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REPRESENTABLE DIVISIBILITY SEMIGROUPS

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To B. H. Neumann on the occasion of his 80th birthday

By a divisibility semigroup we mean an algebra $(S, \cdot, \wedge)$ satisfying (A1) $(S, \cdot)$ is a semigroup; (A2) $(S, \wedge)$ is a semilattice; (A3) $x(a \wedge b)y = xay \wedge xby$; (A4) $a \leqq b \Rightarrow \exists x, y: ax = b = ay$.

A divisibility semigroup is called representable if it admits a subdirect decomposition into totally ordered factors.

In this paper various types of representable divisibility semigroups are investigated and characterized, admitting a representation in general or even a special decomposition, like subdirect sums of archimedean factors, for instance.

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Introduction

A lattice-ordered algebraic structure is called *representable* if it admits a subdirect decomposition into totally ordered factors of similar type. So, the question of representability is of central interest, and there is an abundance of contributions to this topic (cf. [**4**]). In particular one finds a dozen of criteria for lattice-ordered groups to be representable (cf. [**1, 9, 10**]), due to Lorenzen [**15**], Šik [**18**], Byrd [**6**], Fuchs (verbal remark, see [**9**]), and Conrad [**9**], none of which however works in the lattice-semigroup case.

As a matter of fact, a criterion for subdirect products of totally ordered factors has been missing for two decades since L. Fuchs stated his Problem 41 in [**10**], although it had been known for some twenty years (cf. [**11**]), that the subdirect products of totally ordered factors of a class of lattice-ordered algebras form a variety, see also [**12**].

Then, in 1984, an answer was given independently in [**4**] and [**17**] which even turned out to be of symptomatical character [**4**], telling that a lattice-ordered algebra is representable if and only if its linearily composed polynomials satisfy:

$$p(a) \wedge q(b) \leqq p(b) \vee q(a). \tag{0}$$

The proof has to be done via ideal-congruences, and this might be the reason for the solution being so late. A lattice-ordered group is considered as l-group, and not as lattice-g. So congruences are normal subgroups, and nothing else.

In this paper we study divisibility-semigroups, in order to simplify and to replace condition (0) by further equational and also by structural properties. This will lead to several representation theorems, the most interesting seeming to be that a divisibility-semigroup is representable if and only if it satisfies:

$$eae \wedge faf = (e \wedge f)a(e \wedge f)$$

which was stated for lattice-groups by L. Fuchs (cf. [9]).

0. Preliminary notions

By a *divisibility-semigroup* we mean an algebra $(S, \cdot, \wedge)$ of type (2,2) satisfying

(A1) $(S, \cdot)$ is a semigroup.
(A2) $(S, \wedge)$ is a semilattice.
(A3) $x(a \wedge b)y = xay \wedge xby$.
(A4) $a \leqq b \Rightarrow \exists x, y: ax = b = ya$.

Divisibility-semigroups are join-closed (with $(a \wedge b)a' = a \Rightarrow ba' = a \vee b$) and it turns out that the underlying lattice is distributive and that multiplication distributes over meet and join from the right and (by duality) from the left.

A divisibility-monoid is called (*right*) *normal* if it satisfies in addition:

$$\forall a, b \, \exists a', b': a' \wedge b' = 1, (a \wedge b)a' = a, (a \wedge b)b' = b.$$

In what follows we shall sometimes be concerned with *distributive lattice-semigroups*, i.e. lattice-semigroups satisfying the distributive laws mentioned above. They are called *dld-semigroups* in [16].

Let S be a *dld*-semigroup. $a \in S$ is called *positive* if it satisfies $as \geqq s \leqq sa$ for all $s \in S$. Obviously the set S^+ of all positive elements of S is closed w.r.t. $\cdot$, $\wedge$, and $\vee$. S itself is called positive if each of its elements is positive, i.e. if $S = S^+$. As usual S^+ is called the *cone* of S.

In a divisibility-semigroup the elements x, y of (A4) can always be taken from S^+ whence we tacitly shall suppose them to be positive whenever they are involved in calculations.

There is a most important rule of arithmetic.

Lemma 0.1. *In a positive dld-semigroup we have*:

$$a \wedge bc = a \wedge ac \wedge bc = a \wedge (a \wedge b)c = a \wedge b(a \wedge c).$$

Let S be a *dld*-semigroup and let $ea = a = ae$. Then e is called a *unit* of a. The set of all units of a is denoted by $E(a)$. If S is even a divisibility-semigroup no $E(a)$ is empty and in addition one has:

Lemma 0.2. [2]. *Let S be a dld-semigroup. Then each pair a, e with $e \in E(a)$ satisfies*

$$a = (e \wedge a)(e \vee a) = (e \vee a)(e \wedge a).$$

A divisibility-semigroup need not contain an identity element 1. But, every divisibility-semigroup S admits a canonical smallest divisibility-semigroup extension Σ formed by the set of all $(S, \wedge)$-endomorphisms of type fh^{-1} with $f = id$ or $f = f_a: x \to ax$ or $f = \bar{f}_a: x \to x \wedge ax$, and $h = \bar{f}_b$ with suitable elements a, b. This leads in Σ to $\alpha = \beta \Leftrightarrow x \cdot \alpha = x \cdot \beta$ $(\forall x \in S^+)$. Important elements are the idempotents. More precisely we have:

Proposition 0.3. [2]. *In a divisibility-semigroup the idempotents are central and positive.*

A semigroup is called 0-cancellative if it satisfies $ax = ay \neq 0 \Rightarrow x = y$ and $xa = ya \neq 0 \Rightarrow x = y$.

Lemma 0.4. *A divisibility-semigroup S is 0-cancellative iff it satisfies*

$$ae = a \neq 0 \Rightarrow e = 1 \quad \textit{and} \quad ea = a \neq 0 \Rightarrow e = 1,$$

since $ax = ay = a(x \wedge y) \Rightarrow ax = a(x \wedge y)x' = a(x \wedge y)y' = ay$. □

A most important class of divisibility-semigroups is the class of archimedean divisibility-semigroups.

Definition 0.5. A divisibility-semigroup is called *archimedean* if it satisfies

$$t^n \leqq a (\forall n \in \mathbb{N}) \Rightarrow tat \leqq a.$$

In order that a divisibility-semigroup be archimedean it suffices that its cone is archimedean. Furthermore a fundamental result tells:

Theorem 0.6 [3]. *Archimedean divisibility-semigroups are commutative.*

We now turn to properties closely connected with representability, also called the vector property. Here, as an application of (0), we get the criterion:

Proposition 0.7. [4] *A lattice-semigroup is representable if and only if it is a dld-semigroup satisfying $xay \wedge ubv \leqq xby \vee uav$ where x, y, u, v are taken from $S \cup \{1\}$.*

For a divisibility-semigroup S there is no need for an additional element 1 since there are always enough private units. Furthermore a commutative divisibility-semigroup is always representable. However, this fails to be true for *dld*-semigroups in general, consult [16], whereas commutative *dld*-monoids do have the vector property.

Representability depends on the behaviour of certain substructures, the most important being lattice ideals.

Definition 0.8. Let S be a *dld*-semigroup. A nonempty subset A of S is called an *ideal* (*filter*) if it is an ideal (filter) of $(S, \wedge, \vee)$. An ideal (filter) A is called *irreducible* if it cannot be written as intersection of two ideals (filters) different from A. An ideal A is called *m-ideal* if it is multiplicatively closed. It is called *invariant* if it satisfies $xA = Ax$. A filter A is called *Rees-filter* if it satisfies $S \cdot A, A \cdot S \subseteq A$. Finally an ideal is called *positive* if it contains at least one positive element.

By definition A is an irreducible ideal if $S - A$ is an irreducible filter. Furthermore it is folklore that an ideal (filter) P is irreducible if and only if

$$a \wedge b(a \vee b) \in P \Rightarrow a \in P \text{ or } b \in P.$$

Proposition 0.9. *Let S be a dld-semigroup. There are crucial congruences defined via ideals and filters, respectively.*

(*I*) *Let P be an irreducible ideal (filter). Then P generates a congruence via*

$$a \equiv b(P) :\Leftrightarrow xay \in P \leftrightarrow xby \in P,$$

where obviously $\equiv(P)$ *is equal to* $\equiv(S-P)$. *Furthermore S/P is totally ordered if in addition S satisfies* (0).

(*F*) *Let R be a Rees-filter. Then R generates a congruence via*

$$a \equiv b(R) :\Leftrightarrow \exists x \in R : x \wedge a = x \wedge b.$$

This implies that in the positive case every $x \in S$ *generates a congruence mod x by* $a \equiv b(x) \Leftrightarrow x \wedge a = x \wedge b$ *with* $S/\equiv =: S_x$.

(*M*) *Let M be an m-ideal of* S^+. *Then M generates a left congruence via*

$$a \equiv b(M) : \Leftrightarrow \exists e, f \in M : a \leqq be \text{ and } b \leqq af.$$

For the sake of decomposition it is necessary to have enough congruences of a given type, in order to separate each pair a, b, and it is convenient that we may restrict ourselves to pairs $a < b$ in arbitrary lattice-semigroups and even to positive pairs $a < b$ in divisibility-semigroups. Furthermore, with respect to irreducible ideals, we may apply that there are enough regular ideals, i.e. ideals, maximal with respect to not containing a given element a, and that regular ideals are irreducible.

As a further important class of substructures we present:

Definition 0.10. Let S be a divisibility-monoid. By a *solid submonoid* of S we mean a submonoid A whose cone A^+ is an m-ideal of S^+ and whose elements are exactly all

ab^{-1} with $a, b \in A^+$, b invertible. A solid submonoid P of S is called a *prime monoid* of S if it satisfies $A \cap B \subseteq P \Rightarrow A \subseteq P \vee B \subseteq P$ (A, B solid). P is called *regular* if it is maximal with respect to not containing some given element a.

Obviously S itself is solid and with a family A_i of solid submonoids also its intersection is solid. Hence, every subset M of S generates a smallest solid submonoid $C(M)$, which in the case of a positive M turns out to be equal to the set of all $x \leqq m_1 \cdot \ldots \cdot m_n$ ($m_i \in M$). Furthermore in analogy to the l-group case we have the propositions:

Proposition 0.11. *Let S be a divisibility-monoid. Then the set of all solid submonoids forms a distributive lattice and in addition complex-multiplication distributes over meet and join. (For an idea consult* [1]).

Proposition 0.12. *Let S be a divisibility-monoid. Then every direct decomposition of S^+ induces a direct decomposition of the whole in such a way that the direct factors of S are the solid submonoids generated by the direct factors of S^+. (For an idea consult* [4]).

In some theorems of this paper we are concerned with direct factors. For this reason we remark $u \perp v :\Leftrightarrow u \wedge v = 1$.

Definition 0.13. Let S be a divisibility-monoid, and let $A \subseteq S$. Then the *polar* $A^\perp$ of A is defined by

$$A^\perp := \{x \mid \forall a \in A : (1 \vee a)(1 \wedge a)^{-1} \perp (1 \vee x)(1 \wedge x)^{-1}\}.$$

Furthermore the *bipolar* of A is defined by $A^{\perp\perp} := (A^\perp)^\perp$, and the polar of a singleton $\{a\}$ is also written as $a^\perp$, (compare [4]).

Proposition 0.14. *Let S be a divisibility-monoid. Then every polar is solid and moreover a solid submonoid A is a direct factor if and only if $A \cdot A^\perp = S$, and in this case A is equal to $A^{\perp\perp}$.*

Finally we remark on some results which are proved straightforwardly—see also [1].

Lemma 0.15. *Let S be a normal divisibility-monoid. $P \subseteq S$ is a prime submonoid iff P is solid and $a \wedge b = 1 \Rightarrow a \in P$ on $b \in P$.*

Lemma 0.16. *Let S be a normal divisibility-monoid. Then each prime submonoid of S contains a minimal prime submonoid.*

Lemma 0.17. *Let S be a normal divisibility-monoid. Then each minimal prime submonoid M is canonically associated with an ultrafilter of $(S^+, \wedge, \vee)$ by $M \to S^+ \backslash M$ which implies that each minimal prime submonoid M of S is of type $M = \{x^\perp \mid x \notin M\}$.*

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Lemma 0.18. *Let S be a normal divisibility-monoid. Then each regular submonoid is a prime submonoid.*

1. Subdirectly irreducible divisibility semigroups

There is not too much known about subdirectly irreducible divisibility-semigroups in general. In the finite case however the situation is a bit better.

We start with a description of the subdirectly irreducible homomorphic images of arbitrary distributive lattice ordered semigroups.

Proposition 1.1. *If S is a dld-semigroup and S/Θ is subdirectly irreducible, then Θ is generated by an irreducible ideal (filter).*

Proof. Let $a<b$ be a critical pair. We choose an $\bar{a}$ containing, $\bar{b}$ avoiding regular ideal $\bar{M}$ of $\bar{S}:=S/\Theta$ with inverse image M in S. Then $\bar{M}$ is irreducible in $\bar{S}$ whence M is irreducible in S.

Furthermore

$$\bar{x}\equiv\bar{y}\Leftrightarrow\bar{s}\bar{x}\bar{t}\in\bar{M}\Leftrightarrow\bar{s}\bar{y}\bar{t}\in\bar{M}(s,t\in S^1)$$

provides a congruence relation on $\bar{S}$, which according to the subdirect irreducibility of $\bar{S}$ must be the equality relation.

On the other hand we have

$$\bar{s}\bar{x}\bar{t}\in\bar{M}\Leftrightarrow\bar{s}\bar{y}\bar{t}\in\bar{M}\Leftrightarrow sxt\in M\Leftrightarrow syt\in M\ (s,t\in S^1)$$

which yields

$$x\Theta y\Leftrightarrow x\equiv y(M). \qquad \square$$

The next result concerns idempotents in subdirectly irreducible divisibility-semigroups.

Proposition 1.2. *Let S be a subdirectly irreducible divisibility-semigroup. Then S contains at most two idempotents.*

Proof. Let S be subdirectly irreducible and let $u\in S$ be idempotent. We define

$$a\rho b:\Leftrightarrow\exists s\in S: a\wedge su=b\wedge su.$$

and

$$a\sigma b:\Leftrightarrow au=bu.$$

It is easily seen that both definitions provide a congruence, and furthermore we get

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$$au = by \Rightarrow su \vee a = u(su \vee a)$$

$$= u(su \vee b) = su \vee b.$$

But from this follows:

$$\begin{matrix} a\rho b \\ \text{and } a\sigma b \end{matrix} \Rightarrow \begin{matrix} su \wedge a = su \wedge b \\ \text{and } su \vee a = su \vee b \end{matrix} \Rightarrow a = b. \qquad \square$$

We now turn to the positive case, proving as a first general result:

Proposition 1.3. *Let S be a positive subdirectly irreducible dld-semigroup. Then in S there exists a maximum 0 and a unique hyper-atom (co-atom) a which together form a critical pair.*

Proof. Suppose that $a < b$ is critical. Then $x < b$ and $x \nleqq a$ implies $x \wedge a < x = x \wedge b$ whence a and b would be separated in S_x. Therefore we have $b = 0$ and $x < b \Rightarrow x \leqq a$. $\square$

Applying 1.3 to the divisibility case we obtain in particular:

Proposition 1.4. *Let S be a positive subdirectly irreducible divisibility-semigroup. Then S is a normal divisibility-monoid and hence totally ordered or containing an orthogonal pair u^*, v^* with $1 \neq u^* \perp v^* \neq 1$. Verifying these properties it will turn out furthermore that the subset L of all left cancellative elements and the subset R of all right cancellative elements both form an irreducible m-ideal.*

Proof. We start by proving the second assertion. We see immediately that the right and the left units of the hyper-atom a form irreducible m-ideals because of $ax = a$ or $ax = 0$. Furthermore we see that e is a right unit of a iff e is right cancellative, since each right cancellative c satisfies $ac \neq 0c$ and since each right unit e of a produces a congruence separating a and 0, namely $x \equiv y \Leftrightarrow xe = ye$.

Hence L and R form irreducible m-ideals and in addition every unit e of a is cancellative whence S is a monoid.

Suppose now $u, v \leqq a$ and $(u \wedge v)u' = u$, $(u \wedge v)v' = v$, $u^*(u' \wedge v') = u'$ and $v^*(u' \wedge v') = v'$. Then $u^* \wedge v^* = 1$ since $(u^* \wedge v^*)(u' \wedge v') = u' \wedge v'$ and $(u \wedge v)u^* = (u \wedge v)u^*(u' \wedge v') = u$ and $(u \wedge v)v^* = v$. Hence $u^* \wedge v^* \in R \cap L$ whence S is normal on the grounds of right-left-duality. $\square$

Definition 1.5. An ideal is called *co-regular* if it is a complement of a regular filter.

Obviously a co-regular ideal is irreducible and minimal within the set of all irreducible ideals containing a fixed element a.

Proposition 1.6. *For a positive dld-semigroup the subdirectlly irreducible homomorphic images correspond uniquely with the co-regular ideals; and thereby with the regular filters.*

Proof. Let J be co-regular with respect to a and let J not contain b. Then $\bar{a}$ is the uniquely determined hyper-atom in $\bar{S}:=S/J$, since otherwise $S\backslash J$ would not be maximal w.r.t. not containing a. Consider now a subdirectly irreducible homomorphic image $\bar{\bar{S}}$ with $\bar{\bar{a}}\neq\bar{\bar{0}}$. Here $\{\bar{\bar{0}}\}$ is the image of $\{\bar{0}\}$ and both $\{\bar{0}\}$ and $\{\bar{\bar{0}}\}$ are regular filters with respect to the corresponding hyper-atoms. This means $\bar{\bar{S}}\cong S/J\cong\bar{S}$. Hence S/J is subdirectly irreducible.

The rest follows by 1.1. since the inverse image of a filter regular with respect to $\bar{a}$ is a regular filter with respect to a. □

Proposition 1.7. *Let S be a commutative subdirectly irreducible divisibility-semigroup. Then S is a totally ordered,* 0-*cancellative divisibility-monoid.*

Proof. First of all S is totally ordered (cf. the remark following 0.7). Let now $a<b$ be a positive critical pair. Then $S/E(a)\cong S$, whence $E(a)$ is a singleton, say $\{e\}$. We consider $x\leqq a$ and $xu=x$. Then $u\in E(a)$, i.e. $u=e$. Therefore S is a monoid. It remains to verify that $a\leqq y=yu\neq 0$ implies $u=e$. But this follows since the set $F:=\{x\mid E(x)\neq E(a)\}$ is empty or forms a Rees-filter with $S/F\cong S$. □

2. Divisibility semigroups

In this paragraph we give some structure theorems on representation.

Theorem 2.1. *For a divisibility-semigroup S the following are equivalent:*

(i) *S is representable.*

(ii) $xay \wedge ubv \leqq xby \vee uav$.

(iii) *S^+ is representable.*

(iv) *Σ^+ is representable.*

(v) $ax \wedge yb \leqq ay \vee xb$.

(vi) $eae \wedge faf = (e \wedge f)a(e \wedge f)$.

Proof. (i)⇔(ii) is valid on the grounds of 0.7.
(ii)⇔(iii) is evident in one direction.
Assume now (iii) to be true and S to be subdirectly irreducible. We consider

$$xay \wedge ubv,\ xby \vee uav.$$

Obviously (ii) is true, iff for suitable elements a'', b''

$$xa''(a \wedge b)y \wedge ub''(a \wedge b)v \leqq xb''(a \wedge b)y \vee ua''(a \wedge b)v.$$

Therefore by 1.4, (ii) is already valid if it is valid for all orthogonal pairs a, b. Furthermore, choosing suitable elements x', u',

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$$xay \wedge ubv \leqq xby \vee uav$$

can be written as

$$(x \wedge u)x'ay \wedge (x \wedge u)u'bv \leqq (x \wedge u)x'by \vee (x \wedge u)u'av$$

Hence (ii) is already valid if it is valid for all orthogonal pairs $a \perp b$, $x \perp u$ from which it follows that (ii) is already valid if it is valid for all orthogonal pairs $x \perp u$, $a \perp b$, $y \perp v$.

But this means a fortiori that (ii) holds in all of S if it satisfied in S^+.

(iii)⇔(iv) is an immediate consequence of the fact that α and β of Σ are equal if and only if $x \cdot \alpha = x \cdot \beta$ for all $x \in S^+$. To verify this we apply the more general lemma which tells that any identity holding in S^+ is also valid in Σ^+ and which follows from the implication

$$xe = x \Rightarrow x \cdot f(\alpha_1, \ldots, \alpha_n) = x \cdot f(\alpha_1 e, \ldots, \alpha_n e).$$

We continue by considering (ii), (v), (vi).

(ii)⇒(v) is evident.

(v)⇒(vi) follows from

$$eae \wedge faf \leqq eaf \wedge eaf = eaf \quad \text{and} \quad faf \wedge eae \leqq fae \wedge fae = fae$$

since

$$(e \wedge f)a(e \wedge f) = eae \wedge eaf \wedge fae \wedge faf.$$

(vi)⇒(ii). First of all it suffices to consider the positive case. Hence we may start from a positive subdirectly irreducible S with hyper-atom a.

This leads to $L \subseteq R$ or $R \subseteq L$ and thereby to $C = L$ or $C = R$. To see this assume $L \nsubseteq R \nsubseteq L$. Then there exist an $e \in L \backslash R$ and an $f \in R \backslash L$. But this means

$$ea = a = af \quad \text{and} \quad ae = 0 = fa$$

which leads to the contradiction

$$a = (e \wedge f)a(e \wedge f) = eae \wedge faf = 0.$$

So in any case C turns out to be an irreducible m-ideal. In particular this means that $p \perp q$ implies $p \in C$ or $q \in C$.

On the other hand, by the proof of (iii)⇒(i) we may confine ourselves to orthogonal pairs x, u; a, b; y, v. But this means that we may start from the special situation

$$x \perp u, a \perp b, y \perp v \quad \text{and} \quad a \in C.$$

To gain a further reduction we prove that we may assume in addition

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$$(x \wedge a) \wedge y = 1.$$

This can be shown as follows:

$$x \wedge a \wedge y \wedge ybv = 1, \text{ by (0.1.).}$$

Suppose now $(x \wedge a \wedge y)x^* = x$ and $(x \wedge a \wedge y)a^* = a$ and $(x \wedge a \wedge y)y^* = y$. We get $x^* \wedge a^* \wedge y^* = 1$ by $(x \wedge a \wedge y)(x^* \wedge a^* \wedge y^*) = (x \wedge a \wedge y) \in C$ (recall $a \in C$), and moreover we have

$$x^*a^*y^* \wedge ubv = xay \wedge ubv$$

according to 0.1. (Observe $x \wedge a \wedge y \perp ubv$).

Hence

$$x^*a^*y^* \wedge ubv \leqq x^*by^* \vee ua^*v$$

$$\Rightarrow xay \wedge ubv = x^*a^*y^* \wedge ubv$$

$$\leqq x^*by^* \vee ua^*v \leqq xby \vee uav.$$

Summarizing, we have obtained that we may restrict ourselves to the case

$$x \perp u,\ a \perp b,\ y \perp v,\ a \wedge x \perp y \quad \text{and} \quad a \in C.$$

So by symmetry it is enough to consider the three cases

$$(1)\ x, y \in C \quad \text{and} \quad (2)\ x, v \in C \quad \text{and} \quad (3)\ u, v \in C.$$

Before treating these cases we remark as follows. Let d, g be orthogonal. Then

$$c \in C \Rightarrow cd \wedge gc \leqq dcd \wedge gcg = c \Rightarrow c(d \wedge c * gc) = c \Rightarrow d \perp c * gc.$$

Observe that $c * gc$ and $cg{:}c$ are uniquely determined because c is cancellative. This leads, by duality, to the implication

$$c \in C \Rightarrow (d \perp g \Rightarrow d \perp c * gc \text{ and } d \perp cg{:}c) \tag{L}$$

which means: if d and g are orthogonal and c is cancellative then gc is equal to cs for some $s \perp d$ and cg is equal to tc with some $t \perp d$.

Now we are in the position to treat the cases (1) through (3).

Case (1). Since $x, y \in C$ we get by (v) and (L):

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$$\begin{aligned} xay \wedge ubv &= a^*xy \wedge uvb^* \quad (\text{with } a^* \perp b^*) \\ &\leqq a^*(xy \vee uv)a^* \wedge b^*(xy \vee uv)b^* \\ &= xy \vee uv. \end{aligned}$$

Case (2).

$$\begin{aligned} xay \wedge ubv &= xay \wedge (u \wedge xay)b(v \wedge xay) \quad (0.1.) \\ &= xya^* \wedge (u \wedge xay)(v \wedge xay)b^* \quad (\text{with } a^* \perp b^*) \\ &\leqq (xy \vee uv)a^* \wedge (xy \vee uv)b^* \\ &= xy \vee uv. \end{aligned}$$

Case (3). First of all (v) implies $a^2 \wedge x^2 = a \cdot 1 \cdot a \wedge x \cdot 1 \cdot x = (a \wedge x)^2$, which leads by cancellation to $(x * a)(a{:}x) \wedge (a * x)(x{:}a) = 1$. Hence $a * x$ and $a{:}x$ commute. Therefore we can calculate:

$$\begin{aligned} xay \wedge ubv &= (x \wedge a)(a * x)(a{:}x)(a \wedge x)y \wedge ubv \\ &= (x \wedge a)(a{:}x)(a * x)y(x \wedge a) \wedge uvb^* \; (x \wedge a \perp y, b^* \perp a) \\ &\leqq (x \wedge a)(a{:}x)(xy \vee uv)(x \wedge a)(a{:}x) \wedge b^*(xy \vee uv)b^* \\ &= xy \vee uv, \end{aligned}$$

thus completing Case (3) and finishing the proof of 2.1. □

In the preceding theorem representable divisibility-semigroups were characterized by equations. In a further theorem we shall describe representable divisibility-semigroups by special substructure-properties which can be done adequately by studying the cone or more generally by considering the positive case of a divisibility-monoid, since in the positive case S is turned to a divisibility-monoid by merely adjoining an identity 1.

Theorem 2.2. *For a positive divisibility-monoid S the following are equivalent:*

(i) *S is representable.*

(ii) *If J is a co-regular ideal then its kernel*

$$\ker(J) := \{x \mid s \cdot t \in J \Rightarrow s \cdot x \cdot t \in J\}$$

is irreducible.

(iii) *If J is a co-regular ideal the set of all m-ideals between* $\ker(J)$ *and J forms a chain under inclusion.*

(iv) *If J is a co-regular ideal and $x \in S$ then the subsets*

$$X^{\perp} := \{y \mid x \wedge y \in \ker(J)\}$$

and

$$X^{\perp\perp} := \{z \mid \forall y \in X^{\perp} : y \wedge z \in \ker(J)\}$$

satisfy

$$X^{\perp} \cup X^{\perp\perp} = S.$$

(v) *If J is a co-regular ideal then the subsets $X^{\perp}$ and $X^{\perp\perp}$ satisfy*

$$X^{\perp} \cdot X^{\perp\perp} = S.$$

Proof. (i)$\Rightarrow$(ii). If S is representable then S/J is totally ordered and thereby $\bar{1}$ is $\wedge$-irreducible. But $\ker(J)$ is the inverse image of $\bar{1}$. So $\ker(J)$ is irreducible, too.

(ii)$\Rightarrow$(i). If $\ker(J)$ is irreducible then $\bar{1}$ in S/J is $\wedge$-irreducible. Hence S/J is totally ordered on the grounds of 1.4.

(i)$\Rightarrow$(iii). Let J be a co-regular ideal. Then S/J is subdirectly irreducible and hence normal by 1.4.

Consider now two m-ideals A and B between $\ker(J)$ and J with $a \in A \backslash B$, $b \in B$. Since S/J is totally ordered $\ker(J)$ is irreducible. So, choosing orthogonal elements a', b' with $(a \wedge b)a' = a$ and $(a \wedge b)b' = b$ we get $a' \wedge b' \in \ker(J)$ which implies $b' \in \ker(J)$ and thereby $(a \wedge b)b' = b \in A \cap B$, whence B is contained in A.

(iii)$\Rightarrow$(i). On the grounds of (iii) the kernels of co-regular ideals are irreducible. Hence, all we have to show is that there are enough co-regular ideals. But this is evident since there are enough regular filters.

(i)$\Rightarrow$(iv). Let S be representable and let J be a co-regular ideal. Then $S/J =: \bar{S}$ is totally ordered and $\bar{x}^{\perp} \cup \bar{x}^{\perp\perp} = \bar{S}$ which yields condition (iv).

(iv)$\Rightarrow$(i). Let $\bar{S}$ be as above. Then the hyper-atom $\bar{a}$ belongs to $\bar{x}^{\perp}$ or to $\bar{x}^{\perp\perp}$ for each $\bar{x} \in \bar{S}$. But this means $\bar{x} = \bar{1}$ or $\bar{x}^{\perp} = \{1\}$. Consequently there cannot exist an orthogonal pair in $\bar{S}$ whence $\bar{S}$ is totally ordered. Therefore condition (i) holds because S has enough co-regular ideals.

(i)$\Rightarrow$(v). Conclude similarly to (i)$\Rightarrow$(iv).

(v)$\Rightarrow$(i). Assume J to be a co-regular ideal of S and $S/J =: \bar{S}$ not to be totally ordered. Then by (v) the hyper-atom $\bar{a}$ of $\bar{S}$ is a product of an orthogonal pair $\bar{x}, \bar{y}$ which leads to $\bar{x}^2 \leqq \bar{a}, \bar{y}^2 \leqq \bar{a}$ and thereby to the contradiction

$$\bar{a} = \bar{x} \cdot \bar{y} = \bar{x} \vee \bar{y} = \bar{x}^2 \vee \bar{y}^2 = \bar{x}^2 \bar{y}^2 = \bar{a}^2 = \bar{0}.$$

This completes the final part and thereby the whole of the proof. □

We continue our investigation by studying special representable divisibility-semigroups. To this end we give

Definition 2.3. A divisibility-semigroup S is called *real* if it is embeddable in $\mathbf{R}^{\cdot}:=(\mathbf{R}^{\infty},+,\min)$ or $\mathbf{E}:=\mathbf{R}^{\geqq 0}/\{x\,|\,x\geqq 1\}$ or $\mathbf{E}^{\cdot}:=\mathbf{R}^{\geqq 0}/\{x\,|\,x>1\}$.

As is easily seen 1 is a maximum of $\mathbf{E}$ and a hyper-atom of $\mathbf{E}^{\cdot}$.

Definition 2.4. Let S be a divisibility-semigroup and J an ideal of S. J is called *really archimedean* if it satisfies the implication:

$$u\cdot t^n\cdot v\in J(\forall n\in\mathbf{N}) \text{ and } a\cdot b\in J\Rightarrow a\cdot t\cdot b\in J.$$

Let S be as above and let F be a filter. F is called really *primary* if it satisfies:

$$a\cdot t\cdot b\in F\Rightarrow a\cdot b\in F \text{ or } u\cdot t^n\cdot v\in F(\exists u,v\in S,n\in\mathbf{N}).$$

Obviously an irreducible ideal is really archimedean iff its complement $S-J$ is a really primary filter.

Theorem 2.5. *For a divisibility-semigroup S the following are equivalent:*

(i) *S is a subdirect product of real divisibility-semigroups.*

(ii) *S is a subdirect product of totally ordered archimedean divisibility-semigroups.*

(iii) *Every principal ideal is the intersection of a family of really archimedean irreducible ideals.*

(iv) *Every principle filter is the intersection of a family of really primary filtres.*

Proof. (i)⇒(ii) is evident.

(ii)⇒(i). Let $\bar{S}$ be totally ordered and archimedean. Then it is easily checked that every homomorphic image of $\bar{S}$ is totally ordered and archimedean, too. So $\bar{S}$ can be decomposed into 0-cancellative totally ordered archimedean divisibility-semigroups, i.e. according to Hölder [13] and Clifford [7] into subsemigroups of $\mathbf{R}^{\cdot}$ and $\mathbf{E}^{\cdot}$. Observe that subdirectly irreducible positive components have a hyper-atom.

(i) or (ii) ⇒ (iii) and (iv). Let S be a subdirect product of real divisibility-semigroups. Then for every pair $a<b$ there exists an index i with $i(a)<i(b)$, and the ideal $P_i:=\{x\,|\,i(x)\leqq i(a)\}$ is irreducible and really archimedean. Similarly we see that the filter $F_i:=\{x\,|\,i(x)\geqq i(b)\}$ is irreducible and really primary. But this means that there are enough ideals and enough filters to verify (iii) and (iv).

(iii)⇔(iv) is valid by Definition 2.4.

(iii) or (iv)⇒(i) and (ii). We start from (iii). Then S is archimedean and hence commutative. Indeed, $t\in S^+$ and $t^n\leqq a$ $(\forall n\in\mathbf{N})$ and $a<at$ would imply the existence of a really archimedean ideal P with $a\in P$ and (thereby) $t^n\in P$ $(\forall n\in\mathbf{N})$, but $at\notin P$.

Let now P be an irreducible really archimedean ideal of S and suppose $\bar{t}^n \leqq \bar{c}(\forall n \in \mathbf{N})$ in $\bar{S} := S/P$. Then we get

$$(c \cdot s \in P \Rightarrow t^n \cdot s \in P(\forall n \in \mathbf{N})) \Rightarrow (c \cdot s \in P \Rightarrow ct \cdot s \in P)$$

which means $\bar{c} \cdot \bar{t} = \bar{c}$. Thus we get (iii)⇒(ii) whence (iii) or (iv)⇒(i) and (ii). □

3. Divisibility monoids

Up till now we have considered divisibility-semigroups in general. Henceforth we shall consider divisibility-monoids.

This will enable us to apply notions, well-known from lattice-group theory, due to pioneers like Jaffard and Conrad (cf. [14] and [8]), and well discussed above all by Bigard, Keimel and Wolfenstein in [1].

Let G be a lattice-group. Recall that a solid submonoid V of G is called a *value* of a if V is maximal with respect to not containing a. The set of all values of a is denoted by $\mathrm{val}(a)$. G is called *finite-valued* if each $\mathrm{val}(a)$ $(a \in G)$ is finite.

G is called *ortho-finite* if each bounded orthogonal subset $\{a_i \mid i \in I\}$ of G $(a_i = a_j \vee a_i \wedge a_j = 1)$ is finite.

G is called *semi-projectable* if it satisfies $(a \wedge b)^\perp = a^\perp \vee b^\perp$ $(\forall a, b \in G)$. G is called *projectable* if it satisfies $G = a^\perp \times a^{\perp\perp}$ $(\forall a \in G)$. G is called *strongly projectable* if it satisfies $G = C(a) \times C(a)^\perp (\forall a \in G)$. Observe: strongly projectable implies $C(a) = a^{\perp\perp}$.

Obviously each of these notions is based merely on the divisibility-monoid language. Hence we may adopt them once an identity is present.

Theorem 3.1. *For a divisibility-monoid S the following are equivalent:*

(i) *S is a direct sum of totally ordered divisibility-monoids.*

(ii) *S is normal, finite-valued, and semi-projectable.*

(iii) *S is ortho-finite and projectable.*

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii). First of all each prime submonoid contains exactly one minimal prime submonoid. To see this, assume P to be prime and A, B to be minimal prime and contained in P. Then there are elements $a \in A \backslash B$, $b \in B \backslash A$ which yield an orthogonal pair $a' \in A \backslash B$, $b' \in B \backslash A$ such that $a'^\perp \subseteq B$ and $b'^\perp \subseteq A$. But this would lead to

$$S = (a' \wedge b')^\perp = a'^\perp \vee b'^\perp = P.$$

So we get next that S is ortho-finite since $1 \leqq a_i \leqq a$ $(i \in I)$ implies: I is finite or there exists at least one value M containing $a_j^\perp$ and $a_k^\perp$ $(j \neq k)$, a contradiction which is seen as above.

Now we show that any regular $M \in \mathrm{val}(a)$ is a unique value with respect to some c. To this end we start from the family $\{M_i \mid i \in I\}$ of all minimal prime submonoids of S,

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not containing a. This set is finite since each M_i is uniquely associated with some $V_i \in \text{val}(a)$. So we have $\{M_i \mid i \in I\} = \{M_0, M_1, \dots, M_n\}$ with $M_0 \subseteq M$ and $M_i \nsubseteq M$ $(1 \leqq i \leqq n)$. But this leads to some $a_i \in M_i \backslash M$ for each $i \in I$ whence M turns out to be the unique value of $c := a \wedge a_1 \wedge \cdots \wedge a_n$.

Suppose finally $S \neq a^\perp \times a^{\perp\perp}$. Then $a^\perp \times a^{\perp\perp}$ is contained in some M with $\{M\} = \text{val}(c)$, and since $a^{\perp\perp}$ is equal to $\bigcap h^\perp$ $(h \in a^\perp)$ there exists at least one $h^\perp$ not containing c and hence contained in M. But this yields a contradiction, since by $h^\perp \supseteq a^{\perp\perp}$ we get $h \in h^{\perp\perp} \subseteq a^\perp$ which implies

$$S \neq M \supseteq a^\perp \vee h^\perp = (a \wedge h)^\perp = S.$$

So (ii)⇒(iii).

(iii)⇒(i). Suppose $a \in S^+$ and assume $a^{\perp\perp}$ not to be totally ordered. Then there exists an x in $a^{\perp\perp}$ with $\{1\} \neq x^{\perp\perp} \subseteq a^{\perp\perp}$, but $x^{\perp\perp} \neq a^{\perp\perp}$. This leads to

$$a^{\perp\perp} = x^{\perp\perp} \cdot (x^\perp \cap a^{\perp\perp}) \text{ by } (0.11)$$

and thereby to $a = a_1 \cdot a_2$ with $a_1 \in x^{\perp\perp}$ and $a_2 \in x^\perp \cap a^{\perp\perp}$.

We know already $a_1 \perp a_2$. Now we show $a_1 \neq a \neq a_2$. To this end suppose first $a_1 = a$. This implies $x^{\perp\perp} = a^{\perp\perp}$, a contradiction. Suppose next $a_2 = a$. This leads to the implication: $a \in x^\perp \Rightarrow a^\perp \supseteq x^{\perp\perp} \Rightarrow x \in a^\perp \cap a^{\perp\perp}$, once more a contradiction. Therefore the decomposition of a is proper. So, continuing the decomposition procedure, after finitely many steps we arrive at $a = a_1 \cdot a_2 \cdot \ldots \cdot a_n$ with pairwise orthogonal elements a_i, generating totally ordered bipolars $a_i^{\perp\perp}$. Consider now two totally ordered bipolars $x^{\perp\perp} \neq y^{\perp\perp}$. Then $z \in x^{\perp\perp} \cap y^{\perp\perp} \Rightarrow z^{\perp\perp} \subseteq x^{\perp\perp} \cap y^{\perp\perp} \Rightarrow z^{\perp\perp} = \{1\}$, whence $z = 1$. Therefore the family of all totally ordered $x^{\perp\perp}$ can be taken to realize a decomposition of S in the sense of (i). □

For the sake of a further representation theorem we give next:

Definition 3.2. A divisibility-monoid is called *strongly archimedean* if it satisfies:

$$1 < t \Rightarrow \exists n \in \mathbf{N}: t^n \geqq a.$$

Strongly archimedean divisibility-semigroups are totally ordered [5], and according to Hölder's and Clifford's results a (totally ordered) divisibility-monoid is strongly archimedean iff it is embeddable in **R** or **E** or **E**˙.

Now we are ready to present

Theorem 3.3. *For a divisibility-monoid S the following are equivalent:*

(i) *S is a direct sum of strongly archimedean totally ordered divisibility-monoids.*

(ii) *The lattice of solid submonoids of S is boolean.*

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(iii) *S is orthofinite and strongly projectable.*

Proof. (i)⇒(ii) is nearly obvious.

(ii)⇒(iii). If the lattice of solid submonoids is boolean then every solid submonoid is a direct factor. But furthermore S is also ortho-finite, since $C(M)$ cannot be a direct factor if M is an infinite set of pairwise orthogonal elements with $a \in S$ as an upper bound.

(iii)⇒(i). We could apply 3.1. but we wish to give some deeper information. Since every $C(x)$ is a direct factor, S satisfies $a, t \in S^+ \Rightarrow \exists n \in \mathbf{N}$: $a \wedge t^n = a \wedge t^{n+1}$.

Furthermore S is normal. To see this we start from $(a \wedge b)a' = a$ and $(a \wedge b)b' = b$ with $a', b' \in S^+$. It follows $b' = b'_1 b'_2$ with $b'_1 \in C(a')$ and $b'_2 \in C(a')^\perp$. This provides $b'_1 \leqq a'^n$ for some suitable $n \in \mathbf{N}$ which leads to $b'_1 = b'_{11} \cdot b'_{12} \cdot \ldots \cdot b'_{1n}$ with $b'_{1i} \leqq a' \wedge b'$ $(1 \leqq i \leqq n)$. Thus we get $(a \wedge b)b'_1 = a \wedge b$ and thereby $(a \wedge b)a' = a$ and $(a \wedge b)b'_2 = b$ with $a' \perp b'_2$.

Suppose now $1 < x, y < a^n$ and $x \nleqq y \nleqq x$. Then there are orthogonal elements $x', y' \notin \{1\}$ whence $C(a)$ has a direct decomposition, say $C(x') \times D$. This leads to $C(a) = C(a_1) \times C(a_2)$ with $a_1 \perp a_2$, and, by continuing the procedure, after finitely many steps to a direct decomposition $C(a) = \bigtimes C(x_i)$ where the direct factors $C(x_i)$ are directly indecomposable and hence totally ordered. Recall now that the lattice of all solid submonoids is distributive. This yields uniqueness of $\bigtimes C(x_i)$ whence there are only finitely many totally ordered $C(x)$ with $a \wedge x \neq 1$.

So, taking all totally ordered $C(x)$ we get a family of strongly archimedean components in the sense of (i). □

4. Hypernormal divisibility monoids

We continue our studies by considering a class of special normal divisibility-monoids.

Definition 4.1. A divisibility-monoid is called *hypernormal* if it satisfies:

$$x, y \in S^+ \text{ and } ax \wedge ay = a \Rightarrow \exists z \perp x{:}\ ay = az$$

$$x, y \in S^+ \text{ and } xa \wedge ya = a \Rightarrow \exists z \perp x{:}\ ya = za.$$

Lemma 4.2. *A divisibility-monoid is already hypernormal iff it satisfies:*

$$e \in S^+ \text{ and } ae = a \leqq b \Rightarrow \exists x \perp e{:}\ b = ax$$

$$e \in S^+ \text{ and } ea = a \leqq b \Rightarrow \exists x \perp e{:}\ b = xa.$$

Proof. Assume $ax \wedge ay = a$ and $(x \wedge y)y' = y$ $(y' \in S^+)$. Then y' can be replaced by an element $y^* \perp x \wedge y$. Hence $z := y^* \wedge y$ satisfies $az = ay$ $(z \perp x)$. □

The hypernormal divisibility-monoid might be something like an optimal common abstraction of boolean rings (distributive lattices with boolean intervals) and lattice-groups. To have a natural example not boolean and not group-like, consider a Bezout-ring R with identity. Here one has

$$ax \mid a \Rightarrow a = axy \text{ and } az = a(xy - 1 + xyz)$$

whence the principal ideal semigroup of R is a hypernormal divisibility-monoid.

Lemma 4.3. *Let S be a hypernormal divisibility-monoid and let J be an invariant m-ideal of S. Then J generates a congruence and S/J is hypernormal, too.*

Proof. J generates a congruence. Assume now $\bar{a}\bar{u} = \bar{a} \leqq \bar{b}$ and $b = a \vee b$. Then $au \leqq ae$ whence $a(u \wedge e) = a(u \wedge e)u'$ $(u' \in S^+)$ and thereby

$$b = a(u \wedge e)x$$

$$= a(u \wedge e)y' \; (y' \perp u').$$

Hence we get

$$\bar{b} = \bar{a}\overline{((u \wedge e)y')} \; \overline{((u \wedge e)y')} = \bar{y} \perp \bar{u}.$$

The rest follows by duality. □

Obviously 4.3. implies that S/J is 0-cancellative if it is totally ordered. Now we are in the position to prove:

Theorem 4.4. *For a positive hypernormal divisibility-monoid S the following are equivalent:*

(i) *S is representable.*

(ii) $xa \wedge bx \leqq x(a \wedge b) \vee (a \wedge b)x$.

(iii) $a \wedge b = 1 \Rightarrow xa \wedge bx = x$.

(iv) $xa^{\perp} = a^{\perp}x$.

(v) *$a, b \in S$ and $xa \wedge bx = x \Rightarrow \exists c, d \in S$:* $\begin{matrix} c \perp a & \text{and} & cx = bx \\ d \perp b & \text{and} & xd = xa. \end{matrix}$

(vi) *Each minimal prime submonoid of S is invariant (cf.* **[6]**).

(vii) *Each regular invariant m-ideal J of S is prime (cf.* **[9]**).

Proof. (i)⇒(ii)⇒(iii) is obvious.
(iii)⇒(iv). Suppose $a \perp b$. It follows

$$xa \wedge bx = x = xa \wedge xc = x(a \wedge c).$$

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This implies $xc = xc^*$ with $c^* \perp c \wedge a$ whence $z = c^* \wedge c$ satisfies $z \perp a$ and $bx = xz$. Thus we get $a^\perp x \subseteq xa^\perp$ and, by duality, $xa^\perp \subseteq a^\perp x$.

(iv)⇔(v). Suppose $xa \wedge bx = x$. One gets $bx = xu$ and thereby

$$\begin{aligned} xa \wedge bx = x &\Rightarrow xa \wedge xu = x \\ &\Rightarrow xu = xu^*(u^* \perp a) \\ &\Rightarrow bx = xu^* = cx\ (c \perp a). \end{aligned}$$

So (iv) implies (v).

Let now (v) be valid and suppose $a \perp b$ and $xb = dx$. Then we get $xa \wedge dx = x$, whence by (v) there exists an element c such that $a \perp c$ and $cx = dx = xb$. This means $xa^\perp \subseteq a^\perp x$, and, by duality, $a^\perp x \subseteq xa^\perp$.

(iv)⇔(vi). Since each minimal prime submonoid is a union of polars (0.17.) (iv) implies (vi).

On the other hand, if (vi) is valid, then each m-ideal of S separating a and b contains a minimal prime submonoid of S, invariant by (vi). Hence (vi) implies (i) and thereby (iv).

(iv)⇔(vii). Observe that for invariant m-ideals J condition (iv) is carried over from S to S/J. To see this, assume $(a \wedge b)a' = a$, $(a \wedge b)b' = b$, $a' \perp b'$, and $\bar{a} \perp \bar{b}$. One gets

$$\begin{aligned} a \wedge b \in J \Rightarrow xb &= x(a \wedge b)b' \\ &= cx(a \wedge b)(c \perp a') \end{aligned}$$

and thereby $\bar{x}\bar{b} = \bar{c}\bar{x}(\bar{a} \perp \bar{c})$.

But this means that $\bar{x} \perp \bar{y} \Leftrightarrow \bar{x} = \bar{1}$ or $\bar{y} = \bar{1}$ and consequently that J is prime. Thus (iv)⇒(vii).

On the other hand we have (vii)⇒(i)⇒(iv). □

The preceding theorem shows how strong hypernormal divisibility-monoids seem to be. This is confirmed also by the next result, a modification of [**1**, 14.1.2]:

Theorem 4.5. *For a hypernormal divisibility-monoid S the following are equivalent:*

(i) *Each* $a \in S$ *satisfies* $S = C(a) \times C(a)^\perp$. (*Actually any strongly projectable divisibility-semigroup is hypernormal, see above*).

(ii) *S is a subdirect product* $\prod S_i (i \in I)$ *of strongly archimedean factors, satisfying:* $\forall f, g \in S^+ \exists n \in \mathbf{N}: f(x)^n \geqq g(x)\ (\forall x \in \mathrm{supp}(f))$.

(iii) $\forall a, t \in S^+ \exists n \in \mathbf{N}: a \wedge t^n = a \wedge t^{n+1}$.

(iv) *Each prime m-ideal is minimal.*

Proof. (i)⇒(ii). By (i) we have (iii), whence S is commutative. Therefore it suffices to prove that the factors $\bar{S}=S/P$ are strongly archimedean. But this follows from $\bar{t}^n<\bar{a}(\forall n\in\mathbb{N})\Rightarrow\exists m$: $(\bar{t}^m)^2=\bar{t}^m$ since S/P is 0-cancellative for each prime submonoid P.

(ii)⇒(iii) is evident.

(iii)⇒(iv). Each prime submonoid P contains a minimal prime submonoid M. Suppose $M\neq P$. Then there exists an $x\in S^+\backslash P$ satisfying in $S/M=:\bar{S}$ for every arbitrary $y\in P^+$

$$\bar{x}>\bar{y}^n\geqq\bar{1}(\forall n\in\mathbb{N}).$$

But this leads to $\bar{y}=\bar{1}$ as above, which means $y\in M$, and thereby $P=M$.

(iv)⇒(i). Suppose $C(a)\times C(a)^{\perp}\neq S$. Then (iv) implies that $C(a)\times C(a)^{\perp}$ is contained in some minimal prime submonoid M. But by 0.17 each minimal prime submonoid P of S is of type $P=U\{x^{\perp}\mid x\notin P\}$ (cf. [1]). This completes the proof by contradiction. □

5. A final remark

Two natural questions remain unsettled in this paper, namely how to characterize direct products of totally ordered divisibility-monoids and how to characterize irreducible representations of divisibility-monoids. So it should be remarked that a solution of these problems will be given elsewhere in a context which would have extended this paper unduly.

The clue to these results is the fact that the whole of Chapter 4 and nearly all of Chapter 7 of Bigard–Keimel–Wolfenstein carry over to normal divisibility-semigroups.

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