $x \in L$, define a fan of ovoids of Q.*

(iii) *L contains a unique point of $X_{i,C}$ and the remaining points of L belong to $X_{i+1,C}$. In this case, all points $\pi_Q(x)$, $x \in L$, are equal.*

(iv) *L contains a unique point u of $X_{i,C}$ and the remaining points of L belong to $X_{i+1,O}$. In this case, the ovoids O_x, $x \in L \setminus \{u\}$, define a rosette of ovoids through the point $\pi_Q(u)$.*

(v) *L contains a unique point of $X_{i,O}$ and the remaining points of L belong to $X_{i+1,O}$. In this case, all ovoids O_x, $x \in L$, are equal.*

The possible point-quad and line-quad relations were a very important tool in the classification of certain dense near polygons, see e.g. [**1**] and [**4**]. In this paper we shall study the possible relations between a point x and a geodetically closed sub near 2δ-gon F, $\delta \geq 3$. The possible relations are described by the *valuations* of F. Also valuations are an important tool in the classification of near polygons. These objects will be used in [**6**] to classify all dense near octagons with three points per line.

2.2. Definition and elementary properties.

DEFINITION. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2n$-gon. A function f from $\mathcal{P}$ to $\mathbb{N}$ is called a *valuation* if it satisfies the following properties (we call $f(x)$ the *value* of x):

(V_1) there exists at least one point with value 0;

(V_2) every line L of $\mathcal{S}$ contains a unique point x_L with smallest value and $f(x) = f(x_L) + 1$ for every point x of L different from x_L;

(V_3) every point x of $\mathcal{S}$ is contained in a geodetically closed sub near polygon F_x that satisfies the following properties:

- $f(y) \leq f(x)$ for every point y of F_x,
- every point z of $\mathcal{S}$ that is collinear with a point y of F_x and which satisfies $f(z) = f(y) - 1$ also belongs to F_x.

PROPOSITION 2.4. *Let f be a valuation of a dense near $2n$-gon $\mathcal{S}$. Then the following statements hold:*

(i) *for every two points x and y of $\mathcal{S}$, $|f(x) - f(y)| \leq d(x, y)$;*

(ii) *for every point x of $\mathcal{S}$, $f(x) \in \{0, \ldots, n\}$;*

(iii) *if x is a point with value* 0 *and if y is collinear with x, then $f(y) = 1$.*

Proof. (i) This follows from property (V_2).

(ii) This follows from (i) and property (V_1).

(iii) If y were equal to 0, then the line xy cannot contain a unique point with smallest value. □

PROPOSITION 2.5. *Let f be a valuation of a dense near polygon $\mathcal{S}$. Then through every point x of $\mathcal{S}$, there exists exactly one geodetically closed sub near polygon F_x satisfying property* (V_3).

Proof. By [**2**], a geodetically closed sub near polygon F through x is completely determined by the set of lines through x contained in F. Now, by properties (V_2) and (V_3), a line through x belongs to F_x if and only if it contains a point with value $f(x) - 1$. This proves that there exists exactly one geodetically closed sub near polygon F_x satisfying property (V_3). □

The following proposition says that the valuations of a dense near polygon F describe the possible relations between a point of a near polygon $\mathcal{S}$ and any geodetically closed sub near polygon of $\mathcal{S}$ isomorphic to F. The valuations of F give information on how F can be embedded in a larger dense near polygon. That is the reason why these objects are important for classifying near polygons.

PROPOSITION 2.6. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2n$-gon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a geodetically closed sub near 2δ-gon of $\mathcal{S}$. For every point x of $\mathcal{S}$ and for every point y of F, we define $f_x(y) := d(x, y) - d(x, \mathcal{P}')$. Then $f_x : \mathcal{P}' \to \mathbb{N}$ is a valuation of F, for every point x of $\mathcal{S}$.*

Proof. Let y be a point of F such that $d(x, y) = d(x, \mathcal{P}')$. Then $f_x(y) = 0$. Because every line of F contains a unique point nearest to x, also (V_2) is satisfied. For every $y \in F$, we define $F_y := \mathcal{C}(x, y) \cap F$. If $z \in F_y$, then $f_x(z) = d(x, z) - d(x, \mathcal{P}') \leq d(x, y) - d(x, \mathcal{P}') = f_x(y)$. If u is a point of F_y and if u' is a neighbour of u in F with value $f_x(u) - 1$, then $d(x, u') = d(x, u) - 1$, implying that $u' \in \mathcal{C}(x, u) \cap F \subseteq \mathcal{C}(x, y) \cap F = F_y$. This shows that also (V_3) is satisfied. □

We shall now generalize Proposition 2.6, but first we need the following lemma.

LEMMA 2.7. *Let $\mathcal{S}$ be a dense near polygon and let F be a sub near polygon of $\mathcal{S}$ satisfying the following conditions:*

- *F is a subspace of $\mathcal{S}$,*
- *$d_F(x, y) = d_{\mathcal{S}}(x, y)$, for all points x and y of F.*

Then, for every geodetically closed subspace G of $\mathcal{S}$, either $G \cap F = \emptyset$ or $G \cap F$ is a geodetically closed sub near polygon of F.

Proof. Suppose that $G \cap F \neq \emptyset$. As intersection of two subspaces, $G \cap F$ is again a subspace. Let $a, b \in G \cap F$ and let c be a point of F collinear with b such that $d_F(a, c) = d_F(a, b) - 1$. Then $d_{\mathcal{S}}(a, c) = d_{\mathcal{S}}(a, b) - 1$ and so $c \in \mathcal{C}(a, b) \subseteq G$. Hence, $c \in G \cap F$. This proves that $G \cap F$ is geodetically closed. □

PROPOSITION 2.8. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2n$-gon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a sub near 2δ-gon of $\mathcal{S}$ that has the following properties:*

- *F is a dense near polygon,*
- *F is a subspace of $\mathcal{S}$,*
- *if x and y are two points of F, then $d_F(x, y) = d_{\mathcal{S}}(x, y)$.*

For every point x of $\mathcal{S}$ and every point y of F, we define $f_x(y) := d_{\mathcal{S}}(x, y) - d_{\mathcal{S}}(x, \mathcal{P}')$. Then $f_x : \mathcal{P}' \to \mathbb{N}$ is a valuation of F, for every point x of $\mathcal{S}$.

Proof. By Lemma 2.7, $\mathcal{C}(x, y) \cap F$ is a geodetically closed subspace of F for every point x of $\mathcal{S}$ and every point y of F. The proof is now completely similar to the proof of Proposition 2.6. □

Valuations of dense near 0-gons and dense near 2-gons are trivial objects. There is a unique point with value 0 and all other points in the case of near 2-gons have value 1. In the following paragraph we shall show that there are two possible types of valuations in dense generalized quadrangles, corresponding with the two possible point-quad relations given in Proposition 2.2.

2.3. Classical and ovoidal valuations.

PROPOSITION 2.9. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2n$-gon.*
(i) *If y is a point of $\mathcal{S}$, then $f_y : \mathcal{P} \to \mathbb{N};\ x \mapsto d(x, y)$ is a valuation of $\mathcal{S}$.*
(ii) *If $\mathcal{O}$ is an ovoid of $\mathcal{S}$, then $f_{\mathcal{O}} : \mathcal{P} \to \mathbb{N};\ x \mapsto d(x, \mathcal{O})$ is a valuation of $\mathcal{S}$.*

Proof. In both cases, (V_1) and (V_2) are satisfied. In case (i), we put $F_x := \mathcal{C}(x, y)$. In case (ii), we put $F_x := \{x\}$ if $x \in \mathcal{O}$ and $F_x := \mathcal{S}$ otherwise. For these choices of F_x, also (V_3) holds. □

DEFINITION. A valuation of $\mathcal{S}$ is *classical* if it is obtained as in (i) of Proposition 2.9; it is *ovoidal* if it is obtained as in (ii). Classical and ovoidal valuations can be characterized as follows.

Proposition 2.10. *Let f be a valuation of a dense near $2n$-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $n \geq 1$. Then*

(i) $\max\{f(u)|u \in \mathcal{P}\} \leq n$ *with equality if and only if f is classical;*
(ii) $\max\{f(u)|u \in \mathcal{P}\} \geq 1$ *with equality if and only if f is ovoidal.*

Proof. Obviously, the inequalities above hold. If f is a classical valuation, then obviously $\max\{f(u)|u \in \mathcal{P}\} = n$. If f is ovoidal, then $\max\{f(u)|u \in \mathcal{P}\} = 1$.

(i) Suppose that $\max\{f(u)|u \in \mathcal{P}\} = n$. Let x be a point of $\mathcal{S}$ with value 0 and let y be a point with value n. By Proposition 2.4, $\mathrm{d}(x, y) = n$. Let y' be an arbitrary point of $\Gamma_n(x) \cap \Gamma_1(y)$ and let y'' denote the unique point of the line yy' at distance $n - 1$ from x. By Proposition 2.4, it follows that $f(y'') = f(y'') - f(x) \leq n - 1$ and that $f(y'') = f(y) + f(y'') - f(y) \geq n - 1$. Hence, $f(y'') = n - 1$ and by property (V_2), it then follows that $f(y') = n$, so that every point of $\Gamma_n(x) \cap \Gamma_1(y)$ has value n. By the connectedness of $\Gamma_n(x)$, see Proposition 1.1 (iv), it then follows that every point of $\Gamma_n(x)$ has value n. Now, let z be an arbitrary point of $\mathcal{S}$. Then, by [**2**], there exists a path of length $n - \mathrm{d}(x, z)$ between z and a point z' of $\Gamma_n(x)$. From $\mathrm{d}(x, z) \geq |f(z) - f(x)| = f(z)$ and $n - f(z) = |f(z') - f(z)| \leq \mathrm{d}(z, z') = n - \mathrm{d}(x, z)$, it follows that $f(z) = \mathrm{d}(x, z)$. This proves that f is classical.

(ii) Suppose now that $\max\{f(x)|x \in \mathcal{P}\} = 1$. By property (V_2), every line of $\mathcal{S}$ contains a unique point with value 0. Hence the points with value 0 determine an ovoid of $\mathcal{S}$ and f is ovoidal. □

Corollary 2.11. *Every valuation of a dense generalized quadrangle is either classical or ovoidal.*

Any valuation of a dense near polygon $\mathcal{S}$ induces a valuation in every geodetically closed sub near polygon of $\mathcal{S}$.

Proposition 2.12. *Let $\mathcal{S}$ be a dense near polygon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a sub near polygon of $\mathcal{S}$ that has the following properties:*

- *F is a dense near polygon,*
- *F is a subspace of $\mathcal{S}$,*
- *if x and y are two points of F in $\mathcal{S}$, then $d_F(x, y) = d_{\mathcal{S}}(x, y)$.*

Let f denote a valuation of $\mathcal{S}$ and put $m := \min\{f(x)|x \in \mathcal{P}'\}$. Then the map $f_F : \mathcal{P}' \to \mathbb{N}; x \mapsto f(x) - m$ is a valuation of F.

Proof. For every point x of $\mathcal{S}$, let F_x denote the unique geodetically closed sub near polygon of $\mathcal{S}$ for which (V_3) holds with respect to the valuation f. By Lemma 2.7, $F_x \cap F$ is a geodetically closed sub near polygon of F for every point of x of F. Clearly, f_F satisfies properties (V_1) and (V_2). The map f_F also satisfies (V_3) if for every point x of F one takes $F'_x := F_x \cap F$ as a geodetically closed sub near polygon through x. □

Definition. We call f_F an *induced valuation*.

Proposition 2.13. *Let f be a valuation of a dense near polygon $\mathcal{S}$.*

(i) *If every induced quad valuation is classical, then the valuation f itself is classical.*
(ii) *If every induced quad valuation is ovoidal, then the valuation f itself is ovoidal.*

Proof. (i) Suppose that f is a nonclassical valuation of $\mathcal{S}$. Let x denote an arbitrary point with value 0 and let i be the smallest nonnegative integer for which there exists a point y satisfying $i = \mathrm{d}(x, y) \neq f(y)$. Obviously, $i \geq 2$. Choose points $y' \in \Gamma_1(y) \cap \Gamma_{i-1}(x)$ and $y'' \in \Gamma_1(y') \cap \Gamma_{i-2}(x)$. Then $f(y'') = i - 2$, $f(y') = i - 1$ and $f(y) \in \{i - 1, i - 2\}$. Every point of Q collinear with y'' has distance $i - 1$ from x

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and hence has value $i-1$. Since the valuation induced in $\mathcal{C}(y, y'')$ is classical, y'' is the unique point of $\mathcal{C}(y, y'')$ with smallest value and $f(y) = f(y'') + \mathrm{d}(y'', y) = i - 2 + 2 = i$, a contradiction.

(ii) Suppose that f is a nonovoidal valuation of $\mathcal{S}$. Let x denote an arbitrary point with value 0 and let i be the smallest nonnegative integer for which there exists a point y satisfying $i = \mathrm{d}(x, y)$ and $f(y) \geq 2$. Obviously, $i \geq 2$. Choose points $y' \in \Gamma_1(y) \cap \Gamma_{i-1}(x)$ and $y'' \in \Gamma_1(y') \cap \Gamma_{i-2}(x)$. Clearly every point of the line through y' and y'' has value 0 or 1. But then the valuation induced in the quad $\mathcal{C}(y, y'')$ cannot be ovoidal, a contradiction. □

PROPOSITION 2.14. *Let f be a valuation of a dense near polygon $\mathcal{S}$, let O_f denote the set of points of $\mathcal{S}$ with value* 0 *and let x be a point of $\mathcal{S}$. If $d(x, O_f) \leq 2$, then $f(x) = d(x, O_f)$.*

Proof. Obviously, this holds if $\mathrm{d}(x, O_f) \leq 1$. Now, suppose that $\mathrm{d}(x, O_f) = 2$ and let x' denote a point of O_f at distance 2 from x. If the valuation induced in the quad $\mathcal{C}(x, x')$ is ovoidal, then x would be collinear with a point of $O_f \cap \mathcal{C}(x, x')$, a contradiction. Hence, the valuation induced in $\mathcal{C}(x, x')$ is classical and $f(x) = f(x') + \mathrm{d}(x, x') = 2$. □

2.4. The partial linear space G_f. For a valuation f of $\mathcal{S}$, put $O_f = \{x \in \mathcal{S} | f(x) = 0\}$. If $x, y \in O_f$, then by (iii) of Proposition 2.4, $d(x, y) \geq 2$. A quad Q of $\mathcal{S}$ is called *special* if it contains at least two points of O_f. Let G_f be the partial linear space with points the points of O_f, with lines the special quads of $\mathcal{S}$ and with natural incidence. If x and y are two collinear points of G_f, then the line of G_f through x and y corresponds with an ovoid in the special quad of $\mathcal{S}$ through x and y. As a corollary, every line of G_f contains at least 3 points.

2.5. A property of valuations. Let $\mathcal{S}$ be a dense near $2n$-gon and let f be a valuation of $\mathcal{S}$. For every $i \in \mathbb{N}$, we define m_i as the number of points of $\mathcal{S}$ with value i. Obviously, $m_i = 0$ if $i \geq n+1$.

PROPOSITION 2.15. *If $\mathcal{S}$ contains lines of size $s+1$, then $\sum_{i=0}^{\infty} \frac{m_i}{(-s)^i} = 0$.*

Proof. (a) Suppose first that $\mathcal{S}$ has order (s, t). For every line L of $\mathcal{S}$, $\sum_{x \in L} \frac{1}{(-s)^{f(x)}} = \frac{1}{(-s)^{f(x_L)}} + s \frac{1}{(-s)^{f(x_L)+1}} = 0$. Hence,

$$\begin{aligned} 0 &= \sum_{L \in \mathcal{L}} \sum_{x \in L} \frac{1}{(-s)^{f(x)}} \\ &= \sum_{x \in \mathcal{P}} \sum_{L \mathrm{I} x} \frac{1}{(-s)^{f(x)}} \\ &= (t+1) \sum_{x \in \mathcal{P}} \frac{1}{(-s)^{f(x)}} \\ &= (t+1) \sum_{i=0}^{\infty} \frac{m_i}{(-s)^i}. \end{aligned}$$

This shows that the proposition holds if $\mathcal{S}$ has an order.

(b) Suppose next that not every line of $\mathcal{S}$ is incident with the same number of points. Then, by Corollary 1.3, $\mathcal{S}$ has a partition in isomorphic geodetically closed sub near polygons of order (s, t') for some $t' \geq 0$. By (a), the proposition holds for each valuation induced in one of the sub near polygons of the partition. If we add all equations obtained, after multiplying with a suitable power of $-s$, then the required equation is obtained. □

COROLLARY 2.16. *Let f be a valuation of a dense near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. If k different line sizes $s_1 + 1, \ldots, s_k + 1$ occur in $\mathcal{S}$, then* $\max\{f(x) \mid x \in \mathcal{P}\} \geq k$.

Proof. Put $M := \max\{f(x) \mid x \in \mathcal{P}\}$. By Proposition 2.15, the polynomial $p(s) := \sum_{i=0}^{M} m_i(-s)^{M-i} = 0$ has at least k different roots. Hence, $k \leq \deg(f(s)) = M$. □

3. Some classes of valuations. In Section 2.3, classical and ovoidal valuations were discussed. We shall now define several other types of valuations.

3.1. Hybrid valuations. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2n$-gon, $n \geq 2$, let $\delta \in \{2, \ldots, n\}$ and let x be a point of $\mathcal{S}$. Let $\mathcal{A}_{x,\delta}$ be the incidence structure with points the points of $\mathcal{S}$ at distance at least δ from x, with lines the lines of $\mathcal{S}$ at distance at least $\delta - 1$ from x and with natural incidence. By Proposition 1.1 (iv), $\mathcal{A}_{x,\delta}$ is connected. Suppose now that $\mathcal{A}_{x,\delta}$ has an ovoid O. Then the following function $f_{x,O} : \mathcal{P} \to \mathbb{N}$ can be defined: if y is a point of $\mathcal{S}$ at distance at most $\delta - 1$ from x, then we define $f_{x,O}(y) := \mathrm{d}(x, y)$; if y is a point of $\mathcal{S}$ at distance at least δ from x, then we define $f_{x,O}(y) = \delta - 2$ if $y \in O$ and $f_{x,O}(y) = \delta - 1$ otherwise.

PROPOSITION 3.1. *The map $f_{x,O}$ is a valuation of $\mathcal{S}$.*

Proof. Since $f(x) = 0$, property (V_1) holds. Now, let L be an arbitrary line of $\mathcal{S}$. If $\mathrm{d}(x, L) \leq \delta - 2$, then the unique point on L nearest to x is also the unique point on L with smallest value. If $\mathrm{d}(x, L) \geq \delta - 1$, then the unique point of O on L is the unique point of L with smallest value. This proves property (V_2). Now, property $(P3)$ also holds if we make the following choices for F_y, $y \in \mathcal{P}$: $F_y := \mathcal{C}(x, y)$ if $\mathrm{d}(x, y) \leq \delta - 2$, $F_y := \{y\}$ if $y \in O$ and $F_y := \mathcal{S}$ otherwise. □

DEFINITION. A valuation that is obtained as above is called a *hybrid valuation of type δ*. A hybrid valuation of type 2 is just an ovoidal valuation. A hybrid valuation of type n is also called a *semi-classical valuation*. Although not included in the definition, we could regard the classical valuations as hybrid valuations of type $n + 1$.

PROPOSITION 3.2. *If f is a valuation of a dense near $2n$-gon and if x is a point of $\mathcal{S}$ such that $f(y) = d(x, y)$ for every point y at distance at most $n - 1$ from y, then f is either classical or semi-classical.*

Proof. Suppose that f is not classical and consider a point $z \in \Gamma_n(x)$. Every point of $\Gamma_1(z) \cap \Gamma_{n-1}(x)$ has value $n - 1$. Hence by property (V_2) and Proposition 2.10, $f(z) \in \{n - 2, n - 1\}$. By property (V_2), it now follows that the points of $\Gamma_n(x)$ with value $n - 2$ form an ovoid in $\mathcal{A}_{x,n}$. This proves that f is semi-classical. □

PROPOSITION 3.3. *Let $\mathcal{S}$ be a dense near $2n$-gon, $n \geq 2$, of order $(2, t)$ and let x be a point of $\mathcal{S}$. Then there exists a semi-classical valuation f with $f(x) = 0$ if and only if $\Gamma_n(x)$ is bipartite. In this case, there are precisely two semi-classical ovoids with $f(x) = 0$.*

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Proof. Every line of $\mathcal{A}_{x,n}$ contains two points. Hence, $\mathcal{A}_{x,n}$ has ovoids if and only if the graph induced by $\Gamma_n(x)$ is bipartite. □

3.2. Product valuations.

PROPOSITION 3.4. *Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two dense near polygons. If f_i, $i \in \{1, 2\}$, is a valuation of $\mathcal{S}_i$, then the map $f : \mathcal{P}_1 \times \mathcal{P}_2 \mapsto \mathbb{N}$, $(x_1, x_2) \mapsto f_1(x_1) + f_2(x_2)$ is a valuation of $\mathcal{S}_1 \times \mathcal{S}_2$.*

Proof. If x_i, $i \in \{1, 2\}$, is a point of $\mathcal{S}_i$ for which $f_i(x_i) = 0$, then $f[(x_1, x_2)] = 0$. This proves property (V_1). If L is a line of $\mathcal{S}_1 \times \mathcal{S}_2$, then without loss of generality, we may suppose that L is of the form $K \times \{y\}$, with K a line of $\mathcal{S}_1$ and y a point of $\mathcal{S}_2$. Now, $f[(k, y)] = f_1(k) + f_2(y)$ for every point k of K. Property (V_2) now immediately follows: the unique point of L with smallest f-value is the point (x_K, y), where x_K denotes the unique point of K with smallest f_1-value. It remains to check property (V_3). For every point x_i, $i \in \{1, 2\}$, of $\mathcal{S}_i$, let F_{x_i}, $i \in \{1, 2\}$, denote the sub near polygon of $\mathcal{S}_i$ satisfying (V_3). For every point (x_1, x_2) of $\mathcal{S}_1 \times \mathcal{S}_2$, we define $F_{(x_1,x_2)} := \{(a_1, a_2) \mid a_1 \in F_{x_1}$ and $a_2 \in F_{x_2}\}$. If (a_1, a_2) is a point of $F_{(x_1,x_2)}$, then $f[(a_1, a_2)] = f_1(a_1) + f_2(a_2) \leq f_1(x_1) + f_2(x_2) = f[(x_1, x_2)]$. If (a_1, a_2) is a point of $F_{(x_1,x_2)}$ and if (b_1, b_2) is a point of $\mathcal{S}_1 \times \mathcal{S}_2$ collinear with (a_1, a_2) and satisfying $f[(b_1, b_2)] = f(a_1, a_2) - 1$, then without loss of generality, we may suppose that $a_2 = b_2$ and $a_1 \sim b_1$ (in $\mathcal{S}_1$). Then $f_1(b_1) = f[(b_1, b_2)] - f_2(b_2) = f[(a_1, a_2)] - 1 - f_2(a_2) = f_1(a_1) - 1$. Since $a_1 \in F_{x_1}$, the point b_1 also belongs to F_{x_1}. Hence, the point (b_1, b_2) belongs to $F_{(x_1,x_2)}$. This proves property (V_3). □

DEFINITION. A valuation that is obtained as in Proposition 3.4 is called a *product valuation*.

3.3. Extended valuations.

DEFINITION. A geodetically closed sub near polygon F of a dense near polygon $\mathcal{S}$ is called *classical* if, for every point x of $\mathcal{S}$, there exists a (necessarily unique) point $\pi_F(x)$ in F such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$, for every point y of F.

LEMMA 3.5. *If x_1 and x_2 are collinear points of $\mathcal{S}$ such that $d(x_1, F) = d(x_2, F) - 1$, then $\pi_F(x_1) = \pi_F(x_2)$.*

Proof. The point $\pi_F(x_1)$ has distance at most $\mathrm{d}(x_1, \pi_F(x_1)) + \mathrm{d}(x_1, x_2) = \mathrm{d}(x_1, F) + 1 = \mathrm{d}(x_2, F)$ from x_2 and hence coincides with $\pi_F(x_2)$. □

LEMMA 3.6. *Let $\mathcal{S}$ be a dense near polygon, let K be a line of $\mathcal{S}$ and let F denote a geodetically closed sub near polygon of $\mathcal{S}$ that is classical in $\mathcal{S}$. Then one of the following holds.*

- *Every point of K has the same distance from F. In this case we define $\pi_F(K) := \{\pi_F(x) \mid x \in K\}$. Then $\pi_F(K)$ is a line of F parallel with K.*
- *There exists a unique point on K nearest to F. In this case all points $\pi_F(x)$, $x \in K$, are equal.*

Proof. Suppose that all points $\pi_F(x)$, $x \in K$, are equal, to u say. Then there exists a unique point on K nearest to F; namely the unique point of K nearest to u. Suppose therefore that there exist points $x_1, x_2 \in K$ such that $\pi_F(x_1) \neq \pi_F(x_2)$. By Lemma 3.5,

$d(x_1, F) = d(x_2, F)$. Put $i := d(x_1, F)$. Since

$$\begin{aligned} d(\pi_F(x_1), \pi_F(x_2)) &= d(x_1, \pi_F(x_2)) - d(x_1, \pi_F(x_1)) \\ &\leq d(x_1, x_2) + d(x_2, \pi_F(x_2)) - d(x_1, \pi_F(x_1)) \\ &= 1, \end{aligned}$$

$\pi_F(x_1)$ and $\pi_F(x_2)$ are contained in a line K'. If u is a point of K different from x_1 and x_2, then u has distance at most $i+1$ from the points $\pi_F(x_1)$ and $\pi_F(x_2)$ of K'. Hence there exists a point u' on K' at distance at most i from u. By Lemma 3.5, it follows that $d(u, F) = i$ and $\pi_F(u) = u'$. This proves that $\pi_F(K) \subseteq K'$ and that every point of K has the same distance i from F. Suppose now that there exists a point u' in $K' \setminus \pi_F(K)$. Then u' has distance at most $i+1$ from at least two points of K and hence distance at most i from a point u of K, showing that $u' = \pi_F(u)$, a contradiction. □

PROPOSITION 3.7. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near 2n-gon, let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a classical geodetically closed sub near polygon of $\mathcal{S}$ and let f' denote a valuation of F. Then the map $f : \mathcal{P} \mapsto \mathbb{N}, x \to f(x) := d(x, \pi_F(x)) + f'(\pi_F(x))$ is a valuation of $\mathcal{S}$. If f' is a classical valuation, then also f is classical.*

Proof. Obviously, property (V_1) is satisfied. By Lemma 3.6, it easily follows that also property (V_2) is satisfied. For every point x of $\mathcal{S}$, we define $F_x := \mathcal{C}(x, G_x)$, where G_x denotes the unique geodetically closed sub near polygon of F through $\pi_F(x)$ satisfying property (V_3) with respect to the valuation f' of F. Then F_x has the following properties.

• $F_x \cap F = G_x$. Obviously, $G_x \subseteq F_x \cap F$. If y is a point of G_x at distance $\mathrm{diam}(G_x)$ from $\pi_F(x)$ then, since $\pi_F(x)$ is contained in a shortest path between x and y, $G_x = \mathcal{C}(\pi_F(x), y)$ is contained in $\mathcal{C}(x, y)$. Hence, F_x is equal to $\mathcal{C}(x, y)$ and has diameter $d(x, y) = d(x, \pi_F(x)) + \mathrm{diam}(G_x)$. Suppose that there exists a point z in $F_x \cap F$ not contained in G_x. Then $\mathcal{C}(z, G_x)$ has diameter at least $\mathrm{diam}(G_x) + 1$. As before we have that $\mathcal{C}(x, \mathcal{C}(z, G_x))$ has diameter

$$\begin{aligned} d(x, \pi_F(x)) + \mathrm{diam}(\mathcal{C}(G_x, z)) &\geq d(x, \pi_F(x)) + \mathrm{diam}(G_x) + 1 \\ &= \mathrm{diam}(F_x) + 1, \end{aligned}$$

a contradiction, since $F_x = \mathcal{C}(x, \mathcal{C}(z, G_x))$. As a consequence, $F_x \cap F = G_x$.

• For every $y \in F_x$, $\pi_F(y) \in G_x$. Clearly every shortest path between y and a point $z \in G_x$ is contained in F_x. Since the point $\pi_F(y)$ is contained in a shortest path between y and z, the point $\pi_F(y)$ belongs to $F_x \cap F = G_x$.

• For every point y of F_x, $d(y, \pi_F(y)) \leq d(x, \pi_F(x))$. As before, $\mathcal{C}(y, G_x)$ has diameter $d(y, \pi_F(y)) + \mathrm{diam}(G_x)$. Since $\mathcal{C}(y, G_x) \subseteq \mathcal{C}(x, G_x)$, it follows that $d(y, \pi_F(y)) + \mathrm{diam}(G_x) \leq d(x, \pi_F(x)) + \mathrm{diam}(G_x)$, from which the statement follows.

Let u be a point of F_x. Since $\pi_F(u) \in G_x$, $f'(\pi_F(u)) \leq f'(\pi_F(x))$. Hence, $f(u) = d(u, \pi_F(u)) + f'(\pi_F(u)) \leq d(x, \pi_F(x)) + f'(\pi_F(x)) = f(x)$. Let v be a neighbour of u with value $f(u) - 1$. In order to prove property (V_3), we distinguish two possibilities.

– $d(v, \pi_F(v)) \neq d(u, \pi_F(u))$. Then $\pi_F(u) = \pi_F(v)$ by Lemma 3.5. In this case, $d(v, \pi_F(v)) = d(u, \pi_F(u)) - 1$. Hence, v is on a shortest path between u and $\pi_F(u) = \pi_F(v)$. Since $u, \pi_F(u) \in F_x$, also v belongs to F_x.

– $d(v, \pi_F(v)) = d(u, \pi_F(u))$. In this case, $f'(\pi_F(v)) = f'(\pi_F(u)) - 1$. By Lemma 3.6, $d(\pi_F(u), \pi_F(v)) = 1$. From $\pi_F(u) \in G_x$, it then follows that also $\pi_F(v) \in G_x$. Now, v lies on a shortest path between $\pi_F(v)$ and u. Since $\pi_F(v) \in F_x$ and $u \in F_x$, v also belongs to F_x.

If f' is classical valuation of F, then

$$f(x) = \mathrm{d}(x, \pi_F(x)) + f'(\pi_F(x)) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), x^*) = \mathrm{d}(x, x^*),$$

where x^* denotes the unique point of F for which $f'(x^*) = 0$. Hence f is classical if f' is classical. □

DEFINITION. The valuation f is called an *extension* of f'.

3.4. Diagonal valuations.

PROPOSITION 3.8. *Let F be a dense near polygon and let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the direct product $F \times F$. Define $X := \{(x, x) \mid x \in F\}$. Then the function $f : \mathcal{P} \to \mathbb{N}; p \mapsto d(p, X)$ is a valuation of $\mathcal{S}$.*

Proof. For every point (u, v) of $\mathcal{S}$, $f[(u, v)] = \mathrm{d}(u, v)$. Hence, every point of $\mathcal{S}$ has value at most diam(F). Obviously, there exists a point with value 0. Let L denote a line of $\mathcal{S}$. Without loss of generality, we may suppose that $L = \{u\} \times M$. If u' denotes the unique point of M nearest to u, then (u, u') is the unique point of L with smallest value. Now, for every point (u, v) of $\mathcal{S}$, we put $F_{(u,v)} = \mathcal{C}(u, v) \times \mathcal{C}(u, v)$. If $(u_1, v_1) \in F_{(u,v)}$, then $f[(u_1, v_1)] = \mathrm{d}(u_1, v_1) \leq \mathrm{d}(u, v) = f[(u, v)]$. Let $(u_1, v_1) \in F_{(u,v)}$ and let (u_2, v_2) be a point of $\mathcal{S}$ collinear with (u_1, v_1) such that $f[(u_2, v_2)] = f[(u_1, v_1)] - 1$. Without loss of generality, we may suppose that $u_1 = u_2$. Then $v_2 \sim v_1$ and $\mathrm{d}(u_1, v_2) = \mathrm{d}(u_1, v_1) - 1$, so that, $v_2 \in \mathcal{C}(u_1, v_1) \subseteq \mathcal{C}(u, v)$. As a consequence, $(u_2, v_2) \in \mathcal{C}(u, v) \times \mathcal{C}(u, v) = F_{(u,v)}$. This proves that f is a valuation of $\mathcal{S}$. □

DEFINITION. A valuation that is obtained as in Proposition 3.8 is called a *diagonal valuation*.

REMARK. With every set Y of points in $F \times F$, we can associate a matrix M_Y whose rows and columns are indexed by the points of F. If $(u, v) \in Y$, then the (u, v)-th entry of M_Y is equal to 1; otherwise it is equal to 0. The matrix M_X corresponding with the above-mentioned set X gives rise to a matrix with all ones on the diagonal. This explains the name we have given to these valuations.

3.5. Distance-j-ovoidal valuations.

We generalize the notion of distance-j-ovoids in generalized $2n$-gons ([**11**]) to arbitrary near polygons.

DEFINITION. Let $\mathcal{S}$ be a near $2n$-gon, $n \geq 2$. A distance-j-ovoid, $j \in \{2 \dots, n\}$, of $\mathcal{S}$ is a set X of points satisfying

(1) $\mathrm{d}(x, y) \geq j$ for all points $x, y \in X$;

(2) for every point a of $\mathcal{S}$, there exists a point $x \in X$ such that $\mathrm{d}(a, x) \leq \frac{j}{2}$;

(3) for every line L of $\mathcal{S}$, there exists a point $x \in X$ such that $\mathrm{d}(L, x) \leq \frac{j-1}{2}$.

A distance-2-ovoid is just an ovoid. From (1), (2) and (3), we immediately have the following statements.

• If j is odd, then for every point a of $\mathcal{S}$, there exists a unique point $x \in X$ such that $\mathrm{d}(a, x) \leq \frac{j-1}{2}$.

• If j is even, then for every line L of $\mathcal{S}$, there exists a unique point $x \in X$ such that $\mathrm{d}(L, x) \leq \frac{j-2}{2}$.

PROPOSITION 3.9. *If X is a distance-j-ovoid of a dense near $2n$-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $2 \leq j \leq n$ and j even, then the map $f : \mathcal{P} \to \mathbb{N}, x \mapsto d(x, X)$ is a valuation of $\mathcal{S}$.*

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Proof. Since $f(x) = 0$ for every point $x \in X$, property (V_1) holds.

Let L be a line of $\mathcal{S}$. Then there exists a unique point $x^* \in X$ such that $\mathrm{d}(x^*, L) \leq \frac{j-2}{2} = \frac{j}{2} - 1$. Hence, $\mathrm{d}(a, x^*) \leq \frac{j}{2}$ for every point a of L. By property (1), we then have that $\mathrm{d}(a, X) = \mathrm{d}(a, x^*)$ for every point a of L. It is now easily seen that property (V_2) holds: the point x_L is the unique point of L nearest to x^*.

Let x denote an arbitrary point of $\mathcal{S}$. If $\mathrm{d}(x, X) = \frac{j}{2}$, then we define $F_x := \mathcal{S}$. If $\mathrm{d}(x, X) < \frac{j}{2}$, then by property (1), there exists a unique point $x' \in X$ at distance $\mathrm{d}(x, X)$ from x and we define $F_x := \mathcal{C}(x, x')$. Clearly, property (V_3) holds for any point x for which $\mathrm{d}(x, X) = \frac{j}{2}$. Suppose therefore that $\mathrm{d}(x, X) < \frac{j}{2}$ and let x' denote the unique point of X at distance $\mathrm{d}(x, X)$ from x. Then for every point y of F_x, $\mathrm{d}(y, x') \leq \mathrm{d}(x, x') < \frac{j}{2}$, so that $f(y) = \mathrm{d}(y, X) = \mathrm{d}(y, x') \leq f(x)$. Now, let y be a point of F_x and let z be a point of $\mathcal{S}$ collinear with y such that $f(z) = f(y) - 1$. Then there exists a point $x'' \in X$ such that $\mathrm{d}(z, x'') = \mathrm{d}(y, x') - 1$. Since y has distance at most $\mathrm{d}(y, x')$ from x'', x' coincides with x''. Hence, $\mathrm{d}(z, x') = \mathrm{d}(y, x') - 1$ and $z \in F_x$. This proves that also (V_3) holds. □

Definition. A valuation f that is obtained as in Proposition 3.9 is called a *distance-j-ovoidal valuation*. A distance-2-ovoidal valuation is the same as an ovoidal valuation.

3.6. SDPS-valuations. A near polygon is called *classical* if it satisfies the following properties:

- every two points at distance 2 are contained in a unique quad,
- every point-quad relation is classical.

Every near 0-gon, near 2-gon and nondegenerate generalized quadrangle is classical. Every direct product of classical near polygons is again classical. By [**3**], the classical near polygons of diameter at least 2 are precisely the dual polar spaces of rank at least 2. With every polar space P of rank $n \geq 2$ there is associated a dual polar space P^D which is a near $2n$-gon. The points and lines of P^D are the maximal and next-to-maximal totally isotropic subspaces of P. By the classification of polar spaces ([**10**]), every finite dense dual polar space of rank $n \geq 2$ that is not a product near polygon is isomorphic to one of the examples given in the following table.

polar space	dual polar space	quads	(s, t_2)
$Q(2n, q)$	$Q^D(2n, q)$	$W(q)$	(q, q)
$Q^-(2n+1, q)$	$[Q^-(2n+1, q)]^D$	$H(3, q^2)$	(q^2, q)
$H(2n-1, q^2)$	$H^D(2n-1, q^2)$	$Q(5, q)$	(q, q^2)
$H(2n, q^2)$	$H^D(2n, q^2)$	$H^D(4, q^2)$	(q^3, q^2)
$W(2n-1, q)$	$W^D(2n-1, q)$	$Q(4, q)$	(q, q)

Every near $2n$-gon in this table is a regular near polygon with parameters s, t and t_i $(0 \leq i \leq n)$, where $t_i = \frac{t_2^i - t_2}{t_2 - 1}$ and $t = t_n$. In the table, we have made use of the following well-known isomorphisms: $Q^D(4, q) \cong W(q)$ and $[Q^-(5, q)]^D = Q^D(5, q) \cong H(3, q^2)$. See, for example, [**7**].

In [**5**], valuations of dual polar spaces are examined in detail. For completeness, a class of valuations that arises in [**5**] is given here.

Definition. Let $\mathcal{A} = (P, L, \mathrm{I})$ be one of the following classical near $4n$-gons:
(a) a point $(n = 0)$;

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(b) a dense generalized quadrangle ($n = 1$);
(c) $W^D(4n-1, q)$ with $n \geq 2$;
(d) $[Q^-(4n+1, q)]^D$ with $n \geq 2$.
A subset X of P is called an *SDPS-set* of $\mathcal{A}$ if it satisfies the following properties.
(1) No two points of X are collinear in $\mathcal{A}$.
(2) If $x, y \in X$ such that $\mathrm{d}(x, y) = 2$, then $X \cap \mathcal{C}(x, y)$ is an ovoid of the quad $\mathcal{C}(x, y)$.
(3) The point-line incidence structure $\tilde{\mathcal{A}}$ with points the elements of X, with lines the quads of $\mathcal{A}$ containing at least two points of X and with natural incidence is isomorphic to one of the following near $2n$-gons:

- case (a): a point;
- case (b): a line of size at least 2;
- case (c): $W^D(2n-1, q^2)$;
- case (d): $H^D(2n, q^2)$.

(4) For all $x, y \in X$, $\mathrm{d}(x, y) = 2 \cdot \delta(x, y)$, where $\delta(x, y)$ denotes the distance between x and y in the geometry $\tilde{\mathcal{A}}$.

REMARK. The terminology SDPS-set refers to the fact that there is a sub dual polar space associated with each such set. An SDPS-set of the near 0-gon consists of the unique point of the near 0-gon. An SDPS-set of a generalized quadrangle is just an ovoid of that generalized quadrangle. For the cases (c) and (d), examples of SDPS-sets are known. See [**5**].

PROPOSITION 3.10. ([**5**]) *If X is an SDPS-set of the near $4n$-gon $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, then the map $f : \mathcal{P} \mapsto d(x, X)$ is a valuation of $\mathcal{A}$.*

DEFINITION. Any valuation f which can be obtained in the above-mentioned way is called an *SDPS valuation*.

4. Valuations of dense near hexagons. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near hexagon and let f be a valuation of $\mathcal{S}$. There are three possibilities.

- $\max\{f(x)|x \in \mathcal{P}\} = 3$. In this case f is a classical valuation.
- $\max\{f(x)|x \in \mathcal{P}\} = 1$. In this case f is an ovoidal valuation.
- $\max\{f(x)|x \in \mathcal{P}\} = 2$.

PROPOSITION 4.1. *If $|O_f| = 1$, then f is a classical or a semi-classical valuation.*

Proof. This follows directly from Propositions 2.14 and 3.2. □

PROPOSITION 4.2. *Suppose that $|O_f| \geq 2$ and f is not ovoidal. Then every two points of O_f lie at distance 2 from each other. As a consequence, G_f is a linear space.*

Proof. Let x and y denote two distinct points of O_f. Then $\mathrm{d}(x, y) \in \{2, 3\}$. Suppose that $\mathrm{d}(x, y) = 3$ and consider a shortest path x, x_1, x_2, y from x to y. By property (V_2), the points x_1 and x_2 have value 1, and there exists a point p on x_1x_2 with value 0. Let F_{x_1} denote the sub near polygon through x_1 satisfying property (V_3). Now x and p are points with value 0 collinear with x_1 and so $x, p \in F_{x_1}$. Since x_1 and p belong to F_{x_1}, the point x_2 also belongs to F_{x_1}. As y is a point with value 0 collinear with x_2, we also have $y \in F_{x_1}$. Hence, $x, y \in F_{x_1}$ and $\mathcal{C}(x, y) \subseteq F_{x_1}$. Since $\mathrm{d}(x, y) = 3$, $\mathcal{S} = \mathcal{C}(x, y) = F_{x_1}$,

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a contradiction, since every point of F_{x_1} has value at most 1 and $\mathcal{S}$ contains points with value 2. □

PROPOSITION 4.3. *If not every line of a dense near hexagon $\mathcal{S}$ is incident with the same number of points, then f is classical or an extended valuation arising from an ovoidal valuation in a quad of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{S}$ has $k \geq 2$ different line sizes $s_1 + 1, \ldots, s_k + 1$. By Corollary 2.16, f is not ovoidal and $k \leq 3$. If $k = 3$, then by Proposition 1.2, $\mathcal{S}$ is the direct product of three lines of different sizes. Any quad of $\mathcal{S}$ is then a nonsymmetrical grid and hence does not contain ovoids. Hence, every induced quad valuation is classical. By Proposition 2.13, it then follows that the valuation f itself is classical, and so we may suppose that $k = 2$. By Proposition 1.2, it follows that $\mathcal{S}$ is the direct product of a line L and a generalized quadrangle Q. Without loss of generality, we may suppose that L has size $s_1 + 1$ and that Q has order (s_2, t_2) for a certain $t_2 \in \mathbb{N} \setminus \{0\}$. Since f is not ovoidal, $\mathcal{S}$ contains points with value 2. If f contains points with value 3, then f is classical by Proposition 2.10. Hence, we may suppose that there are only points with value 0, 1 or 2. There are $(t_2 + 1)(s_2 t_2 + 1)$ quads in $\mathcal{S}$ isomorphic to a $(s_1 + 1) \times (s_2 + 1)$-grid. The induced valuation in each such quad cannot be ovoidal and hence is classical. As a consequence, each such quad contains a unique point of O_f. Since any point of $\mathcal{S}$ is contained in precisely $t_2 + 1$ $(s_1 + 1) \times (s_2 + 1)$-grids, $|O_f| = \frac{(t_2+1)(s_2 t_2+1)}{t_2+1} = s_2 t_2 + 1 \geq 2$. We can now apply Proposition 4.2 and we find that any two points of O_f lie at distance 2 from each other. Since f is not classical, there exists a quad R such that the valuation induced in R is ovoidal. See Proposition 2.13. Obviously, the quad R is isomorphic with Q. For any point x of $\mathcal{S}$ outside Q, there always exists a point of the ovoid $O_f \cap R$ at distance 3 from x by Proposition 1.1 (iii). Hence, $f(x) \neq 0$ and $O_f \subset R$. By Proposition 1.1 (iii) and Proposition 2.14, it now follows that $f(x) = \mathrm{d}(x, O_f) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), O_f)$ for every point x of $\mathcal{S}$, so that f is the extension of an ovoidal valuation in R. □

If all lines of $\mathcal{S}$ are incident with $s + 1$ points, then by Proposition 2.15, $m_0 - \frac{m_1}{s} + \frac{m_2}{s^2} = 0$, where m_i, $i \in \{0, 1, 2\}$, denotes the total number of points with value i.

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