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RADICAL PAIRS

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The search for new radicals goes on. Recently R. L. Snider ([**6**], see page 216) introduced the following notion. Let α and β be any two radicals. A ring R will be said to be an $(\alpha : \beta)$ ring if for any ideal A of R, we have $\alpha(R/A) \geqq \beta(R/A)$.

In the special case when $\alpha \leqq \beta$, the class $(\alpha : \beta)$ consists of all rings R for which the α radical coincides with the β radical in all homomorphic images of R. If α is the Baer radical and β the Jacobson radical then $(\alpha : \beta)$ rings are known as Jacobson rings. When α is Baer and β is the Brown-McCoy radical then $(\alpha : \beta)$ rings are known as Brown-McCoy rings. Jacobson rings play an important role in commutative ring theory (see [**4**]). Noncommutative Jacobson rings and noncommutative Brown-McCoy rings have been studied by Procesi [**5**] and Watters [**7**; **8**]. Thus $(\alpha : \beta)$ generalizes well known notions and in fact leads to new radical properties.

It is clear that $(\alpha : \beta)$ is a homomorphically closed class and that $\alpha \leqq (\alpha : \beta)$. If we let 1 represent the radical for which all rings are radical then it is clear that $(\alpha : 1) = \alpha$. Furthermore if $\beta \leqq \alpha$ then $(\alpha : \beta) = 1$. Thus $(\alpha : \beta)$ is a homomorphically closed class of rings that lies somewhere between the class of all α rings and the class of all rings.

Two obvious questions arise. When is $(\alpha : \beta)$ a radical and when is $(\alpha : \beta)$ an hereditary class? Snider [**6**] proved:

THEOREM 1. *If α and β are both hereditary radicals then $(\alpha : \beta)$ is itself a radical.*

Snider characterized this radical in the following way:

THEOREM 2. *If α and β are both hereditary radicals then the radical $(\alpha : \beta)$ is the largest radical such that $(\alpha : \beta)(R) \cap \beta(R) \leqq \alpha(R)$ for every ring R.*

He also gave the following

Example 1. Let α be the torsion (re addition) radical, i.e. $\alpha(R)$ = the additive torsion subgroup of R. Then α is an hereditary radical. Let 0 represent the trivial radical, i.e. $\{0\}$ is the only radical ring. Then 0 is hereditary and $(0 : \alpha)$ is a radical. Now $(0 : \alpha)$ is the class of all strongly α-semi simple rings i.e. α-semi simple rings which remain α-semi simple under all homomorphisms. The point of this example is that $(0 : \alpha)$ is itself not an hereditary radical. To see this, let A be a two dimensional algebra over the rationals Q with basis 1 and a, where $a^2 = 0$. Then A has the ideal Aa which is isomorphic to the additive

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group of Q, with zero multiplication. Now $\alpha(A) = 0$, $(0 : \alpha)(A) = A$ but $(0 : \alpha)(Aa) = 0$. Thus the radical $(0 : \alpha)$ is not hereditary.

In general, $(\alpha : \beta)$ is not a radical. To this end we consider

Example 2. Let β be the upper radical determined by all nilpotent rings i.e. all nilpotent rings are β-semi simple. This β is not an hereditary radical. Then $(0 : \beta)$ is not a radical. To see this, consider (see [**2**, pages 19/20, example 3]) the well known algebra R with basis $\{x_\alpha : 0 < \alpha < 1\}$ where multiplication is defined by

$$\begin{aligned} x_\alpha x_\beta &= x_{\alpha+\beta} \text{ if } \alpha + \beta < 1 \\ &= 0 \quad \text{ if } \alpha + \beta \geqq 1. \end{aligned}$$

Then R is a union of nilpotent ideals i.e. of $(0 : \beta)$ rings. Now if $(0 : \beta)$ was a radical property then R would have to be in $(0 : \beta)$. However since $R^2 = R$, it is clear that $\beta(R) = R$ and thus R is not in $(0 : \beta)$.

We *can* say something about $(0 : \beta)$ in general i.e. even if β is not hereditary.

LEMMA 1. *If both R/I and I are in $(0 : \beta)$ then R is in $(0 : \beta)$.*

Proof. If R is not in $(0 : \beta)$ then some homomorphic image R/A has a nonzero β radical say K/A. If $I \leqq A$ then $R/A = R/I/A/I$ is β-semi simple because R/I is strongly β-semi simple. This forces $I \nleqq A$. In that case $(A + I)/A = I/(A \cap I)$ is nonzero. This is β-semi simple since I is strongly β-semi simple. Consider $R/(I + A) = (R/I)/[(A + I)/I]$. This is β-semi simple since R/I is strongly β-semi simple. Also $R/(I + A) = (R/A)/[(A + I)/A]$ is β-semi simple. Therefore $K/A \leqq (A + I)/A$. But $(A + I)/A$ is β-semi simple. Thus $K/A = 0$ and the lemma is proved.

To gain more information about $(\alpha : \beta)$ we make the following

Definition. Let α be a radical and let A be an ideal of R. Then by $\sqrt[\alpha]{A}$ we shall mean the ideal of R such that $\sqrt[\alpha]{A}/A = \alpha(R/A)$. We shall call $\sqrt[\alpha]{A}$ the *radical closure* of A.

Definition. An ideal A of R will be called an $\bar{\alpha}$ *ideal* if $\alpha(R/A) = 0$.

Definition. A radical α will be called *superior* if every α-semi simple ring can be expressed as a subdirect sum of subdirectly irreducible rings R_t where each R_t is itself α-semi simple.

Remark. It is easy to check that the Brown-McCoy radical is superior. Neither the Baer radical nor the Jacobson radical nor any radical between them is superior (see [**2**, pages 113–115]).

Definition. A class $\mathscr{N}$ of rings will be called *regular* is every nonzero ideal of a ring in $\mathscr{N}$ can be homomorphically mapped onto a nonzero ring in $\mathscr{N}$.

LEMMA 2. *If α is a superior radical and β an hereditary radical then the class $\mathscr{M}$ of all α-semi simple subdirectly irreducible rings with β radical hearts is regular.*

Proof: Let A be a nonzero ideal of R where R is in $\mathscr{M}$. Since R is α-semi simple, A must be α-semi simple. Since α is superior, A is equal to a subdirect sum $\sum_t A_t$ where $\alpha(A_t) = 0$ and where each A_t is subdirectly irreducible. Let $H(A_t)$ be the heart of A_t for each t. Now A contains $H(R)$, the heart of R, which is a nonzero β radical ring. Thus $\beta(A) \neq 0$ and $\beta(A_t) \neq 0$ for some t, for otherwise A would be a subdirect sum of β-semi simple rings and such a sum is known to be β-semi simple. Then $0 \neq H(A_t) \leqq \beta(A_t)$. Since β is hereditary, $H(A_t)$ is a β ring. Thus A_t is an α-semi simple, subdirectly irreducible ring with a β radical heart, and certainly A can be homomorphically mapped onto this A_t. Thus $\mathscr{M}$ is regular.

LEMMA 3. *If α is superior, β hereditary and if A is an ideal of R, then if $\sqrt[\alpha]{A}$ can be represented as an intersection of $\bar{\alpha}$ ideals I_t of R such that each R/I_t is subdirectly irreducible with β-semi simple heart, we can conclude that $\alpha(R/A) \geqq \beta(R/A)$.*

Proof. Suppose that $\beta(R/\sqrt[\alpha]{A}) \neq 0$. By assumption $R/\sqrt[\alpha]{A}$ is a subdirect sum $\sum_t R/I_t$ where the R/I_t are α-semi simple and subdirectly irreducible with β-semi simple hearts. There must exist a t for which $\beta(R/I_t) \neq 0$. Then $0 \neq H(R/I_t) \leqq \beta(R/I_t)$. Since β is hereditary, $H(R/I_t)$ is a β ring which is impossible. Thus $\beta(R/\sqrt[\alpha]{A}) = 0 = \beta((R/A)/(\sqrt[\alpha]{A}/A))$. Therefore $\beta(R/A) \leqq \sqrt[\alpha]{A}/A = \alpha(R/A)$.

Lemma 2 tells us that the class $\mathscr{M}$ is regular and thus it easily defines an upper radical $\mathscr{U}_{\mathscr{M}}$.

THEOREM 3. *If α is superior and β is hereditary then $(\alpha : \beta) = \mathscr{U}_{\mathscr{M}}$, and in particular, $(\alpha : \beta)$ is a radical.*

Proof. Take R in $(\alpha : \beta)$. If R is not in $\mathscr{U}_{\mathscr{M}}$ then some nonzero homomorphic image R/A must be in $\mathscr{M}$. Let $H(R/A)$ be the heart of R/A. Then we have $0 = \alpha(R/A) \geqq \beta(R/A) \geqq H(R/A) \neq 0$. This is impossible and thus R is in $\mathscr{U}_{\mathscr{M}}$. Note that this half of the theorem, $(\alpha : \beta) \leqq \mathscr{U}_{\mathscr{M}}$, holds for arbitrary radicals α and β.

Conversely suppose that R is in $\mathscr{U}_{\mathscr{M}}$. Let A be any ideal of R. Since α is superior, $\sqrt[\alpha]{A}$ is the intersection of $\bar{\alpha}$ ideals I_t of R such that the R/I_t are subdirectly irreducible. If for one of these $\beta(R/I_t) \neq 0$ then $0 \neq H(R/I_t) \leqq \beta(R/I_t)$ and the heart is β radical since β is hereditary. Thus R/I_t is in $\mathscr{M}$. This cannot happen since R is in $\mathscr{U}_{\mathscr{M}}$. Thus each $\beta(R/I_t) = 0$ and each heart $H(R/I_t)$ is β-semi simple. By Lemma 3, $\alpha(R/A) \geqq \beta(R/A)$ and thus R is in $(\alpha : \beta)$.

THEOREM 4. *If α is superior and β is hereditary then R is in $(\alpha : \beta)$ if and only if for every ideal A of R, $\sqrt[\alpha]{A}$ can be represented as an intersection of $\bar{\alpha}$ ideals I_t of R such that the R/I_t are subdirectly irreducible with β-semi simple hearts.*

Proof. Let R be in $(\alpha : \beta)$ and let A be an ideal of R. Since α is superior $\sqrt[\alpha]{A}$ can be represented as an intersection of $\bar{\alpha}$ ideals I_t of R such that the R/I_t are

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subdirectly irreducible. For each I_t, $0 = \alpha(R/I_t) \geqq \beta(R/I_t)$ and thus each heart $H(R/I_t)$ is β-semi simple. The converse follows from Lemma 3.

THEOREM 5. *If α is superior and β is hereditary then R is in $(\alpha : \beta)$ if and only if for any ideal A of R, $R/\sqrt[\alpha]{A}$ can be represented as a subdirect sum of α-semi simple, subdirectly irreducible rings with β-semi simple hearts.*

Proof. This is immediate from Theorem 4.

Theorems 3, 4 and 5 generalize the following result known to Andrunakievic [**1**].

COROLLARY. *Let β be an hereditary radical. Then the following are equivalent:*
(i) *R is in $(0 : \beta)$.*
(ii) *Every homomorphic image of R can be represented as a subdirect sum of subdirectly irreducible rings with β-semi simple hearts.*
(iii) *Every ideal A of R can be represented as an intersection of ideals I_t of R such that the rings R/I_t are subdirectly irreducible with β-semi simple hearts.*
(iv) *R is in $\mathscr{U}_{\mathscr{M}}$ where $\mathscr{M}$ is the class of all subdirectly irreducible rings with β radical hearts.*

Proof. We note that the trivial radical 0 is superior since every ring can be represented as a subdirect sum of subdirectly irreducible rings. Furthermore we note that for any ideal A of R, $\sqrt[0]{A} = A$.

Now we wish to consider when $(\alpha : \beta)$ is an hereditary class. To this end we wish to consider radicals that contain or are equal to the Baer radical. Unfortunately the word supernilpotent is used to mean an hereditary radical which contains or equals the Baer radical. So we shall use the term *weakly supernilpotent* to mean a radical which is not necessarily hereditary but which does contain or is equal to the Baer radical.

LEMMA 4. *Let α be a radical and let A be an ideal of R such that $\alpha(A) = A$. Then $\alpha(R/A) = \alpha(R)/A$.*

Proof. First we note that $A = \alpha(A) \subseteq \alpha(R)$. Then $\alpha(R)/A$ is an α ring and it is an ideal of R/A. Therefore it is $\subseteq \alpha(R/A)$. On the other hand $\alpha(R/A) = K/A$ for some ideal K of R. Now K/A is an α ring and so is A and thus K itself is an α ring and therefore $K \subseteq \alpha(R)$. Thus $\alpha(R/A) \subseteq \alpha(R)/A$. This proves the lemma.

THEOREM 6. *If α is an hereditary radical and if β is a weakly supernilpotent radical then $(\alpha : \beta)$ is an hereditary class.*

Proof. Take R in $(\alpha : \beta)$, let A be an ideal of R and suppose that A is not in $(\alpha : \beta)$. Then there exists an ideal W of A such that $\beta(A/W) \nless \alpha(A/W)$. Then $\beta(A/W)$ is not an α ring. Let $\bar{W}$ be the ideal of R generated by W. Then we know that $\bar{W}^3 \leqq W \leqq \bar{W} \leqq A \leqq R$. We know that $R/\bar{W}$ is in $(\alpha : \beta)$ and thus $\alpha(R/\bar{W}) \geqq \beta(R/\bar{W})$. Also $R/\bar{W}^3$ is in $(\alpha : \beta)$ and thus $\alpha(R/\bar{W}^3) \geqq \beta(R/\bar{W}^3)$.

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Consider $\beta(A/\bar{W}) = \beta((A/W)/(\bar{W}/W))$. Since $\bar{W}/W$ is nilpotent it is a β ring. Then by Lemma 4, $\beta((A/W)/(\bar{W}/W)) = \beta(A/W)/(\bar{W}/W)$. This is a β ring and an ideal of $R/\bar{W}$ and thus it is $\leqq \beta(R/\bar{W}) \leqq \alpha(R/\bar{W})$. Since α is hereditary, $\beta(A/W)/(\bar{W}/W)$ is an α ring.

Next we show that $\bar{W}/W$ is an α ring. To see this we note that $\bar{W}/\bar{W}^3$ is nilpotent and thus a β ring and it is $\leqq \beta(R/\bar{W}^3) \leqq \alpha(R/\bar{W}^3)$. Since α is hereditary, $\bar{W}/\bar{W}^3$ is an α ring. Then $\bar{W}/W = (\bar{W}/\bar{W}^3)/(W/\bar{W}^3))$ is an α ring since it is a homomorphic image of an α ring. Thus both $(\beta(A/W))/(\bar{W}/W)$ and $\bar{W}/W$ are α rings and therefore $\beta(A/W)$ is an α ring. But it is not. This contradiction proves the theorem.

COROLLARY 1. *If α is hereditary and β is supernilpotent then $(\alpha : \beta)$ is an hereditary radical.*

COROLLARY 2. *If α is hereditary and β is supernilpotent then $((\alpha : \beta) : \beta) = (\alpha : \beta)$.*

Proof. $((\alpha : \beta) : \beta)(R) \cap \beta(R) \leqq (\alpha : \beta)(R)$ and it is in $\beta(R)$ and thus it is in $(\alpha : \beta)(R) \cap \beta(R) \leqq \alpha(R)$ by Theorem 2. Since $(\alpha : \beta)$ is maximal with respect to this property, we must have $((\alpha : \beta) : \beta) \leqq (\alpha : \beta)$. However $(\alpha : \beta) \leqq ((\alpha : \beta) : \beta)$ and thus they are equal.

Some well behaved radicals are hereditary. Many radicals are not hereditary and some are very far from being hereditary. We wish to select those radicals which are extremely far away. To this end we make the following

Definition. A radical α will be called *mutagenic* if there exists a ring R such that

(i) R is the union of an ascending chain of ideals I_t,

$$0 < I_1 < I_2 < \ldots < I_t < \ldots < R = \bigcup_t I_t$$

where each I_t is in $(0 : \alpha)$, and

(ii) $\alpha(R) \neq 0$ i.e. R is not only not in $(0 : \alpha)$ but it is not even α-semi simple.

THEOREM 7. $(0 : \alpha)$ *is a radical if and only if α is not mutagenic.*

Proof. If α is mutagenic then the given ring R is not in $(0 : \alpha)$. However every nonzero homomorphic image of R contains a nonzero ideal in $(0 : \alpha)$ namely a homomorphic image of some I_n. Therefore $(0 : \alpha)$ is not a radical.

Conversely suppose that $(0 : \alpha)$ is not a radical. Since we know that $(0 : \alpha)$ is homomorphically closed, there must exist a ring R which is not in $(0 : \alpha)$ and yet every nonzero homomorphic image of R contains a nonzero ideal in $(0 : \alpha)$. Since R is not in $(0 : \alpha)$ there must exist a nonzero homomorphic image of R which is not α-semi simple. Thus we may assume without loss of generality that $\alpha(R) \neq 0$.

Now R itself must have a nonzero ideal I_1 in $(0 : \alpha)$. Since R is not in $(0 : \alpha)$, $I_1 \neq R$. Then R/I_1 must have a nonzero ideal I_2/I_1 in $(0 : \alpha)$. By Lemma 1, I_2 is in $(0 : \alpha)$. We continue in this way and obtain $0 < I_1 < I_2 < \ldots < I_n < \ldots < \bigcup I_n$, where each I_n is in $(0 : \alpha)$. If $\bigcup I_n = R$ we are done. If not, then either $\bigcup I_n$ is in $(0 : \alpha)$ and we carry on extending the chain of ideals, or $\bigcup I_n$ is not in $(0 : \alpha)$. In this latter case we can use $\bigcup I_n$ if $\alpha(\bigcup I_n) \neq 0$ or we can use some homomorphic image of $\bigcup I_n$ which is not α-semi simple. Thus α is mutagenic and the theorem is established.

In the case when α is weakly supernilpotent, the definition of mutagenic can be made neater.

THEOREM 8. *If α is a weakly supernilpotent radical then α is mutagenic if and only if there exists an α ring R such that*

$$0 < J_1 < J_2 < \ldots < J_t < \ldots < R = \bigcup_t J_t$$

where each J_t is in $(0 : \alpha)$.

Proof. One direction is trivial. For the other direction suppose α is mutagenic. Then there exists a ring $R = \bigcup_t I_t$, $0 < I_1 < \ldots < I_t < \ldots$, each I_t is in $(0 : \alpha)$ and $\alpha(R) \neq 0$. Now $\alpha(R) = R \cap \alpha(R) = \bigcup [I_t \cap \alpha(R)]$. Let $J_t = I_t \cap \alpha(R)$. Theorem 6 tells us that $(0 : \alpha)$ is an hereditary class. Therefore J_t must be in $(0 : \alpha)$. Therefore the ring $\alpha(R) = \bigcup J_t$ with each J_t in $(0 : \alpha)$, and of course $\alpha(R)$ is an α ring.

COROLLARY. *If α is a weakly supernilpotent radical and if there does not exist an α ring which is the union of an ascending chain of $(0 : \alpha)$ ideals then $(0 : \alpha)$ is an hereditary radical.*

Example 3. To find an example of a mutagenic radical we can use Example E in [**3**, page 688]. Let α be the lower radical determined by $\mathscr{S}_{w0}$. Then every ideal $\mathscr{S}_n$ is in $(0 : \alpha)$ basically because $\mathscr{S}_{w0}$ has no maximal ideals where every subideal of $\mathscr{S}_n$ does have a maximal ideal. This α is mutagenic but is not weakly supernilpotent. We can extend α to say β, the lower radical determined by $\mathscr{S}_{w0}$ and by all nilpotent rings. Then $\mathscr{S}_{w0}$ is a β ring and all the ideals $\mathscr{S}_n$ are in $(0 : \beta)$. Thus β is a mutagenic, weakly supernilpotent radical. In particular of course, β is not hereditary.

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