# UNION CURVES OF A HYPERSURFACE

C. E. SPRINGER

**1. Introduction.** A curve on an ordinary surface is a union curve[1] if its osculating plane at each point contains the line of a specified rectilinear congruence through the point. The author[2] has obtained the differential equations of union curves on a metric surface in ordinary space and has exhibited certain generalizations for union curves of known results concerning geodesic curves on a surface. It is the purpose of the present paper to develop the differential equations of the union curves of a hypersurface $V_n$ immersed in a Riemannian manifold $V_{n+1}$ of $n + 1$ dimensions. The osculating plane to a curve on a surface is generalized to a totally geodesic surface the straight lines of which are geodesics in the space $V_{n+1}$. A formula is given for the union curvature vector of a curve in $V_n$.

**2. Vector field in $V_n$.** If $y^\alpha$ $(\alpha = 1, \ldots, n + 1)$ denote the coordinates of a point in $V_{n+1}$, and $x^i$ $(i = 1, \ldots, n)$ the coordinates of a point in $V_n$, the equations of the hypersurface $V_n$ may be written in the form

$$y^\alpha = y^\alpha (x^1, \ldots, x^n). \tag{1}$$

For points in the $V_n$ the functional matrix $\|\partial y^\alpha/\partial x^i\|$ is of rank $n$. Let the metric of $V_n$ be denoted by $g_{ij}dx^idx^j$ and that of $V_{n+1}$ by $a_{\alpha\beta}dy^\alpha dy^\beta$. These metrics are assumed to be positive definite. It follows that

$$a_{\alpha\beta}y^\alpha{}_{,i}\, y^\beta{}_{,j} = g_{ij}\ , \tag{2}$$

where $y^\alpha{}_{,i}$ denotes the covariant derivative of $y^\alpha$ with respect to $x^i$. (Greek indices always have the range $1, \ldots, n + 1$ and Latin indices the range $1, \ldots, n$.) If $N^\alpha$ denote the components of a unit vector in $V_{n+1}$ normal to $V_n$, then

$$a_{\alpha\beta}y^\alpha{}_{,i}\, N^\beta = 0 \qquad (i = 1, \ldots, n), \tag{3}$$

and

$$a_{\alpha\beta}N^\alpha N^\beta = 1. \tag{4}$$

If a vector field in $V_n$ has components $U^\alpha$ in the $y$'s and components $u^i$ in the $x$'s, then the relation

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[1]P. Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, Amer. J. of Math., vol. 40 (1928), p. 213.

[2]C. E. Springer, *Union curves and union curvature*, Bull. Amer. Math. Soc., vol. 51 (1945), pp. 686-691.

$$U^\alpha = y^\beta{}_{,i}\, u^i \tag{5}$$

must obtain. If $q^\alpha$ are the contravariant components in the $y$'s of the derived vector relative to $V_{n+1}$ of a vector of the field along a curve $C$ in $V_n$, and if $p^i$ are the contravariant components in the $x$'s of the derived vector relative to $V_n$ of the same vector along $C$, it can be shown[3] that

$$q^\alpha = \Omega_{ij}\, u^i \frac{dx^j}{ds} N^\alpha + y^\alpha{}_{,i}\, p^i, \tag{6}$$

where $\Omega_{ij}\, dx^i\, dx^j$ is the second fundamental form for $V_n$.

**3. Totally geodesic surface in $V_{n+1}$.** As an analogue for the osculating plane in ordinary space a totally geodesic surface in $V_{n+1}$ is introduced. It is determined by the tangent to the curve $C$ with equations $x^i = x^i(s)$ in $V_n$, $s$ denoting arc length, and by the first curvature vector in $V_{n+1}$ of the curve $C$. Let $\lambda^\alpha$ be the contravariant components in the $y$'s of a unit vector in the direction of a curve of a congruence of curves, one curve of which passes through each point of $V_n$. The vector with components $\lambda^\alpha$ is, in general, not normal to $V_n$, and may be specified by

$$\lambda^\alpha = t^i\, y^\alpha{}_{,i} + rN^\alpha, \tag{7}$$

where $t^i$ and $r$ are parameters. Because $\lambda^\alpha$ represent a unit vector $a_{\alpha\beta}\lambda^\alpha\lambda^\beta = 1$, and it follows by use of equations (3), (4), (7) that

$$t_i t^i = 1 - r^2.$$

If the geodesic in $V_{n+1}$ in the direction of the curve of the congruence with direction $\lambda^\alpha$ is to be a geodesic of the totally geodesic surface, then it is necessary that $\lambda^\alpha$ be a linear combination of $y^\alpha{}_{,i}\, u^i$ and $q^\alpha$. Hence,

$$t^i y^\alpha{}_{,i} + rN^\alpha = vy^\alpha{}_{,i}\, u^i + wq^\alpha, \tag{8}$$

wherein $v$ and $w$ are to be determined, the $u^i$ of equations (5) are now $dx^i/ds$, and $q^\alpha$ are given by

$$q^\alpha = \Omega_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} N^\alpha + y^\alpha{}_{,i}\, p^i, \tag{9}$$

and $p^i$ are given by

$$p^i = \frac{d^2x^i}{ds^2} + \left\{ {i \atop jk} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds}\,. \tag{10}$$

If $K_n$ is written for $\Omega_{ij} \dfrac{dx^i}{ds} \dfrac{dx^j}{ds}$, which is the normal component of the curvature vector of the curve $C$ in $V_{n+1}$, equations (8) take the form

[3] C. E. Weatherburn, *Riemannian Geometry and the Tensor Calculus* (Cambridge University Press, 1938).

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$$t^i y^\alpha{}_{,i} + rN^\alpha = vy^\alpha{}_{,i} \frac{dx^i}{ds} + w(K_n N^\alpha + y^\alpha{}_{,i} p^i). \tag{11}$$

Multiplication of equations (11) by $a_{\alpha\beta} y^\beta{}_{,j}$, summation with respect to $\alpha$, and use of equations (2), (3) yield the $n$ equations

$$g_{ij} t^i = v g_{ij} \frac{dx^i}{ds} + w g_{ij} p^i. \tag{12}$$

If equations (11) are multiplied by $a_{\alpha\beta} N^\alpha$, summation on $\alpha$ and use of (4) give the relation

$$r = wK_n. \tag{13}$$

The solution of (12) for $v$ is effected by multiplying by $\frac{dx^j}{ds}$ and summing on $j$. Because $g_{ij} p^i \frac{dx^j}{ds} = 0$, it follows that

$$v = g_{ij} t^i \frac{dx^j}{ds}. \tag{14}$$

Therefore, on using the values of $v$ and $w$ from (13) and (14), the $n$ equations (12) take the form

$$g_{ij} t^i = g_{ij} \frac{dx^i}{ds} g_{lm} t^l \frac{dx^m}{ds} + \frac{r}{K_n} g_{ij} p^i. \tag{15}$$

Multiplication of equations (15) by $g^{jk}$, summation on $j$, and the replacement of $t^k/r$ by $l^k$ lead to

$$p^k - K_n\left(l^k - g_{im} l^i \frac{dx^m}{ds} \frac{dx^k}{ds}\right) = 0 \qquad (k = 1, \ldots, n), \tag{16}$$

wherein $p^k$ are given by equations (10).

**4. Union curves in $V_n$.** For a congruence specified by the parameters $l^k$, the solutions of the $n$ equations (16) determine the union curves in $V_n$ relative to that congruence. The parameter $r$ can not vanish under the assumption that the direction $\lambda^\alpha$ is not in the $V_n$. The left members of equations (16) may be denoted by $\eta^k$, which we shall call the contravariant components of the union curvature vector in $V_{n+1}$. A union curve of $V_n$ with respect to a congruence determined by the parameters $l^k$ may therefore be defined as a curve along which the union curvature vector is a null vector.

By use of (10) and the fact that $g_{ij} dx^i dx^j = ds^2$, equations (16) can be written in the form

$$\eta^k \equiv p^k - K_n \nu^k = 0, \tag{17}$$

where the vector $\nu^k$ is defined by

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$$\nu^k = g_{ij} \frac{dx^i}{ds} \left( l^k \frac{dx^j}{ds} - l^j \frac{dx^k}{ds} \right).$$

From equations (17) it follows that if the curve $C$ is an asymptotic curve in $V_n$, in which case $K_n = 0$ along the curve, then for a union curve ($\eta^k = 0$), $p^k = 0$ and the curve is a geodesic. Hence, if a union curve is an asymptotic curve, it is a geodesic. Furthermore, if a union curve is a geodesic, then it is either an asymptotic curve or the vector of components $\nu^k$ is a null vector.

The magnitude $K_U$ of the vector $\eta^k$ is given by $K_U{}^2 = g_{ij}\eta^i\eta^j$. From equations (7) it is seen that the angle $\phi$ between the vectors $\lambda^a$ and $N^a$ in $V_{n+1}$ is given by $\cos\phi = r$, and because $t^k/r = l^k$ and $t_i t^i = 1 - r^2$, it follows that $g_{ij}l^i l^j = \tan^2\phi$. The angle $\alpha$ between the vector $l^k$ and the tangent vector to $C$ is given by $\cos\alpha = g_{ik}l^i \dfrac{dx^k}{ds}$. In terms of $\phi$ and $\alpha$, the magnitude $K_U$ of the union curvature vector can be shown to be given by

$$K_U = K_g - K_n \tan\phi \sin\alpha,$$

where $K_g$ is the geodesic curvature of the curve $C$ in $V_n$. It is to be observed that if $\phi = 0$, the union curve is a geodesic.

*University of Oklahoma*

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