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RESEARCH ARTICLE

The Quantitative Kurosh Problem

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Abstract

We prove a strong quantitative version of the Kurosh Problem, which has been conjectured by Zelmanov, up to a mild polynomial error factor, thereby extending all previously known growth rates of algebraic algebras. Consequently, we provide the first counterexamples to the Kurosh Problem over any field with known subexponential growth, and the first examples of finitely generated, infinite-dimensional, nil Lie algebras with known subexponential growth over fields of characteristic zero.

We also widen the known spectrum of the Gel'fand–Kirillov dimensions of algebraic algebras, improving the answer of Alahmadi–Alsulami–Jain–Zelmanov to a question of Bell, Smoktunowicz, Small and Young. Finally, we prove improved analogous results for graded-nil algebras.

Contents

1. Introduction

One of the most tantalizing problems in combinatorial algebra over the 20th century was the Kurosh Problem:

The Kurosh Problem (A. Kurosh, 1941). Is every finitely generated algebraic algebra finite-dimensional?

The Kurosh Problem is tightly connected to the (general) Burnside Problem from 1902, which asks whether every finitely generated periodic group is finite. In 1964, Golod (using the Golod-Shafarevich theorem) constructed the first counterexamples to the Kurosh Problem. He constructed finitely generated, infinite-dimensional, *nil* algebras – namely, non-unital algebras in which every element is nilpotent; such algebras – as well as their unital hulls – are immediately algebraic. Using these methods, Golod and Shafarevich solved the aforementioned Burnside Problem and Hilbert's Class Field Tower Problem on the existence of finite extensions of number fields with class number one [13, 14, 15]. Later on, other long-standing open questions have been resolved using Golod-Shafarevich algebras and groups; see [12, 26].

Golod-Shafarevich algebras are infinite-dimensional in the 'strongest possible' sense: they have exponential growth. The growth function of a finitely generated algebra A over a field k, generated by a finite-dimensional subspace V, is defined as

$$\gamma_A(n) = \dim_k(k + V + \cdots + V^n).$$

The growth functions of Golod-Shafarevich algebras are exponential, and so are the growth functions of the periodic groups resulting from them. This naturally led to the question of whether the solution to the Kurosh Problem is affirmative under growth restrictions; that is, must every finitely generated algebraic algebra of subexponential (or: polynomial) growth be finite-dimensional? This question has been posed and repeated by many experts; see [24] and references therein. This variation of the Kurosh Problem had been open for many years until Lenagan and Smoktunowicz constructed in 2007 an infinite-dimensional nil algebra of polynomially bounded growth over any countable fields [24] (see also [25]). (For more examples of algebraic algebras of polynomially bounded growth, see [2, 7].) Bell and Young [8] constructed infinite-dimensional nil algebras over arbitrary fields whose growth is bounded above by an arbitrarily slow super-polynomial function. Later on, Smoktunowicz [34] constructed nil algebras of intermediate (that is, subexponential and super-polynomial) growth over arbitrary fields. Recently, the author and Zelmanov [17, Theorems A1, A2] utilized matrix wreath products (previously developed in [2]) to prove that every increasing and submultiplicative function can be approximated by the growth function of a nil algebra, up to certain 'distortion' factors.

In view of the flexible nature of the space of growth rates of algebraic algebras, Zelmanov posed the following conjecture, which can be thought of as a strong quantitative version of the Kurosh Problem:

Conjecture 1.1 (E. Zelmanov, 2017, [38, 3]). *The following classes of functions coincide, up to asymptotic equivalence:*

$$\left\{\textit{Growth functions* of algebras}\right\} = \left\{\textit{Growth functions of nil algebras}\right\}.$$

**Except for algebras of linear growth*

In other words, Conjecture 1.1 asserts that counterexamples to the Kurosh Problem are 'as ubiquitous as possible' and occur within any possible growth rate.[1]

Proving Conjecture 1.1 in its full generality is considered an extremely difficult task since even some of its (very) special cases are wide open. For instance, no finitely generated, infinite-dimensional algebraic algebras of *known* subexponential growth have appeared so far: the constructions in [2, 7, 8, 17, 24, 25, 34] all have different bounds on their growth rates, but to the best of our knowledge, no single concrete function has been realized as the growth function of any counterexample to the Kurosh Problem so far. Another evidence to the difficulty of Conjecture 1.1 occurs within polynomial growth rates. Recall that the Gel'fand-Kirillov (GK) dimension of a finitely generated algebra A is given by $\mathrm{GKdim}(A) = \limsup_{n\to\infty} \frac{\log \gamma_A(n)}{\log n}$ – namely, the (optimal) degree of polynomial growth of

[1] Linear growth functions are excluded since finitely generated algebras of linear growth satisfy a polynomial identity [33], and for such algebras, the answer to the Kurosh Problem is affirmative [22].

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A. In particular, $\mathrm{GKdim}(A) < \infty$ if and only if A has polynomially bounded growth. The Lenagan-Smoktunowicz algebra from [24] has GK-dimension at most 3 (see [25]), although the precise GK-dimension is unknown; every real number in the interval $[8, \infty)$ is realizable as the GK-dimension of some nil algebra [2]. It is a wide open problem to determine the spectrum of possible GK-dimensions of finitely generated algebraic algebras, or even if there is a finitely generated algebraic algebra of GK-dimension 2 (the Kurosh Problem has an affirmative answer for algebras of GK-dimension smaller than 2); see open questions in [7, 8, 24, 25]. It is also unknown if there exist finitely generated infinite-dimensional algebraic algebras of polynomially bounded growth over uncountable fields; see open questions in [7, 8, 24, 25, 39].

It is interesting to compare the Kurosh Problem under growth restrictions with the Burnside Problem under growth restrictions. By Gromov's celebrated theorem, finitely generated groups of polynomial growth are virtually nilpotent [21], and therefore, the answer to the Burnside Problem is affirmative for such groups. The first examples of groups of intermediate growth were constructed by Grigorchuk [18]; these groups are furthermore periodic. However, their precise growth rate is still unknown (see [11] for a recent major progress). The first examples of finitely generated, infinite periodic groups of known subexponential growth were given in [4], almost 30 years after (periodic) groups of intermediate growth first appeared.

In this paper, we give an affirmative answer to Conjecture 1.1 up to a mild polynomial error term (or an arbitrarily slow super-polynomial error factor in the case of algebras over uncountable fields). Notice that for sufficiently regular super-polynomial functions, a polynomial error factor is negligible. This improves the main results [17, Theorem A1, A2] and extends all previously known classes of growth functions of algebraic algebras, including [2, 8, 34].

Theorem 1.2. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be the growth function of any finitely generated infinite-dimensional algebra, not of linear growth. Then there exists a finitely generated nil graded algebra A over any countable field such that*

$$f(n) \leq \gamma_A(n) \leq n^{4+\varepsilon} f(n)$$

for all $\varepsilon > 0$, and a finitely generated nil graded algebra over an arbitrary field such that

$$f(n) \leq \gamma_A(n) \leq n^{\omega(n)} f(n)$$

for any arbitrarily slow given function $\omega(n)$ tending to infinity.

As an application, we obtain the first counterexamples to the Kurosh Problem (namely, infinite-dimensional algebraic algebras) of known subexponential growth, and the first examples of nil Lie algebras of known subexponential growth, over any field of characteristic zero (Petrogradsky, Shestakov and Zelmanov constructed finitely generated nil Lie algebras of known polynomial growth over fields of positive characteristic in [29, 30, 32]). A Lie algebra L is nil if all of its elements are ad-nilpotent; that is, for every element $a \in L$, the adjoint operator $\mathrm{ad}_a = [a, -]$ is nilpotent.

Corollary 1.3. *For every $\alpha \in (0, 1)$, there exists a finitely generated nil graded algebra A and a finitely generated nil graded Lie algebra over an arbitrary field such that*

$$\gamma_A(n) \sim \gamma_L(n) \sim \exp(n^{\alpha}).$$

Moreover, we are able to construct nil algebras of an arbitrary GK-dimension ≥ 6, strengthening the solution of [2] to [7, Question 1] and [8, Question 2]; our algebras are furthermore naturally graded.

Corollary 1.4. *For every $\alpha \geq 6$, there exists a finitely generated nil graded algebra A and a finitely generated nil graded Lie algebra over any countable field such that* $\mathrm{GKdim}(A) = \mathrm{GKdim}(L) = \alpha$.

A graded associative (resp. Lie) algebra $A = \bigoplus_{n=1}^{\infty} A_n$ is *graded-nil* if all of its homogeneous elements are nilpotent (resp. ad-nilpotent). Such algebras naturally arise from residually finite periodic groups (e.g., [27]) and from filtered algebraic algebras [31] and play a major role in Zelmanov's solution of the Restricted Burnside Problem (see [37]). We prove improved analogs of the above results, for graded-nil algebras, where only a linear error factor is needed.

Theorem 1.5. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be the growth function of any finitely generated infinite-dimensional algebra. Then there exists a finitely generated graded-nilpotent algebra A over an arbitrary field such that*

$$f(n) \preceq \gamma_A(n) \preceq nf(n).$$

Corollary 1.6. *For every real number $\alpha \geq 3$, there exists a graded-nil associative algebra A and a graded-nil Lie algebra L over an arbitrary field such that* $\mathrm{GKdim}(A) = \mathrm{GKdim}(L) = \alpha$.

To the best of our knowledge, this provides the first examples of graded-nil algebras of an arbitrary (in particular, non-integer) GK-dimension, over an uncountable field.

Finally, in Section 6, we revisit Theorem 1.2 and show how a slightly weaker version of it (namely, slightly weakening the polynomial error factor and omitting the grading assumption) can be constructed using matrix wreath products, which have been utilized in [2, 17].

2. Growth of algebras

Let A be a finitely generated associative algebra over a field k. Let V be a finite-dimensional generating subspace – namely, $A = k\langle V\rangle$. The growth of A with respect to V is the function

$$\gamma_{A,V}(n) = \dim_k\Big(k + V + V^2 + \cdots + V^n\Big).$$

The definition applies for non-unital algebras too (whose growth coincides with the growth of their unital hulls). If $1 \in V$, then equivalently, $\gamma_{A,V}(n) = \dim_k V^n$.

We write $f \preceq g$ if $f(n) \leq Cg(Dn)$ for some $C, D > 0$ and for all $n \in \mathbb{N}$, and we say that f is asymptotically equivalent to g, denoted $f \sim g$, if $f \preceq g$ and $g \preceq f$. The function $\gamma_{A,V}(n)$ is independent of the choice of the generating subspace up to asymptotic equivalence, and so we write $\gamma_A(n)$ for the growth function of the algebra A, considered up to asymptotic equivalence. Bell and Zelmanov [9] characterized growth functions of algebras by means of combinatorial conditions on their discrete derivatives, from which it follows that every increasing, submultiplicative function is equivalent to a growth function, up to a linear error term (see [16, Proposition 3.1]). The Gel'fand-Kirillov (GK) dimension of A is

$$\mathrm{GKdim}(A) = \limsup_{n\to\infty} \frac{\log \gamma_A(n)}{\log n}.$$

It follows from the definition that $\mathrm{GKdim}(A) < \infty$ if and only if $\gamma_A(n)$ is polynomially bounded. If A is commutative, then $\mathrm{GKdim}(A)$ coincides with the classical Krull dimension of A. The possible values of $\mathrm{GKdim}(A)$ are $0, 1, [2, \infty]$. For more on the growth of algebras and the Gel'fand-Kirillov dimension, see [23].

If $A = \bigoplus_{n=0}^{\infty} A_n$ is graded with finite-dimensional homogeneous components, then $\gamma_A(n) \sim \dim_k \bigoplus_{i=0}^{n} A_i$. For a graded algebra A and a subset $S \subseteq A$, we let S_n denote the intersection of S with the degree-n homogeneous component of A, and $S_{\leq n} := S \cap \left(\bigoplus_{i\leq n} A_i\right)$. For a non-unital graded algebra $A = \bigoplus_{n=1}^{\infty} A_n$, we let $A^1 = k + A$ denote its unital hull.

If L is a finitely generated Lie algebra (over a field of characteristic $\neq 2$) generated by a finite-dimensional subspace V, then its growth function is defined as

$$\gamma_{L,V}(n) = \dim_k \mathrm{Span}_k\{d\text{-fold Lie brackets of elements from } V \text{ for } d \leq n\},$$

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again, considered up to asymptotic equivalence. The definition of GK-dimension applies similarly for Lie algebras.

Let $\Sigma = \{x_1, \ldots, x_m\}$ be a finite alphabet. We denote by Σ^* the free monoid generated by Σ. Let $I \triangleleft k\langle\Sigma\rangle$ be an ideal generated by monomials. Then the quotient ring $A = k\langle\Sigma\rangle/I$ is called a monomial algebra. For a subset S of a monomial algebra, we let $\mathrm{Mon}(S)$ denote the set of nonzero monomials of S. Thus, for every monomial algebra A and monomial ideal $I \triangleleft A$, $\mathrm{Mon}(A) = \mathrm{Mon}(I) \cupdot \mathrm{Mon}(A/I)$.

Every monomial algebra A is naturally graded by assigning $\deg(x_1) = d_1, \ldots, \deg(x_m) = d_m$, for any choice of natural numbers $d_1, \ldots, d_m \in \mathbb{N}$. Let $\mathcal{L} \subseteq \Sigma^*$ be a hereditary language – namely, a nonempty set closed under taking subwords. Associated with $\mathcal{L}$ is a monomial algebra:

$$A_{\mathcal{L}} := k\langle\Sigma\rangle/\langle u \mid u \text{ is not a factor of a word from } \mathcal{L}\rangle.$$

A special case is where $\mathcal{L}$ consists of the subwords (also called factors) of a given (right) infinite word $w \in \Sigma^{\mathbb{N}}$, and we denote

$$A_w := k\langle\Sigma\rangle/\langle u \mid u \text{ is not a factor of } w\rangle.$$

For a hereditary language $\mathcal{L}$ (resp. infinite word w), we let $p_{\mathcal{L}}(n)$ (resp. $p_w(n)$) be its complexity function, counting its length-n subwords. If the letters of Σ are assigned with degree 1, then $\gamma_{A_{\mathcal{L}}}(n) = \sum_{i=0}^{n} p_{\mathcal{L}}(i)$ and $\gamma_{A_w}(n) = \sum_{i=0}^{n} p_w(i)$; if the monomial generators are assigned with arbitrary degrees (not necessarily 1), then these equations hold up to asymptotic equivalence. In any case, $\gamma_{A_w}(n) \sim np_w(n)$.

Let $A = \bigoplus_{n=0}^{\infty} A_n = k\langle\Sigma\rangle/I$ be a monomial algebra whose monomial generators are assigned with degree 1. For any $x \in \Sigma$ and $0 \neq f \in A$, let

$$\mathrm{pre}_x(f) = \max\{i \geq 0 \mid f \in x^i A\}, \quad \mathrm{suf}_x(f) = \max\{i \geq 0 \mid f \in Ax^i\}.$$

Let

$$A^x(n;i) = \mathrm{Span}_k\{w \in \mathrm{Mon}(A)_n \mid \mathrm{pre}_x(f) = i\}$$
$$A^x_y(n;i,j) = \mathrm{Span}_k\{w \in \mathrm{Mon}(A)_n \cap \langle y\rangle \mid \mathrm{pre}_x(w) = i, \ \mathrm{suf}_x(w) = j\}$$

and

$$A^x(\leqslant n;i) = \bigoplus_{l \leq n} A^x(l;i), \quad A^x_y(\leqslant n;i,j) = \bigoplus_{l \leq n} A^x_y(l;i,j)$$

so $A_{\leq n} = \bigoplus_{i=0}^{n} A^x(\leqslant n;i)$ and $\langle y\rangle_n = \bigoplus_{i+j\leq n} A^x_y(n;i,j)$.

3. Extensions of algebras

Theorem 3.1. *Let $R = k\langle x,y\rangle/I$ be an infinite-dimensional monomial algebra in which $y^2 = 0$ and y generates a locally nilpotent ideal. Let A be a graded, infinite-dimensional, nil algebra which is finitely generated in degree 1. Then there exists a finitely generated nil, graded algebra $\widehat{A}$ such that*

$$\dim_k \widehat{A}_{\leq n} = \dim_k A_{\leq n} + \sum_{i+j\leq n} \dim_k R^x_y(\leqslant n;i,j) \cdot \dim_k A_i \cdot \dim_k A_j$$

(we formally let $\dim_k A_0 = 1$.)

Proof. Observe that $R_n = \left(\bigoplus_{i+j\leq n} R^x_y(n;i,j)\right) \oplus kx^n$. Fix a k-linear basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots$ of A, consisting of monomials in some homogeneous generating set of A (say, a basis of A_1). For each $i \geq 1$, fix a basis element $w[i] \in \mathcal{B}_i = \mathcal{B} \cap A_i$ and let

$$W = \mathrm{Span}_k\{w[i] \mid i = 1, 2, \dots\},\ W^\perp = \mathrm{Span}_k(\mathcal{B} \setminus \{w[i] \mid i = 1, 2, \dots\}),$$

so $A = W \oplus W^\perp$. Consider the (non-unital) free product

$$A *_k \left(k[y]/\langle y^2\rangle\right) = A \oplus A^1 y A^1 \oplus A^1 y A y A^1 \oplus \cdots$$

with basis $\{b_0, b_0 y, y b_0, b_0 y b_1, b_0 y b_1 y, y b_0 y b_1, b_0 y b_1 y b_2, \dots \mid b_0, b_1, b_2, \dots \in \mathcal{B}\}$. For each y^2-free monomial divisible by y (namely, a nonzero monomial in $\langle y\rangle \triangleleft k\langle x, y\rangle/\langle y^2\rangle$),

$$\xi = x^{i_0} y x^{i_1} \cdots x^{i_{s-1}} y x^{i_s},$$

and for every $w_0 \in \mathcal{B}_{i_0}, w_1 \in \mathcal{B}_{i_s}$, consider the following element in the above free product:

$$\pi_{w_0,w_1}(\xi) = w_0 y w[i_1] \cdots w[i_{s-1}] y w_1.$$

In case that $\xi = x^{i_0}$, we let $\pi_{w_0,w_1}(\xi) = w_0$, and if $i_0 = 0$ or $i_s = 0$, we let $\pi_{w_0,w_1}(\xi) = y w[i_1] \cdots w[i_{s-1}] y w_1, w_0 y w[i_1] \cdots w[i_{s-1}] y$, respectively. We formally set $\pi_{w_0,w_1}(0) = 0$.

Remark 3.2. Notice that for any pair of disjoint sets of monomials $X, Y \subseteq k\langle x, y\rangle$, we have that $\mathrm{Span}_k \pi_{*,*}(X) \cap \mathrm{Span}_k \pi_{*,*}(Y) = 0$. Indeed, let $\theta\colon A *_k \left(k[y]/\langle y^2\rangle\right) \to R$ be the linear map defined by $\theta(b_0 y b_1 \cdots y b_r) = x^{\deg(b_0)} y x^{\deg(b_1)} \cdots y x^{\deg(b_r)}$ on basis elements and extended by linearity. Then $\theta \circ \pi_{w_0,w_1}(\xi) = \xi$ for any w_0, w_1 of degrees compatible with $\mathrm{pre}_x(\xi), \mathrm{suf}_x(\xi)$.

Consider the ideals

$$J = \langle y W^\perp y\rangle = \langle y u y \mid u \in \mathcal{B} \setminus \{w[1], w[2], \dots\}\rangle \triangleleft A *_k \left(k[y]/\langle y^2\rangle\right)$$

and

$$\widehat{I} = \left\langle \pi_{w_0,w_1}(\xi) \mid \xi \in \mathrm{Mon}(I),\ w_0 \in \mathcal{B}_{\mathrm{pre}_x(\xi)},\ w_1 \in \mathcal{B}_{\mathrm{suf}_x(\xi)} \right\rangle \triangleleft A *_k \left(k[y]/\langle y^2\rangle\right).$$

Notice that $I \subseteq \langle y\rangle \triangleleft k\langle x, y\rangle$, for otherwise, since I is a monomial ideal of $k\langle x, y\rangle$, it would contain some x^n, making $\langle y\rangle$ finite-codimensional in $R = k\langle x, y\rangle/I$; since by assumption, y generates a locally nilpotent ideal in R, it follows that R is finite-dimensional, a contradiction to the assumption.

We also note that every element in $\widehat{I}$ is a linear combination of elements of the form $\pi_{w_0,w_1}(\xi)$, $\xi \in I$ and elements from J. Indeed, fix $\xi = x^{i_0} y \cdots y x^{i_s} \in \mathrm{Mon}(I)$ and $w_0 \in \mathcal{B}_{\mathrm{pre}_x(\xi) = i_0}, w_1 \in \mathcal{B}_{\mathrm{suf}_x(\xi) = i_s}$:

- Given any $u \in \mathcal{B}_d$, we can write $u w_0 = \sum_{i=1}^n c_i w_i'$ for some $w_i' \in \mathcal{B}_{d+i_0}, c_i \in k$, since u, w_0 are homogeneous; we compute that
  $u \pi_{w_0,w_1}(\xi) = u w_0 y w[i_1] \cdots w[i_{s-1}] y w_1 = \sum_{i=1}^n c_i w_i' y w[i_1] \cdots w[i_{s-1}] y w_1 = \sum_{i=1}^n c_i \pi_{w_i',w_1}(x^d \xi)$.
  An analogous outcome holds for $\pi_{w_0,w_1}(\xi) u$.
- We compute $y \pi_{w_0,w_1}(\xi) = y w_0 y w[i_1] \cdots w[i_{s-1}] y w_1$, which is either in J if $w_0 \neq w[i_0]$, or otherwise, $w_0 = w[i_0]$ and $y \pi_{w_0,w_1}(\xi) = \pi_{w_0,w_1}(y\xi)$. An analogous outcome holds for $\pi_{w_0,w_1}(\xi) y$.

It follows by induction that every element in $\widehat{I}$ takes the desired form, and moreover, for every monomials ξ, ξ' and $w_0, w_1, w_0', w_1' \in \mathcal{B}$ of compatible degrees, we have that

$$\pi_{w_0,w_1}(\xi) \pi_{w_0',w_1'}(\xi') \equiv \sum_{i=1}^n c_i \pi_{u_0^i,u_1^i}(\xi\xi') \mod J \tag{3.1}$$

for suitable $u_0^i, u_1^i \in \mathcal{B}$ and $c_i \in k$.

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Finally, let

$$\widehat{A} = \frac{A *_k \left(k[y]/\langle y^2\rangle\right)}{J + \widehat{I}}.$$

Note that $\widehat{A}$ is naturally graded, extending the grading from A by letting $\deg(y) = 1$; indeed, this defines a grading on $A *_k \left(k[y]/\langle y^2\rangle\right)$, and both $J, \widehat{I}$ are homogeneous with respect to this grading, which thus induces a grading on the quotient ring. Furthermore, $\deg_{\widehat{A}}(\pi_{w_0,w_1}(\xi)) = \deg_R(\xi)$.

Consider the following homogeneous set:

$$\widehat{\mathcal{B}} = \{\pi_{w_0,w_1}(\xi) \mid \xi \in \mathrm{Mon}(R) \setminus \{1\},\ w_0 \in \mathcal{B}_{\mathrm{pre}_x(\xi)},\ w_1 \in \mathcal{B}_{\mathrm{suf}_x(\xi)}\} \subseteq \widehat{A}. \tag{3.2}$$

Notice that $\mathcal{B} \subseteq \widehat{\mathcal{B}}$ as seen by considering $\xi = x^n$, $n \in \mathbb{N}$. We claim that $\widehat{\mathcal{B}}$ spans $\widehat{A}$. Indeed, $A *_k \left(k[y]/\langle y^2\rangle\right)/J$ is spanned by

$$\mathcal{B} \cup \left\{\pi_{w_0,w_1}(\xi) \;\middle|\; \xi \in \mathrm{Mon}\Big(k\langle x,y\rangle/\langle y^2\rangle\Big) \cap \langle y\rangle,\ w_0 \in \mathcal{B}_{\mathrm{pre}_x(\xi)},\ w_1 \in \mathcal{B}_{\mathrm{suf}_x(\xi)}\right\},$$

and modulo $\widehat{I}$, we may further assume that $\xi \notin I$ – namely, $\xi \in \mathrm{Mon}(R)$.

We now claim that $\widehat{\mathcal{B}}$ is linearly independent. We first observe that $\widehat{\mathcal{B}}$ is a linearly independent subset of $A *_k \left(k[y]/\langle y^2\rangle\right)$. Next, if some nontrivial linear combination $\sum_{i=1}^n \alpha_i \pi_{w_0^i,w_1^i}(\xi_i)$ with $\xi_i \in \mathrm{Mon}(R)$ belongs to $\widehat{I} + J$ (as an element of $A *_k \left(k[y]/\langle y^2\rangle\right)$), then we claim that it in fact belongs to $\widehat{I}$. Recall that every element of $\widehat{I}$ is a linear combination of elements of the form $\pi_{w_0,w_1}(\xi)$, where $\xi \in I$ (for possibly different w_0, w_1, ξ) and elements from J. Write

$$\sum_{i=1}^n \alpha_i \pi_{w_0^i,w_1^i}(\underbrace{\xi_i}_{\in\mathrm{Mon}(R)}) = \underbrace{f}_{\in J} + \sum_{j=1}^m \beta_j \pi_{u_0^j,u_1^j}(\underbrace{\xi_j'}_{\in\mathrm{Mon}(I)})$$

so

$$\sum_{i=1}^n \alpha_i \pi_{w_0^i,w_1^i}(\xi_i) - \sum_{j=1}^m \beta_j \pi_{u_0^j,u_1^j}(\xi_j') = f \in J.$$

However, the left-hand side belongs to the vector space

$$A \oplus A^1 y A^1 \oplus A^1 y W y A^1 \oplus A^1 y W y W y A^1 \oplus \cdots$$

while

$$J \subseteq \bigoplus_{i,j\geq 1} A^1 \underbrace{yA\cdots Ay}_{i \text{ times } y} W^\perp \underbrace{yA\cdots Ay}_{j \text{ times } y} A^1,$$

so f must vanish and $\sum_{i=1}^n \alpha_i \pi_{w_0^i,w_1^i}(\xi_i) = \sum_{j=1}^m \beta_j \pi_{u_0^j,u_1^j}(\xi_j')$. Since $\mathrm{Mon}(I), \mathrm{Mon}(R)$ are disjoint sets of monomials in $k\langle x,y\rangle$, we have by Remark 3.2 that $\sum_{i=1}^n \alpha_i \pi_{w_0^i,w_1^i}(\xi_i) = 0$ in the free product $A *_k \left(k[y]/\langle y^2\rangle\right)$, a contradiction.

Let us now provide a formula for products of basis elements from $\widehat{\mathcal{B}}$. Let $\psi_n : A \to k$ be the linear functional projecting an element to the (possibly zero) coefficient of $w[n]$ of it, when written uniquely

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as a linear combination of $\mathcal{B}$. If $w_1, w_2 \in A$ multiply as $w_1 w_2 = \sum_{b\in\mathcal{B}} \alpha_b b$, then

$$w_0 y w[i_1] \cdots w[i_{s-1}] y w_1 * w_2 = \sum_{b\in\mathcal{B}} \alpha_b w_0 y w[i_1] \cdots w[i_{s-1}] y b$$

$$w_1 * w_2 y w[i_1] \cdots w[i_{s-1}] y w_3 = \sum_{b\in\mathcal{B}} \alpha_b b y w[i_1] \cdots w[i_{s-1}] y w_3$$

and

$$\begin{aligned} &w_0 y w[i_1] \cdots w[i_{s-1}] y w_1 * w'_0 y w[j_1] \cdots w[j_{t-1}] y w'_1 \\ &\quad = \psi_p(w_1 w'_0) \cdot w_0 y w[i_1] \cdots y w[p] y \cdots w[j_{t-1}] y w'_1, \end{aligned}$$

where $p = \deg(w_1) + \deg(w'_0)$.

Let $\sum_{i=1}^{l} c_i \pi_{w_0^i, w_1^i}(\xi_i) \in \langle y \rangle \triangleleft \widehat{A}$. Then $\xi_1, \ldots, \xi_l \in \langle y \rangle \triangleleft R$. By assumption, $\langle y \rangle$ is a locally nilpotent ideal of R, so there exists some d such that

$$\forall 1 \leq i_1, \ldots, i_d \leq l,\ \xi_{i_1} \cdots \xi_{i_d} = 0,$$

and it follows that

$$\left(\sum_{i=1}^{l} c_i \pi_{w_0^i, w_1^i}(\xi_i) \right)^d \in \operatorname{Span}_k \{ \pi_{*,*}(\xi_{i_1} \cdots \xi_{i_d}) \mid 1 \leq i_1, \ldots, i_d \leq l \} = 0$$

as seen by (3.1). Consequently, $\langle y \rangle \triangleleft \widehat{A}$ is a nil ideal. Notice that $\widehat{A}/\langle y \rangle \cong A$, which is nil. Therefore,

$$0 \to \langle y \rangle \to \widehat{A} \to A \to 0$$

is an extension of two nil algebras; hence, $\widehat{A}$ is nil.

As seen by (3.2),

$$\widehat{A}_n \cong A_n \oplus \bigoplus_{i+j \leq n} A_i \otimes_k R_y^x(n; i, j) \otimes_k A_j$$

and

$$\widehat{A}_{\leq n} \cong A_{\leq n} \oplus \bigoplus_{i+j \leq n} A_i \otimes_k R_y^x(\leqslant n; i, j) \otimes_k A_j$$

isomorphisms of vector spaces (formally taking $A_0 = k$); hence,

$$\dim_k \widehat{A}_{\leq n} = \dim_k A_{\leq n} + \sum_{i+j \leq n} \dim_k R_y^x(\leqslant n; i, j) \cdot \dim_k A_i \cdot \dim_k A_j,$$

as claimed. □

Proof of Theorem 1.2. Let $f\colon \mathbb{N} \to \mathbb{N}$ be the growth function of any finitely generated infinite-dimensional algebra, not of linear growth. We may assume that f is subexponential (nil algebras of exponential growth exist by the Golod-Shafarevich theorem, and all of the exponential growth functions are equivalent to each other). By [9, Theorem 1.1], f satisfies $f'(n) \geq n + 1$ and $f'(n) \leq f'(m)^2$ for every $m \leq n \leq 2m$. Moreover, by the same theorem, f is then equivalent to the growth function of an algebra R explicitly constructed therein. Let us recall now the general structure of that algebra. This is

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a monomial algebra $R = k\langle x, y\rangle/I$ in which the set of nonzero monomials is

$$\bigcup_{n\geq 1} T(d_n, n) \cup \bigcup_{n\geq 1} x^{e_n} T(d_n - 1, n - e_n),$$

where

$$T(d, n) = \{x^n\} \cup \{x^i yx^{a_1} yx^{a_2} \cdots x^{a_s} yx^j \text{ of length } n \text{ with } a_1, \dots, a_s \geq d\}$$

and for some sequences $\{d_n\}_{n=1}^{\infty}, \{e_n\}_{n=1}^{\infty}$ of positive integers, explicitly constructed in the proof of [9, Theorem 1.1]. Furthermore, the sequence $\{d_n\}_{n=1}^{\infty}$ tends to infinity since the realized growth function f is subexponential (see [9, Page 691]). Notice also that $y^2 = 0$. Since every monomial w in the ideal $\langle y\rangle^2 = RyRyR$ contains an occurrence of $yx^i y$ for some $i > 0$, it follows that for some n (in fact, for every n such that $d_n > i + 1$), $wR_n = R_n w = 0$, and therefore, $\langle y\rangle$ is locally nilpotent. Furthermore,

$$\dim_k \langle y\rangle_{\leq n} = \dim_k R_{\leq n} - \dim_k (k + kx + \cdots + kx^n) = \dim_k R_{\leq n} - (n + 1). \tag{3.3}$$

Let us now conclude the proof, using Theorem 3.1.

Case I: Countable base fields. Assume that k is countable. We use R from the above discussion as a base ring for the construction given in Theorem 3.1, along with the nil graded algebra A from [25], constructed over an arbitrary countable field. As proven in [25, Theorem 7.5], $\dim_k A_n \leq Cn^2 \log^6 n$ for all n and for some constant $C > 0$. Consider the finitely generated nil graded algebra $\widehat{A}$ resulting by applying Theorem 3.1 to this setting. Then, for every $\varepsilon > 0$,

$$\begin{aligned} \dim_k \widehat{A}_{\leq n} &= \dim_k A_{\leq n} + \sum_{i+j\leq n} \dim_k R_y^x(\leqslant n; i, j) \cdot \dim_k A_i \cdot \dim_k A_j \\ &\leq Cn^3 \log^6 n + \sum_{i+j\leq n} \dim_k R_y^x(\leqslant n; i, j) \cdot Ci^2 \log^6 i \cdot Cj^2 \log^6 j \\ &\leq 2C^2 n^4 \log^{12} n \dim_k R_{\leq n} \leq n^{4+\varepsilon} f(n). \end{aligned}$$

However, by Theorem 3.1,

$$\dim_k \widehat{A}_{\leq n} \geq \dim_k A_{\leq n} + \sum_{i+j\leq n} \dim_k R_y^x(\leqslant n; i, j) \geq n + \langle y\rangle_{\leq n} = \dim_k R_{\leq n} - 1 \sim f(n).$$

Case II: Arbitrary base fields. Assume that k is of arbitrary cardinality. We use R from the above discussion as a base ring for the construction given in Theorem 3.1, along with a nil graded algebra A from [8], constructed over an arbitrary base field. As proven in [25], for any super-polynomial function, one can construct a nil graded algebra A over k whose growth is bounded from above by that function. Given a function $\omega(n) \xrightarrow{n\to\infty} \infty$, let A be a nil graded algebra over k whose growth satisfies $\gamma_A(n) \leq n^{\frac{1}{2}\omega(n)}$. Consider the finitely generated nil graded algebra $\widehat{A}$ resulted by applying Theorem 3.1 to this setting. Then

$$\begin{aligned} \dim_k \widehat{A}_{\leq n} &= \dim_k A_{\leq n} + \sum_{i+j\leq n} \dim_k R_y^x(\leqslant n; i, j) \cdot \dim_k A_i \cdot \dim_k A_j \\ &\leq \dim_k A_{\leq n} + \sum_{i+j\leq n} \dim_k R_y^x(\leqslant n; i, j) \cdot \left(n^{\frac{1}{2}\omega(n)}\right)^2 \\ &\leq n^{\frac{1}{2}\omega(n)} + n^{\omega(n)} \dim_k R_{\leq n} \sim n^{\omega(n)} f(n). \end{aligned}$$

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However, by Theorem 3.1,

$$\dim_k \widehat{A}_{\leq n} \geq \dim_k A_{\leq n} + \sum_{i+j\leq n} \dim_k R_y^x(\leqslant n; i, j) \geq n + \langle y \rangle_{\leq n} = \dim_k R_{\leq n} - 1 \sim f(n).$$

The proof is completed. □

4. Applications

Let A be a finitely generated associative algebra. Then A, equipped with the Lie brackets $[a, b] := ab - ba$, is a Lie algebra, denoted by $A^{(-)}$. Notice that if A is graded, then so are $A^{(-)}$ and all of its Lie derived powers. Furthermore,

Lemma 4.1. *Let A be a nil (resp., graded-nil) associative algebra. Then $A^{(-)}$ is a nil (resp., graded-nil) Lie algebra.*

Proof. Let A be a nil associative algebra. For every $a \in A$, there is some d such that $a^d = 0$. Since

$$\mathrm{ad}_a^m(x) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} a^i x a^{m-i}$$

for every $m \geq 0$, we obtain that $\mathrm{ad}_a^{2d+1}(x)$ is a linear combination of monomials of the form $a^i x a^j$ with $i + j = 2d + 1$, so $\mathrm{ad}_a^{2n+1}(x) \in a^d A + A a^d = 0$. It follows that L is a nil Lie algebra. If $A = \bigoplus_{n=0}^{\infty} A_n$ is a graded-nil associative algebra, then the above argument applies for every homogeneous element a (and arbitrary x), proving that L is a graded-nil Lie algebra. □

In general, finite generation of A as an associative algebra does not guarantee the finite generation of the Lie algebras $A^{(-)}$ and $[A^{(-)}, A^{(-)}]$; but if A is generated by finitely many nilpotent elements, then $[A^{(-)}, A^{(-)}]$ is finitely generated as well and, furthermore, grows asymptotically as A:

Theorem 4.2 [1, Theorem 2]. *Let A be a graded associative algebra generated by finitely many nilpotent elements. Then $[A^{(-)}, A^{(-)}]$ is a finitely generated Lie algebra and*

$$\gamma_{[A^{(-)},A^{(-)}]}(n) \sim \gamma_A(n).$$

Proof of Corollary 1.3. Fix an arbitrary $\alpha \in (0, 1)$. There exists a finitely generated algebra B whose growth function is $\gamma_B(n) \sim \exp(n^\alpha)$ (see, for example, [35, Corollary D]). By Theorem 1.2, it follows that, for every increasing, super-polynomial function $\omega\colon \mathbb{N} \to \mathbb{N}$, there exists a finitely generated nil graded algebra over an arbitrary base field whose growth is

$$\gamma_B(n) \preceq \gamma_A(n) \preceq \omega(n)\gamma_B(n).$$

Taking $\omega(n) = n^{\log n}$, we get, for $n \gg_\alpha 1$, that

$$\begin{aligned}\exp(n^\alpha) \preceq \gamma_A(n) &\preceq n^{\log n} \exp(n^\alpha)\\ &\preceq \exp((2n)^\alpha) \sim \exp(n^\alpha),\end{aligned}$$

as claimed.

Now let $L = [A^{(-)}, A^{(-)}]$. By [1, Theorem 2], L is a finitely generated graded Lie algebra and $\gamma_L(n) \sim \gamma_A(n) \sim \exp(n^\alpha)$, and by Lemma 4.1, L is a nil Lie algebra. □

Proof of Corollary 1.4. Fix an arbitrary $\alpha \geq 6$. Let A be a finitely generated nil graded algebra of $\mathrm{GKdim}(A) \leq 3$ (see [25]); denote $\rho := \mathrm{GKdim}(A)$. Let $d \in \mathbb{Z}_{\geq 0}$ and $0 < \beta \leq 1$ be such that $\alpha = 2\rho + d\beta$ (in particular, if $\alpha > 6$, then $d = \lceil \alpha - 2\rho \rceil$, and $\beta = \frac{\alpha - 2\rho}{d}$ and if $\alpha = 6$, take $d = 0, \beta = 1$). Let $S = \{\lceil n^{1/\beta} \rceil \mid n \in \mathbb{N}\}$ and observe that $\#(S \cap [1, n]) = \lfloor n^\beta \rfloor$.

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Consider the monomial algebra

$$R = \frac{k\langle x, y\rangle}{\left\langle y^2\right\rangle + \langle y\rangle^{d+2} + \langle yx^i y \mid i \notin S\rangle}.$$

Notice that the ideal $\langle y\rangle \triangleleft R$ is nilpotent. Now for any i, j,

$$\begin{aligned}\dim_k R_y^x(\leqslant n; i, j) &\leq \sum_{c=1}^{d+1} \#\left\{\vec{t} \in S^{c-1} \;\middle|\; \sum_{i=1}^{c-1} t_i \leq n\right\} \\ &\leq \sum_{c=1}^{d+1} \#(S \cap [1, n])^{c-1} \leq (d+1) \cdot \#(S \cap [1, n])^d \leq (d+1)n^{d\beta}.\end{aligned} \tag{4.1}$$

Moreover, if $i + j \leq 2n$, then, given $\vec{t} \in (S \cap [1, n])^d$, we have a nonzero monomial $\xi_{\vec{t}} \in R_y^x(\leqslant Kn; i, j)$

$$x^i yx^{t_1} y \cdots yx^{t_d} yx^j$$

for some constant K (in fact, one can take any K such that $Kn \geq dn + (d + 1) + i + j$, so we can take $K = 5d$), and the assignment $\vec{t} \mapsto \xi_{\vec{t}}$ is injective. It follows that

$$\begin{aligned}\dim_k R_y^x(\leqslant Kn; i, j) &\geq \#(S \cap [1, n])^d \\ &\geq \left(\lfloor n^\beta \rfloor\right)^d \geq 2^{-d} n^{d\beta}\end{aligned} \tag{4.2}$$

(the last inequality follows since $\lfloor x\rfloor \geq \frac{1}{2}x$ for every $x \geq 1$.) Thus, by Theorem 3.1, there exists a finitely generated nil graded algebra $\widehat{A}$ such that

$$\dim_k \widehat{A}_{\leq n} = \dim_k A_{\leq n} + \sum_{i+j\leq n} \dim_k R_y^x(\leqslant n; i, j) \cdot \dim_k A_i \cdot \dim_k A_j. \tag{4.3}$$

Hence, by (4.1),

$$\begin{aligned}\dim_k \widehat{A}_{\leq n} &\leq \dim_k A_{\leq n} + \sum_{i+j\leq n} (d+1)n^{d\beta} \cdot \dim_k A_i \cdot \dim_k A_j \\ &\leq \dim_k A_{\leq n} + (d+1)n^{d\beta}(\dim_k A_{\leq n} + 1)^2 \\ &\leq (d+2)n^{d\beta}(\dim_k A_{\leq n} + 1)^2,\end{aligned} \tag{4.4}$$

and by (4.2),

$$\begin{aligned}\dim_k \widehat{A}_{\leq Kn} &\geq \sum_{i+j\leq 2n} \dim_k R_y^x(\leqslant Kn; i, j) \cdot \dim_k A_i \cdot \dim_k A_j \\ &\geq 2^{-d} n^{d\beta} \sum_{i+j\leq 2n} \dim_k A_i \cdot \dim_k A_j \\ &\geq 2^{-d} n^{d\beta} (\dim_k A_{\leq n})^2.\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), it follows that

$$\mathrm{GKdim}(\widehat{A}) = \limsup_{n\to\infty} \frac{\log \dim_k \widehat{A}_{\leq n}}{\log n} = 2\rho + d\beta = \alpha,$$

as required.

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Now let $L = [A^{(-)}, A^{(-)}]$. By [1, Theorem 2], L is a finitely generated graded Lie algebra and $\mathrm{GKdim}(L) = \mathrm{GKdim}(A) = \alpha$, and by Lemma 4.1, L is a nil Lie algebra. □

5. Graded-nil algebras

5.1. Morphic words and their algebras

Let $\Sigma = \{a, b\}$ and let $\phi\colon \Sigma^* \to \Sigma^*$ be an endomorphism of the free monoid $\Sigma^* = \langle a, b\rangle$. Assume that $\phi(a), \phi(b) \neq 1$ for all $i \geq 1$. If $\phi(a) = au$ for some $u \in \Sigma^*$, then there is a (unique) fixed point of ϕ in its action on the set of (right) infinite words $\Sigma^{\mathbb{N}}$:

$$w = au\phi(u)\phi^2(u)\cdots$$

Consider the monomial algebra

$$A_w = k\langle\Sigma\rangle/\langle v \in \Sigma^* \mid v \text{ is not a factor of } w\rangle.$$

In [5, Proposition 2.1], we proved that, under some technical assumptions on ϕ, if we assign certain degrees to the monomial generators of A_w, then the positive part $A_w^+ = \bigoplus_{n=1}^{\infty}(A_w)_n$ is graded-nilpotent; that is, every subalgebra of A_w^+ generated by homogeneous elements of an equal degree is nilpotent (and hence finite-dimensional). This property is stronger than being graded-nil. Let us focus on a concrete example, which will serve as a 'basis' for the construction in Theorem 1.5.

Example 5.1 [5, Example 2.2]. Let $\Sigma = \{a, b\}$ and let $\phi\colon \Sigma^* \to \Sigma^*$ be given by $\phi(a) = ab$, $\phi(b) = b^2a$. Then ϕ satisfies the conditions of [5, Proposition 2.1], and consequently, the resulting infinite-word

$$w = au\phi(u)\phi^2(u)\cdots = abbbabbabbaab\cdots$$

gives rise to a monomial algebra A_w which, when endowed with the grading $\deg(a) = 1, \deg(b) = 2$, has a graded-nilpotent positive part A_w^+. Furthermore, by [6, Proposition 3.1], the algebra A_w has a quadratic growth; that is, $c_1n^2 \leq \gamma_{A_w}(n) \leq c_2n^2$ for some constants $c_1, c_2 > 0$. Equivalently, the complexity function $p_w(n) = \dim_k(A_w)_n$ grows linearly with n, say, $c_1 n \leq p_w(n) \leq c_2(n+1)$ for some $c_1, c_2 > 0$, for all $n \geq 0$ (we put $c_2(n+1)$ on the right-hand side since $p_w(0) = 1$).

5.2. Growth of graded-nilpotent algebras

Theorem 5.2. *Let $R = k\langle x, y\rangle/I$ be an infinite-dimensional monomial algebra in which $y^2 = 0$ and y generates a locally nilpotent ideal. Then there exists a finitely generated graded-nil algebra $\widehat{A}$ such that*

$$c_1 \cdot \sum_{i=0}^{n} i \cdot \dim_k R^x(\leqslant n; i) \leq \dim_k \widehat{A}_{\leq n} \leq c_2 \cdot \sum_{i=0}^{n} (i+1) \cdot \dim_k R^x(\leqslant n; i)$$

for some $c_1, c_2 > 0$.

Proof. Let w be the infinite word from Example 5.1 and let A_w be its associated monomial algebra. Let $w[i]$ denote the length-i prefix of w (we think of $w[0]$ as the empty monomial '1') – for example,

$$w[0] = 1,\ w[1] = a,\ w[2] = ab,\ w[3] = abb, \ldots$$

Consider the following formal language $\mathcal{L} \subseteq \{a, b, y\}^*$:

$$\mathcal{L} = \{uyw[i_1]y\cdots yw[i_{s+1}] \mid u \in \mathrm{Mon}(A_w),\ |u| = i_0,\ x^{i_0}yx^{i_1}y\cdots yx^{i_{s+1}} \in \mathrm{Mon}(R)\}. \tag{5.1}$$

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(We include the cases where the monomial in $\mathrm{Mon}(R)$ is x^{i_0}, and i_0, i_{s+1} may be zero.) Notice that $\mathcal{L}$ is a hereditary language. Indeed, consider a word

$$W = uyw[i_1]y \cdots yw[i_{s+1}],$$

where u is a factor of w of length i_0 and $x^{i_0}yx^{i_1}y \cdots yx^{i_{s+1}}$ is a nonzero monomial in R. Then every subword W_0 of W takes one of the forms

$$v,\ vy,\ yv',\ vyw[i_p]y \cdots yw[i_q]yv'$$

where $1 \le p \le q \le s$ and v is either a suffix of u of length $0 \le l \le i_0$, or a suffix of $w[i_{p-1}]$ of length $l \le i_{p-1}$; and v' is a prefix of $w[i_{q+1}]$ of length $0 \le l' \le i_{q+1}$; hence, $v' = w[l']$. Since $x^{i_0}yx^{i_1}y \cdots yx^{i_{s+1}}$ is a nonzero monomial in R, so is $x^l yx^{i_p}y \cdots yx^{i_q}yx^{l'}$, and thus, by (5.1), it follows that $W_0 \in \mathcal{L}$.

Let $A_{\mathcal{L}} = k\langle a, b, y\rangle/J$ be the monomial algebra associated with $\mathcal{L}$. Endow $A_{\mathcal{L}}$ with the grading induced by $\deg(a) = 1, \deg(b) = 2, \deg(y) = 1$. Let $A_{\mathcal{L}}^+ = \bigoplus_{n=1}^{\infty}(A_{\mathcal{L}})_n$ be the positive part of $A_{\mathcal{L}}$. Then $A_{\mathcal{L}}^+$ is finitely generated (by a, b, y) and fits into

$$0 \to \langle y\rangle \to A_{\mathcal{L}}^+ \to A_w^+ \to 0. \tag{5.2}$$

Define a surjective map $\theta\colon \mathcal{L} \to \mathrm{Mon}(R)$ by

$$\theta(uyw[i_1]y \cdots yw[i_{s+1}]) = x^{|u|}yx^{i_1}y \cdots yx^{i_{s+1}}.$$

Notice that $\theta(\mathcal{L} \cap \langle y\rangle) = \mathrm{Mon}(R) \cap \langle y\rangle$ and that $\theta(W_1W_2) = \theta(W_1)\theta(W_2)$ if $W_1W_2 \neq 0$. We claim that $\langle y\rangle \triangleleft A_{\mathcal{L}}$ is locally nilpotent. Indeed, for any finite set of monomials from $\mathcal{L}$, each of which containing an occurrence of y, say $W_1, \ldots, W_t$, consider $\theta(W_1), \ldots, \theta(W_t) \in \langle y\rangle \triangleleft R$. Since $\langle y\rangle \triangleleft R$ is locally nilpotent by assumption, there exists some N such that $\theta(W_{\mu(1)}) \cdots \theta(W_{\mu(N)}) = 0$ for every $\mu\colon \{1, \ldots, N\} \to \{1, \ldots, t\}$. If $W_{\mu(1)} \cdots W_{\mu(n)} \neq 0$, then $\theta\big(W_{\mu(1)} \cdots W_{\mu(n)}\big) = \theta(W_{\mu(1)}) \cdots \theta(W_{\mu(N)}) = 0$, a contradiction (recall that $\theta\colon \mathcal{L} \to \mathrm{Mon}(R)$ and $0 \notin \mathrm{Mon}(R)$). It follows that $\langle y\rangle \triangleleft A_{\mathcal{L}}$ is a locally nilpotent ideal. Let us now prove that $A_{\mathcal{L}}^+$ is graded-nilpotent. Fix $d \ge 1$ and let $(A_{\mathcal{L}})_d = \mathrm{Span}_k\{f_1, \ldots, f_m\}$. Since A_w^+ is graded-nilpotent, it follows from (5.2) that there exists some N_1 such that every product of N_1 elements among $f_1, \ldots, f_m$ vanishes in A_w^+. Again by the exact sequence (5.2), it follows that every product of N_1 elements among $f_1, \ldots, f_m$ lies in $\langle y\rangle \triangleleft A_{\mathcal{L}}$. By the local nilpotency of $\langle y\rangle \triangleleft A_{\mathcal{L}}$, it follows that there is some N_2 such that every product of N_2 elements among $\{f_1, \ldots, f_m\}^{N_1}$ vanishes. Therefore, $(A_{\mathcal{L}})_d^{N_1N_2} = 0$, showing that $A_{\mathcal{L}}^+$ is graded-nilpotent.

Let us analyze the growth function of $A_{\mathcal{L}}^+$. By (5.1),

$$\begin{aligned}\dim_k(A_{\mathcal{L}})_n &= \sum_{W \in \mathrm{Mon}(R)_n} \#\theta^{-1}(W) \\ &= \sum_{i=0}^{n} p_w(i) \cdot \#\{W \in \mathrm{Mon}(R)_n \mid \mathrm{pre}_x(W) = i\} \\ &= \sum_{i=0}^{n} p_w(i) \cdot \dim_k R^x(n; i)\end{aligned}$$

so

$$c_1 \cdot \sum_{i=0}^{n} i \cdot \dim_k R^x(\leqslant n; i) \le \dim_k(A_{\mathcal{L}}^+)_{\le n} \le c_2 \cdot \sum_{i=0}^{n}(i+1) \cdot \dim_k R^x(\leqslant n; i),$$

for c_1, c_2 from Example 5.1, as claimed. □

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Proof of Theorem 1.5. Let $f\colon \mathbb{N} \to \mathbb{N}$ be the growth function of any finitely generated infinite-dimensional algebra; we may assume that f is not of linear growth. As in the proof of Theorem 1.2, we may assume that f is subexponential. Using [9, Theorem 1.1], let $R = k\langle x,y\rangle/I$ be a monomial algebra of growth $\gamma_R \sim f$. As observed in the proof of Theorem 1.2, $\langle y\rangle \triangleleft R$ is locally nilpotent. Applying Theorem 5.2, we obtain a finitely generated, graded-nilpotent algebra $\widehat{A} = \bigoplus_{n=1}^{\infty} \widehat{A}_n$ such that

$$\begin{aligned}\dim_k \widehat{A}_{\leq n} &\leq c_2 \cdot \sum_{i=0}^{n} (i+1)\cdot \dim_k R^x(\leqslant n; i) \\ &\leq 2c_2 n \cdot \sum_{i=0}^{n} \dim_k R^x(\leqslant n; i) = 2c_2 n \dim_k R_{\leq n} \sim nf(n)\end{aligned}$$

and

$$\begin{aligned}\dim_k \widehat{A}_{\leq n} &\geq c_1 \cdot \sum_{i=0}^{n} i\cdot \dim_k R^x(\leqslant n; i) \\ &\geq c_1 \cdot \sum_{i=0}^{n} \dim_k R^x(\leqslant n; i) = c_1 \cdot \dim_k R_{\leq n} \sim f(n),\end{aligned}$$

so $f(n) \leq \gamma_A(n) \leq nf(n)$. The proof is completed. □

Proof of Corollary 1.6. In the proof of Corollary 1.4 we constructed, for each $d \in \mathbb{Z}_{\geq 0}$, $0 < \beta \leq 1$, a monomial algebra

$$R = \frac{k\langle x,y\rangle}{\left\langle y^2\right\rangle + \langle y\rangle^{d+2} + \langle yx^i y \mid i \notin S\rangle}$$

satisfying, for some constants $K, \lambda_1, \lambda_2 > 0$ (in fact, $\lambda_1 = 2^{-d}, \lambda_2 = d+1$ and $K = 5d$),

- ○ $\langle y\rangle \triangleleft R$ is locally nilpotent;
- ○ For every $i + j \leq 2n$, we have that $\dim_k R_y^x(\leqslant Kn; i,j) \geq \lambda_1 n^{d\beta}$;
- ○ For every i,j,n, we have that $\dim_k R_y^x(\leqslant n; i,j) \leq \lambda_2 n^{d\beta}$.

It follows that for every $i \leq n$, we have

$$\dim_k R^x(\leqslant n; i) = 1 + \sum_{j=0}^{n} \dim_k R_y^x(\leqslant n; i,j) \leq 1 + (n+1)\cdot \lambda_2 n^{d\beta} \leq \lambda_2' n^{d\beta+1}$$

for some constant $\lambda_2' > 0$. In addition, we have, for $i \leq n$,

$$\dim_k R^x(\leqslant Kn; i) \geq \sum_{j=1}^{n} \dim_k R_y^x(\leqslant Kn; i,j) \geq \sum_{j=1}^{n} \lambda_1 n^{d\beta} = \lambda_1 n^{d\beta+1}.$$

We now apply Theorem 5.2 to the algebra R. As a result, we obtain a finitely generated, graded-nilpotent algebra $\widehat{A} = \bigoplus_{n=1}^{\infty} \widehat{A}_n$ whose growth function satisfies

$$\dim_k \widehat{A}_{\leq n} \leq c_2 \cdot \sum_{i=0}^{n} (i+1)\cdot \dim_k R^x(\leqslant n; i) \leq 2c_2 n \cdot \sum_{i=0}^{n} \lambda_2' n^{d\beta+1} \leq \lambda_2'' n^{d\beta+3}$$

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(for some $\lambda_2'' > 0$) and

$$\dim_k \widehat{A}_{\leq Kn} \geq c_1 \cdot \sum_{i=0}^{Kn} i \cdot \dim_k R^x(\leqslant Kn; i)$$
$$\geq c_1 \cdot \sum_{i=\lfloor n/2 \rfloor}^{n} i \cdot \dim_k R^x(\leqslant Kn; i) \geq c_1 \left\lfloor \frac{n}{2} \right\rfloor \cdot \frac{n}{2} \cdot \lambda_1 n^{d\beta+1} \geq \lambda_1' n^{d\beta+3}$$

for some $\lambda_1' > 0$ (and for all $n > 2$). It follows that $\mathrm{GKdim}(\widehat{A}) = d\beta + 3$. Since for any $\alpha \geq 3$ we can find $d \in \mathbb{Z}_{\geq 0}$, $0 < \beta \leq 1$ such that $\alpha = d\beta + 3$, the claim follows. □

6. Matrix wreath products

6.1. Matrix wreath products

Let A, B be k-algebras. We let their matrix wreath product be

$$A \wr B = B + \mathrm{Lin}_k(B^1, B^1 \otimes_k A).$$

Given a linear map $\phi\colon B^1 \to A$ defined on B^1, the unital hull of B, we let $c_\phi\colon b \mapsto 1 \otimes \phi(b)$ and consider the 'restricted' matrix wreath product:

$$A \wr_\phi B = k\langle B, c_\phi \rangle \subseteq A \wr B.$$

If B is a finitely generated k-algebra generated by some finite-dimensional subspace $V \leq B$, then $A \wr_\phi B$ is the finitely generated k-subalgebra of $A \wr B$ generated by $V + kc_\phi$. These have proven to be extremely useful to construct algebras with prescribed growth and desired algebraic properties such as algebraicity and primeness [2, 17]. In particular, if A is stably nil (that is, $M_n(A)$ is nil for every n) and B is nil, then $A \wr_\phi B$ is nil too [2, §4]. Let

$$\gamma_\phi(n) = \dim_k \sum_{i_1+\cdots+i_s \leq n} \phi(V^{i_1}) \cdots \phi(V^{i_s})$$

be the growth function of ϕ. By [17, Lemma 2.2],

$$\gamma_\phi(n) \leq \gamma_{A \wr_\phi B}(n) \leq \gamma_B(n)^2 \gamma_\phi(n), \tag{6.1}$$

and if in addition, ϕ satisfies some 'density' conditions, then $\gamma_{A\wr_\phi B}(n) \sim \gamma_\phi(n)\gamma_B(n)^2$ [2, Lemma 3.9].

6.2. Theorem 1.2 revisited

Fix an arbitrary base field k. Let $f\colon \mathbb{N} \to \mathbb{N}$ be the growth function of a finitely generated infinite-dimensional algebra, not of linear growth. In particular, $f'(n) \geq n + 1$; as in the proof of Theorem 1.2, we may assume that f is subexponential.

Let $R = k\langle x, y\rangle/I$ be the monomial algebra constructed in [9] of growth $\gamma_R \sim f$. As in the proof of Theorem 1.2, the ideal $J := \langle y \rangle \triangleleft R$ is locally nilpotent, and by (3.3), $\dim_k J_{\leq n} = \dim_k R_{\leq n} - (n + 1)$. Since $y^2 \in I$, then, as a k-algebra, J is generated by the set $\{x^i y x^j\}_{i,j\geq 0}$. Let $B = \bigoplus_{n=1}^{\infty} B_n$ be a finitely generated infinite-dimensional nil, graded k-algebra; then $\dim_k B_n \geq n$ for every $n \geq 1$, for otherwise, B has a linear growth by [23, Lemma 2.4], which implies that B satisfies a polynomial identity [33]; but a finitely generated nil algebra satisfying a polynomial identity is finite-dimensional [22, Theorem 5].

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Thus, we can define a system of (arbitrary) surjective linear maps

$$\phi_n : B_n \twoheadrightarrow \mathrm{Span}_k\{x^i y x^{n-1-i} \mid 0 \le i \le n-1\} \subseteq J \ \text{ for } \ n = 1, 2, \ldots$$

and let

$$\phi = \oplus_n \phi_n : B^1 \to J,$$

setting $\phi(1) = y$. This defines a restricted matrix wreath product $J \wr_\phi B$. Since B is a finitely generated nil algebra and J is locally nilpotent (and hence stably nil), it follows that $J \wr_\phi B$ is a finitely generated nil algebra.

Let us estimate $\gamma_\phi(n)$. We have

$$\sum_{i_1+\cdots+i_s \le n} \phi(B_{i_1}) \cdots \phi(B_{i_s}) \subseteq J \cap \left(\sum_{i_1+\cdots+i_s \le n} R_{i_1} \cdots R_{i_s} \right) = J_{\le n}. \tag{6.2}$$

Conversely, let $\xi \in \mathrm{Mon}(R) \cap J_{\le n}$

$$\xi = x^{i_0} y x^{i_1} \cdots x^{i_{m-1}} y x^{i_m}$$

for some $m \ge 1,\ i_0, \ldots, i_{m+1} \ge 0$, and if $m > 1$, then $i_1, \ldots, i_{m-1} > 0$. Decompose

$$\begin{aligned} \xi &= (x^{i_0} y)(x^{i_1} y) \cdots (x^{i_{m-1}} y x^{i_m}) \\ &= \phi_{i_0+1}(b_0)\phi_{i_1+1}(b_1) \cdots \phi_{i_{m-1}+i_m+1}(b_{m-1}) \\ &= \phi(b_0)\phi(b_1) \cdots \phi(b_{m-1}) \end{aligned} \tag{6.3}$$

for suitable $b_0 \in B_{i_0+1}, b_1 \in B_{i_1+1}, \ldots, b_{m-1} \in B_{i_{m-1}+i_m+1}$, where

$$(i_0 + 1) + (i_1 + 1) + \cdots + (i_{m-1} + i_m + 1) = |\xi| \le n;$$

hence, $\xi \in \sum_{p_1+\cdots+p_s \le n} \phi(B_{p_1}) \cdots \phi(B_{p_s})$. Therefore,

$$J_{\le n} = \mathrm{Span}_k(\mathrm{Mon}(R) \cap J_{\le n}) \subseteq \sum_{p_1+\cdots+p_s \le n} \phi(B_{p_1}) \cdots \phi(B_{p_s}).$$

Together with (6.2), we conclude that

$$\gamma_\phi(n) = \dim_k J_{\le n} = \gamma_R(n) - (n+1) \sim \gamma_R(n) \sim f(n). \tag{6.4}$$

Case I: Countable base fields. Let k be a countable field. We specify B to be a nil algebra of GK-dimension ≤ 3, which can be constructed over any countable field [24, 25]. By (6.1) and (6.4), we obtain, for every $\varepsilon > 0$, that

$$f(n) \sim \gamma_\phi(n) \preceq \gamma_{J \wr_\phi B}(n) \preceq \gamma_B(n)^2 \gamma_\phi(n) \preceq n^{6+\varepsilon} f(n).$$

Case II: Arbitrary base fields. Let k be an arbitrary field. Let $\omega(n) \xrightarrow{n\to\infty} \infty$. We specify B to be the Bell-Young nil algebra [8] of growth $\gamma_B(n) \preceq n^{\frac{1}{2}\omega(n)}$. By (6.1) and (6.4), we obtain that

$$f(n) \sim \gamma_\phi(n) \preceq \gamma_{J \wr_\phi B}(n) \preceq \gamma_B(n)^2 \gamma_\phi(n) \preceq n^{\omega(n)} f(n).$$

Thus, every growth function of an algebra is realizable as the growth of a nil matrix wreath product over any countable field, up to a polynomial error factor, and over an arbitrary field, up to an arbitrarily slow super-polynomial error factor.

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