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REPRESENTATION OF A CERTAIN CLASS OF POLYNOMIALS

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In this note, we discuss a representation of the class of polynomials with real coefficients having all zeros in a given disk of the complex plane **C**, in terms of convex combinations of certain extremal polynomials of this class. The result stated in the theorem is known [1] for polynomials having n real zeros in the interval [a.b.]. In the following z will be a complex number and $D[(a+b)/2, (b-a)/2]$ the closed disk of the complex plane centered at the real point $(a+b)/2$ and having radius $(b-a)/2$.

LEMMA 1. *Let* $T:\mathbf{R}^2 \to \mathbf{R}^2$

$$(\alpha, \beta) \mapsto (-(\alpha+\beta), \alpha\beta)$$

then the image of the square $[a, b]\times[a, b]$ *is given by the region determined by the following conditions*

(1) $y \le x^2/4$
(2) $y \ge -a(x+2a)+a^2$
(3) $y \ge -b(x+2b)+b^2$

The shaded area in Fig. 2 illustrates this region. To establish this result, first observe that T has a certain symmetry expressed by

$$T(\alpha, \beta) = T(\beta, \alpha)$$

so to find the image of the square, we can restrict ourselves to the shaded triangle of Fig. 1. To find the image of this triangle, we work with the three sides expressed in parametric form respectively by

$$(t, t),\ (b, t) \quad \text{and } (t, a) \quad \text{with } a \le t \le b$$

But

$$T(t, t) = (-2t, t^2) \quad \text{belongs to the curve } y = x^2/4$$
$$T(b, t) = (-b-t, bt) \quad \text{belongs to the line } y = -b(x+2b)+b^2$$

and

$$T(t, a) = (-a-t, at) \quad \text{to the line } y = -a(x+2a)+a^2.$$

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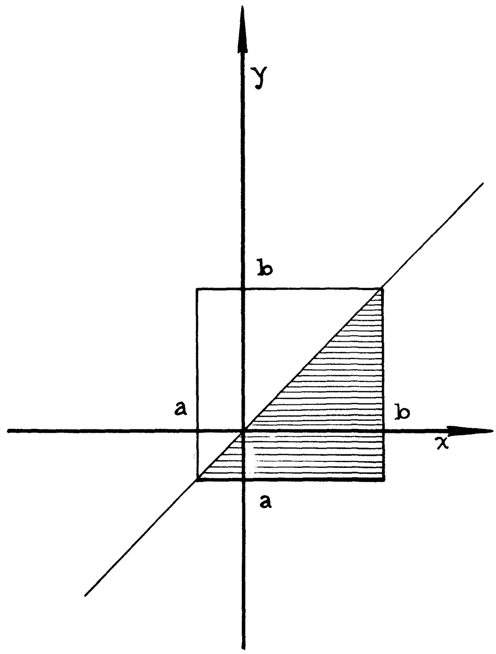

Figure 1

With this in mind, using the continuity of T and the connexity and compacity of the triangle, we can easily complete the proof.

LEMMA 2. *Let a complex number $z \in D[(a+b)/2, (b-a)/2]$. Then the point $(-2 \operatorname{Re} z, |z|^2)$ is in the region determined by the conditions*
(4) $y \geq x^2/4$ *and* $y \leq -(a+b)(x+2a)/2 + a^2$ (5)
Note that this region corresponds to the upper part of triangle ABC in Fig. 2.

Proof. Clearly condition (4) is satisfied by $(-2 \operatorname{Re} z, |z|^2)$ for any $z \in \mathbf{C}$.
Let $z \in D[(a+b)/2, (b-a)/2]$ then

$$\left|z - \frac{a+b}{2}\right|^2 \leq \left(\frac{b-a}{2}\right)^2$$
$$\Updownarrow$$
$$z\bar{z} - 2 \operatorname{Re} z \frac{a+b}{2} + \left(\frac{a+b}{2}\right)^2 \leq \left(\frac{b-a}{2}\right)^2$$
$$\Updownarrow$$
$$|z|^2 \leq 2 \operatorname{Re} z\left(\frac{a+b}{2}\right) - ab$$

and this last expression is precisely condition (5) for the point $(-2 \operatorname{Re} z, |z|^2)$.

Let $[a, b]$ be a real interval and P_n be the set of polynomials with real

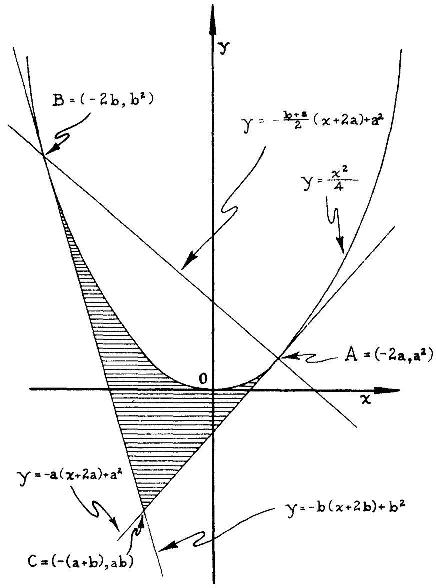

Figure 2

coefficients of the form

$$p(z)=z^n+\sum_{k=1}^{n} c_k z^{n-k}, \qquad c_k\in\mathbf{R}$$

having all n roots in the closed disk $D[(a+b)/2,(b-a)/2]$ of the complex plane centered in the middle of the real interval $[a, b]$.

Let $E_n=\{g_0^n, g_1^n,\ldots, g_n^n\}$ where $g_k^n(z)=(z-a)^{n-k}(z-b)^k$ $k=0, 1, 2,\ldots, n$.

THEOREM. *P_n is a subset of* co E_n, *the convex hull of E_n.*

Proof (by induction). It would be sufficient to prove this with $[a, b]=[-1, 1]$, but we find it more revealing to treat the general interval.

(i) case $n=1$. Let $p\in P_1$, then $p(z)=z-c_1$, $c_1\in[a, b]$.
Since $[a, b]$ is compact and convex, $c_1=\lambda a+(1-\lambda)b(0\le\lambda\le1)$. Then, with $g_0^1(z)=z-a$, $g_1^1(z)=(z-b)$ we have $p=\lambda g_0^1+(1-\lambda)g_1^1$ and $P_1\subseteq \text{co } E_1$.

(ii) case $n=2$. Let $p\in P_2$, then $p(z)=(z-\alpha)(z-\beta)$, $\alpha,\beta\in[a, b]$ or $p(z)=(z-z_1)(z-\bar z_1)$, $z_1\in D[(a+b)/2,(b-a)/2]$.
If we identify a polynomial $z^2+c_1z+c_2$ with its coefficients (c_1, c_2)

$$z^2+c_1z+c_2\leftrightarrow(c_1, c_2)$$

then we have

$$\begin{aligned}
g_0^2(z)&=(z-a)^2\leftrightarrow(-2a, a^2)=A\in\mathbf{R}^2\\
g_1^2(z)&=(z-a)(z-b)\leftrightarrow(-(a+b), ab)=C\\
g_2^2(z)&=(z-b)^2\leftrightarrow(-2b, b^2)=B
\end{aligned}$$

thus $p(z)$ is identified either with

$$P=(-(\alpha+\beta), \alpha\beta) \quad \text{or} \quad P=(-2 \operatorname{Re} z_1, |z_1|^2).$$

By Lemma 1 and 2, in both cases, the point P belongs to the triangle ABC, The convex hull of the set $\{(-2a, a^2), (-(a+b), ab), (-2b, b^2)\}$. Hence

$$P=\lambda_0 A+\lambda_1 C+\lambda_2 B$$

where

$$0 \leq \lambda_i \leq 1 \qquad (i=0,1,2), \qquad \lambda_0+\lambda_1+\lambda_2=1.$$

We can immediately verify that $p(z)=\lambda_0 g_0^2(z)+\lambda_1 g_1^2(z)+\lambda_2 g_2^2(z)$. Thus $P_2 \subseteq$ co E_2.

(iii) One can verify the following relation

$$g_k^n \cdot g_l^m = g_{k+l}^{n+m}$$

Now suppose the theorem true for $n (n \geqq 2)$ and let $p \in P_{n+1}$. Then p has either a real root or two conjugate complex roots. Thus

$$p(z)=(z-\alpha)q(z), \qquad \alpha \in [a, b], q \in P_n,$$

or

$$p(z)=(z-z_1)(z-\bar{z}_1)q(z), . \qquad z_1 \in D\left[\frac{a+b}{2}, \frac{b-a}{2}\right], q \in P_{n-1}.$$

In the first case using (i) we have

$$p(z)=[\lambda g_0^1(z)+(1-\lambda)g_1^1(z)]q(z)(0 \leq \lambda \leq 1),$$

and by hypothesis

$$q(z)=\sum_{k=0}^{n} \lambda_k g_k^n(z), \qquad 0 \leq \lambda_k \leq 1, \sum \lambda_k = 1.$$

Hence rearranging terms we can write

$$p=\sum_k \lambda\lambda_k g_{k+1}^{n+1}+\sum_k (1-\lambda)\lambda_k g_k^{n+1} (k=0,1,\ldots,n).$$

where

$$0 \leq \lambda\lambda_k \leq 1, \qquad 0 \leq (1-\lambda)\lambda_k \leq 1, \qquad \sum \lambda\lambda_k + \sum (1-\lambda)\lambda_k = 1.$$

Thus p is in the convex hull of E_{n+1}.

The second case can be worked out in a similar way using (ii).

A few questions arise at this point. Is the disk the largest set which can be used in this theorem? The answer is clearly yes for $n=2$. However, one can verify that $p(t)=t^3+2\frac{1}{3}t^2+3t+\frac{7}{9}$ has $-\frac{1}{3}$, $-1 \pm 2i/\sqrt{3}$ as zeros, which are not all in $D(0, 1)$, and yet

$$p(t) = \tfrac{1}{9}(t-1)^3 + \tfrac{8}{9}(t+1)^3.$$

Thus if the disk is maximal, the theorem still falls short of characterizing co E_n completely.

Bibliography

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