Mathematika 60 (2014) 307–320 © University College London
doi:10.1112/S0025579314000114

ON THE METRIC THEORY OF CONTINUED FRACTIONS IN POSITIVE CHARACTERISTIC

POJ LERTCHOOSAKUL AND RADHAKRISHNAN NAIR

Abstract. Let $\mathbb{F}_q$ be the finite field of q elements. An analogue of the regular continued fraction expansion for an element α in the field of formal Laurent series over $\mathbb{F}_q$ is given uniquely by

$$\alpha = A_0(\alpha) + \cfrac{1}{A_1(\alpha) + \cfrac{1}{A_2(\alpha) + \ddots}},$$

where $(A_n(\alpha))_{n=0}^{\infty}$ is a sequence of polynomials with coefficients in $\mathbb{F}_q$ such that $\deg(A_n(\alpha)) \geqslant 1$ for all $n \geqslant 1$. We first prove the exactness of the continued fraction map in positive characteristic. This fact implies a number of strictly weaker properties. Particularly, we then use the weak-mixing property and ergodicity to establish various metrical results regarding the averages of partial quotients of continued fraction expansions. A sample result that we prove is that if $(p_n)_{n=1}^{\infty}$ denotes the sequence of prime numbers, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \deg(A_{p_j}(\alpha)) = \frac{q}{q-1}$$

for almost every α with respect to Haar measure. In the case where the sequence $(p_n)_{n=1}^{\infty}$ is replaced by $(n)_{n=1}^{\infty}$, this result is due to V. Houndonougbo, V. Berthé and H. Nakada. Our proofs rely on pointwise subsequence and moving average ergodic theorems.

§1. *Introduction.* Let $\mathbb{F}_q$ denote the finite field of q elements, where q is a power of a prime p. If Z is an indeterminate, we denote by $\mathbb{F}_q[Z]$ and $\mathbb{F}_q(Z)$ the ring of polynomials in Z with coefficients in $\mathbb{F}_q$ and the quotient field of $\mathbb{F}_q[Z]$, respectively. For each $P, Q \in \mathbb{F}_q[Z]$ with $Q \neq 0$, define $|P/Q| = q^{\deg(P)-\deg(Q)}$ and $|0| = 0$. The field $\mathbb{F}_q((Z^{-1}))$ of a formal Laurent series is the completion of $\mathbb{F}_q(Z)$ with respect to the valuation $|\cdot|$. That is,

$$\mathbb{F}_q((Z^{-1})) = \{a_n Z^n + a_{n-1}Z^{n-1} + \cdots + a_0 + a_{-1}Z^{-1} + \cdots : n \in \mathbb{Z},\ a_i \in \mathbb{F}_q\}$$

and we have $|a_n Z^n + a_{n-1}Z^{n-1} + \cdots| = q^n$ ($a_n \neq 0$) and $|0| = 0$, where q is the number of elements of $\mathbb{F}_q$. It is worth keeping in mind that $|\cdot|$ is a

Received 5 June 2013, published online 27 May 2014.
MSC (2010): 11K50 (primary), 28D99 (secondary).

non-Archimedean norm, since $|\alpha + \beta| \leqslant \max(|\alpha|, |\beta|)$. In fact, $\mathbb{F}_q((Z^{-1}))$ is the non-Archimedean local field of positive characteristic p. As a result, there exists a unique, up to a positive multiplicative constant, countably additive Haar measure μ on the Borel subsets of $\mathbb{F}_q((Z^{-1}))$. In [17, pp. 65–70], Sprindžuk finds a characterization of Haar measure on $\mathbb{F}_q((Z^{-1}))$ by its value on the balls $B(\alpha; q^n) = \{\beta \in \mathbb{F}_q((Z^{-1})) : |\alpha - \beta| < q^n\}$. Indeed, it was shown that the equation $\mu(B(\alpha; q^n)) = q^n$ completely characterizes Haar measure.

As in the classical context of real numbers, we have a continued fraction algorithm in $\mathbb{F}_q((Z^{-1}))$. Note that, in the field of the formal Laurent series case, we shall be considering that the roles of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ in the classical theory of continued fractions are played by $\mathbb{F}_q[Z]$, $\mathbb{F}_q(Z)$ and $\mathbb{F}_q((Z^{-1}))$, respectively. For each $\alpha \in \mathbb{F}_q((Z^{-1}))$, we can uniquely write

$$\alpha = A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \ddots}} = [A_0; A_1, A_2, \dots],$$

where $(A_n)_{n=0}^{\infty}$ is a sequence of polynomials in $\mathbb{F}_q[Z]$ with $|A_n| > 1$ for all $n \geqslant 1$. Note that, in the context of continued fractions, we shall often deal with the set $\mathbb{F}_q[Z]^* = \{A \in \mathbb{F}_q[Z] : |A| > 1\}$. As in the classical theory, we define recursively the two sequences of polynomials $(P_n)_{n=0}^{\infty}$ and $(Q_n)_{n=0}^{\infty}$ by

$$P_n = A_n P_{n-1} + P_{n-2} \quad \text{and} \quad Q_n = A_n Q_{n-1} + Q_{n-2},$$

with the initial conditions $P_0 = A_0$, $Q_0 = 1$, $P_1 = A_1 A_0 + 1$ and $Q_1 = A_1$. Then we have $Q_n P_{n-1} - P_n Q_{n-1} = (-1)^n$, and hence P_n and Q_n are coprime. In addition, we have $P_n / Q_n = [A_0; A_1, \dots, A_n]$. For a general reference on this subject, the reader should consult [10] and [16].

In this setting, we wish to investigate some metrical questions regarding the averages of partial quotients of continued fraction expansions. Indeed, we try to answer the following types of question. Let $(a_n)_{n=1}^{\infty}$ be any sequence of positive integers. For a typical point $\alpha = [A_0(\alpha); A_1(\alpha), A_2(\alpha), \dots]$, we would like to identify the limits:

(1) $\lim_{n\to\infty} (1/n) \sum_{j=1}^{n} \deg(A_{a_j}(\alpha))$;
(2) for each $A \in \mathbb{F}_q[Z]^*$, $\lim_{n\to\infty} (1/n) \cdot \#\{1 \leqslant j \leqslant n : A_{a_j}(\alpha) = A\}$;
(3) for each $m \in \mathbb{N}$, $\lim_{n\to\infty} (1/n) \cdot \#\{1 \leqslant j \leqslant n : \deg(A_{a_j}(\alpha)) = m\}$;
(4) let $(b_n)_{n=1}^{\infty}$ be another sequence of positive integers. We also ask about the moving averages of the same quantities as in (1)–(3). For instance, what is the limit,

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} \deg(A_{a_n + j}(\alpha)).$$

In order to calculate these averages for a large class of the sequences $(a_n)_{n=1}^{\infty}$, we shall use pointwise subsequence and moving average ergodic theorems. In the special case where $(a_n)_{n=1}^{\infty} = (n)_{n=1}^{\infty}$, we note that Houndonougbo [8] and

Berthé and Nakada [2] gave almost complete answers to the first three questions by using Birkhoff's pointwise ergodic theorem.

We finally introduce the continued fraction map in positive characteristic, which shall be used with the ergodic theorems to calculate the averages. Define T on the unit ball $B(0; 1) = \{a_{-1}Z^{-1} + a_{-2}Z^{-2} + \cdots : a_i \in \mathbb{F}_q\}$ by

$$T\alpha = \left\{\frac{1}{\alpha}\right\} \quad \text{and} \quad T0 = 0,$$

where $\{a_n Z^n + \cdots + a_0 + a_{-1}Z^{-1} + \cdots\} = a_{-1}Z^{-1} + a_{-2}Z^{-2} + \cdots$ denotes its fractional part. We note that if $\alpha = [0; A_1(\alpha), A_2(\alpha), \ldots]$, then we have, for all $m, n \geqslant 1$,

$$T^n\alpha = [0; A_{n+1}(\alpha), A_{n+2}(\alpha), \ldots] \quad \text{and} \quad A_m(T^n\alpha) = A_{n+m}(\alpha).$$

We now summarize the contents of this paper. In §2, we establish some essential mixing properties of the continued fraction map. In §3, we summarize the relevant portions of subsequence pointwise ergodic theory. In §4, we summarize the relevant material about the moving average ergodic theorem. In §5, subsequence pointwise ergodic theory is used to prove results about various averages of convergents. Finally, in §6, analogous results for moving averages are given.

§2. *Exactness and weak mixing.* In [8, Théorème II.3.1], Houndonougbo proved that the dynamical system $(B(0; 1), \mathcal{B}, \mu, T)$ is measure-preserving and ergodic. Nevertheless, in order to calculate the more general averages of convergents of continued fraction expansions we need subsequence pointwise ergodic theory, which requires a stronger property of the dynamical system, called weak mixing. Indeed, we shall systematically prove that the continued fraction map in positive characteristic is exact with respect to Haar measure. This fact of exactness implies all mixing properties and ergodicity.

Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system consisting of a set X with the σ-algebra $\mathcal{B}$ of its subsets, a probability measure μ, and a transformation $T : X \to X$. We say that $(X, \mathcal{B}, \mu, T)$ is *measure-preserving* if, for all $E \in \mathcal{B}$, $\mu(T^{-1}E) = \mu(E)$. Let $\mathcal{N} = \{E \in \mathcal{B} \colon \mu(E) = 0 \text{ or } \mu(E) = 1\}$ denote the trivial σ-algebra of subsets of $\mathcal{B}$ of either null or full measure. We say that the measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is *exact* if

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \mathcal{N},$$

where $T^{-n}\mathcal{B} = \{T^{-n}E \colon E \in \mathcal{B}\}$.

THEOREM 1. *The dynamical system* $(B(0; 1), \mathcal{B}, \mu, T)$ *is exact.*

To proceed, the following notation is useful.

Recall that $\mathbb{F}_q[Z]^* = \{A \in \mathbb{F}_q[Z] \colon |A| > 1\}$. Let n be a natural number, and let $A_1, \ldots, A_n \in \mathbb{F}_q[Z]^*$. The *cylinder* $\Delta_{A_1,\ldots,A_n}$ of length n is defined to be the set of all points in $B(0; 1)$ whose continued fraction expansions are of the form $[0; A_1, \ldots, A_n, \ldots]$. That is,

$$\Delta_{A_1,\ldots,A_n} = \{[0; A_1, \ldots, A_{n-1}, A_n + \beta] \colon \beta \in B(0; 1)\}.$$

We give some relationship between a cylinder and a ball by the following lemma. This is crucial for calculating the measure of each cylinder.

LEMMA 2. *Let n be a natural number, and let $A_1, \ldots, A_n \in \mathbb{F}_q[Z]^*$. We have*

$$\Delta_{A_1,\ldots,A_n} = B([0; A_1, \ldots, A_n]; |A_1 \cdots A_n|^{-2}).$$

Proof of Lemma 2. First we show that the cylinder $\Delta_{A_1,\ldots,A_n}$ belongs to the ball $B([0; A_1, \ldots, A_n]; |A_1 \cdots A_n|^{-2})$. Let $\alpha = [0; A_1, \ldots, A_{n-1}, A_n + \beta]$, where $\beta \in B(0; 1)$, and let $P_n/Q_n = [0; A_1, \ldots, A_n]$. Then we have

$$\begin{aligned} \left|\alpha - \frac{P_n}{Q_n}\right| &= \left|\frac{(A_n + \beta)P_{n-1} + P_{n-2}}{(A_n + \beta)Q_{n-1} + Q_{n-2}} - \frac{P_n}{Q_n}\right| \\ &= \left|\frac{\beta(P_{n-1}Q_n - P_nQ_{n-1})}{Q_n(Q_n + \beta Q_{n-1})}\right| = \frac{|\beta|}{|Q_n||Q_n + \beta Q_{n-1}|} \\ &< \frac{1}{|Q_n|^2} = \frac{1}{|A_1 \cdots A_n|^2}. \end{aligned}$$

This shows that $\alpha \in B([0; A_1, \ldots, A_n]; |A_1 \cdots A_n|^{-2})$.

To prove the converse, suppose that $\alpha \notin \Delta_{A_1,\ldots,A_n}$. Then we can write α as the continued fraction $[0; B_1, \ldots, B_{n-1}, B_n + \gamma]$, where $\gamma \in B(0; 1)$ and $B_i \neq A_i$ for some $i = 1, \ldots, n$. Let j be the first position where $B_j \neq A_j$, so $\alpha = [0; A_1, \ldots, A_{j-1}, B_j, \ldots, B_{n-1}, B_n + \gamma]$. If $P_j/Q_j = [0; A_1, \ldots, A_j]$, then

$$\begin{aligned} &|[0; A_1, \ldots, A_{j-1}, B_j, \ldots, B_{n-1}, B_n + \gamma] - [0; A_1, \ldots, A_{j-1}, A_j, \ldots, A_n]| \\ &= \left|\frac{[B_j; \ldots, B_{n-1}, B_n + \gamma]P_{j-1} + P_{j-2}}{[B_j; \ldots, B_{n-1}, B_n + \gamma]Q_{j-1} + Q_{j-2}} - \frac{[A_j; \ldots, A_n]P_{j-1} + P_{j-2}}{[A_j; \ldots, A_n]Q_{j-1} + Q_{j-2}}\right| \\ &= \frac{|[B_j; \ldots, B_{n-1}, B_n + \gamma] - [A_j; \ldots, A_n]|}{|[B_j; \ldots, B_{n-1}, B_n + \gamma]Q_{j-1}||[A_j; \ldots, A_n]Q_{j-1}|} \\ &= \frac{|A_j - B_j|}{|A_j||B_j||Q_{j-1}|^2} = \frac{1}{\min(|A_j|, |B_j|)|Q_{j-1}|^2} \geqslant \frac{1}{|Q_n|^2}. \end{aligned}$$

This shows that $\alpha \notin B([0; A_1, \ldots, A_n]; |A_1 \cdots A_n|^{-2})$, as required. □

From Lemma 2, it follows immediately that $\mu(\Delta_{A_1,\ldots,A_n}) = |A_1 \cdots A_n|^{-2}$. We note also that two cylinders $\Delta_{A_1,\ldots,A_n}$ and $\Delta_{B_1,\ldots,B_n}$ are disjoint if and only if $A_j \neq B_j$ for some $1 \leqslant j \leqslant n$.

Let $\mathcal{A}$ denote the algebra of finite unions of cylinders. Then $\mathcal{A}$ generates the Borel σ-algebra of the dynamical system $(B(0; 1), \mathcal{B}, \mu, T)$. This follows from the fact that the cylinders are clearly Borel sets themselves and that they separate points, that is, if $\alpha \neq \beta$, then there exist disjoint cylinders Δ_1 and Δ_2 such that $\alpha \in \Delta_1$ and $\beta \in \Delta_2$.

In order to prove the exactness, we need the following three lemmas. Note that the first two lemmas appear in [8] in slightly different language.

LEMMA 3. *The dynamical system $(B(0; 1), \mathcal{B}, \mu, T)$ is measure-preserving.*

Proof of Lemma 3. By the Kolmogorov extension theorem, it suffices to show that, for any cylinder $\Delta_{A_1,\ldots,A_n}$, we have $\mu(T^{-1}\Delta_{A_1,\ldots,A_n}) = \mu(\Delta_{A_1,\ldots,A_n})$. First, we note that $\mu(\Delta_{A_1,\ldots,A_n}) = |A_1 \cdots A_n|^{-2}$. Then we notice that

$$T^{-1}\Delta_{A_1,\ldots,A_n} = \bigcup_{A \in \mathbb{F}_q[Z]^*} \Delta_{A,A_1,\ldots,A_n}. \tag{2.1}$$

Note that, for each $j \geqslant 1$, $\#\{A \in \mathbb{F}_q[Z]^* : |A| = q^j\} = (q-1)q^j$. Now, by the disjointness of cylinders, it follows from (2.1) that

$$\begin{aligned}
\mu(T^{-1}\Delta_{A_1,\ldots,A_n}) &= \sum_{A \in \mathbb{F}_q[Z]^*} |AA_1 \cdots A_n|^{-2} = |A_1 \cdots A_n|^{-2} \sum_{A \in \mathbb{F}_q[Z]^*} |A|^{-2} \\
&= |A_1 \cdots A_n|^{-2} \sum_{j=1}^{\infty} \frac{(q-1)q^j}{q^{2j}} = |A_1 \cdots A_n|^{-2} \sum_{j=1}^{\infty} \frac{q-1}{q^j} \\
&= |A_1 \cdots A_n|^{-2} = \mu(\Delta_{A_1,\ldots,A_n}).
\end{aligned}$$

This shows that the continued fraction map preserves Haar measure. □

LEMMA 4. *For the dynamical system $(B(0; 1), \mathcal{B}, \mu, T)$, suppose that $E \in \mathcal{B}$. Then, for any $n \geqslant 1$ and any cylinder $\Delta_{A_1,\ldots,A_n}$, we have*

$$\mu(\Delta_{A_1,\ldots,A_n} \cap T^{-n}E) = \mu(\Delta_{A_1,\ldots,A_n})\mu(E).$$

Proof of Lemma 4. By the Kolmogorov extension theorem, we need only to prove the case that $E = \Delta_{B_1,\ldots,B_m}$ is any cylinder. We first observe that

$$T^{-n}\Delta_{B_1,\ldots,B_m} = \bigcup_{C_1,\ldots,C_n \in \mathbb{F}_q[Z]^*} \Delta_{C_1,\ldots,C_n,B_1,\ldots,B_m}.$$

By the disjointness of cylinders, it follows immediately that

$$\Delta_{A_1,\ldots,A_n} \cap T^{-n}\Delta_{B_1,\ldots,B_m} = \Delta_{A_1,\ldots,A_n,B_1,\ldots,B_m}.$$

Therefore, we conclude that

$$\begin{aligned}
\mu(\Delta_{A_1,\ldots,A_n} \cap T^{-n}\Delta_{B_1,\ldots,B_m}) &= |A_1 \cdots A_n B_1 \cdots B_m|^{-2} \\
&= \mu(\Delta_{A_1,\ldots,A_n})\mu(\Delta_{B_1,\ldots,B_m}).
\end{aligned}$$

This completes the proof of Lemma 4. □

LEMMA 5. *Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $E \in \mathcal{B}$. Suppose that $\mathcal{A} \subseteq \mathcal{B}$ is an algebra that generates $\mathcal{B}$. Suppose further that there exists*

https://doi.org/10.1112/S0025579314000114 Published online by Cambridge University Press

an $\omega > 0$ *such that*

$$\mu(E \cap \Delta) \geqslant \omega\mu(E)\mu(\Delta)$$

for all $\Delta \in \mathcal{A}$. *Then either* $\mu(E) = 0$ *or* $\mu(E) = 1$.

Proof of Lemma 5. Let $\epsilon > 0$. As $\mathcal{A}$ generates $\mathcal{B}$, there exists a $\Delta \in \mathcal{A}$ such that $\mu((E^c \setminus \Delta) \cup (\Delta \setminus E^c)) < \epsilon$. Hence, $|\mu(E^c) - \mu(\Delta)| < \epsilon$. Note that $E \cap \Delta \subseteq (E^c \setminus \Delta) \cup (\Delta \setminus E^c)$ so that $\mu(E \cap \Delta) < \epsilon$. It now follows that

$$\begin{aligned}\mu(E)\mu(E^c) &< \mu(E)(\mu(\Delta) + \epsilon) \leqslant \mu(E)\mu(\Delta) + \epsilon \leqslant \frac{1}{\omega}\mu(E \cap \Delta) + \epsilon \\ &< \left(\frac{1}{\omega} + 1\right)\epsilon.\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we have $\mu(E)\mu(E^c) = 0$. Thus, $\mu(E) = 0$ or 1, and this completes the proof of Lemma 5. □

We are now in a position to prove that the continued fraction map T is exact with respect to Haar measure.

Proof of Theorem 1. It is not hard to check that we need only to prove the inclusion $\bigcap_{n=1}^{\infty} T^{-n}\mathcal{B} \subseteq \mathcal{N}$. Let $E \in \bigcap_{n=1}^{\infty} T^{-n}\mathcal{B}$. It follows immediately that, for each $n \geqslant 1$, there exists an $E_n \in \mathcal{B}$ such that $E = T^{-n}E_n$ and $\mu(E_n) = \mu(E)$. Then, for each cylinder $\Delta_{A_1,\ldots,A_n}$ of length n, we always have

$$\mu(E \cap \Delta_{A_1,\ldots,A_n}) = \mu(T^{-n}E_n \cap \Delta_{A_1,\ldots,A_n}) = \mu(E)\mu(\Delta_{A_1,\ldots,A_n}).$$

It follows that $\mu(E) = 0$ or 1, so $E \in \mathcal{N}$. This proves the exactness. □

If $(X, \mathcal{B}, \mu, T)$ is exact, then a number of strictly weaker properties arise. Firstly, for any natural number n and any $E_0, E_1, \ldots, E_n \in \mathcal{B}$, we have

$$\lim_{j_1,\ldots,j_n\to\infty} \mu(E_0 \cap T^{-j_1}E_1 \cap \cdots \cap T^{-(j_1+\cdots+j_n)}E_n) = \mu(E_0)\mu(E_1)\cdots\mu(E_n).$$

This is called *mixing of order n*. Mixing of order $n = 1$ is

$$\lim_{j\to\infty} \mu(E_0 \cap T^{-j}E_1) = \mu(E_0)\mu(E_1),$$

and this is called *strong mixing*, which in turn implies

$$\lim_{m\to\infty} \frac{1}{m}\sum_{j=1}^{m} |\mu(E_0 \cap T^{-j}E_1) - \mu(E_0)\mu(E_1)| = 0,$$

which is called *weak mixing*. The weak-mixing property implies the condition that if $E \in \mathcal{B}$ and if $T^{-1}E = E$, then either $\mu(E) = 0$ or $\mu(E) = 1$. This last property is referred to as *ergodicity* in measurable dynamics. All these implications are known to be strict in general, see [6, pp. 22–26].

§3. *Subsequence ergodic theory.* In this section, we describe the arithmetic and number-theoretic context in which the results of §5 are proved. The two issues here are which sequences of integers satisfy a pointwise ergodic theorem and calculating the limit of the ergodic averages in the instances where this limit exists. We begin with some formal definitions for describing the framework in which this is done.

A sequence of integers $(a_n)_{n=1}^{\infty}$ is called L^p-*good universal* if, for each dynamical system $(X, \mathcal{B}, \mu, T)$ and $f \in L^p(X, \mathcal{B}, \mu)$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(T^{a_j - 1}\alpha)$$

exists μ-almost everywhere. Recall that we say that a sequence of real numbers $(x_n)_{n=1}^{\infty}$ is *uniformly distributed modulo* 1 if, for each interval $I \subseteq [0, 1)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \cdot \#\{1 \leqslant j \leqslant n : \{x_j\} \in I\} = |I|,$$

where $|I|$ denotes the length of I and $\{x_j\}$ denotes the fractional part of x_j. Also, we say that a sequence of integers $(a_n)_{n=1}^{\infty}$ is *uniformly distributed on* $\mathbb{Z}$ if, for each $m \in \mathbb{N} \setminus \{1\}$ and $k \in [0, m-1] \cap \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \cdot \#\{1 \leqslant j \leqslant n : a_j \equiv k \bmod m\} = \frac{1}{m}.$$

A sequence of integers $(a_n)_{n=1}^{\infty}$ is said to be *Hartman uniformly distributed on* $\mathbb{Z}$ if $(\{a_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo 1 for each irrational γ and if $(a_n)_{n=1}^{\infty}$ is uniformly distributed on $\mathbb{Z}$. See [**11**] for further background.

Now we give some examples of L^p-good universal sequences for some $p \geqslant 1$. The examples 1 and 2 are not in general Hartman uniformly distributed. The examples 3–6 are Hartman uniformly distributed.

(1) *The natural numbers.* The sequence $(n)_{n=1}^{\infty}$ is L^1-good universal. This is Birkhoff's pointwise ergodic theorem.

(2) *Polynomial like sequences.* If $\phi(x)$ is a polynomial such that $\phi(\mathbb{N}) \subseteq \mathbb{N}$ and $p > 1$, then $(\phi(n))_{n=1}^{\infty}$ and $(\phi(p_n))_{n=1}^{\infty}$, where p_n is the nth prime, are L^p-good universal sequences. See [**4**, **5**, **12**].
Note that if $n \in \mathbb{N}$, then $n^2 \not\equiv 3 \bmod 4$, so in general the sequences $(\phi(n))_{n=1}^{\infty}$ and $(\phi(p_n))_{n=1}^{\infty}$ are not Hartman uniformly distributed. We do, however, know that if $\gamma \in \mathbb{R} \setminus \mathbb{Q}$, then $(\phi(n)\gamma)_{n=1}^{\infty}$ and $(\phi(p_n)\gamma)_{n=1}^{\infty}$ are uniformly distributed modulo 1 from [**19**].

(3) *Condition* H. Sequences $(a_n)_{n=1}^{\infty}$ that are both L^p-good universal and Hartman uniformly distributed can be constructed as follows. Denote by $[x]$ the integer part of a real number x. Set $a_n = [\tau(n)]$ $(n = 1, 2, \ldots)$, where $\tau : [1, \infty) \to [1, \infty)$ is a differentiable function whose derivative increases with its argument. Let Ω_n denote the cardinality of the set $\{n : a_n \leqslant n\}$, and suppose, for some function $\varphi : [1, \infty) \to [1, \infty)$ increasing to infinity as its argument does, that we set

$$\varrho(m) = \sup_{\{z\} \in [1/\varphi(m), \frac{1}{2})} \left| \sum_{n:\, a_n \leqslant m} \mathrm{e}(z a_n) \right|,$$

https://doi.org/10.1112/S0025579314000114 Published online by Cambridge University Press

where $\mathrm{e}(x) = \mathrm{e}^{2\pi \mathrm{i} x}$ for a real x. Suppose also, for some decreasing function $\rho : [1, \infty) \to [1, \infty)$ and some positive constant $\omega > 0$, that

$$\frac{\varrho(m) + \Omega_{[\varphi(m)]} + m/\varphi(m)}{\Omega_m} \leqslant \omega\rho(m).$$

Then if we have

$$\sum_{n=1}^{\infty} \rho(\theta^n) < \infty$$

for all $\theta > 0$, we say that $(a_n)_{n=1}^{\infty}$ satisfies condition H, see [14]. Sequences satisfying condition H are both Hartman uniformly distributed and L^p-good universal. Specific sequences of integers that satisfy condition H include $a_n = [\tau(n)]$ $(n = 1, 2, \dots)$ where:

(I) $\tau(n) = n^{\gamma}$ if $\gamma > 1$ and $\gamma \notin \mathbb{N}$;

(II) $\tau(n) = \mathrm{e}^{\log^{\gamma} n}$ for $\gamma \in (1, \frac{3}{2})$;

(III) $\tau(n) = b_k n^k + \cdots + b_1 n + b_0$ for $b_k, \dots, b_1$ not all rational multiples of the same real number;

(IV) *Hardy fields.* By a Hardy field, we mean a closed subfield (under differentiation) of the ring of germs at $+\infty$ of continuous real-valued functions with addition and multiplication taken to be pointwise. Let $\mathcal{H}$ denote the union of all Hardy fields. If $(a_n)_{n=1}^{\infty} = ([\psi(n)])_{n=1}^{\infty}$, where $\psi \in \mathcal{H}$ satisfies the condition that, for some $k \in \mathbb{Z}$ and $k \geqslant 2$,

$$\lim_{x\to\infty} \frac{\psi(x)}{x^{k-1}} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{\psi(x)}{x^{k}} = 0,$$

then $(a_n)_{n=1}^{\infty}$ satisfies condition H. This example is given in [3].

(4) *A random example.* Suppose that $S = (b_n)_{n=1}^{\infty}$ is a strictly increasing sequence of natural numbers. By identifying S with its characteristic function χ_S, we may view it as a point in $\Lambda = \{0, 1\}^{\mathbb{N}}$, the set of maps from $\mathbb{N}$ to $\{0, 1\}$. We may endow Λ with a probability measure by viewing it as a Cartesian product $\Lambda = \prod_{n=1}^{\infty} X_n$, where, for each natural number n, we have $X_n = \{0, 1\}$ and specify the probability ν_n on X_n by $\nu_n(\{1\}) = \omega_n$ with $0 \leqslant \omega_n \leqslant 1$ and $\nu_n(\{0\}) = 1 - \omega_n$ such that $\lim_{n\to\infty} \omega_n n = \infty$. The desired probability measure on Λ is the corresponding product measure $\nu = \prod_{n=1}^{\infty} \nu_n$. The underlying σ-algebra $\mathcal{A}$ is that generated by the cylinders

$$\{(\Delta_n)_{n=1}^{\infty} \in \Lambda \colon \Delta_{n_1} = \alpha_{n_1}, \dots, \Delta_{n_k} = \alpha_{n_k}\}$$

for all possible choices of $n_1, \dots, n_k$ and $\alpha_{n_1}, \dots, \alpha_{n_k}$. Then almost every point $(a_n)_{n=1}^{\infty}$ in Λ, with respect to the measure ν, is Hartman uniformly distributed, [4].

(5) *Block sequences.* Suppose that $(a_n)_{n=1}^{\infty} = \bigcup_{n=1}^{\infty}[d_n, e_n]$ is ordered by absolute value for disjoint $([d_n, e_n])_{n=1}^{\infty}$ with $d_{n-1} = O(e_n)$ as n tends to infinity. Note that this allows the possibility that $(a_n)_{n=1}^{\infty}$ is zero density. This example is an immediate consequence of Tempelman's semigroup ergodic theorem [18, p. 218].

(6) *Random perturbation of good sequences.* Suppose that $(a_n)_{n=1}^{\infty}$ is an L^p-good universal sequence which is also Hartman uniformly distributed. Let $\theta = (\theta_n)_{n=1}^{\infty}$ be a sequence of $\mathbb{N}$-valued independent, identically distributed random variables with basic probability space $(Y, \mathcal{A}, \mathcal{P})$, and a $\mathcal{P}$-complete σ-field $\mathcal{A}$. Let $\mathbb{E}$ denote expectation with respect to the basic probability space $(Y, \mathcal{A}, \mathcal{P})$. Assume that there exist $0 < \alpha < 1$ and $\beta > 1/\alpha$ such that

$$a_n = O(\mathrm{e}^{n^{\alpha}}) \quad \text{and} \quad \mathbb{E}\log_+^{\beta}|\theta_1| < \infty.$$

Then $(a_n + \theta_n(\omega))_{n=1}^{\infty}$ is both L^p-good universal and Hartman uniformly distributed [15].

We introduce the following two pointwise subsequence ergodic theorems. The first lemma, which was proved in [13], enables us to calculate the limit of the ergodic averages for an L^p-good universal sequence. This lemma is what makes it possible to make the calculations in §5 given an L^p-good universal sequence.

LEMMA 6. *Let $(X, \mathcal{B}, \mu, T)$ be a weak-mixing dynamical system. Suppose that $(a_n)_{n=1}^{\infty}$ is L^2-good universal. Also suppose that, for any irrational number γ, the sequence $(\{a_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo 1. Then, for any $f \in L^2(X, \mathcal{B}, \mu)$,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f(T^{a_j}\alpha) = \int_X f\,d\mu$$

μ-almost everywhere.

The second lemma, which was proved in [7], shows that, for all but example (2) in our list of examples, we only need ergodicity of the continued fraction map in positive characteristic. This lemma, while informative, is not strictly necessary for the calculations in §5.

LEMMA 7. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. Suppose that $(a_n)_{n=1}^{\infty}$ is an L^2-good universal sequence which is also Hartman uniformly distributed on $\mathbb{Z}$. Then, for any $f \in L^2(X, \mathcal{B}, \mu)$,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f(T^{a_j}\alpha) = \int_X f\,d\mu$$

μ-almost everywhere.

Note that these two lemmas extend readily to $p > 1$ by approximation by L^2 functions. We forego the details as we do not use this degree of generality in our applications.

§4. *Moving averages.* We begin by introducing some notation. Let Z be a collection of points in $\mathbb{Z} \times \mathbb{N}$, and let

$$Z^h = \{(m, n) \in Z : n \geqslant h\},$$
$$Z_\alpha^h = \{(x, y) \in \mathbb{Z}^2 : |x - m| < \alpha(y - n) \text{ for some } (m, n) \in Z^h\},$$
$$Z_\alpha^h(k) = \{x : (x, k) \in Z_\alpha^h\} \quad (k \in \mathbb{N}).$$

https://doi.org/10.1112/S0025579314000114 Published online by Cambridge University Press

Geometrically, we can think of Z^1_α as the lattice points contained in the union of all solid cones with aperture α and vertex contained in $Z^1 = Z$. We say that a sequence of pairs of natural numbers $(a_n, b_n)_{n=1}^\infty$ is *Stoltz* if there exist a collection of points Z in $\mathbb{Z} \times \mathbb{N}$ and a function $h = h(t)$ tending to infinity with t such that $(a_n, b_n)_{n=t}^\infty \in Z^{h(t)}$, and if there exist h_0, α_0, and $c > 0$ such that, for all $k \in \mathbb{N}$, we have the cardinality $\#Z^{h_0}_{\alpha_0}(k) \leqslant ck$. This technical condition is interesting because of the following lemma from [1].

LEMMA 8. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. Suppose that $(a_n, b_n)_{n=1}^\infty$ is a Stoltz sequence. Then, for any $f \in L^1(X, \mathcal{B}, \mu)$, the limit*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} f(T^{a_n+j-1}\alpha)$$

exists μ-almost everywhere.

Note that if we set

$$M_f(\alpha) = \lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} f(T^{a_n+j-1}\alpha) \quad \text{and} \quad M_{n,f}(\alpha) = \frac{1}{b_n} \sum_{j=1}^{b_n} f(T^{a_n+j-1}\alpha)$$

and observe that

$$M_{n,f}(T\alpha) - M_{n,f}(\alpha) = \frac{1}{b_n}(f(T^{a_n+b_n}\alpha) - f(T^{a_n}\alpha)),$$

then we can see that $M_f(T\alpha) = M_f(\alpha)$ μ-almost everywhere. A standard fact in ergodic theory is that if $(X, \mathcal{B}, \mu, T)$ is ergodic and if $M_f(T\alpha) = M_f(\alpha)$ μ-almost everywhere, then $M_f(\alpha) = \int_X f\, d\mu$ μ-almost everywhere, [6, p 14]. We have the following lemma that will be used in §6.

LEMMA 9. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. Suppose that $(a_n, b_n)_{n=1}^\infty$ is a Stoltz sequence. Then, for any $f \in L^1(X, \mathcal{B}, \mu)$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} f(T^{a_n+j-1}\alpha) = \int_X f\, d\mu$$

μ-almost everywhere.

We note that the term "Stoltz" is used here because the condition on $(a_n, b_n)_{n=1}^\infty$ is analogous to the condition required in the classical non-radial limit theorem for harmonic functions, also called a Stoltz condition, which suggested Lemma 8 to the authors of [1]. Averages where $a_n = 1$ for all n will be called *non-moving*. This is as opposed the more general *moving* averages which are averages along intervals whose initial element, i.e. a_n, may not be 1. Moving averages satisfying the above hypothesis can be constructed by taking, for instance, $a_n = 2^{2^n}$ and $b_n = 2^{2^{n-1}}$.

§5. *Application of the pointwise subsequence ergodic theorems.* In this section we assume the sequence $(a_n)_{n=1}^\infty$ is L^2-good universal. We also suppose for any irrational number γ that the sequence $(\{a_n\gamma\})_{n=1}^\infty$ is uniformly distributed modulo 1.

Recall the elementary identities

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \quad \text{and} \quad \sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3} \quad \text{for } |x| < 1.$$

Also, as is easily verified, a simple computation shows that

$$\mu(\{\alpha \in B(0;1) \colon |\alpha| = q^{-n}\}) = \frac{q-1}{q^n} \quad (n = 1, 2, \ldots).$$

From this, we get

$$\int_{B(0;1)} |\alpha| \, d\mu = \sum_{n=1}^{\infty} n \cdot \mu(\{\alpha \in B(0;1) \colon |\alpha| = q^{-n}\}) = \frac{1}{q-1},$$
$$\int_{B(0;1)} |\alpha|^2 \, d\mu = \sum_{n=1}^{\infty} n^2 \cdot \mu(\{\alpha \in B(0;1) \colon |\alpha| = q^{-n}\}) = \frac{q(q+1)}{(q-1)^2}.$$

These two identities, in the light of the results of this section, also indicate the relation between the expectation of the variable $|\alpha|$ and the frequency with which it takes a specific value for almost all α. Analogous observations hold for other variables in this section. In particular, the valuation $|\cdot|$ is in $L^2(B(0;1), \mathcal{B}, \mu)$, and so we have the following results.

THEOREM 10. *Suppose that $F : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is a continuous increasing function such that*

$$\int_{B(0;1)} |F(|A_1(\alpha)|)|^2 \, d\mu < \infty.$$

For each $n \in \mathbb{N}$ and arbitrary non-negative real numbers $d_1, \ldots, d_n$, we define

$$M_{F,n}(d_1, \ldots, d_n) = F^{-1}\left(\frac{F(d_1) + \cdots + F(d_n)}{n}\right).$$

Then we have

$$\lim_{n\to\infty} M_{F,n}(|A_{a_1}(\alpha)|, \ldots, |A_{a_n}(\alpha)|) = F^{-1}\left(\int_{B(0;1)} F(|A_1(\alpha)|) \, d\mu\right)$$

almost everywhere with respect to Haar measure.

Proof. Apply Lemma 6 with $f(\alpha) = F(|A_1(\alpha)|)$. □

THEOREM 11. *Suppose that $H : \mathbb{N}^m \to \mathbb{R}$ is a function such that*

$$\int_{B(0;1)} |H(|A_1(\alpha)|, \ldots, |A_m(\alpha)|)|^2 \, d\mu < \infty.$$

Then we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} H(|A_{a_j}(\alpha)|, \ldots, |A_{a_j+m-1}(\alpha)|)$$
$$= \sum_{(i_1,\ldots,i_m)\in\mathbb{N}^m} H(q^{i_1}, \ldots, q^{i_m}) \left(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}}\right)$$

almost everywhere with respect to Haar measure.

Proof. Apply Lemma 6 with $f(\alpha) = H(|A_1(\alpha)|, \ldots, |A_m(\alpha)|)$. □

Theorems 10 and 11 are general results for calculating means. They both readily extend from L^2 to L^p for $p \in (1, 2]$ whenever $(a_n)_{n\geqslant 1}$ is L^p good universal, though this is primarily of technical interest. Specializing for instance to the case where $F(x) = \log_q x$, we recover the positive characteristic analogue of Khinchin's famous result that

$$\lim_{n\to\infty} |A_1(\alpha)\cdots A_n(\alpha)|^{1/n} = q^{q/(q-1)}$$

almost everywhere with respect to Haar measure, [**2**] and [**10**]. Results for means other than the geometric mean can be obtained by making different choices of F and H, see [**9**, p 230–232] for more details. In addition, the following three theorems can be viewed as corollaries of Theorem 10.

THEOREM 12. *We have*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \deg(A_{a_j}(\alpha)) = \frac{q}{q-1}$$

almost everywhere with respect to Haar measure.

Proof. Apply Lemma 6 with $f(\alpha) = \sum_{n=1}^{\infty} n \cdot \chi_{\{q^n\}}(|A_1(\alpha)|)$. □

THEOREM 13. *For any $A \in \mathbb{F}_q[Z]^*$, we have*

$$\lim_{n\to\infty} \frac{1}{n}\cdot \#\{1 \leqslant j \leqslant n\colon A_{a_j}(\alpha) = A\} = |A|^{-2}$$

almost everywhere with respect to Haar measure.

Proof. Apply Lemma 6 with $f(\alpha) = \chi_{\{A\}}(A_1(\alpha))$. □

THEOREM 14. *For any natural numbers $k < l$, we have*

$$\lim_{n\to\infty} \frac{1}{n}\cdot \#\{1 \leqslant j \leqslant n\colon \deg(A_{a_j}(\alpha)) = l\} = \frac{q-1}{q^l},$$

$$\lim_{n\to\infty} \frac{1}{n}\cdot \#\{1 \leqslant j \leqslant n\colon \deg(A_{a_j}(\alpha)) \geqslant l\} = \frac{1}{q^{l-1}},$$

$$\lim_{n\to\infty} \frac{1}{n}\cdot \#\{1 \leqslant j \leqslant n\colon k \leqslant \deg(A_{a_j}(\alpha)) < l\} = \frac{1}{q^{k-1}}\left(1 - \frac{1}{q^{l-k}}\right)$$

almost everywhere with respect to Haar measure.

Proof. Apply Lemma 6 with $f_1(\alpha) = \chi_{\{q^l\}}(|A_1(\alpha)|)$, $f_2(\alpha) = \chi_{[q^l,\infty)}(|A_1(\alpha)|)$, and $f_3(\alpha) = \chi_{[q^k,q^l)}(|A_1(\alpha)|)$, respectively. □

§6. *Application of the moving average pointwise ergodic theorem.* In this section, we state moving average variants of the results in the previous section. The proofs, which are very similar to those in the previous section, are foregone. Note that we use Lemma 9 for the calculations in this section.

THEOREM 15. *Let $(a_n, b_n)_{n=1}^{\infty}$ be a Stoltz sequence. Suppose $F : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is a continuous increasing function such that*

$$\int_{B(0;1)} |F(|A_1(\alpha)|)|\, d\mu < \infty.$$

For each $n \in \mathbb{N}$ and arbitrary non-negative real numbers $d_1, \ldots, d_n$, we define

$$M_{F,n}(d_1, \ldots, d_n) = F^{-1}\left(\frac{F(d_1) + \cdots + F(d_n)}{n}\right).$$

Then we have

$$\lim_{n\to\infty} M_{F,b_n}(|A_{a_n+1}(\alpha)|, \ldots, |A_{a_n+b_n}(\alpha)|) = F^{-1}\left(\int_{B(0;1)} F(|A_1(\alpha)|)\, d\mu\right)$$

almost everywhere with respect to Haar measure.

THEOREM 16. *Let $(a_n, b_n)_{n=1}^{\infty}$ be a Stoltz sequence. Suppose $H : \mathbb{N}^m \to \mathbb{R}$ is a function such that*

$$\int_{B(0;1)} |H(|A_1(\alpha)|, \ldots, |A_m(\alpha)|)|\, d\mu < \infty.$$

Then we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} & H(|A_{a_n+j}(\alpha)|, \ldots, |A_{a_n+j+m-1}(\alpha)|) \\ &= \sum_{(i_1,\ldots,i_m)\in\mathbb{N}^m} H(q^{i_1}, \ldots, q^{i_m}) \left(\frac{(q-1)^m}{q^{i_1+\cdots+i_m}}\right)\end{aligned}$$

almost everywhere with respect to Haar measure.

THEOREM 17. *Suppose that $(a_n, b_n)_{n=1}^{\infty}$ is Stoltz. Then*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{b_n} \deg(A_{a_n+j}(\alpha)) = \frac{q}{q-1}$$

almost everywhere with respect to Haar measure.

THEOREM 18. *Let $A \in \mathbb{F}_q[Z]^*$, and suppose that $(a_n, b_n)_{n=1}^{\infty}$ is Stoltz. Then*

$$\lim_{n\to\infty} \frac{1}{b_n} \cdot \#\{1 \leqslant j \leqslant b_n : A_{a_n+j}(\alpha) = A\} = |A|^{-2}$$

almost everywhere with respect to Haar measure.

THEOREM 19. *Let $k, l \in \mathbb{N}$, and suppose that $(a_n, b_n)_{n=1}^{\infty}$ is Stoltz. Then*

$$\lim_{n\to\infty} \frac{1}{b_n} \cdot \#\{1 \leqslant j \leqslant b_n : \deg(A_{a_n+j}(\alpha)) = l\} = \frac{q-1}{q^l},$$

$$\lim_{n\to\infty} \frac{1}{b_n} \cdot \#\{1 \leqslant j \leqslant b_n : \deg(A_{a_n+j}(\alpha)) \geqslant l\} = \frac{1}{q^{l-1}},$$

$$\lim_{n\to\infty} \frac{1}{b_n} \cdot \#\{1 \leqslant j \leqslant b_n : k \leqslant \deg(A_{a_n+j}(\alpha)) < l\} = \frac{1}{q^{k-1}}\left(1 - \frac{1}{q^{l-k}}\right)$$

almost everywhere with respect to Haar measure.

References

1. A. Bellow, R. Jones and J. Rosenblatt, Convergence for moving averages. *Ergodic Theory Dynam. Systems* **10**(1) (1990), 43–62.
2. V. Berthé and H. Nakada, On continued fraction expansions in positive characteristic: equivalence relations and some metric properties. *Expo. Math.* **18**(4) (2000), 257–284.
3. M. Boshernitzan, G. Kolesnik, A. Quas and M. Wierdl, Ergodic averaging sequences. *J. Anal. Math.* **95** (2005), 63–103.
4. J. Bourgain, On the maximal ergodic theorem for certain subsets of the integers. *Israel J. Math.* **61**(1) (1988), 39–72.
5. J. Bourgain, Pointwise ergodic theorems for arithmetic sets. *Publ. Math. Inst. Hautes Études Sci.* **69** (1989), 5–45.
6. I.P. Cornfeld, S.V. Fomin and Ya.G. Sinaĭ, *Ergodic Theory* (*Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **245**), Springer (New York, 1982).
7. J. Hančl, A. Jaššová, P. Lertchoosakul and R. Nair, On the metric theory of p-adic continued fractions. *Indag. Math. (N.S.)* **24**(1) (2013), 42–56.
8. V. Houndonougbo, Développement en fractions continues et répartititon modulo 1 dans un corps de séries formelles. In *Thése de troisiéme cycle*, Université de Bordeaux I (1979).
9. M. Iosifescu and C. Kraaikamp, *Metrical Theory of Continued Fractions* (*Mathematics and its Applications,* **547**), Kluwer Academic Publishers (Dordrecht, 2002).
10. A.Ya. Khinchin, *Continued Fractions*, Dover Publications Inc. (Mineola, New York, 1997).
11. L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience [John Wiley & Sons] (New York, 1974).
12. R. Nair, On polynomials in primes and J. Bourgain's circle method approach to ergodic theorems. II. *Studia Math.* **105**(3) (1993), 207–233.
13. R. Nair, On the metrical theory of continued fractions. *Proc. Amer. Math. Soc.* **120**(4) (1994), 1041–1046.
14. R. Nair, On uniformly distributed sequences of integers and Poincaré recurrence. II. *Indag. Math. (N.S.)* **9**(3) (1998), 405–415.
15. R. Nair and M. Weber, On random perturbation of some intersective sets. *Indag. Math. (N.S.)* **15**(3) (2004), 373–381.
16. W.M. Schmidt, On continued fractions and Diophantine approximation in power series fields. *Acta Arith.* **95**(2) (2000), 139–166.
17. V.G. Sprindžuk, *Mahler's Problem in Metric Number Theory* (*Translations of Mathematical Monographs* **25**), American Mathematical Society (Providence, RI, 1969).
18. A. Tempelman, *Ergodic Theorems for Group Actions: Informational and Thermodynamical Aspects* (*Mathematics and Its Applications* **78**), Kluwer Academic Publishers Group (Dordrecht, 1992).
19. H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins. *Math. Ann.* **77**(3) (1916), 313–352.

Poj Lertchoosakul,
Institute of Mathematics,
Polish Academy of Sciences,
ul. Śniadeckich 8,
00-656 Warsaw,
Poland
E-mail: p.lertchoosakul@impan.pl

Radhakrishnan Nair,
Mathematical Sciences,
The University of Liverpool,
Peach Street,
Liverpool L69 7ZL,
U.K.
E-mail: nair@liv.ac.uk