FINITE GROUPS OF OUTER AUTOMORPHISMS OF FREE GROUPS

by BRUNO ZIMMERMANN

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1. Introduction. Let F_r denote the free group of rank r and $\text{Out}\, F_r := \text{Aut}\, F_r/\text{Inn}\, F_r$ the outer automorphism group of F_r (automorphisms modulo inner automorphisms). In **[10]** we determined the maximal order $2^r r!$ (for $r > 2$) for finite subgroups of $\text{Out}\, F_r$ as well as the finite subgroup of that order which, for $r > 3$, is unique up to conjugation. In the present paper we determine all maximal finite subgroups (that is not contained in a larger finite subgroup) of $\text{Out}\, F_3$, up to conjugation (Theorem 2 in Section 3). Here the considered case $r = 3$ serves as a model case: our method can be applied for other small values of r (in principle for any value of r) but the computations become considerably longer and are more apt for a computer then; the method can also be applied to determine the maximal finite subgroups of the automorphism group $\text{Aut}\, F_r$ of F_r. Note that the canonical projection $\text{Aut}\, F_r \to \text{Out}\, F_r$ is injective on finite subgroups of $\text{Aut}\, F_r$; however, not all finite subgroups of $\text{Out}\, F_r$ lift to finite subgroups of $\text{Aut}\, F_r$.

Our method is based on the following realization result observed in [**11**, p. 478], [**12**, Theorem 2.3.4], see [**2**, Theorem 4.1] for the second part.

THEOREM 1. *Let G be a finite subgroup of* $\text{Out}\, F_r$. *Then there exists a finite connected graph Γ, with $\pi_1\Gamma \cong F_r$, and an action of G on Γ inducing on its fundamental group the given action of G on F_r. Moreover G is the isomorphic image of a finite subgroup of* $\text{Aut}\, F_r$ *if and only if the action of G on Γ has a fixed point in Γ (and then any action realizing G has a fixed point).*

For the convenience of the reader, we shall indicate the proof at the end of the next section.

In Section 4 we discuss a number $\rho(G)$ associated to a finite group G which we call the *graph rank* of the group. Recall that the symmetric genus of a finite group is the minimal genus of a closed orientable surface such that the group acts faithfully on the surface as a group of homeomorphisms. In the definition of $\rho(G)$ instead of closed surfaces and their genera we take finite graphs without vertices of valence one and their ranks (where the rank of a graph is defined as the rank of its fundamental group), so $\rho(G)$ is the minimal rank of a finite graph without vertices of valence one on which the group acts faithfully. For finite groups which are not cyclic or dihedral, it is also the minimal rank of a free group F_r such that G embeds into $\text{Out}\, F_r$.

2. Preliminaries. Let G be a finite subgroup of $\text{Out}\, F_r$. By Theorem 1, G acts on a finite connected graph Γ inducing the given action on F_r (a *geometric realization* of G). By deleting all *free edges* (edges which have a vertex of valence 1) we may assume that Γ has no free edges, or equivalently, no vertices of valence 1. Also, we may delete all vertices of valence 2, amalgamating the adjacent edges into a single edge. Note that after this G may act with inversions: an *inversion* is an edge and an element of G mapping the edge to itself acting as a reflection in the midpoint of this edge. The following Lemma may be considered a converse to Theorem 1.

LEMMA 1. *Let Γ be a connected graph of rank $r > 1$ without vertices of valence* 1 *and* G *a finite group acting faithfully on* Γ. *Then the induced action on* $\pi_1\Gamma$ *injects into* Out $\pi_1\Gamma$.

Proof. Note that the induced action of G on $\pi_1\Gamma$ is defined only up to inner automorphisms because no base point is fixed, in general. Suppose $g \in G$ acts trivially on $\pi_1\Gamma$, i.e. induces an inner automorphism. Then g acts trivially on the homology of Γ, so its Lefschetz number is $1 - r$ which, by the Hopf trace formula, is also equal to the Euler characteristic of the fixed point set of g which is a subgraph of Γ (assume here that G acts without inversions, by subdividing edges). But then the fixed point set of g is equal to the whole graph Γ because there are no free edges, therefore g is the identity and the Lemma is proved.

A consequence of the proof of Lemma 1 is the following well-known result (see also [**11**, Lemma 1.3]).

COROLLARY 1. *The canonical projection* $\text{Out}\, F_r \to \text{GL}(r, \mathbb{Z})$ *is injective on finite subgroups of* $\text{Out}\, F_r$.

In particular, the whole *automorphism* or *symmetry group* $\text{Aut}\,\Gamma$ of a finite graph Γ without free edges injects into Out $\pi_1\Gamma$. As noted above, we may suppose that the graph Γ has no edges of valence 2; in addition, we may assume that the graph Γ has no $\text{Aut}\,\Gamma$-invariant forest (a disjoint union of trees), by contracting each tree of such a forest to a point. We call a finite connected graph *admissible* if it has no vertices of valences 1 or 2 and no $\text{Aut}\,\Gamma$-invariant subforests. Then in Theorem 1 we may restrict ourselves to admissible graphs Γ, and we have the following consequence of Theorem 1 and Lemma 1.

PROPOSITION 1. *Up to conjugation, the maximal finite subgroups of* $\text{Out}\, F_r$, *for* $r > 1$, *are among the automorphism groups of finite admissible graphs of rank* r, *by taking induced actions on fundamental groups.*

However, it is not clear that all these groups are really maximal since it may happen that one is conjugate in $\text{Out}\, F_r$ to a subgroup of another.

As an easy example we consider the case $r = 2$. There are exactly two admissible graphs of rank 2 which are shown in Figure 1.

Their automorphism groups are the direct resp. semidirect product

$$S_3 \times \mathbb{Z}_2 \cong D_6 \quad \text{resp.} \quad (\mathbb{Z}_2)^2 \rtimes S_2 \cong D_4$$

(where D_n denotes the dihedral group of order $2n$ and S_n the symmetric group of order $n!$), therefore up to conjugation these are exactly the maximal finite subgroups of $\text{Out}\, F_2$.

Figure 1.

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COROLLARY 2. *Up to conjugation, the maximal finite subgroups of* Out F_2 *are* D_6 *and* D_4.

Of course this follows also from the well-known isomorphisms

$$\mathrm{Out}\, F_2 \simeq \mathrm{GL}(2, \mathbb{Z}) \cong D_6 *_{D_2} D_4$$

and the fact that every finite subgroup of a free product with amalgamation is conjugate into one of the factors.

In order to enumerate all admissible graphs of a certain rank r the following Lemma is useful.

LEMMA 2. *Let* Γ *be a finite graph of rank* $r > 0$ *without vertices of valence* 1. *Let* e *resp.* v *denote the number of edges resp. vertices of* Γ, *and let* v_i *be the number of vertices of valence* i. *Then*

$$2(r-1) = \sum (i-2)v_i.$$

Proof. This follows from

$$\chi(\Gamma) = 1 - r = v - e, \qquad \sum iv_i = 2e \quad \text{and} \quad \sum v_i = v,$$

where $\chi(\Gamma)$ denotes the Euler characteristic of Γ.

In order to show that two finite subgroups of Out F_r are not conjugate we shall use the extensions of F_r determined by them. For $r > 1$, each subgroup G of Out F_r determines a group extension, unique up to equivalence of extensions,

$$1 \to F_r \hookrightarrow E \to G \to 1$$

determined by the following property: the given action of $G \subset \mathrm{Out}\, F_r$ on F_r can be recovered from the extension by taking conjugations of F_r by preimages of elements of G in E. The extension E can be defined as the preimage of $G \subset \mathrm{Out}\, F_r$ in Aut F_r under the canonical projection from Aut F_r onto Out F_r, noting that its kernel Inn F_r is isomorphic to F_r. Then, by a short calculation, we have the following

LEMMA 3. *Two subgroups* G *and* G' *of* Out F_r *are conjugate if and only if the corresponding extensions* E *and* E' *of* F_r *are isomorphic by an isomorphism mapping* F_r *to itself.*

Now suppose that $G \subset \mathrm{Out}\, F_r$ is finite and that G acts on a finite connected graph Γ, with $\pi_1\Gamma \cong F_r$, realizing the given action of $G \subset \mathrm{Out}\, F_r$ on F_r. Then the extension E associated to G can be constructed in the following geometric way. Let $\tilde{\Gamma}$ be the universal covering of Γ, so $\tilde{\Gamma}$ is a tree. Then the group of automorphisms of $\tilde{\Gamma}$ consisting of all lifts of elements of G to $\tilde{\Gamma}$ is isomorphic to the extension E, where $F_r \subset E$ corresponds to the group of covering transformations; we denote this group of automorphisms of $\tilde{\Gamma}$ also by E.

In the next section, we shall represent the extension E as the fundamental group of a finite graph of finite groups $\mathcal{G}$ which can be constructed as follows. The underlying graph is the quotient graph $\bar{\Gamma} := \tilde{\Gamma}/E = \Gamma/G$ (here we suppose, by subdividing edges, that G resp. E act without inversions so that the quotient is again a graph). In all examples in the next section, the quotient graph $\bar{\Gamma}$ will be a tree so we restrict to this case here. Then $\bar{\Gamma}$ can be lifted isomorphically to Γ and from there to $\tilde{\Gamma}$, and we associate to the vertices and edges of $\bar{\Gamma}$ their stabilizers in G or, what is the same (as F_r operates without fixed points), in E. These groups are called the *vertex* resp. *edge groups* of the graph of groups. Note also that we have canonical inclusions of the edge groups into adjacent vertex groups. The result is a finite graph of finite groups $\mathscr{G}$, and the main theorem of the Bass-Serre theory of groups acting on trees says that E is isomorphic to the fundamental group $\pi_1\mathscr{G}$ of this graph of groups (see [**9**], [**8**] or [**12**]). In our case where $\bar{\Gamma}$ is a tree this fundamental group is the iterated free product with amalgamation of the vertex groups amalgamated over the edge groups.

Following [**8**] we call a graph of groups *minimal* if it has no trivial edges where an edge is called *trivial* if it has distinct vertices and the canonical monomorphism (inclusion) from the edge group to one of the two adjacent vertex groups is an isomorphism.

LEMMA 4. (a) *Let E be the fundamental group of a finite minimal graph of finite groups, associated to an action of E on a tree $\tilde{\Gamma}$. Then each maximal finite subgroup of E has a unique fixed point in $\tilde{\Gamma}$. The conjugacy classes of maximal finite subgroups of E correspond bijectively to the vertex groups of the graph of groups.*

(b) *Let $\mathscr{G}$ and $\mathscr{G}'$ be two finite graphs of finite groups which are minimal and have isomorphic fundamental groups. Then there is a bijection between the vertices of $\mathscr{G}$ and $\mathscr{G}'$ such that corresponding vertex groups are isomorphic.*

Proof. (see also [**8**, Lemma 7.6]).

Each finite group acting on a tree has a fixed point (consider the invariant subtree generated by an orbit and delete external edges in an equivariant way). A maximal finite subgroup M of E has a unique fixed point in $\tilde{\Gamma}$ because otherwise the edges of the unique edge path (segment) in the tree $\tilde{\Gamma}$ connecting two different fixed points would also be fixed by M; then M would occur also as an edge group and the associated graph of groups would not be minimal. By a similar argument, each vertex group is a maximal finite subgroup of E. This proves part (a) of the Lemma. Part (b) follows from the fact that, by the Bass-Serre theory of groups acting on trees, each graph of groups is associated to an action of its fundamental group on a tree.

In [**4**], [**6**] (see also the review of [**6**] in Zentralblatt Math. 793, 200017 (1994) and [**1**] a more general result has been proved which takes care also of the edge groups and gives an algorithm to decide whether the fundamental groups of two finite graphs of finite groups are isomorphic.

Finally, we sketch the

Proof of Theorem 1. Let E be the extension of F_r associated to $G \subset \text{Out}\, F_r$. By [**5**] (or [**12**, Theorem 2.3.1]), a finite extension E of a free group F_r is isomorphic to the fundamental group $\pi_1\mathscr{G}$ of a finite graph of finite groups $\mathscr{G}$. By the Bass-Serre theory of groups acting on trees the graph of groups $\mathscr{G}$ is associated to an action of the group $E = \pi_1\mathscr{G}$ on a tree $\tilde{\Gamma}$; in particular, $F_r \subset E$ acts freely. Then $G = E/F_r$ acts on the quotient graph $\Gamma := \tilde{\Gamma}/F_r$ and realizes the given action of $G \subset \text{Out}\, F_r$ on $\pi_1\Gamma \cong F_r$.

Moreover, G lifts to $\text{Aut}\, F_r$ if and only if the finite group G is a subgroup of the extension E (that is the extension splits), which in turn happens if and only if G is a

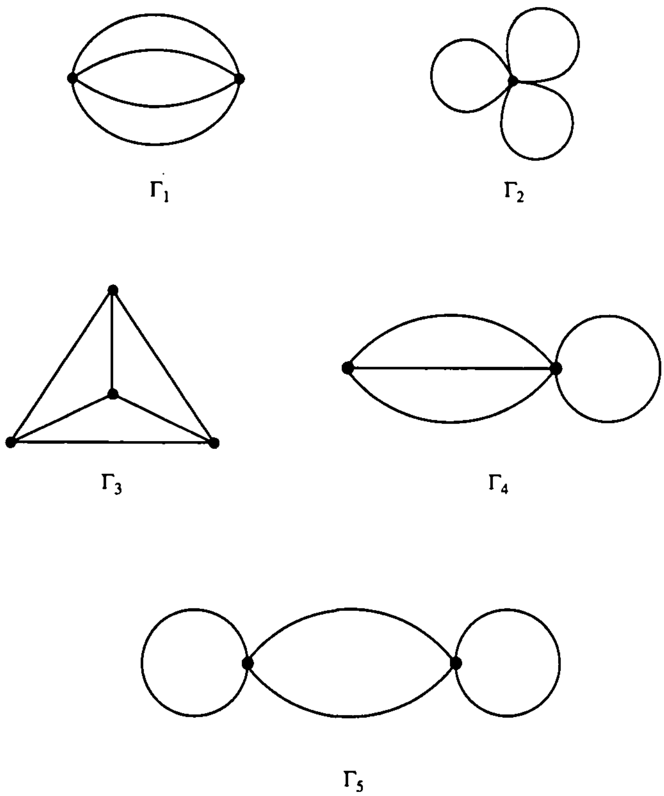


Figure 2.

vertex group of the graph of groups $\mathcal{G}$ (using that every finite group acting on a tree has a fixed point).

3. Maximal finite subgroups of Out F_3**.** Using, for example, Lemma 2 it is easy to find all admissible graphs of rank 3.

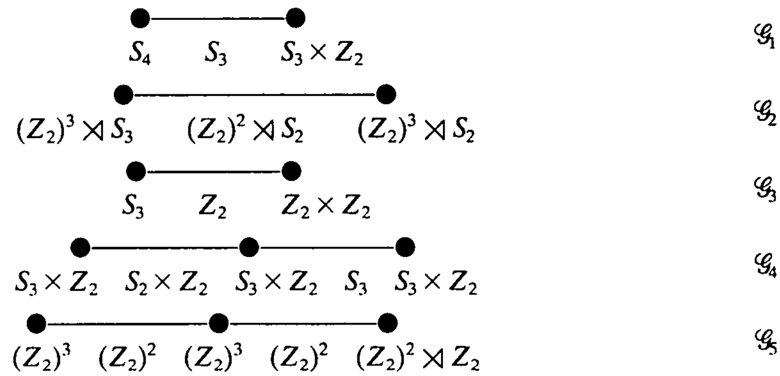
LEMMA 5. *There are exactly five admissible graphs* $\Gamma_1, \ldots, \Gamma_5$ *of rank* 3 *which are shown in Figure* 2. *Their automorphism groups* $G_i := \operatorname{Aut}\Gamma_i$ *are*

$$G_1 = S_4 \times Z_2,\ G_2 = (Z_2)^3 \rtimes S_3,\ G_3 = S_4,\ G_4 = S_3 \times \mathbb{Z}_2 \text{ and } G_5 = (Z_2)^3 \rtimes Z_2$$

(*note that* $G_1 \cong G_2$).

By subdividing edges such that G_i acts on Γ_i without inversions and then constructing the quotient of that action we get the following.

Lemma 6. *The graphs of group* $\mathcal{G}_i$ *associated to the* G_i*-actions on* Γ_i *are as follows*:



The groups G_i determine subgroups of G_i of Out F_3 denoted by the same letter.

Theorem 2. *Up to conjugation, the maximal finite subgroups of* Out F_3 *are exactly the following groups*:

$$G_1 = S_4 \times Z_2, \quad G_2 = (Z_2)^3 \rtimes S_3, \quad G_3 = S_4, \quad G_4 = S_3 \times Z_2 \quad \text{and} \quad G_5 = (Z_2)^3 \rtimes Z_2.$$

Proof. By Proposition 1 it remains to show that no G_i is conjugate in Out F_3 to a subgroup of some other G_j. Denote by $E_i := \pi_1 \mathcal{G}_i$ the extension of F_3 determined by G_i. By Lemma 4 the extensions E_1 and E_2 are not isomorphic because they have non-isomorphic vertex groups, therefore by Lemma 3 the groups G_1 and G_2 are not conjugate in Out F_3. In a basically equivalent way, the group G_1 has a fixed point on Γ_1 and therefore lifts from Out F_3 to Aut F_3 whereas G_2 has no fixed point on Γ_2 and does not lift, see Theorem 1. Similarly, the subgroup A_4 of $G_3 = S_4$ has no fixed points in Γ_4 and therefore does not occur as a vertex group of the graph of groups associated to the A_4-action on Γ_4 whereas each subgroup A_4 of G_1 or G_2 has a fixed point and occurs as a vertex group, therefore G_3 is not conjugate to a subgroup of G_1 or G_2.

It remains to show that G_4 and G_5 are not conjugate to subgroups of G_1 or G_2 (note that for order reasons they cannot be conjugate to a subgroup of G_3). Now $G_4 = S_3 \times Z_2$ has three different fixed points on Γ_4 corresponding to the three vertex groups $S_3 \times Z_2$ of the graph of groups $\mathcal{G}_4$. On the other hand, a subgroup $G \cong S_3 \times Z_2$ of G_1 resp. G_2 with a fixed point has exactly one fixed point, therefore the corresponding graph of groups associated to the G-action on Γ_1 resp. Γ_2 has exactly one vertex group isomorphic to $S_3 \times Z_2$. It follows that the extensions of F_3 associated to G_4 and to G are not isomorphic and consequently G_4 is not conjugate to a subgroup of G_1 or G_2.

Now consider $G_5 = (Z_2)^3 \rtimes Z_2$ which has no fixed points in Γ_5. As G_2 has a fixed point, G_5 is not conjugate to a subgroup of G_2. The subgroup $(Z_2)^3$ of G_5 has four fixed points in Γ_5. As there is no subgroup $(Z_2)^3$ of G_1 with a fixed point (because $(Z_2)^3$ is not a subgroup of one of the two vertex groups of $\mathcal{G}_1$) it follows that G_5 is not conjugate to a subgroup of G_1. This completes the proof of Theorem 2.

4. The graph rank of a finite group. Let G be a finite group. The (strong) *symmetric genus* of G is defined as the minimal genus of a closed orientable surface on which G acts by (orientation preserving) homeomorphisms (see e.g. [**3**]). By analogy, we define the *graph rank* of G, denoted by $\rho(G)$, as the minimal rank of a finite connected graph Γ without vertices of valence one (or equivalently, without free edges) such that G acts on Γ as a group of automorphisms. Then only the trivial group has graph rank zero, and the finite groups of graph rank one are exactly the cyclic and dihedral groups (because the only finite graphs of rank one without free edges are subdivided circles). As consequences of Theorem 1 and Lemma 1 resp. Corollary 2 and Theorem 2 we have

COROLLARY 3. *The graph rank of a finite group G which is not cyclic or dihedral is the minimal number r such that G embeds into the outer automorphism group* Out F_r *of the free group of rank r.*

COROLLARY 4. *The finite groups of graph rank one are exactly the cyclic and dihedral groups. There are no finite groups of graph rank two. The finite groups of graph rank* 3 *are* $S_4 \times Z_2 \cong (Z_2)^3 \rtimes S_3$ *and its subgroups which are not cyclic or dihedral.*

By [**10**] the maximal order of a finite subgroup G of Out F_r is $2^r r!$, for $r > 2$. For $r > 3$, the unique (up to conjugation) maximal subgroup of Out F_r is $(Z_2)^r \rtimes S_r$ acting on the graph with a single vertex and r edges (a "bouquet" of r circles), therefore

$$\rho((Z_2)^n \rtimes S_n) = n.$$

The possible isomorphism types of finite subgroups of Out F_r have been determined in [**7**]; in principle, this allows an algorithmic computation of the graph rank of a finite group G. For example, it follows from the results in [**7**] that, for $n > 3$

$$\rho(S_n \times Z_2) = \rho(S_n) = \rho(A_n) = n - 1;$$

note that $S_n \times Z_2$ acts on the graph of rank $n-1$ consisting of 2 vertices and n edges connecting these vertices.

Suppose the finite group G acts on an admissible graph of rank r. As above, G determines a finite group of finite groups $\mathcal{G}$ and an extension

$$1 \to F_r \hookrightarrow E = \pi_1 \mathcal{G} \to G \to 1.$$

Define the *Euler number* of $\mathcal{G}$ as

$$\chi(\mathcal{G}) := \sum 1/|G_e| - \sum 1/|G_v|$$

where the sum is extended over all edge groups G_e and all vertex groups of $\mathcal{G}$. Then we have the formula

$$r - 1 = |G|\,\chi(\mathcal{G})$$

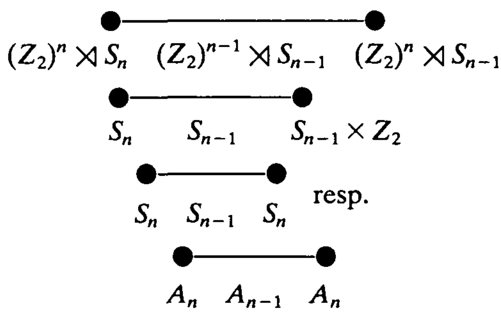
(see [**5**], [**11**] or [**12**, Prop. 2.3.3]). Conversely, given a finite graph of finite groups $\mathcal{G}$ and a surjection of $E = \pi_1 G$ onto G with torsionfree kernel, this kernel is a free group F_r of rank r where r is determined by the above formula (it is free because it acts freely on a tree). Suppose in addition that E has no non-trivial finite normal subgroups. Then, by taking conjugations of $F_r \subset E$ with preimages of elements of G in E, we get an inclusion of G into Out F_r. On the other hand, if E has a finite normal subgroup, then the elements of that subgroup commute with the elements of the normal subgroup F_r of E and the induced homomorphism from G to Out F_r is not injective. Thus we have

PROPOSITION 2. *The graph rank of a finite group G which is not cyclic or dihedral is equal to $|G|\,\chi(\mathcal{G}) - 1$ where $\chi(\mathcal{G})$ is the minimal positive Euler number of a finite graph of* *finite groups $\mathcal{G}$ such that $\pi_1(\mathcal{G})$ has no non-trivial finite normal subgroups and such that there exists a surjection of $\pi_1\mathcal{G}$ onto G with torsionfree kernel.*

For example, for the finite groups

$$(Z_2)^n \rtimes S_n, \qquad S_n \times Z_2, \qquad S_n \text{ resp. } A_n$$

this minimum is obtained for the following graphs of groups:



These graphs of groups are associated to the action of the group $(Z_2)^n \rtimes S_n$ on the graph of rank n which is a bouquet of n circles, and to the action of $S_n \times Z_2$ and its subgroups S_n and A_n on the graph of rank $n-1$ with two vertices and n connecting edges.

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Università Degli Studi di Trieste
Dipartimento di Scienze Matematiche
34100 Trieste
Italy
e-mail: zimmer@univ.trieste.it