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RESEARCH ARTICLE

Orthogonal roots, Macdonald representations, and quasiparabolic sets

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Abstract
Let W be a simply laced Weyl group of finite type and rank n. If W has type E_7, E_8 or D_n for n even, then the root system of W has subsystems of type nA_1. This gives rise to an irreducible Macdonald representation of W spanned by n-roots, which are products of n orthogonal roots in the symmetric algebra of the reflection representation. We prove that in these cases, the set of all maximal sets of orthogonal positive roots has the structure of a quasiparabolic set in the sense of Rains–Vazirani. The quasiparabolic structure can be described in terms of certain quadruples of orthogonal positive roots which we call crossings, nestings and alignments. This leads to nonnesting and noncrossing bases for the Macdonald representation, as well as some highly structured partially ordered sets. We use the 8-roots in type E_8 to give a concise description of a graph that is known to be non-isomorphic but quantum isomorphic to the orthogonality graph of the E_8 root system.

Contents

1. Introduction

If W is a finite simply laced Weyl group, then it is possible to find a basis of the reflection representation V of W that consists of orthogonal positive roots when W has type E_7, E_8 or D_n for n even. The goal of this paper is to demonstrate that the set $X = X(W)$ of such bases has a rich combinatorial structure, both by identifying X with a subset of a Macdonald representation of W and by regarding X as a quasiparabolic set in the sense of Rains and Vazirani [43]. A quasiparabolic set for a Weyl group W is a W-set equipped with an integer-valued height function satisfying two axioms that specify how the action of a reflection changes the height. The axioms generalize properties satisfied by quotients of arbitrary Coxeter groups by their parabolic subgroups that allow one to deform the action of Coxeter group on the quotient to create a corresponding module for the Iwahori–Hecke algebra of the group.

Let W be a finite Weyl group with root system Φ and Dynkin diagram Γ. Let V be the reflection representation of W defined over $\mathbb{Q}$ (see Section 2.1), and let V^* be the dual of V over $\mathbb{Q}$. The space of all rational-valued polynomial functions on V is then the symmetric algebra $\mathrm{Sym}(V^*)$, which is a W-module via the contragredient action $(w \cdot \phi)(x) = \phi(w^{-1}(x))$. Let Ψ be a *subsystem* of Φ, meaning that $\emptyset \neq \Psi \subset \Phi$ and that Ψ is also a root system, and let Φ^+ and Ψ^+ be the set of positive roots in Φ and Ψ, respectively. Following [39], we use the (positive definite) inner product on V to identify V with V^*, and we define the *Macdonald representation* $j_\Psi^\Gamma(\mathrm{sgn}) = j_\Psi^\Phi(\mathrm{sgn})$ of W given by Ψ to be the cyclic $\mathbb{Q}W$-submodule of $\mathrm{Sym}(V^*) \cong \mathrm{Sym}(V)$ generated by π_Ψ, where

$$\pi_\Psi = \prod_{\alpha \in \Psi^+} \alpha.$$

A short argument shows that the Macdonald representation is an absolutely irreducible W-module.

When W has type E_7, E_8 or D_n for n even, Φ contains subsystems of type nA_1. The set Ψ^+ for any such subsystem Ψ consists of n orthogonal positive roots, and we call an element of the form $w.\pi_\Psi$ (for $w \in W$) an *n-root*. Thus, an n-root has the form $\alpha = \prod_{i=1}^n \beta_i$ where the elements β_i are orthogonal roots. Conversely, since W acts transitively on the set of maximal sets of orthogonal roots (Lemma 3.2), every product of n orthogonal roots is an n-root. The transitivity of this W-action also implies that any two subsystems of type nA_1 in Φ give rise to the same Macdonald representation, which we will thus simply denote as $j_{nA_1}^\Phi(\mathrm{sgn})$. The representation $j_{nA_1}^\Phi(\mathrm{sgn})$ and the n-roots within it are the central objects of study in this paper, and we summarize their definition below.

Definition 1.1. Let Φ be a root system of type E_7, E_8 or D_n for n even. Let W be the Weyl group of Φ, and let V be the reflection representation of W. We denote the Macdonald representation $j_\Psi^\Phi(\mathrm{sgn}) \subset \mathrm{Sym}(V^*) \cong \mathrm{Sym}(V)$ arising from any subsystem Ψ of type nA_1 in Φ by $j_{nA_1}^\Phi(\mathrm{sgn})$. We call each element of the form $\alpha = \prod_{i=1}^n \beta_i \in j_{nA_1}^\Phi(\mathrm{sgn})$, where $\beta_1, \cdots, \beta_n$ are orthogonal roots of Φ an *n-root* of W.

Given an n-root $\alpha = \prod_{i=1}^n \beta_i$, the factors β_i are unique up to reordering and multiplication by nonzero scalars because they are the irreducible factors of α in the unique factorization domain $\mathbb{Q}[\alpha_1, \alpha_2, \ldots, \alpha_n]$,

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where the α_i correspond to the simple roots of Φ. We say α is *positive* if all the factors β_i may be taken to be positive or, equivalently, if evenly many of the components are negative. If $\alpha = \prod_{i=1}^n \beta_i$ is a positive root with all the β_i positive, we call the roots β_i the *components* of α. An n-root α is *negative* if $-\alpha$ is positive. It is immediate from the definitions that if α is an n-root, then $-\alpha$ is also an n-root, and that precisely one of α and $-\alpha$ is positive, similarly to how roots appear in positive-negative pairs in ordinary root systems. If α is either a root or an n-root, we define the *absolute value* of α, denoted $|\alpha|$, to be the positive element in the pair $\{\alpha, -\alpha\}$. (We may view both ordinary roots and n-roots as special cases of *k-roots*, by which we mean products in $\mathrm{Sym}(V^*)$ of k orthogonal roots of W for any fixed integer $1 \leq k \leq n$. The notion of k-roots plays an important role in our previous papers [30] and [28], and we will occasionally speak of 4-roots, even when $n \neq 4$, in this paper.)

The set Φ_n^+ of all positive n-roots admits a natural W-action given by $w(\alpha) = |w(\alpha)|$. Similarly, the set X of sets of n orthogonal roots admits a natural W-action given by $w(\{\beta_1, \cdots, \beta_n\}) = \{|w(\beta_1)|, \cdots, |w(\beta_n)|\}$. The map sending each set $\{\beta_1, \cdots, \beta_n\} \in X$ to the product $\prod_{i=1}^n \beta_i \in \Phi_n^+$ respects these two W-actions, and we use it to identify X with Φ_n^+. In other words, we identify each positive n-root with its set of components.

We show that X has the structure of a quasiparabolic set under a suitable height function λ (Theorem 4.5). As we explain in sections 3 and 4, to understand this structure, it is useful to consider quadruples $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ of four orthogonal roots with the property that $(\beta_1 + \beta_2 + \beta_3 + \beta_4)/2$ is also a root. We call such quadruples *coplanar quadruples* and show that they fall into three distinct types, called *crossings*, *nestings* and *alignments*. The height function λ is given by $\lambda(\gamma) = C(\gamma) + 2N(\gamma)$, where $C(\gamma)$ and $N(\gamma)$ are the numbers of crossings and nestings in γ, respectively, for each $\gamma \in X$. The terms 'crossing', 'nesting' and 'alignment' are motivated by the theory of perfect matchings (Remark 3.11).

As a quasiparabolic set, the set X is equipped with a partial order $\leq_Q$, which is the weakest partial order such that $x \leq_Q rx$ whenever r is a reflection such that $\lambda(x) \leq \lambda(rx)$. We use the theory of quasiparabolic sets to prove that X has a unique maximally aligned n-root, θ_A, and a unique maximally nesting n-root, θ_N; these two elements are the unique minimal and maximal element of X with respect to $\leq_Q$, respectively (Proposition 4.11). The n-roots that avoid alignments, or the *alignment-free* n-roots, form a quasiparabolic set $X_I \subset X$ of a certain maximal parabolic subgroup W_I of W. The corresponding partial order on X_I allows us to show that X also has a maximally crossing element, θ_C, and that it is the unique minimal element of X_I (Proposition 5.2 (iii)). The set X_I has a natural bipartite structure, with the n-roots in X_I with even levels and those with odd levels partitioning X_I into two equal-sized components that are interchanged by every reflection in W_I (Remark 5.3).

The alignment-free n-roots in X are one of three families that avoid a particular type of coplanar quadruple, the other two being those that avoid crossings and avoid nestings. Section 5 studies these three families together. We use a version of Bergman's diamond lemma to show that the noncrossing elements of X form a basis for the Macdonald representation, as do the nonnesting positive n-roots (Theorem 5.5). In addition, the noncrossing basis behaves somewhat like a simple system in a root system (Theorem 5.7) and may be viewed as a canonical basis (Remark 5.8). The nonnesting basis is naturally parametrized by a particular interval $[1, w_N]$ in the weak Bruhat order of W and has the structure of a distributive lattice (Theorem 5.13). The element w_N, which we call the *nonnesting element* of W, is the unique shortest element taking the maximally crossing n-root θ_C to the maximally aligned element θ_A. We note that the noncrossing basis is essentially the same as a basis that appears in the work of Fan [23, Section 6] and others, although the realization of the basis as polynomials in roots seems to be new.

We say that two positive n-roots are *sum equivalent* or *σ-equivalent* if their sets of components have the same sum. We show that the σ-equivalence classes are in canonical bijection with the nonnesting and the noncrossing elements of X in the following way: each σ-equivalence class is an interval $[\beta_1, \beta_2]_Q = \{\gamma \in X : \beta_1 \leq_Q \gamma \leq_Q \beta_2\}$ in the quasiparabolic order $\leq_Q$, where the minimal element β_1 and maximal element β_2 are the unique nonnesting and noncrossing n-roots in the class, respectively. The alignment-free elements in X form a single σ-equivalence class that is maximal with respect to a natural order (Proposition 5.15). Any set of σ-equivalence class representatives forms a basis for the Macdonald representation, and the change of basis matrix between any two such bases, such as the

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one between the nonnesting and noncrossing basis, is always unitriangular with integer entries once the bases are suitably ordered (Theorem 5.16).

As we explain in Section 7.3, the feature-avoiding n-roots and σ-equivalence classes in the set X can all be characterized abstractly using the quasiparabolic structure of X, without using the combinatorics of sets of roots. This raises the possibility of extending the notions of alignment-free, noncrossing and nonnesting elements to more general quasiparabolic sets.

The theory developed in this paper has various natural connections to many previous works. In type D_n for an even integer $n = 2k$, the n-roots correspond naturally to perfect matchings of the set $[n] = \{1, 2, \ldots, n\}$, and the crossings, nestings and alignments in n-roots recover the corresponding notions in the theory of matchings. Besides matchings, the quasiparabolic set X can be identified with the set of fixed-point free involutions in S_n, which is one of the original motivating examples of a quasiparabolic set [43, Section 4]. The level function $\lambda = C + 2N$ appears as a useful statistic on matchings in [52, 20, 15] and has a natural interpretation in the context of combinatorial game theory [36]; see Section 6.1.

The Macdonald representation $j^{\Phi}_{nA_1}(\mathrm{sgn})$ in type D_n for n even recovers a Specht module in a very natural way: the action of the Weyl group W factors through an obvious sign-forgetting map (Equation (2.2)) to induce an S_n-module structure on the Macdonald representation for the symmetric group S_n, and the resulting module is isomorphic to a realization of the Specht module corresponding to the two-row partition (k, k) due to Rhoades [47] (Proposition 6.2). The noncrossing bases and nonnesting bases have been studied extensively as the web basis and the Specht basis, respectively, of the Specht module [48, 35, 34, 32]; see Section 6.1.

In type E_7, the Macdonald representation contains 135 positive 7-roots and has degree 15 [19, Proposition 4.12]. This representation has a long history, going back the work of Coble in 1916 [16, (65)] on the Göpel variety. There are also applications of 7-roots to quantum information theory and supergravity [12, Section IV G], [22]. In this case, the elements of the quasiparabolic set X_I are in canonical bijection with the 30 distinct labellings of the Fano plane, and the maximal and minimal elements are given by $\{136, 145, 127, 235, 246, 347, 567\}$ and $\{123, 145, 246, 257, 347, 356, 167\}$, respectively (Proposition 6.5).

In type E_8, there are 2025 positive 8-roots. The bases of orthogonal roots have applications to physics, where they can be used to prove the Kochen–Specker theorem in quantum mechanics [57] (Section 7.2). The Macdonald representation in this case has degree 50 but seems not to have been studied much before. The quasiparabolic set X_I in this case is a bipartite structure with 240 elements. As we explain in Section 6.3, either partite component can be used to define a graph that has an interesting relationship with two strongly regular graphs studied recently by Schmidt [51] (Remark 6.13). Those two graphs each have 120 vertices, and they have the remarkable property of being quantum isomorphic (in the sense of [3]) but not isomorphic.

The properties of n-roots summarized in the last three paragraphs are explained in more detail in Section 6. It is worth noting that while these properties are type-specific, we have attempted to develop the theory of n-roots in a type-independent way in the other parts of the paper in general. In particular, we give a uniform proof for the fact that the positive n-roots form a quasiparabolic set in types E_7, E_8, and D_{2k} (Theorem 4.5). While it is possible to verify the theorem for types E_7 and E_8 using direct computation (which we did, using the software SageMath [49]) and then separately deduce the theorem for type D_{2k} by considering the S_{2k}-action on its fixed-point free involutions, our uniform proof of the theorem relying on Proposition 4.7 has the advantage of being more conceptual and revealing more details about the action of reflections on n-roots. Some of these details will be further used in a forthcoming paper [31], where we will generalize aspects of this paper and study quasiparabolic sets arising from k-roots for more general values of k.

The rest of the paper is organized as follows. We recall the basics of root systems in Section 2. Section 3 introduces the key notions of crossings, nestings and alignments in an n-root, and we connect them to the theory of quasiparabolic sets in Section 4. Section 5 studies the alignment-free, noncrossing and nonnesting n-roots. Section 6 discusses the details of n-roots in the types D_n with n even, E_7 and E_8.

Section 7 concludes the paper and includes discussions of the Poincaré polynomial of the set X and of orbits of n-roots under the action of Coxeter elements.

2. Review of root systems

In this section, we recall the basic properties of simply laced root systems of finite type. We will mostly follow the notation of the first two chapters of [33], except in the case of type E_7, where we follow [26, Section 4].

2.1. Weyl groups, root systems and reflection representations

The root systems in this paper will be irreducible simply laced root systems of finite type, whose Dynkin diagrams are shown in Figure 1. The vertices of the Dynkin diagram Γ index the *simple roots* $\Pi = \{\alpha_i : i \in \Gamma\}$. The *root lattice* $\mathbb{Z}\Pi$ is the free $\mathbb{Z}$-module on Π. We define a $\mathbb{Z}$-bilinear form B on $\mathbb{Z}\Pi \times \mathbb{Z}\Pi$ by

$$B(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i \text{ and } j \text{ are adjacent in } \Gamma; \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha_i \in \Pi$, then we define the *simple reflection* $s_i = s_{\alpha_i}$ to be the $\mathbb{Z}$-linear operator $\mathbb{Z}\Pi \longrightarrow \mathbb{Z}\Pi$ given by

$$s_i(\beta) = \beta - B(\alpha_i, \beta)\alpha_i.$$

The *Weyl group* $W = W(\Gamma)$ is the finite group generated by the simple reflections.

The *root system* of W is the set $\Phi = \{w(\alpha_i) : \alpha_i \in \Pi, w \in W\}$. Each element of Φ is called a *root*. The group W acts transitively on Φ, and the form B is W-invariant in the sense that $B(\alpha, \beta) = B(w(\alpha), w(\beta))$ for all $w \in W$ and all $\alpha, \beta \in \Phi$. We say two roots are $\alpha, \beta \in \Phi$ are *orthogonal* if $B(\alpha, \beta) = 0$.

Each root $\alpha \in \Phi$ gives rise to a *reflection* in W, which is the self-inverse $\mathbb{Z}$-linear operator $s_\alpha : \mathbb{Z}\Pi \longrightarrow \mathbb{Z}\Pi$ generalizing simple reflections and given by the formula

$$s_\alpha(\beta) = \beta - B(\alpha, \beta)\alpha. \tag{2.1}$$

The reflections in W form a single conjugacy class. The $\mathbb{Q}$-vector space $V := \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}\Pi$ affords the *reflection representation* of W, where each reflection s_α acts by Equation (2.1).

A subset Ψ of Φ is called a *subsystem* if Ψ is itself a root system (in the sense of [33, Section 1.2]). For each root $\alpha \in \Phi$, the set $\Phi_\alpha := \{\beta \in \Phi : B(\alpha, \beta) = 0\}$ is automatically a subsystem.

2.2. Positive and simple systems

A subset Δ of a root system Φ is called a *simple system* if Δ is a vector space basis for V and every root is a linear combination of Δ with coefficients of like sign. Given such a system Δ, we say a root $\alpha = \sum_{i \in \Gamma} c_i \alpha_i$ is *positive* (with respect to Δ) if $c_i \geq 0$ for all i, and we call α *negative* if $c_i \leq 0$ for all i. The sets of positive and negative roots are denoted by Φ^+_Δ and Φ^-_Δ, and they are setwise negations of each other. For each root α, the integer $\mathrm{ht}(\alpha) = \sum_{i \in \Gamma} c_i$ is called the *height* of α. The set $\{\alpha_i : c_i \neq 0\}$ is called the *support* of α (with respect to Δ).

Each simple system Δ also gives rise to a partial order $\leq_\Delta$ on Φ, which is defined by the condition that $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a nonnegative linear combination of Δ. With respect to this partial order, Φ has a unique maximal element, θ_Δ, called the *highest root*.

The set of simple roots Π is an example of a simple system, and the corresponding set of positive roots is an example of a *positive system*. Recall that each positive system P contains a unique simple

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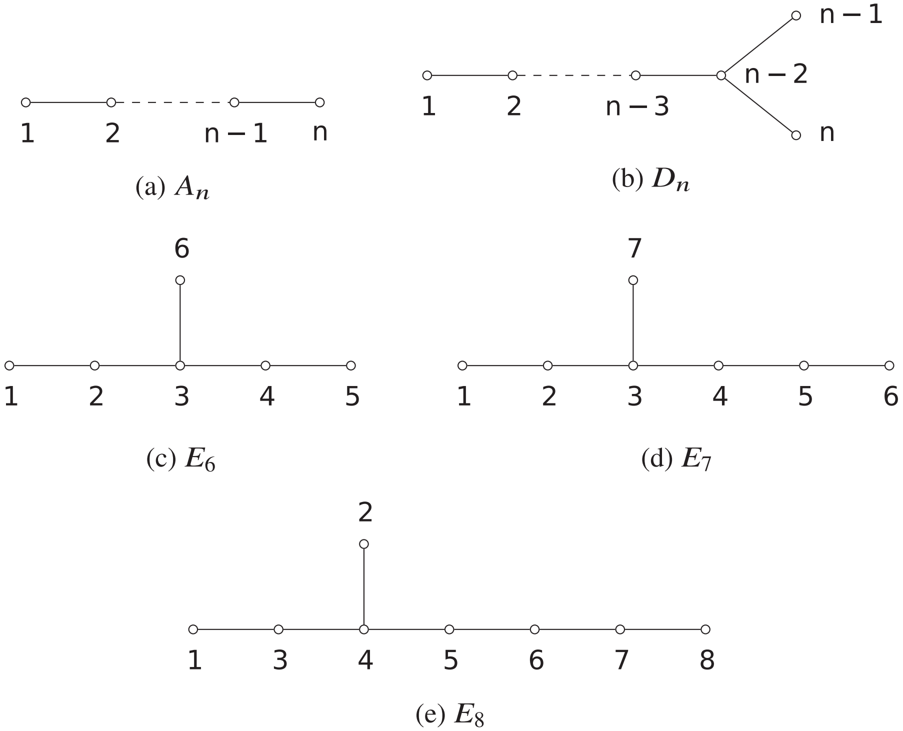

Figure 1. *Dynkin diagrams of irreducible simply laced Weyl groups.*

system, which is the set Δ_P of all elements in P that cannot be expressed as positive linear combinations of other elements of P. The maps $\Delta \mapsto \Phi^+_\Delta$ and $P \mapsto \Delta_P$ are mutually inverse bijections between the sets of simple systems and positive systems in Φ [33, Theorem 1.3]. The simple systems of Φ are conjugate to each other under the action of W, as are the positive systems.

From now on, we choose Π as the default simple system of Φ and choose Φ^+_Π as the default positive system. For each notion defined above relative to a general simple system Δ, an omission of the subscript in the corresponding notation will indicate that Π is chosen as Δ. For example, the set of positive roots with respect to Π will be denoted by Φ^+.

For any subsystem Ψ of Φ, the set $\Psi^+ := \Psi \cap \Phi^+$ is automatically a positive system of Ψ. We call Ψ^+ the *induced positive system* of Ψ with respect to Φ^+. We call the corresponding simple system of Ψ the *induced simple system* of Ψ.

If Ψ is a subsystem of the form Φ_α (i.e., if Ψ is the subsystem of roots orthogonal to a root α), then we denote the induced simple system by Π_α. The elements of Π_α are thus the positive roots orthogonal to α that cannot be expressed as a nonnegative linear combination of other positive roots orthogonal to α. Note that the elements of Π_α may not all lie in Π, but every simple root $\alpha_i \in \Pi$ that lies in Ψ and is orthogonal to α will lie in Π_α. We denote the Weyl group corresponding to Π_α by W_α, so that Φ_α is the root system of W_α. It is known that W_α is the stabilizer of α under the action of W and W_α is a direct product of irreducible simply laced Weyl groups [7, 1]. We will recall the known Dynkin type of the groups W_α in the next subsection.

Example 2.1. Let W be the Weyl group of type D_5 and let α_2 be the simple root of W under the labelling used in Figure 1. The induced simple system of Φ_{α_2} is given by the disjoint union

$$\Pi_{\alpha_2} = \{\beta_1 = \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5\} \sqcup \{\beta_2 = \alpha_1 + \alpha_2 + \alpha_3, \beta_3 = \alpha_4, \beta_4 = \alpha_5\},$$

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where each root in one part of the union is orthogonal to every root in the other part. The Weyl group W_{α_2} corresponding to Φ_{α_2} is the direct product $W(A_1) \times W(A_3)$ of the Weyl groups of types A_1 and A_3, generated respectively by the sets $\{s_{\beta_1}\}$ and $\{s_{\beta_2}, s_{\beta_3}, s_{\beta_4}\}$.

2.3. Explicit constructions

We now recall well-known explicit realizations of the root systems of types A, D and E in coordinate systems. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the usual standard basis of the Euclidean space $\mathbb{R}^n$. The vectors $\{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n\}$ form a root system of type A_{n-1}. The simple roots $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. A root $\varepsilon_i - \varepsilon_j$ is positive if $i < j$ and negative if $i > j$. The highest root (with respect to Π) is $\varepsilon_1 - \varepsilon_n$. The bilinear form B is the Euclidean inner product on $\mathbb{R}^n$, and two roots are orthogonal if and only if they have disjoint support. The Weyl group is isomorphic to S_n and acts by permuting the standard basis $\varepsilon_1, .., \varepsilon_n$. The stabilizer W_α of each root α is a Weyl group of type A_{n-3}, which is trivial if $n \leq 3$.

The vectors $\{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\}$ form a root system of type D_n. The simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < n$, and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. If $i < j$, then the roots $\varepsilon_i \pm \varepsilon_j$ are positive, and the roots $-\varepsilon_i \pm \varepsilon_j$ are negative. The highest root is $\varepsilon_1 + \varepsilon_2$. The bilinear form B is the Euclidean inner product, and two roots α and β are orthogonal if and only if either (a) α and β have disjoint support or (b) α and β have the same support and $\alpha \neq \pm\beta$. The Weyl group has order $2^{n-1}n!$ and acts by signed permutations of the standard basis, with the restriction that each element effects an even number of sign changes [33, Section 2.10]. The stabilizer of a root is a Weyl group of type $A_1 + D_{n-2}$, meaning $W(A_1) \times W(D_{n-2})$, where we interpret D_3 as A_3 and D_2 as $A_1 + A_1$. There is a well-known homomorphism ϕ from $W(D_{2n})$ to the symmetric group S_{2n} resulting from forgetting the signs in a signed permutation; it is given by the following formula:

$$\phi(s_i) = \begin{cases} (i, i+1) & \text{if } i < 2n; \\ (2n-1, 2n) & \text{if } i = 2n. \end{cases} \tag{2.2}$$

Let $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_7$ be the standard basis of $\mathbb{R}^8$. The root system of type E_7 may be regarded as a subset of $\mathbb{R}^8$ as follows. There are 56 roots of the form $\pm 2(\varepsilon_i - \varepsilon_j)$ where $0 \leq i \neq j \leq 7$, and there are 70 roots of the form $\sum_{i=0}^{7} \pm\varepsilon_i$, where the signs are chosen so that there are four + and four −. The simple roots are $\alpha_1, \alpha_2, \ldots, \alpha_7$, where $\alpha_i = 2(\varepsilon_i - \varepsilon_{i+1})$ for $1 \leq i \leq 6$, and

$$\alpha_7 = -\varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7.$$

A root of the form $2(\varepsilon_i - \varepsilon_j)$ is positive if $0 < i < j$ or $j = 0$, and negative otherwise. A root of the form $\sum_{i=0}^{7} \pm\varepsilon_i$ is positive if and only if ε_0 occurs with negative coefficient. The highest root is $2(\varepsilon_1 - \varepsilon_0)$. The bilinear form B is $1/4$ of the Euclidean inner product. The stabilizer of a root is a Weyl group of type D_6. We call the coordinates ε_i *Fano coordinates* because they are particularly compatible with the combinatorics of the Fano plane; this will be important in Section 6.2.

Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_8$ be the standard basis of $\mathbb{R}^8$. The root system of type E_8 may be regarded as a subset of $\mathbb{R}^8$ as follows. There are 112 roots of the form $\pm 2(\varepsilon_i \pm \varepsilon_j)$ where $1 \leq i \neq j \leq 8$, and there are 128 roots of the form $\sum_{i=1}^{8} \pm\varepsilon_i$, where the signs are chosen so that the total number of − is even. The simple roots are $\alpha_1, \alpha_2, \ldots, \alpha_8$, where

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8,$$

$\alpha_2 = \varepsilon_1 + \varepsilon_2$, and $\alpha_i = \varepsilon_{i-1} - \varepsilon_{i-2}$ for all $3 \leq i \leq 8$. If k is the largest integer such that ε_k appears in α with nonzero coefficient c, then α is positive if and only if $c > 0$. The highest root is $2(\varepsilon_7 + \varepsilon_8)$. The bilinear form B is $1/4$ of the Euclidean inner product. The stabilizer of a root is a Weyl group of type E_7. We call the coordinates ε_i the *standard coordinates* for E_8.

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3. Combinatorics of coplanar quadruples

A *matching* of $[2n] := \{1, 2, \ldots, 2n\}$ is a collection of pairwise disjoint size-2 subsets, or *2-blocks*, of $[2n]$. The matching is *perfect* if the union of the 2-blocks is the whole of $[2n]$. If $1 \le a < b < c < d \le 2n$, then a *crossing* is a subset of the matching of the form $\{\{a, c\}, \{b, d\}\}$, a *nesting* is a subset of the form $\{\{a, d\}, \{b, c\}\}$, and an *alignment* is a subset of the form $\{\{a, b\}, \{c, d\}\}$. For convenience, we will often denote each 2-block $\{a, b\}$ in a matching simply by ab from now on.

In this section, we generalize crossings, nestings and alignments to the notion of coplanar quadruples in the context of orthogonal sets of roots (Definition 3.9, Remark 3.11). As explained in the introduction, we can naturally identify each n-root $\alpha = \prod_{i=1}^{n} \beta_i$ in the Macdonald representation $j_{nA_1}^{\Phi}(\mathrm{sgn})$ (Definition 1.1) with the set of its orthogonal components, and it turns out that coplanar quadruples are very useful for understanding the action of W on the orthogonal sets arising this way. We develop some key properties of coplanar quadruples in Theorem 3.10, which is the main result of Section 3. We also show that crossings, nestings and alignments can be distinguished from each other based on the heights of the roots that they contain (Proposition 3.13), and we give a precise description of the ways in which two coplanar quadruples can overlap (Proposition 3.20).

3.1. Coplanar quadruples

We gather a few facts about n-roots and define coplanar quadruples in this subsection. The following two results on maximal orthogonal sets of roots are well known, but we include proofs for ease of reference.

Lemma 3.1. *Let W be a Weyl group of type D_n for n even. Suppose that $n = 2k \ge 4$.*

(i) *Every maximal orthogonal set of roots is of the form*

$$\{\pm(\varepsilon_{i_1} + \varepsilon_{j_1}),\ \pm(\varepsilon_{i_1} - \varepsilon_{j_1}),\ \pm(\varepsilon_{i_2} + \varepsilon_{j_2}),\ \pm(\varepsilon_{i_2} - \varepsilon_{j_2}),\ \ldots,\quad \pm(\varepsilon_{i_k} + \varepsilon_{j_k}),\ \pm(\varepsilon_{i_k} - \varepsilon_{j_k})\},$$

where we have $\{i_1, j_1, \ldots, i_k, j_k\} = \{1, 2, \ldots, 2k-1, 2k\}$ as sets and the signs are chosen independently.

(ii) *Every maximal orthogonal set of positive roots is of the form*

$$\{\varepsilon_{i_1} + \varepsilon_{j_1},\ \varepsilon_{i_1} - \varepsilon_{j_1},\ \varepsilon_{i_2} + \varepsilon_{j_2},\ \varepsilon_{i_2} - \varepsilon_{j_2},\ \ldots, \varepsilon_{i_k} + \varepsilon_{j_k},\ \varepsilon_{i_k} - \varepsilon_{j_k}\},$$

and these sets are in bijection with perfect matchings $\{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_k, j_k\}\}$ of the set $[n]$ that satisfy $i_r < j_r$ for all $1 \le r \le k$.

Proof. Let R be a maximal orthogonal set of roots. By symmetry, we may reduce to the case where all the roots in R are positive. If R contains the root $\varepsilon_i \pm \varepsilon_j$, then R must also contain the root $\varepsilon_i \mp \varepsilon_j$ because otherwise $R \cup \{\varepsilon_i \mp \varepsilon_j\}$ would be a set of orthogonal roots that was larger than R. It follows that R consists of $n/2$ pairs of roots such that each pair has the same support, and roots from distinct pairs have disjoint supports. This completes the proof of (i).

Part (ii) follows from (i) and the fact that if $1 \le i < j \le n$, then the roots $\varepsilon_i \pm \varepsilon_j$ are positive. □

It follows from Lemma 3.1 (ii) that a maximal orthogonal set of positive roots in type D_n (for n even) contains the root $\varepsilon_i - \varepsilon_j$ if and only if it contains the root $\varepsilon_i + \varepsilon_j$. We will call such a pair of roots $\{\varepsilon_i \pm \varepsilon_j\}$ a *collinear pair* of roots.

Lemma 3.2. *If W is a Weyl group of type E_7, E_8 or D_n with $n \ge 4$ even, then W acts transitively on the set $\mathcal{M}(W)$ of unordered maximal sets of orthogonal roots of W.*

Proof. Recall from Section 2 that the group $W(D_n)$ can be regarded as the group of signed permutations of n objects in which there is an even number of sign changes. Such a group acts transitively on the set described in Lemma 3.1 (i).

Now suppose W has type E_7, and let α be a root of W. Then by Section 2, the stabilizer W_α is a Weyl group of type D_6 whose root system is the set Φ_α of roots that are orthogonal to α. Since W_α acts transitively on Φ_α, it follows that there is a well-defined bijection $[R] \mapsto [R \cup \{\alpha\}]$ from the set of W_α-orbits on $\mathcal{M}(W_\alpha)$ to the set of W-orbits on $\mathcal{M}(W)$, where the orbit $[R]$ of every 6-tuple $R \in \mathcal{M}(W_\alpha)$ is sent to the orbit $[R \cup \{\alpha\}]$ of the 7-tuple $R \cup \{\alpha\}$. It then follows that W acts transitively on $\mathcal{M}(W)$, as desired.

Finally, if W has type E_8, then for each root α of W, the stablizer W_α is of type E_7. A similar argument to the one above shows that W acts transitively on $\mathcal{M}(W)$ because W_α acts transitively on $\mathcal{M}(W_\alpha)$. □

We are ready to define coplanar quadruples. The following proposition offers multiple equivalent characterizations of them.

Proposition 3.3. *Let $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ be a set of four mutually orthogonal roots for a simply laced Weyl group W with root system Φ, and let $\gamma = (\beta_1 + \beta_2 + \beta_3 + \beta_4)/2$. The following are equivalent:*

(i) *γ is a root (i.e., the elements of Q sum to twice a root);*
(ii) *Q is contained in a subsystem Ψ of type D_4;*
(iii) *there is a unique subsystem Ψ of type D_4 such that $(Q \cup \{\gamma\}) \subset \Psi \subseteq \Phi$, and we have*

$$\Psi = \{\pm\beta_1,\ \pm\beta_2,\ \pm\beta_3,\ \pm\beta_4,\ (\pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4)/2\},$$

where all the signs are chosen independently.

Proof. We first prove that (i) implies (iii). Assume that γ is a root. Any root subsystem containing $Q \cup \{\gamma\}$ also contains $s_{\beta_i}(\beta_i) = -\beta_i$ for each i, as well as all roots of the form

$$s_{\beta_1}^{\epsilon_1} s_{\beta_2}^{\epsilon_2} s_{\beta_3}^{\epsilon_3} s_{\beta_4}^{\epsilon_4}(\gamma),$$

where we have $\epsilon_i \in \{0, 1\}$ for all i. The 16 roots listed above can also be expressed as

$$(\pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4)/2.$$

We have constructed all 24 roots in the set Ψ listed in the statement, and this is the cardinality of a root system of type D_4. To prove (iii), it now suffices to show that Ψ is a root system of type D_4. Because the elements of Q are orthogonal vectors of the same length, we may choose Euclidean coordinates $\beta_1 = \varepsilon_1 - \varepsilon_2$, $\beta_2 = \varepsilon_1 + \varepsilon_2$, $\beta_3 = \varepsilon_3 - \varepsilon_4$ and $\beta_4 = \varepsilon_3 + \varepsilon_4$. With respect to these coordinates, we have

$$\Psi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le 4\},$$

which indeed forms a root system of type D_4, as desired.

It is immediate that (iii) implies (ii).

In the usual notation for the simple roots of type D_4, the orthogonal roots α_1, α_3, α_4, and $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ sum to 2α, where α is the root $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Lemma 3.2 applied to a root system of type D_4 then implies that the sum of every orthogonal quadruple of roots in a root system of type D_4 is equal to $2\alpha'$ for some root α'. It follows that (ii) implies (i), which completes the proof. □

Definition 3.4. A set Q of four mutually orthogonal roots for a simply laced Weyl group is called a *coplanar quadruple* if it satisfies the equivalent conditions of Proposition 3.3. In this case, we call the set Ψ from Proposition 3.3 *the D_4-subsystem associated to Q.*

Coplanar quadruples can be described explicitly in coordinates in type D:

Lemma 3.5. *Let W be a Weyl group of type D_n for n even and $n \ge 4$. Then four positive roots of W form a coplanar quadruple if and only if they consist of two collinear pairs of roots (i.e., if and only if they are of the form $\varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j, \varepsilon_k + \varepsilon_l, \varepsilon_k - \varepsilon_l$ for four distinct indices i, j, k, l where $i < j$ and $k < l$).*

Remark 3.6. In the setting of Lemma 3.5, we may naturally identify the coplanar quadruple $\{\varepsilon_i \pm \varepsilon_j, \varepsilon_k \pm \varepsilon_l\}$ with the matching $\{ij, kl\}$ of the set $\{i, j, k, l\}$.

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Remark 3.7. Recall that reflections in $W(D_n)$ act on the reflection representation as signed permutations, with $s_\alpha(\varepsilon_i) = \varepsilon_j$ if $\alpha = \varepsilon_i - \varepsilon_j$ and $s_\alpha(\varepsilon_i) = -\varepsilon_j$ if $\alpha = \varepsilon_i + \varepsilon_j$. It follows that $W(D_n)$ acts on the set $\{\varepsilon_1^2, \cdots, \varepsilon_n^2\}$ as ordinary permutations, with $s_\alpha(\varepsilon_i^2) = \varepsilon_j^2$ for all distinct i, j. When n is even, it then follows from Lemmas 3.1 and 3.5 that the action of $W(D_n)$ on n-roots factors through the map $\phi : W(D_n) \to S_n$ from Equation (2.2) to induce an action of S_n on n-roots. In particular, each reflection $r \in W(D_n)$ acts in the same way as $\phi(r)$ on every n-root of $W(D_n)$.

Proof of Lemma 3.5. The 'if' implication holds since the four roots in the given form sum to twice the positive root $\varepsilon_i + \varepsilon_k$. To prove the 'only if' implication, let $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ be a coplanar quadruple. Recall from Section 2 that two roots in type D_n are orthogonal if and only if they have the same or disjoint support. It follows that if no two roots in Q have the same support, then the supports of $\beta_1, \beta_2, \beta_3, \beta_4$ contain a total of eight distinct coordinates ε_i, in which case the sum $\gamma = \beta_1 + \beta_2 + \beta_3 + \beta_4$ cannot be twice a root. We may therefore assume, without loss of generality, that $\{\beta_1, \beta_2\} = \{\varepsilon_i \pm \varepsilon_j\}$ for some $i < j$. This implies $\beta_1 + \beta_2 = 2\varepsilon_i$. The condition that Q is an orthogonal set summing to twice a root then forces us to have $\{\beta_3, \beta_4\} = \{\varepsilon_k \pm \varepsilon_l\}$ for some elements k, l distinct from i and j with $k < l$. □

The next proposition shows that the action of W on n-roots is local to coplanar quadruples in the following sense: whenever a reflection in W does not fix a maximal orthogonal set R of roots, it must change exactly four elements of R that form a coplanar quadruple, and it changes these four elements to another coplanar quadruple with the same associated D_4-subsystem.

Proposition 3.8. *Let W be a Weyl group of type E_7, E_8 or D_n for n even, let α be a root, and let R be a maximal set of orthogonal positive roots. Suppose that neither α nor $-\alpha$ is an element of R.*

- (i) *The root α is orthogonal to all but precisely four elements $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ of R. The elements of Q form a coplanar quadruple, and we have $2\alpha = \pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4$ for suitable choices of signs.*
- (ii) *Let Ψ be the D_4 subsystem associated to Q. Then we have $\alpha \in \Psi$, and the set $s_\alpha(Q) = \{s_\alpha(\beta_i) : 1 \leq i \leq 4\}$ is also a coplanar quadruple whose associated D_4-subsystem is Ψ.*

Proof. To prove (i), it suffices by Lemma 3.2 to do so for a fixed R. Suppose first that W is of type D_n, and choose

$$R = \{\varepsilon_1 + \varepsilon_2,\ \varepsilon_1 - \varepsilon_2,\ \varepsilon_3 + \varepsilon_4,\ \varepsilon_3 - \varepsilon_4,\ \ldots,\ \varepsilon_{n-1} + \varepsilon_n,\ \varepsilon_{n-1} - \varepsilon_n\}.$$

The root α must be of the form $\pm\varepsilon_i \pm \varepsilon_j$, where i and j come from different parts of the partition $\{\{1, 2\}, \{3, 4\}, \ldots, \{n-1, n\}\}$. It follows that the support of α has one element in common with the support of each of precisely four elements of R making up two collinear pairs, and that α is orthogonal to all the other elements of R. Furthermore, the roots β_1, β_2, β_3 and β_4 that are not orthogonal to α are of the form $\pm\varepsilon_h \pm \varepsilon_i$ and $\pm\varepsilon_j \pm \varepsilon_k$, where $|\{h, i, j, k\}| = 4$. It follows that 2α can be written in the form $\pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4$ for suitable choices of signs.

Next, suppose that W has type E_8, and choose

$$R = \{2(\varepsilon_1 + \varepsilon_2),\ 2(\varepsilon_1 - \varepsilon_2),\ 2(\varepsilon_3 + \varepsilon_4),\ 2(\varepsilon_3 - \varepsilon_4),\ 2(\varepsilon_5 + \varepsilon_6),\ 2(\varepsilon_5 - \varepsilon_6),\ 2(\varepsilon_7 + \varepsilon_8),\ 2(\varepsilon_7 - \varepsilon_8)\}.$$

If α has the form $2(\pm\varepsilon_i \pm \varepsilon_j)$, then the proof is completed using the same argument as in type D_8. The other possibility is that we have $\alpha = \sum_{i=1}^{8} \pm\varepsilon_i$, where the signs are chosen so that there is an even number of minus signs. In this case, α is orthogonal to precisely one of the roots $\{2(\varepsilon_j - \varepsilon_{j+1}),\ 2(\varepsilon_j + \varepsilon_{j+1})\}$, according as ε_j and ε_{j+1} occur in α with the same or with opposite coefficients. It follows that α is orthogonal to precisely four elements of R and that 2α can be expressed in the required form.

Now suppose that W has type E_7. By Section 2.3, we may identify the root system of W with the set of roots orthogonal to the highest root θ in the root system of type E_8, so that $R \cup \{\theta\}$ is a maximal set of orthogonal roots in type E_8. By the above paragraph, α is orthogonal to four of the roots in $R \cup \{\theta\}$, but one of these roots is θ itself. It follows that α is orthogonal to three elements of R and that 2α can be expressed as a signed sum of the other four elements of R, as required. This completes the proof of (i).

It follows from (i) that we have $\alpha \in \Psi$, because condition (i) implies condition (iii) in Proposition 3.3. The element $\gamma = (\beta_1+\beta_2+\beta_3+\beta_4)/2$ is a root because Q is a coplanar quadruple, and the set $s_\alpha(Q)$ is a coplanar quadruple because its elements sum to twice $s_\alpha(\gamma)$, which is a root because γ is. We also have $s_\alpha(Q) \subseteq \Psi$ because both α and Q are in Ψ. This implies that Ψ must be the D_4-subsystem associated to $s_\alpha(Q)$, proving (ii). □

3.2. Crossings, nestings and alignments

We examine coplanar quadruples more closely in this subsection and classify them into three types – namely, crossings, nestings and alignments. As we will explain in Remark 3.11, our terminology comes from the theory of matchings, but the following definition makes sense for all the simply laced root systems considered in this paper.

Definition 3.9. Let $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ be a coplanar quadruple of positive orthogonal roots, let Ψ be the D_4-subsystem associated to Q, let $\leq$ be the partial order on Ψ relative to the induced simple roots of Ψ, and let γ be the root $(\beta_1+\beta_2+\beta_3+\beta_4)/2$. We say that Q is

(i) a *crossing* if $\beta_i \leq \gamma$ for all i and Q contains the unique $\leq$-maximal element of $Q \cup (-s_\gamma(Q))$;
(ii) a *nesting* if $\beta_i \leq \gamma$ for all i and Q contains the unique $\leq$-minimal element of $Q \cup (-s_\gamma(Q))$;
(iii) an *alignment* otherwise.

We also call each crossing, nesting and alignment a *feature* of *type C*, *type N* and *type A*, respectively.

Theorem 3.10. *Let Φ be a root system for a Weyl group of type E_7, E_8 or D_n for n even. Let Q be a coplanar quadruple of positive roots of Φ, let Ψ be the associated D_4-subsystem, and let $\Psi^+ = \Psi \cap \Phi^+$ be the induced positive system of Ψ.*

(i) *The set Ψ^+ contains precisely three distinct quadruples of mutually orthogonal roots. These quadruples are pairwise disjoint and partition Ψ^+.*
(ii) *The three quadruples of orthogonal roots in Ψ^+ are all coplanar. Among them there is exactly one crossing, Ψ^+_C, exactly one nesting, Ψ^+_N, and exactly one alignment, Ψ^+_A. In particular, the quadruple Q cannot be both a crossing and a nesting, and the three conditions in Definition 3.9 are mutually exclusive.*
(iii) *Each quadruple in $\{\Psi^+_C, \Psi^+_N, \Psi^+_A\}$ uniquely determines both of the other two.*
(iv) *If R is a set of mutually orthogonal roots that is disjoint from Ψ, then either each of the three sets $\{R \cup \Psi^+_C,\ R \cup \Psi^+_N,\ R \cup \Psi^+_A\}$ consists of mutually orthogonal roots, or none of them does.*
(v) *The crossing Ψ^+_C contains no root from the induced simple system of Ψ.*
(vi) *For each $x \in \{C, N, A\}$, let γ_x be the product of the roots in Ψ^+_x, and let $\sigma(\gamma_x)$ be the sum of the components of γ_x. Then we have $\sigma(\gamma_A) < \sigma(\gamma_N) = \sigma(\gamma_C)$ and $\gamma_C = \gamma_N + \gamma_A$. Moreover, if α is any component in one of the three 4-roots γ_C, γ_N and γ_A, then the reflection s_α sends the other two 4-roots to each other; for example, if $\alpha \in \Psi^+_C$, then s_α sends γ_N and γ_A to each other.*

Proof. By Section 2.3, the roots orthogonal to a given root in type D_4 form a subsystem of type $3A_1$. Therefore, each positive root lies in a unique quadruple of mutually orthogonal positive roots, which proves (i).

Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the induced simple roots of Ψ, with α_2 corresponding to the branch node in the Dynkin diagram. Then the three quadruples from (i) are given by

$$\begin{aligned}
\Psi^+_1 &= \{\alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_2+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4\},\\
\Psi^+_2 &= \{\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_4,\ \alpha_2+\alpha_3+\alpha_4\}, \qquad \text{and}\\
\Psi^+_3 &= \{\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_1+2\alpha_2+\alpha_3+\alpha_4\}.
\end{aligned}$$

The roots in Ψ^+_3 add up to twice the root $\alpha = \alpha_1+\alpha_2+\alpha_3+\alpha_4$. The root α is strictly lower in the $\leq$ order than one of the roots in Ψ^+_3; therefore, Ψ^+_3 is an alignment by Definition 3.9. The roots in Ψ^+_1 and Ψ^+_2 both

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add up to 2θ, where $\theta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. The root θ is strictly higher in the $\leq$ order than each element of Ψ_1^+ and Ψ_2^+. Note that we have $s_\theta(\Psi_1^+) = -\Psi_2^+$ and $s_\theta(\Psi_2^+) = -\Psi_1^+$. Furthermore, the roots α_2 and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ are the unique $\leq$-minimal and unique $\leq$-maximal elements of $\Psi_1^+ \cup \Psi_2^+$, respectively. This implies that Ψ_1^+ is a crossing and that Ψ_2^+ is a nesting, and it also follows that none of Ψ_1^+, Ψ_2^+ and Ψ_3^+ is both a crossing and a nesting. The quadruple Q must be one of Ψ_1^+, Ψ_2^+ and Ψ_3^+, and (ii) follows.

Part (iii) follows from (ii) since each of Ψ_C^+, Ψ_N^+ and Ψ_A^+ uniquely determines Ψ^+ as its associated D_4-subsystem by Proposition 3.3 (iii).

Part (iv) holds as each of the quadruples in $\{\Psi_C^+, \Psi_N^+, \Psi_A^+\}$ is a basis for the span of Ψ^+, so that any root that is orthogonal to every element of one quadruple is also orthogonal to every element of each of the other quadruples.

Finally, the claims in (v) and (vi) can all be verified by inspection or direct computation based on the description of $\Psi_C^+ = \Psi_1^+, \Psi_N^+ = \Psi_2^+$ and $\Psi_A^+ = \Psi_3^+$. For the equation $\gamma_C = \gamma_N + \gamma_A$ and the assertion about s_α in (vi), one can alternatively prove them using the usual realizations of the root system and Weyl group of type D_4, where the simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_3 = \varepsilon_3 - \varepsilon_4$ and $\alpha_4 = \varepsilon_3 + \varepsilon_4$ and the group W acts as signed permutations. Under this realization, we have

$$\gamma_C = (\varepsilon_1^2 - \varepsilon_3^2)(\varepsilon_2^2 - \varepsilon_4^2),\ \gamma_N = (\varepsilon_1^2 - \varepsilon_4^2)(\varepsilon_2^2 - \varepsilon_3^2),\ \text{and } \gamma_A = (\varepsilon_1^2 - \varepsilon_2^2)(\varepsilon_3^2 - \varepsilon_4^2), \tag{3.1}$$

and the equation $\gamma_C = \gamma_N + \gamma_A$ follows as the terms expressed in coordinates satisfy the Ptolemy relation

$$(A - C)(B - D) = (A - D)(B - C) + (A - B)(C - D). \qquad \square$$

Remark 3.11. In the setting of Theorem 3.10, the coordinate forms of the 4-roots $\gamma_C, \gamma_N, \gamma_A$ given in Equation (3.1) correspond via the bijection of Lemma 3.1 (ii) to the perfect matchings of the set $[4]$ given by the crossing $m_C = \{\{1,3\},\{2,4\}\}$, the nesting $m_N = \{\{1,4\},\{2,3\}\}$, and the alignment $m_A = \{\{1,2\},\{3,4\}\}$, respectively. Definition 3.9 generalizes the notion of crossings, nestings and alignments in the sense that a coplanar quadruple in the sense of the definition is a crossing, nesting or alignment if and only if the corresponding perfect matching of the set $[4]$ is a crossing, nesting or alignment, respectively, in the context of matchings.

Remark 3.12. Let R be a maximal orthogonal set of positive roots and let α be a positive root not in R. Proposition 3.8 shows that the reflection s_α changes precisely four elements in R which form a coplanar quadruple Q and that $Q' = s_\alpha(Q)$ is another coplanar quadruple with the same associated D_4-subsystem as Q. Theorem 3.10 (vi) reveals more about α, Q and Q': it shows that Q and Q' are two distinct features from the set $\{\Psi_C^+, \Psi_N^+, \Psi_A^+\}$ inside the D_4-subsystem Ψ associated to Q, while α is in the remaining feature. We will say that s_α *moves* Q in this case; we will also say that s_α (or α) *moves an* X *to a* Y and call s_α an XY *move*, where X and Y are the distinct types of Q and Q', respectively. Note that knowledge of Q and Y is enough to determine Q' by Theorem 3.10 (iii), even if α is not known. Note also that Theorem 3.10 (vi) guarantees that XY moves exist for any distinct elements X, Y in $\{C, N, A\}$, so any two coplanar quadruples sharing the same D_4-subsystem can be connected, up to sign, by a reflection.

The next proposition shows how to distinguish crossings, nestings and alignments from each other using only the heights of their components. Recall from Section 2.2 that in a root system with simple system $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, the height of each root $\alpha = \sum_{i=1}^n c_i\alpha_i$ is the integer $\mathrm{ht}(\alpha) = \sum_{i=1}^n c_i$.

Proposition 3.13. *Let W be a Weyl group of type E_7, E_8 or D_n for n even. Let $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ be a coplanar quadruple of orthogonal positive roots, ordered so that we have $h_1 \leq h_2 \leq h_3 \leq h_4$ where $h_i = ht(\beta_i)$ for each i.*

(i) *We have $h_1 + h_2 + h_3 \neq h_4$ and $h_2 + h_3 \neq h_1 + h_4$.*
(ii) *If $h_1 + h_2 + h_3 < h_4$, then Q is an alignment.*
(iii) *If $h_1 + h_2 + h_3 > h_4$ and $h_2 + h_3 > h_1 + h_4$, then Q is a nesting. In this case, we also have $h_1 < h_2$.*
(iv) *If $h_1 + h_2 + h_3 > h_4$ and $h_2 + h_3 < h_1 + h_4$, then Q is a crossing. In this case, we also have $h_1 > 1$ and $h_3 < h_4$.*

Proof. Let Ψ be the type D_4 subsystem associated to Q, and let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the simple system of Ψ induced by Φ^+, with α_2 corresponding to the branch node in the Dynkin diagram. Then as in the proof of Theorem 3.10, the set Ψ^+ decomposes into the union of the crossing Ψ_C^+, nesting Ψ_N^+ and alignment Ψ_A^+ given below:

$$\begin{aligned}\Psi_C^+ &= \{\alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_2+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4\},\\ \Psi_N^+ &= \{\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_4,\ \alpha_2+\alpha_3+\alpha_4\}, \qquad \text{and}\\ \Psi_A^+ &= \{\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_1+2\alpha_2+\alpha_3+\alpha_4\}.\end{aligned}$$

If $Q = \Psi_A^+$, then we have $\beta_4 - \beta_1 - \beta_2 - \beta_3 = 2\alpha_2$, and $\beta_4 + \beta_1 - \beta_2 - \beta_3 = 2\alpha_2 + 2\alpha_i$ for some $i \in \{1, 3, 4\}$.

If $Q = \Psi_N^+$, then we have $\beta_1 < \beta_j$ for all $j \in \{2, 3, 4\}$. We also have $\beta_4 - \beta_1 - \beta_2 - \beta_3 = -2\alpha_i - 2\alpha_2$ for some $i \in \{1, 3, 4\}$, and $\beta_4 + \beta_1 - \beta_2 - \beta_3 = -2\alpha_i$ for some $i \in \{1, 3, 4\}$.

If $Q = \Psi_C^+$, then we have $\beta_4 > \beta_j$ for all $j \in \{1, 2, 3\}$. We also have $\beta_4 - \beta_1 - \beta_2 - \beta_3 = -2\alpha_2$ and $\beta_4 + \beta_1 - \beta_2 - \beta_3 = 2\alpha_i$ for some $i \in \{1, 3, 4\}$. Furthermore, none of the β_i is a simple root of Φ because none of the β_i is even a simple root of Ψ.

All the claims in the proposition follow from the above observations. Note that the conditions in (ii), (iii), and (iv) are exclusive and exhaustive because of part (i). $\square$

Remark 3.14. It can be shown that if W is a Weyl group of type E_7, E_8 or D_n for n even and the components of a positive n-root β have heights $h_1, h_2, \ldots, h_n$, then the number $\sum_{i=1}^n h_i^2$ depends only on W and is independent of the choice of β.

3.3. *Intersections of coplanar quadruples*

Let R be a maximal orthogonal set of roots of W. In this subsection, we first focus on type E_8 and show that coplanar quadruples in R gives rise to a Steiner quadruple system in this type. We will then use this result to count coplanar quadruples in R and deduce how coplanar quadruples in R can overlap with each other, in the general case.

Definition 3.15. A *Steiner system* $S(t, k, N)$ is a collection B of k-element subsets of the set $[N] = \{1, 2, \ldots, N\}$ with the property that every t-element subset is contained in a unique element of B. The elements of B are called *blocks*, and we write each block $\{a, b, c, \ldots\}$ where $a < b < c < \ldots$ as $abc\ldots$. We call B a *Steiner triple system* if $k = 3$ and a *Steiner quadruple system* if $k = 4$.

Remark 3.16. A Steiner system $S(t, k, N)$ is also known as a t-$(N, k, 1)$ *design*, which is a special kind of t-designs [18, Section 4.1]. It is well known that, up to isomorphism (by permutations), there is a unique Steiner triple system $S(2, 3, 7)$ and a unique Steiner quadruple system $S(3, 4, 8)$ [4]. The following 14 quadruples form an example of a Steiner quadruple system, and removing the element 8 from all the quadruples on the left results in a Steiner triple system.

1 2 4 8	3 5 6 7
2 3 5 8	1 4 6 7
3 4 6 8	1 2 5 7
4 5 7 8	1 2 3 6
1 5 6 8	2 3 4 7
2 6 7 8	1 3 4 5
1 3 7 8	2 4 5 6

Lemma 3.17. *Let $\beta_1, \beta_2, \beta_3$ be three mutually orthogonal positive roots of type E_8.*

(i) *There exists a unique positive root β_4 such that $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a coplanar quadruple.*
(ii) *If R is a maximal orthogonal set of positive roots of type E_8, and $\{\beta_1, \beta_2, \beta_3\} \subset R$, then we have $\beta_4 \in R$.*

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Proof. By [11, Lemma 11 (iii)] and its proof, if $W = W(E_8)$, then W acts transitively on ordered triples of orthogonal roots, and the set of roots orthogonal to three given mutually orthogonal roots is a root system of type $A_1 + D_4$. Since the action of orthogonal triples of roots is transitive, it suffices to prove (i) for a fixed choice of β_1, β_2 and β_3. If we choose $\beta_1 = \alpha_3 = 2(\varepsilon_2 - \varepsilon_1)$, $\beta_2 = \alpha_5 = 2(\varepsilon_4 - \varepsilon_3)$ and $\beta_3 = \alpha_7 = 2(\varepsilon_6 - \varepsilon_5)$, then it follows from the explicit description of the root system in Section 2.3 that the only positive root that forms a coplanar quadruple with β_1, β_2 and β_3 is the root $\beta_4 = 2(\varepsilon_8 - \varepsilon_7)$. This proves (i).

The uniqueness property of (i) proves that the set of coplanar quadruples corresponds to the A_1 summand of the $A_1 + D_4$ subsystem. This implies that if β is any positive root that is orthogonal to all of β_1, β_2 and β_3, then either $\beta_4 = \beta$, or β_4 is orthogonal to β. The maximality of R in the statement of (ii) implies that β_4 cannot be orthogonal to all elements of R. It follows that $\beta_4 \in R$, which proves (ii). ◻

Lemma 3.18. *Let W be a Weyl group of type E_8, and let R be a maximal orthogonal set of roots.*

(i) *The coplanar quadruples of R endow R with the structure of a Steiner quadruple system $S(3,4,8)$.*
(ii) *Any two coplanar quadruples of R intersect in either* 0, 2, *or* 4 *elements.*

Proof. Part (i) is immediate from Lemma 3.17. To prove part (ii), we need to show that any two distinct quadruples from the Steiner quadruple system are either disjoint or overlap in precisely two elements. This can be proved by an exhaustive check, or by arguing as follows.

The quadruples in the left column of the table consist of the element 8 together with three points forming a line in the Fano plane (see Section 6.2). Any two such quadruples intersect in two elements: the element 8 and the unique point on the intersection of the two lines of the Fano plane. The general case follows by combining this observation with the fact that each quadruple in the right column is the complement of the corresponding quadruple in the left column. ◻

Corollary 3.19. *Let W be a Weyl group of type E_7, E_8 or D_n for n even and $n = 2k \geq 4$. Let R be a maximal orthogonal set of positive roots. Then the number M of coplanar quadruples contained in R does not depend on R. We have $M = \binom{k}{2}$ if W has type D_{2k}, $M = 7$ if W has type E_7, and $M = 14$ if W has type E_8.*

Proof. If W has type D_{2k}, then R determines a perfect matching of the set $\{1, 2, \dots, 2k\}$ with k blocks by Lemma 3.1, and the coplanar quadruples in R correspond bijectively to pairs of these blocks by Remark 3.6. It follows that M does not depend on R and equals $\binom{k}{2}$.

In type E_8, the result follows from Lemma 3.18 (i).

If W has type E_7, then as in the proof of Proposition 3.8, we may again identify the root system of W with the set of roots orthogonal to the the highest root θ in the root system of type E_8. The set $R \cup \{\theta\}$ is then a maximal orthogonal set of roots in type E_8. The coplanar quadruples of R are in bijection with the quadruples of $S(3,4,8)$ that exclude a fixed element, and there are 7 such quadruples, so we are done. ◻

Proposition 3.20. *Let W be a Weyl group of type E_7, E_8 or D_n for n even. Let R be a maximal orthogonal subset of positive roots, and suppose Q_1 and Q_2 are coplanar quadruples of roots contained in R.*

(i) *The intersection $Q_1 \cap Q_2$ has size 0, 2 or 4.*
(ii) *If $|Q_1 \cap Q_2| = 2$, then there is a root subsystem $\Psi \subseteq \Phi$ of type D_6 that contains both Q_1 and Q_2. In this case, each of the sets Q_1, Q_2, $Q_1 \cap Q_2$, and $Q_1 \cup Q_2$ consists of collinear pairs of roots with respect to Ψ, and the symmetric difference $Q_1 \,\Delta\, Q_2$ is also a coplanar quadruple.*

Proof. If W has type D_n, then the assertions follow from Lemma 3.5.

Suppose that W has type E_8. In this case, part (i) follows from Lemma 3.18 (ii). If $|Q_1 \cap Q_2| = 2$, then $|Q_1 \cup Q_2| = 6$, and there are precisely two elements $\alpha, \beta \in R$ that are orthogonal to every root in $Q_1 \cup Q_2$. Let Ψ be the set of roots in Φ that are orthogonal to α and β. Then Ψ forms a root system of type D_6 by Section 2.3, proving the first assertion of (ii), and the sets $Q_1, Q_2, Q_1 \cap Q_2$ and $Q_1 \cup Q_2$ all lie in Ψ. The other assertions of (ii) now follow by applying the type D_n case of the result to Ψ.

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Finally, suppose that W has type E_7. We identify the root system with the set of roots orthogonal to the the highest root θ in the root system of type E_8 as usual. Then $R \cup \{\theta\}$ is a maximal orthogonal set of roots in type E_8. The assertions in this case follow by applying the argument of the previous paragraph with $\alpha = \theta$. □

4. Quasiparabolic structure

Let $X = X(W)$ be the set of all maximal orthogonal sets of positive roots of W, and recall from the introduction that W acts on X naturally via the action $w(\{\beta_1, \cdots, \beta_n\}) = \{|w(\beta_1)|, \cdots, |w(\beta_n)|\}$. In this section, we recall the notion of a quasiparabolic set as defined by Rains and Vazirani [43], and we use the concepts of crossings and nestings to endow the W-set X with a quasiparabolic structure.

4.1. Quasiparabolic sets

Quasiparabolic sets were introduced by Rains and Vazirani for a general Coxeter system as follows.

Definition 4.1 [43, Section 2, Section 5]. Let W be a Coxeter group with generating set S and set of reflections T. A *scaled W-set* is a pair $(\mathcal{X}, \lambda)$, where $\mathcal{X}$ is a W-set and $\lambda : \mathcal{X} \to \mathbb{Z}$ is a function satisfying $|\lambda(sx) - \lambda(x)| \leq 1$ for all $s \in S$. An element $x \in \mathcal{X}$ is *W-minimal* if $\lambda(sx) \geq \lambda(x)$ and is *W-maximal* if $\lambda(sx) \leq \lambda(x)$ for all $s \in S$.

A *quasiparabolic set* for W is a scaled W-set $\mathcal{X}$ satisfying the following two properties:

(QP1) for any $r \in T$ and $x \in \mathcal{X}$, if $\lambda(rx) = \lambda(x)$, then $rx = x$;
(QP2) for any $r \in T$, $x \in \mathcal{X}$, and $s \in S$, if $\lambda(rx) > \lambda(x)$ and $\lambda(srx) < \lambda(sx)$, then $rx = sx$.

For a quasiparabolic set $\mathcal{X}$, we define $\leq_Q$ to be the weakest partial order such that $x \leq_Q rx$ whenever $x \in \mathcal{X}$, $r \in T$, and $\lambda(x) \leq \lambda(rx)$.

Rains and Vazirani call $\lambda(x)$ the *height* of x, and $\leq_Q$ the *Bruhat order*, but we will refer to them as the *level* of x and the *quasiparabolic order* because of the potential for confusion in the context of this paper. It follows from [43, Proposition 5.16] that λ is a rank function with respect to the partial order $\leq_Q$, so that every covering relation $x <_Q y$ satisfies $\lambda(y) = \lambda(x) + 1$.

We will show that the set $X = X(W)$ forms a quasiparabolic set for the Weyl group W in type E_7, E_8 or D_n with n even under a suitable level function defined in terms of coplanar quadruples. We define the level function and some other useful statistics below.

Definition 4.2. Let W be a Weyl group of type E_7, E_8 or D_n for n even. Let R be a set of mutually orthogonal roots of W, and let β be a positive n-root of W.

(i) We define the *crossing number* $C(R)$, the *nesting number* $N(R)$ and the *alignment number* $A(R)$ of R to be the numbers of crossings, nestings and alignments contained in R^+, respectively.
(ii) We define the *type* of R to be the monomial $A^{A(R)}C^{C(R)}N^{N(R)}$, and define the *level* $\lambda(R)$ of R to be $C(R) + 2N(R)$.
(iii) If R is a maximal orthogonal set of roots, then we say that R is *noncrossing*, *nonnesting* and *alignment-free* if $C(R) = 0$, $N(R) = 0$ and $A(R) = 0$, respectively; we also call R *maximally crossing*, *maximally nesting* or *maximally aligned* if we have $N(R) = A(R) = 0$, $C(R) = A(R) = 0$ or $C(R) = N(R) = 0$, respectively.

We also apply all the above definitions to β by applying them to the set of components of β. (Note that the definitions of type in (ii) and in Definition 3.9 are consistent.)

Remark 4.3. By Corollary 3.19, when R is a maximal orthogonal set of positive roots, the sum of the numbers $C(R), N(R)$ and $A(R)$ is a constant depending only on W and not on R; therefore, each of these numbers achieves the maximal possible value when the other two equal zero. This justifies the terms 'maximally crossing', 'maximally nesting' and 'maximally aligned' in Definition 4.2.(iii).

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Example 4.4. Suppose that W has type D_6, and that $R = \{\varepsilon_1 \pm \varepsilon_2, \varepsilon_3 \pm \varepsilon_6, \varepsilon_4 \pm \varepsilon_5\}$ is the maximal orthogonal set of positive roots corresponding to the matching $\{12, 36, 45\}$ (via the natural bijection of Lemma 3.1 (ii)). In this case, R contains two alignments, corresponding to the pairs $\{12, 45\}$ and $\{12, 36\}$, and one nesting, corresponding to the pair $\{36, 45\}$. The type of R is therefore $A^2C^0N^1$ (or AAN).

We now state the main theorem of this section. Its proof will occupy the next subsection.

Theorem 4.5. *Let W be a Weyl group of type E_7, E_8 or D_n for n even, and let X be the set of maximal orthogonal sets of positive roots of W, regarded as a W-set under the action*

$$w(\{\beta_1, \cdots, \beta_n\}) = \{|w(\beta_1)|, \cdots, |w(\beta_n)|\}.$$

Then the set (X, λ) is a quasiparabolic set for W, where $\lambda : X \to \mathbb{Z}$ is the level function $\lambda(x) = C(x) + 2N(x)$.

4.2. Proof of Theorem 4.5

We will prove Theorem 4.5 by showing that the set X is a scaled W-set satisfying the axioms (QP1) and (QP2) of Definition 4.1. To this end, we first study how the action of a reflection s_α corresponding to a root α can affect the level of a maximal orthogonal set R of roots. Recall from Remark 3.12 that s_α must replace a coplanar quadruple in R with a feature of a different type whenever R does not contain $\pm\alpha$. We will therefore examine how such feature replacements affect the level function λ. Also recall from Section 2 that the root system Φ is equipped with a natural partial order $\leq$ defined by the condition that $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a nonnegative linear combination of simple roots. We will frequently use the order $\leq$ throughout the proofs.

Example 4.6. Let $W = W(D_6)$ and $R = \{\varepsilon_1 \pm \varepsilon_2, \varepsilon_3 \pm \varepsilon_6, \varepsilon_4 \pm \varepsilon_5\}$ be as in Example 4.4. Let $\alpha = \varepsilon_2 - \varepsilon_4$, so that s_α acts as the transposition $(2, 4)$. In this case, the set $s_\alpha(R)$ corresponds to the matching $\{14, 25, 36\}$ and has type CCC. The reflection s_α changes the alignment $Q = \{\varepsilon_1 \pm \varepsilon_2,\ \varepsilon_4 \pm \varepsilon_5\}$ to the crossing $Q' = s_\alpha(Q) = \{\varepsilon_1 \pm \varepsilon_4,\ \varepsilon_2 \pm \varepsilon_5\}$, so α moves an A to a C. Note that while s_α changes the quadruple Q from an A to a C locally, globally s_α does not change the type of R from AAN to ACN but to CCC. This is because after the application of s_α, each collinear pair of roots in Q becomes a new collinear pair that forms a new type of coplanar quadruple with the collinear pair of roots $\varepsilon_3 \pm \varepsilon_6$ outside Q.

Part (ii) of the next proposition, however, will imply that if α is minimal among roots moving an A to a C, then the global change in the type of R will mirror this local change, so that if R has type $A^pC^qN^r$, then $s_\alpha(R)$ has type $A^{p-1}C^{q+1}N^r$. In our example, the root $\alpha' = \varepsilon_2 - \varepsilon_3$ satisfies the minimality condition since it is simple. The reflection $s_{\alpha'}$ changes the coplanar quadruple $\{\varepsilon_1 \pm \varepsilon_2, \varepsilon_3 \pm \varepsilon_6\}$ of type A to the coplanar quadruple $\{\varepsilon_1 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_6\}$ of type C, and changes the set R of type AAN to a maximal orthogonal set of type ACN.

Proposition 4.7. *Let W be a Weyl group of type E_7, E_8 or D_n with n even, and let R be a maximal set of orthogonal positive roots of type $A^pC^qN^r$. Let $\lambda = C + 2N$ be the level function from Definition 4.2.*

(i) *If α_i is a simple root, then either $\alpha_i \in R$, or $\lambda(s_{\alpha_i}(R)) \neq \lambda(R)$. If $\lambda(s_{\alpha_i}(R)) > \lambda(R)$, then we have $\lambda(s_{\alpha_i}(R)) = \lambda(R) + 1$, and either (1) s_{α_i} moves an A to a C and $s_{\alpha_i}(R)$ has type $A^{p-1}C^{q+1}N^r$, or (2) s_{α_i} moves a C to an N and $s_{\alpha_i}(R)$ has type $A^pC^{q-1}N^{r+1}$.*

(ii) *If Q is an alignment in R, Q' is the corresponding crossing quadruple, and $R' = (R \setminus Q) \cup Q'$, then $d = \lambda(R') - \lambda(R)$ is a positive odd number. If there is a positive root α such that $s_\alpha(Q) = Q'$ but no positive root $\alpha' < \alpha$ moves an A in R to a C, then R' has type $A^{p-1}C^{q+1}N^r$.*

(iii) *If Q is a crossing in R, Q' is the corresponding nesting quadruple, and $R' = (R \setminus Q) \cup Q'$, then $A(R) = A((R \backslash Q) \cup Q')$ and $d = \lambda(R') - \lambda(R)$ is a positive odd number. If there is a positive root*

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α such that $s_\alpha(Q) = Q'$ but no positive root $\alpha' < \alpha$ moves a C in R to an N, then R' has type $A^pC^{q-1}N^{r+1}$.

(iv) *If Q is an alignment in R, Q' is the corresponding nesting quadruple, and $R' = (R \setminus Q) \cup Q'$, then $d = \lambda((R\setminus Q) \cup Q') - \lambda(R)$ is a strictly positive even number.*

Lemma 4.8. *Proposition 4.7 holds if W has type D_6.*

Proof of Lemma 4.8. Throughout the proof, we will identify both R and the coplanar quadruples within R with their corresponding matchings (as in Lemma 3.1 and Remark 3.6). Recall that we will often write a 2-block $\{a, b\}$ in a matching as ab.

Suppose that α_i is a simple root, so that the reflection s_{α_i} acts as the transposition $(i, i+1)$. We will assume that $\alpha_i \notin R$, so that we have $R = \{ai, (i+1)b, ef\}$ and the coplanar quadruple moved by s_{α_i} is $Q = \{ai, (i+1)b\}$. Let Q' be any other coplanar quadruple in R. Then Q' is of the form $\{xy, ef\}$ with $x \in \{a, b\}$ and $y \in \{i, i+1\}$, and we have $s_{\alpha_i}(Q') = \{\{x, s_{\alpha_i}(y)\}, ef\}$. Since the numbers i and $i+1$ are only distance 1 apart, the four elements $x, s_{\alpha_i}(y), e, f$ appearing in Q' have the same relative order as the numbers x, y, e, f; therefore, Q' and $s_{\alpha_i}(Q')$ have the same type. It follows that Q is the only quadruple in R that is changed to a quadruple of another type by s_{α_i}. Note that Q will be an alignment if $a < i < i+1 < b$; Q will be a crossing if $a < b < i$ or $i+1 < a < b$ or $b < i < i+1 < a$; and Q will be a nesting if $b < a < i$ or $i+1 < b < a$. We have $\lambda(s_{\alpha_i}(R)) = \lambda(R) + 1$ in the first three of these six cases and $\lambda(s_{\alpha_i}(R)) = \lambda(R) - 1$ in the last three cases. The first of the six cases corresponds to the situation in (1), and the second and third cases correspond to the situation in (2). Part (i) follows.

Suppose that Q and Q' are as in the statement of (ii), with Q being the alignment $\{a_1a_2, b_1b_2\}$ for some $a_1 < a_2 < b_1 < b_2$. If $b_1 = a_2 + 1$, then the simple root s_{a_2} moves an A to a C by moving Q, and we have $R' = s_{\alpha_2}(R)$; therefore, R' has type $A^{p-1}C^{q+1}N^r$ by (i). If $b_1 > a_2 + 1$, then Q must be one of the following five quadruples:

$$\{12, 56\}, \quad \{12, 45\}, \quad \{12, 46\}, \quad \{23, 56\}, \quad \text{and} \quad \{13, 56\}.$$

Direct computation shows that d equals 5, 1, 3, 1 and 3 in these cases, respectively. It follows that d is a positive odd number.

To prove the second assertion in (ii), we prove its contrapositive. If R' does not have type $A^{p-1}C^{q+1}N^r$, then Q must be one of five quadruples listed in the last paragraph. We claim that in each case, for every positive root α such that $s_\alpha(Q) = Q'$, there exists a positive root $\alpha' < \alpha$ that moves another alignment in R other than Q to a crossing. Specifically, we may always take α' to be $\varepsilon_2 - \varepsilon_3$ in the first three cases and $\varepsilon_4 - \varepsilon_5$ in the last two cases. For example, the only possibilities for α if $Q = \{12, 56\}$ are $\varepsilon_2 \pm \varepsilon_5$ and $\varepsilon_1 \pm \varepsilon_6$, and for all these possibilities the root $\alpha' = \varepsilon_2 - \varepsilon_3$ is smaller than α and moves an alignment in R other than Q to a crossing. This completes the proof of the desired contrapositive.

A similar argument proves (iii). This time, we have $Q = \{a_1a_2, b_1b_2\}$ for some $a_1 < b_1 < a_2 < b_2$. If we have $b_1 = a_1 + 1$ or $b_2 = a_2 + 1$, then the simple root α_{a_1} or α_{a_2} moves Q to a nesting Q', so R' has type $A^pC^{q-1}N^{r+1}$ by (i). The only remaining possibility for Q is $\{14, 36\}$. If α is a positive root such that $s_\alpha(Q) = Q'$, then $\alpha \in \{\varepsilon_1 \pm \varepsilon_3, \varepsilon_4 \pm \varepsilon_6\}$. Direct computation shows that any simple root $\alpha' < \alpha$ moves some crossing in R other than Q to a nesting.

Finally, (iv) follows by combining (ii) and (iii) since R' can be obtained from by first replacing Q in R with its corresponding crossing Q'' and then replacing Q'' in the result with Q'. □

Proof of Proposition 4.7. If W has type D_4 then there are only three possibilities for R, and all the assertions follow by direct verification. By Lemma 4.8, we may therefore assume that the rank of W is at least 7.

For each coplanar quadruple $Q \subseteq R$, we define H_Q to be the set of all 6-subsets H of R such that there exists a D_6-subsystem of Φ containing both Ψ_Q and H, where Ψ_Q is the D_4-subsystem associated with Q. By Proposition 3.20, if Q_1 is any coplanar quadruple in R, then either $Q_1 = Q$, or $Q_1 \cap Q = \emptyset$, or $|Q_1 \cap Q| = 2$ and there is a unique element of H_Ψ that contains both Q and Q_1.

We now prove (iii). By tracking the contributions towards the crossing number made by the three types of coplanar quadruples $Q_1 \subseteq R$ just mentioned, we note that

$$C(R') - C(R) = -1 + \sum_{H \in H_Q} \big(C((H \backslash Q) \cup Q') - C(H) + 1 \big). \tag{4.1}$$

Here, the term -1 comes from the case $Q_1 = Q$ since the crossing Q in R is replaced by the non-crossing feature Q' as we change R to R'. In the second case where $Q_1 \cap Q = \emptyset$, the quadruple Q_1 lies in both R and R' and thus does not contribute to the difference $C(R') - C(R)$. Finally, every Q_1 with $|Q_1 \cap Q| = 2$ appears together with Q in a unique element H of H_Q and contributes a term in the sum over H_Q, where we have added 1 to the difference $C((H \backslash Q) \cup Q') - C(H)$ to account for the fact that the change from Q to Q' in H has been recorded by the term -1 in the first case.

Similar arguments based on the facts that $N(Q') = N(Q) + 1$ and $A(Q') = A(Q)$ show that

$$N(R') - N(R) = 1 + \sum_{H \in H_Q} \big(N((H \backslash Q) \cup Q') - N(H) - 1 \big) \tag{4.2}$$

and

$$A(R') - A(R) = \sum_{H \in H_Q} \big(A((H \backslash Q) \cup Q') - A(H) \big). \tag{4.3}$$

Since $\lambda = C + 2N$, it follows from Equations (4.1) and (4.2) that

$$\lambda(R') - \lambda(R) = 1 + \sum_{H \in H_Q} \big(\lambda((H \backslash Q) \cup Q') - \lambda(H) - 1 \big). \tag{4.4}$$

By Lemma 4.8.(iii), each summand in the sum over H_Q is zero in Equation (4.3) and is a nonnegative even number in Equation (4.4); therefore, we have $A(R') = A(R)$, and the number $d = \lambda(R') - \lambda(R)$ is a positive odd number.

To prove the last assertion in (ii), suppose that $s_\alpha(Q) = Q'$ for some positive root α, but no positive root $\alpha' < \alpha$ moves a C to an N in R. The same minimality condition then applies if R is replaced by an element of H_Q, so every summand in the sums over H_Q in Equations (4.1) and (4.2) is zero by Lemma 4.8 (iii). It follows that R' has type $A^p C^{q-1} N^{r+1}$, which proves Proposition 4.7 (iii).

The proof of Proposition 4.7 (ii) follows by a similar but shorter argument, and the proof of Proposition 4.7 (iv) follows by combining parts (ii) and (iii).

Finally, to prove Proposition 4.7 (i), assume that α_i is a simple root such that $\alpha_i \notin R$. We have already proved part (i) if W has type D_4, and this implies that either α_i moves an A to a C or a C to an N. In the former case, the conclusions follow from part (ii), and in the latter case, they follow from part (iii), in each case because the simple root α_i is minimal in the order $\leq$. □

Proof of Theorem 4.5. We first prove (i). Proposition 4.7 (i) proves that (X, λ) is a scaled W-set, so it suffices to show (X, λ) satisfies the axioms (QP1) and (QP2). We do so by induction on n.

If $n = 4$ or $n = 6$, then W has type D, and the axioms (QP1) and (QP2) can be proved by direct verification or as follows. Suppose $n = 2k$. By Remark 3.7, it suffices to show that (X, λ) is a quasiparabolic set for the symmetric group S_n. We may identify the set X with the set X' of fixed-point free involutions in $S_n = W(A_{n-1})$, with each collinear pair $\{\varepsilon_i \pm \varepsilon_j\}$ in a maximal set $R \in X$ corresponding to a factor (i, j) in an involution $\iota \in X'$. Under this identification, the actions of S_n on X' and X coincide with each other, so it suffices to show that X' is a quasiparabolic set for S_n under the level function λ. Rains and Vazirani [43, Section 4] proved that X' is a quasiparabolic set for S_n under the level function h given by $h(\iota) = (\ell(\iota) - k)/2$, where ℓ denotes Coxeter length, so it further suffices to show that whenever an involution $\iota \in S_n$ corresponds to a maximal set R of positive orthogonal roots, we have $(\ell(\iota)-k)/2 = \lambda(R)$. This can be proved by an exhaustive check or by induction on $\lambda(R)$ by using the first three cases in the second paragraph of the proof of Lemma 4.8. For example, the involution

$$\iota = (13)(26)(45) = s_2s_1s_3s_5s_4s_3s_5s_4s_2 \in S_6$$

corresponds to the set $R = \{13, 26, 45\}$ of type ACN. In this case, we have $(\ell(\iota)-k)/2 = (9-3)/2$ and $\lambda(R) = 3$, as required.

Now assume $n \geq 7$. Let $r = s_\alpha \in T$ be the reflection corresponding to a root α, and let $x \in X$. If $rx \neq x$, then $\pm\alpha$ are not in x, so r moves an A to a C, or a C to an N, or an A to an N, or vice versa by Remark 3.12. It follows from Proposition 4.7 (ii), (iii) and (iv) that $\lambda(rx) > \lambda(x)$ in all the first three cases and $\lambda(rx) < \lambda(x)$ in the last three cases; therefore, axiom (QP1) holds.

To prove axiom (QP2), assume that we have $r \in T$, $x \in X$, $s \in S$, $\lambda(rx) > \lambda(x)$ and $\lambda(srx) < \lambda(sx)$. Then the definition of scaled W-sets forces $\lambda(rx) = \lambda(sx) = \lambda(x)+1$, so each of r and s must be an AC or CN move by Proposition 4.7 (ii), (iii) and (iv). Let Q_1 and Q_2 be the coplanar quadruples of roots in R moved by r and s, respectively. Then Q_1 and Q_2 are disjoint, or coincide with each other, or intersect in two elements by Proposition 3.20. If Q_1 and Q_2 were disjoint, we would have $Q_2 \subseteq r(R)$, so that s would move Q_2 in $r(R)$, and Proposition 4.7 (ii) and (iii) would imply that $\lambda(srx) = \lambda(rx)+1 > \lambda(rx) = \lambda(sx)$, contradicting the assumption that $\lambda(srx) < \lambda(sx)$. If $Q_1 = Q_2$, then the fact that $\lambda(rx) = \lambda(sx) = \lambda(x)+1$ implies that s and r must both be AC moves or both be CN moves, according as $Q_1 = Q_2$ is an alignment or crossing, by Proposition 4.7. It follows from Remark 3.12 that $rx = sx$. Finally, if $|Q_1 \cap Q_2| = 2$, then Proposition 3.20 implies that there is a subsystem Σ of type D_6 containing both Q_1 and Q_2. Applying the inductive hypothesis to Σ proves that $rx = sx$, which completes the proof. □

We end this subsection by recording some useful consequences of Proposition 4.7 concerning sums of n-roots:

Corollary 4.9. *Let W be a Weyl group of type E_7, E_8 or D_n with n even.*

(i) *If $\beta \leq_Q \gamma$ are two positive n-roots that are comparable in the quasiparabolic order, then we have $\sigma(\beta) \leq \sigma(\gamma)$, with equality if and only if we have $A(\beta) = A(\gamma)$.*
(ii) *If α_i is a simple root and R is a maximal orthogonal set of positive roots, then we have*

$$\sigma(s_i(R)) = \begin{cases} \sigma(R) - 2\alpha_i & \text{if } \alpha_i \in R \text{ or } \alpha_i \text{ moves a } C \text{ to an } A, \\ \sigma(R) + 2\alpha_i & \text{if } \alpha_i \text{ moves an } A \text{ to a } C, \\ \sigma(R) & \text{otherwise.} \end{cases}$$

(iii) *If α_i is a simple root and β is a positive n-root of type $A^pC^qN^r$ such that $\sigma(s_{\alpha_i}(\beta)) > \sigma(\beta)$, then $s_{\alpha_i}(\beta)$ is a positive n-root of type $A^{p-1}C^{q+1}N^r$, and we have $\sigma(s_{\alpha_i}(\beta)) - \sigma(\beta) = 2\alpha_i$. If β is nonnesting, then so is $s_{\alpha_i}(\beta)$.*

Proof. All the claims can be proved by examining the effects of various types of (simple) reflections on sums and levels of n-roots recorded in Theorem 3.10 and Proposition 4.7. We first prove (i). By the definition of $\leq_Q$, it is enough to consider the case where $r \in T$ is a reflection and $\gamma = r(\beta)$ satisfies $\lambda(\beta) < \lambda(\gamma)$. By Proposition 4.7, this implies that r is an AC, CN or AN move. The proof of (i) then follows from Theorem 3.10 (vi).

We now prove (ii). If $\alpha_i \in R$, then $\sigma(s_i(R)) = \sigma(R) - 2\alpha_i$ because $s_i(\alpha_i) = -\alpha_i$. If α_i is a CN or NC move, Theorem 3.10 (vi) implies that $\sigma(s_{\alpha_i}(R)) = \sigma(R)$. It follows from Proposition 4.7 (i) that the only other possibility is that α_i moves an A to a C or vice versa. By Theorem 3.10 (vi), this can only happen if α_i is the unique simple root in the corresponding nesting (i.e., the root α_2 in the explicitly constructed sets $\Psi_2^+ = \Psi_N^+$ in the proof of that theorem). Explicit computation shows that root sums in this case differ by $2\alpha_i$ in the precise manner described in the statement, which completes the proof of (ii).

The first part of (iii) follows immediately from (ii) and Proposition 4.7 (i). The assertion about nonnesting n-roots follows from the special case $r = 0$. □

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Remark 4.10. Burns and Pfeiffer [9, Theorem 1.2] prove that if T is a maximal order abelian subgroup of one of the groups W in Theorem 4.5, then T is elementary abelian of order 2^n, where n is the rank of W. They also prove that the set of all such subgroups forms a single conjugacy class [9, Theorem 3.1]. It follows that the stabilizers of the elements $x \in X$ can be defined abstractly from the group structure of W: they are the normalizers of the maximal order abelian subgroups of W.

4.3. Extremal elements

In this subsection, we identify X with the set Φ_n^+ of positive n-roots as usual and discuss an application of Theorem 4.5 concerning the maximally aligned and maximally nested n-roots, which turn out to be the unique W-minimal and W-maximal elements of the set X. The uniqueness of the maximally aligned and maximally nested n-root is not a priori clear, but it will follow conveniently from the general theory of quasiparabolic sets.

Proposition 4.11. *Let W be a Weyl group of rank n of types E_7, E_8 or D_n for n even, and let M be the number of coplanar quadruples in a positive n-root.*

(i) *There is a unique positive n-root, θ_A, of type A^M, and it corresponds to the unique W-minimal element of the quasiparabolic set X of Theorem 4.5.*
(ii) *There is a unique positive n-root, θ_N, of type N^M, and it corresponds to the unique W-maximal element of the quasiparabolic set X of Theorem 4.5.*

Proof. We recall that by Theorem 2.8, Remark 2.9 and Corollary 2.10 of [43], every orbit of a quasiparabolic set contains at most one W-minimal and at most one W-maximal element. If such a W-minimal or W-maximal element exists, then it can be identified as the unique element in the orbit with the minimal or maximal possible level, respectively.

The set X is finite, so it has at least one W-maximal and one W-minimal element. Since X consists of a single W-orbit by Lemma 3.2, it follows from the paragraph preceding this proposition that X has a unique W-maximal and a unique W-minimal element, and that they are the unique elements with the minimal and maximal possible level.

Let R be a maximal orthogonal set of positive roots. If R contains any coplanar quadruple Q that is a crossing or a nesting, then by Remark 3.12, we can find a reflection s_α that moves Q to an alignment or a crossing, respectively, and in both cases, Proposition 4.7 implies that $\lambda(s_\alpha(R)) < \lambda(R)$. Iterating this procedure proves the existence of an n-root of type A^M, which achieves the lowest possible value of λ. The uniqueness property of the previous paragraph then completes the proof of (i). Part (ii) can be proved similarly, by using the fact that any alignment or crossing in R would induce a level-increasing AC or CN move. □

We will prove shortly, in Proposition 5.2, that W also has a unique maximally crossing element, θ_C. The element θ_C will be the unique minimal element in a quasiparabolic set of a parabolic subgroup of W.

Remark 4.12. In Theorem 5.7 below, we will introduce another partial order, $\leq_{\mathcal{B}}$, on the positive n-roots. The argument of [30, Section 6] can be adapted to show that, under suitable identifications, $\leq_{\mathcal{B}}$ refines the *monoidal order* introduced by Cohen, Gijsbers and Wales [17, Section 3]. The quasiparabolic order has θ_N and θ_A as its unique maximal and unique minimal elements, whereas the monoidal order (and $\leq_{\mathcal{B}}$) has θ_C as its unique maximal element and has multiple minimal elements.

5. Feature-avoiding elements

In this section, we develop the properties of n-roots that avoid features of a given type: the alignment-free, noncrossing and nonnesting elements. We show that the alignment-free elements form a quasiparabolic set X_I of a maximal standard parabolic subgroup W_I of W, and that the unique maximally crossing n-root θ_C is the unique W_I-minimal element of X_I. We also show that the sets of noncrossing elements

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and of nonnesting elements both form bases of the Macdonald representation $j_{nA_1}^{\Phi}(\mathrm{sgn})$ (Definition 1.1). Moreover, the basis of noncrossing elements may be viewed as a canonical basis and behaves in a way that is reminiscent of the set of simple roots of a root system (Theorem 5.7). The basis of nonnesting elements admits an interesting combinatorial characterization: it is a distributive lattice induced by a suitable Bruhat order (Theorem 5.13). Finally, we introduce the notion of σ-equivalence classes to tie together the alignment-free, noncrossing and nonnesting elements. These equivalence classes turn out to be intervals with respect to the quasiparabolic order on X, and the set X_I of alignment-free elements form the top class with respect to a natural partial order. Any set of σ-equivalence class representatives forms a basis of the Macdonald representation, and the change of basis matrices between any pair of such bases, including the noncrossing and nonnesting bases, are unitriangular (Theorem 5.16).

Throughout the rest of this section, we assume that we are working with a Weyl group W of rank n and type E_7, E_8 or D_n for n even. All results hold independently of the rank and type of W, and we shall omit the statement of the above assumption except in the main theorems. We define the *sum* of each positive n-root γ to be the sum of the components of γ, and we denote it by $\sigma(\gamma)$.

5.1. Alignment-free elements

Recall from Proposition 4.11 that W has a unique positive n-root θ_N that avoids both alignments and crossings. We will use θ_N to help study general alignment-free elements.

Proposition 5.1. *Let θ_N be the unique positive n-root of type N^M.*

(i) *A noncrossing n-root (i.e., one of type A^pN^r) has a simple component.*
(ii) *The n-root θ_N has a unique simple component, α_x.*
(iii) *If α_i is a simple root, then $B(\sigma(\theta_N), \alpha_i) \geq 0$, where equality holds if and only if $\alpha_i \neq \alpha_x$.*
(iv) *Let W_I be the parabolic subgroup of W generated by the set $S\backslash\{\alpha_x\}$. Then the stabilizer of $\sigma(\theta_N)$ is precisely W_I, and we have $B(\sigma(\theta_N), \alpha_i) \geq 0$ for all simple roots α_i.*

Proof. Let γ be a noncrossing n-root of type A^pN^r, and let R be the set of components of γ. Assume for a contradiction that R contains no simple root, and let β be a root of minimal height in R. The bilinear form B has the property that $B(\alpha, \alpha') \in \{-2, -1, 0, 1, 2\}$ for any roots α, α', with $B(\alpha, \alpha') = 2$ if and only if $\alpha = \alpha'$. Furthermore, for any positive root α, there exists a simple root α_i such that $B(\alpha, \alpha_i) > 0$ [33, Theorem 1.5]. It follows that there exists a simple root α_i such that $B(\beta, \alpha_i) = 1$.

Since α_i is not in R, it moves a coplanar quadruple $Q \subseteq R$, and we have $\beta \in Q$ since $B(\beta, \alpha_i) \neq 0$. Let Ψ be the D_4-subsystem associated to Q. By hypothesis, Q is either an alignment or a nesting, and β is an element of Q of minimal height. It follows from the explicit description of the sets $\Psi_A^+ = \Psi_3^+$ and $\Psi_N^+ = \Psi_2^+$ in the proof of Theorem 3.10 that β is a root in the induced simple system of Ψ. However, since α_i is simple root of W, α_i is also in this induced simple system. This is a contradiction because we cannot have $B(\gamma_1, \gamma_2) > 0$ for two simple roots γ_1 and γ_2 in a root system. This completes the proof of (i).

Now suppose further that $\gamma = \theta_N$. It follows from (i) that R contains a simple root, so assume for a contradiction that R contains two simple roots, α_x and α_y. Let P be a path from x to y in the Dynkin diagram Γ, and let β be the root $\sum_{p\in P}\alpha_p$. Note that $B(\beta, \alpha_x) = B(\beta, \alpha_y) = 1$, so that α_x and α_y are both elements of the coplanar quadruple Q consisting of the roots moved by β. Let Ψ be the D_4-subsystem associated with Q. Then α_i, α_j are both induced simple roots of Ψ since they are simple roots of W. However, the type of R is N^M, so Q is a nesting and thus contains a unique minimal root by the description of the set $\Psi_N^+ = \Psi_2^+$ in the proof of Theorem 3.10. This is a contradiction, and (ii) follows.

To prove (iii), let α_i be a simple root. If $\alpha_i = \alpha_x$, then α_x is a component of θ_N and we have

$$s_i(\sigma(\theta_N)) = \sigma(\theta_N) - 2\alpha_i.$$

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If $\alpha_i \neq \alpha_x$, then α_i is not a component of θ_N, and Proposition 4.7 (i) implies that α_i moves an N to a C. Corollary 4.9 (ii) then implies that

$$s_i(\sigma(\theta_N) = \sigma(\theta_N).$$

Equation 2.1 implies that $B(\alpha_i, \sigma(\theta_N)) \geq 0$ for all i, with equality holding if and only if $\alpha_i \neq \alpha_x$.

Part (iv) follows from (iii) by [33, Theorem 1.12 (a)], which says that the stabilizer of $\sigma(\theta_N)$ in W is generated by the simple reflections it contains. □

Proposition 5.2. *Let θ_N be the unique positive n-root of type N^M, let α_x be the unique simple component of θ_N, and let W_I be the parabolic subgroup of W generated by the set $S \backslash \{\alpha_x\}$.*

(i) *The W_I-orbit of positive n-roots that contains θ_N is a quasiparabolic set (X_I, λ_I) for W_I, where λ_I is the restriction of λ to X_I.*
(ii) *The following are equivalent for a positive n-root β:*
   (1) *β has type $C^q N^r$ for some q and r;*
   (2) *$\sigma(\beta) = \sigma(\theta_N)$;*
   (3) *β is an element of X_I.*

   In particular, the elements of the quasiparabolic set X_I are precisely the alignment-free positive n-roots.
(iii) *There is a unique positive n-root, θ_C, of type C^M, and it corresponds to the unique W-minimal element of the quasiparabolic set X_I.*
(iv) *If α is a root of W_I and β is an n-root in X_I whose components do not contain $\pm\alpha$, then $\lambda(s_\alpha(\beta)) = \lambda(\beta) + 1 \bmod 2$.*

Proof. Part (i) follows from Theorem 4.5 by restriction.

We now prove the implication (1) ⇒ (2) of part (ii). Let β be an n-root of type $C^q N^r$. If $q = 0$, then $\beta = \theta_N$ by Proposition 4.11 (ii) and (2) follows immediately, so suppose that $q > 0$. By Remark 3.12 and Proposition 4.7 (iii), there exists a reflection α that moves a crossing to a nesting in such a way that the n-root $\beta' = s_\alpha(\beta)$ has type $C^{q-1}N^{r+1}$. Corollary 4.9 (ii) proves that $\sigma(\beta') = \sigma(\beta)$. It now follows from induction on q that $\sigma(\beta) = \sigma(\theta_N)$, proving (2).

To prove (2) ⇒ (3), assume that $\sigma(\beta) = \sigma(\theta_N)$. By Lemma 3.2, there exists $w \in W$ such that $w(\theta_N) = \beta$, so we have

$$w\sigma(\theta_N) = \sigma(w(\theta_N)) = \sigma(\beta) = \sigma(\theta_N).$$

It follows that $w \in W_I$ and $\beta \in X_I$, which proves (3).

To prove (3) ⇒ (1), let $\beta = w(\theta_N)$ for some $w \in W_I$ and let $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced word of w. Then each simple reflection s_{i_j} fixes $\sigma(\theta_N)$, so it follows from Corollary 4.9 (ii) that s_{i_j} is a CN or NC move. It follows from Proposition 4.7 (i) that β has type $C^q N^r$, which implies (1) and completes the proof of (ii).

To prove (iii), note that the quasiparabolic set X_I is finite and transitive, so it follows, as in the proof of Proposition 4.11 (i), that X_I has a unique minimal element (with respect to the quasiparabolic order) – namely, the unique element having minimal level in X_I. The elements of X_I are all of type $C^q N^r$ by (ii) where $q + r = M$ and M is as in Corollary 3.19, so we have $\lambda_I(\gamma) \geq M$ for any element $\gamma \in X_I$, with equality holding if and only if γ has type C^M. To prove (iii), it now remains to show that such an element exists.

Let β be an n-root in X_I, and suppose β has type $C^q N^r$ where $r > 0$. Then $\sigma(\beta) = \sigma(\theta_N)$, and β admits an NC move by a reflection s_α corresponding to some root α. Theorem 3.10 (vi) implies that $\sigma(s_\alpha(\beta)) = \sigma(\beta) = \sigma(\theta_N)$, so $s_\alpha(\beta)$ is in X_I and has type $C^a N^b$ by (ii). Proposition 4.7 (iii) implies that $a > q$ and $b < r$, and that $s_\alpha(\beta)$ has a lower level than β. It follows that X_I has an element of type C^M, which completes the proof of (iii).

Suppose that α and β are as in the statement of (iv). Since neither of $\pm\alpha$ is a component of β, the reflection s_α must move a C to an N or vice versa by Remark 3.12 and (ii). Proposition 4.7 (iii) then implies that $\lambda(\beta)$ and $\lambda(s_\alpha(\beta))$ have opposite parities. □

Remark 5.3. With some work, it can be shown that the positive n-roots in X_I are also exactly the positive n-roots β with the property that every component α of β has x-height 1, in the sense that α expands into a linear combination of simple roots where the simple root α_x appears with coefficient 1. Since the simple roots of W_I do not include α_x, it follows that no root of W_I divides any n-root in X_I. In other words, the 'if' condition in Proposition 5.2 (iv) in fact holds for every root α of W_I and every n-root $\beta \in X_I$. This implies that the sets of all n-roots in X_I with even levels and of all n-roots in X_I with odd levels are interchanged by s_α for every $\alpha \in W_I$. In particular, these two sets have the same cardinality.

5.2. *Two bases*

The goal of this subsection is to prove that the noncrossing n-roots and the nonnesting n-roots each form a basis for the Macdonald representation $j_{nA_1}^{\Phi}(\mathrm{sgn})$. The proof is based on a commutative version of Bergman's diamond lemma [5], which is a special case of Newman's diamond lemma [41]. We define the *crossing order*, $\leq_C$, on the set of positive n-roots by declaring that $\beta \leq_C \gamma$ if either $\sigma(\beta) < \sigma(\gamma)$, or both $\sigma(\beta) = \sigma(\gamma)$ and $\beta \geq_Q \gamma$, where $\leq_Q$ is the quasiparabolic order. Similarly, we define the *nesting order*, $\leq_N$, on the set of positive n-roots by declaring that $\beta \leq_N \gamma$ if either $\sigma(\beta) < \sigma(\gamma)$ (with respect to the order $\leq$ on roots), or both $\sigma(\beta) = \sigma(\gamma)$ and $\beta \leq_Q \gamma$.

Given any relation of the form $\gamma\gamma_C = \gamma\gamma_N + \gamma\gamma_A$ among three n-roots in the setting of Theorem 3.10 (where γ_C, γ_N and γ_A are the crossing, nesting and alignment corresponding to the same type-D_4 subsystem of the root system of W, respectively), we have $\gamma\gamma_A <_C \gamma\gamma_C$ and $\gamma\gamma_A <_N \gamma\gamma_N$ because $\sigma(\gamma_A) < \sigma(\gamma_C) = \sigma(\gamma_N)$ by Theorem 3.10 (vi). We also have $\gamma_C <_N \gamma_N$ and $\gamma_N <_C \gamma_C$ by Proposition 4.7 (iii) and the definition of $\leq_Q$, because for any component α of θ_A, the reflection s_α moves the components of γ_C to those of γ_N by Theorem 3.10 (vi). It also follows that $\lambda(\gamma\gamma_C) < \lambda(\gamma\gamma_N)$ and $\gamma\gamma_C \leq_Q \gamma\gamma_N$. We may therefore regard the relations $\gamma\gamma_C = \gamma\gamma_N + \gamma\gamma_A$ as directed *reduction rules*, each of which operates on a single term $\lambda_i\beta_i$ in a linear combination $\sum_i \lambda_i\beta_i$, where the β_i are positive n-roots. Each reduction rule can be used either (a) to express a positive n-root $\gamma\gamma_C$ containing a crossing as the sum of two other positive n-roots $\gamma\gamma_A, \gamma\gamma_N$ that are strictly lower than it in the crossing order, or (b) to express a positive n-root $\gamma\gamma_N$ containing a nesting as a linear combination of two other positive n-roots $\gamma\gamma_A, \gamma\gamma_C$ that are strictly lower than it in the nesting order.

In order to apply the diamond lemma, we need to know (a) that it is never possible to apply an infinite sequence of reduction rules to a linear combination of n-roots, and (b) that the reduction rules are *confluent*. The latter condition means that if m is a linear combination of n-roots and if f_1 and f_2 are two different reductions that can be applied to m, then the linear combinations $f_1(m)$ and $f_2(m)$ themselves have a common reduction, m'. In other words, it is possible to reduce $f_1(m)$ to m' by applying a suitable sequence of reductions, and it is possible to reduce $f_2(m)$ to the same m' by applying a possibly different sequence of reductions. If these two conditions hold, the conclusion of the diamond lemma is that every element of the module may be uniquely expressed as an element to which no reduction rules may be applied – in other words, a unique linear combination of noncrossing n-roots, or a unique linear combination of nonnesting n-roots.

Conversely, the diamond lemma guarantees that if each element m can be uniquely expressed as a linear combination of nonnesting (or noncrossing) n-roots, then the reduction relations are confluent.

Lemma 5.4. *There are* 2 *nonnesting positive* 4*-roots in type* D_4*, and* 5 *nonnesting positive* 6*-roots in type* D_6*. The nonnesting positive n-roots are linearly independent in the Macdonald representation* $j_{nA_1}^{\Phi}(\mathrm{sgn})$ *in each case.*

Proof. In type D_4, the set in question is $\{(\varepsilon_1^2 - \varepsilon_2^2)(\varepsilon_3^2 - \varepsilon_4^2),\ (\varepsilon_1^2 - \varepsilon_3^2)(\varepsilon_2^2 - \varepsilon_4^2)\}$, which is clearly linearly independent. In type D_6, the nonnesting positive 6-roots correspond to the matchings

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$$\{14, 25, 36\},\ \{13, 25, 46\},\ \{13, 24, 56\},\ \{12, 35, 46\}, \text{ and } \{12, 34, 56\}.$$

One can check that this set is linearly independent by comparing coefficients of $\varepsilon_1^2\varepsilon_2^2\varepsilon_3^2$, $\varepsilon_1^2\varepsilon_2^2\varepsilon_4^2$, $\varepsilon_1^2\varepsilon_2^2\varepsilon_5^2$, $\varepsilon_1^2\varepsilon_3^2\varepsilon_4^2$ and $\varepsilon_1^2\varepsilon_3^2\varepsilon_5^2$. □

Theorem 5.5. *Let W be a Weyl group of type E_7, E_8 or D_n for n even. Let $j_{nA_1}^{\Phi}(\mathrm{sgn})$ be the Macdonald representation of W.*

(i) *The nonnesting positive n-roots form a $\mathbb{Q}$-basis for $j_{nA_1}^{\Phi}(\mathrm{sgn})$.*
(ii) *The noncrossing positive n-roots form a $\mathbb{Q}$-basis for $j_{nA_1}^{\Phi}(\mathrm{sgn})$.*
(iii) *The alignment-free positive n-roots span $j_{nA_1}^{\Phi}(\mathrm{sgn})$.*

Proof. We first prove (i) by using the reduction rule $\gamma\gamma_N = \gamma\gamma_C - \gamma\gamma_A$ of Theorem 3.10 (vi) to express an n-root that contains a nesting as a linear combination of n-roots that are strictly lower in the nesting order. There are no infinite descending chains in the crossing order because there are only finitely many n-roots. It remains to show that the reductions f_i are confluent, by induction on the rank n. By Lemma 5.4, this is already known to be the case in types D_4 and D_6, so we assume from now on that we have $n > 6$.

If two reductions, f_i and f_j, affect different terms $\lambda_i\beta_i$ in the linear combination $m = \sum_i \lambda_i\beta_i$, or if f_i and f_j affect disjoint components of the same term $\lambda_i\beta_i$, then f_i and f_j commute. It is then immediate that $f_i(m)$ and $f_j(m)$ have a common reduction – namely, $f_i f_j(m) = f_j f_i(m)$. The proof of confluence now reduces to proving that if f_i and f_j change at least one component in the same n-root β, then $f_i(\beta)$ and $f_j(\beta)$ have a common reduction. In this case, if Q_i and Q_j are the sets of components of β that are moved by f_i and f_j respectively, then Proposition 3.20 (i) implies that either $Q_i = Q_j$, or $|Q_i \cap Q_j| = 2$. In the first case, we have $f_i = f_j$, and there is nothing to prove. In the latter case, Proposition 3.20 (ii) implies that there is a root subsystem Ψ of type D_6 that contains Q_i and Q_j as coplanar quadruples. Confluence now follows by applying the inductive hypothesis to Ψ, which completes the proof of (i).

We now prove (ii) by using the reduction rule $\gamma\gamma_C = \gamma\gamma_N + \gamma\gamma_A$ of Theorem 3.10 (vi) to express an n-root that contains a crossing as a sum of n-roots that are strictly lower in the crossing order $\leq_C$. It follows that the noncrossing positive n-roots form a spanning set. There are 2 noncrossing positive 4-roots in type D_4, corresponding to the matchings $\{12, 34\}$ and $\{14, 23\}$, and 5 noncrossing positive 6-roots in type D_6, corresponding to the matchings

$$\{16, 25, 34\},\ \{16, 23, 45\},\ \{14, 23, 56\},\ \{12, 36, 45\}, \text{ and } \{12, 34, 56\}.$$

These spanning sets are bases of $j_{nA_1}^{\Phi}(\mathrm{sgn})$ by (i), and the rest of the argument used to prove (i) now applies *mutatis mutandis*.

Part (iii) follows by expressing the reduction rule in the form $\gamma\gamma_A = \gamma\gamma_C - \gamma\gamma_N$. By Theorem 3.10 (vi), we have $\sigma(\gamma\gamma_A) < \sigma(\gamma\gamma_N) = \sigma(\gamma\gamma_C)$. This implies that the relation can only be applied finitely many times before the procedure terminates, and (iii) follows. □

We will refer to the bases of nonnesting and noncrossing positive n-roots as the *nonnesting basis* and *noncrossing basis* of the Macdonald representation.

5.3. Properties of the noncrossing basis

In this subsection, we show that the noncrossing basis behaves in the Macdonald representation in many ways like a simple system in the reflection representation. In particular, every n-root decomposes into the noncrossing basis with coefficients of like sign, and the noncrossing n-roots are precisely the minimal ones that are minimal in the sense that they are not further decomposable. This minimality property yields an elementary algebraic characterization. We also show that the maximally crossing n-root θ_C has a maximal decomposition into the noncrossing basis in a natural sense, and that simple reflections act on the noncrossing basis in a way reminiscent of the way they act on a simple system in the reflection representation. In addition, as we explain in Remark 5.8, the noncrossing basis is a sign-coherence basis

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in the sense of cluster algebras, and it also essentially agrees with an IC basis in the sense of Du [21]. For the above reasons, we may think of the noncrossing basis as the canonical basis of the Macdonald representation.

Lemma 5.6. *If β and $\lambda\beta$ are both n-roots for some scalar λ, then we must have $\lambda = \pm 1$.*

Proof. Theorem 5.5 implies that the scalar λ in (ii) lies in $\mathbb{Q}$. Lemma 3.2 implies that there exists $w \in W$ such that $w(\beta) = \lambda\beta$. Because w has finite order, it follows that λ is a root of unity, and this forces $\lambda = \pm 1$. □

Theorem 5.7. *Let W be a Weyl group of type E_7, E_8 or D_n for n even, and let $\mathcal{B}$ be the set of noncrossing positive n-roots.*

(i) *Every n-root is a $\mathbb{Z}$-linear combination of elements of $\mathcal{B}$, with coefficients of like sign. This sign is positive if the n-root is positive, and is negative if the n-root is negative.*
(ii) *A positive n-root is noncrossing if and only if it is not a positive linear combination of other positive n-roots.*
(iii) *Define $\gamma \leq_{\mathcal{B}} \gamma'$ for positive n-roots $\gamma = \sum_{\beta\in\mathcal{B}} c_\beta \beta$ and $\gamma' = \sum_{\beta\in\mathcal{B}} c'_\beta \beta$ whenever $c_\beta \leq d_\beta$ for all $\beta \in \mathcal{B}$. Then $\leq_{\mathcal{B}}$ is a partial order on the set X of positive n-roots. The maximally crossing element θ_C is the unique maximal element of X with respect to $\leq_{\mathcal{B}}$.*
(iv) *If $\gamma \in \mathcal{B}$ and α_i is a simple root, then we have*

$$s_{\alpha_i}(\gamma) = \begin{cases} -\gamma & \text{if } \alpha_i|\gamma; \\ \gamma + \gamma' & \text{otherwise, for some } \gamma' \in \mathcal{B} \text{ such that } \alpha_i|\gamma'. \end{cases}$$

Proof. Let β be a positive n-root. By the proof of Theorem 5.5, the result of applying reductions of the form $\gamma\gamma_C = \gamma\gamma_N + \gamma\gamma_A$ to β until this is no longer possible has the effect of expressing β as a positive integer linear combination of noncrossing n-roots, and this procedure will always terminate after finitely many steps. This proves (i) for positive n-roots, and the statement for negative n-roots follows because n-roots occur in positive-negative pairs.

If β is a positive n-root that contains a crossing, then β is a positive linear combination of other positive n-roots by applying the reduction rule in the first paragraph. Conversely, suppose that β is a noncrossing n-root and that $\beta = \sum_i \lambda_i \beta_i$, where $\lambda_i > 0$ and β_i is a positive n-root that is different from β for each i. Part (i) implies that each of the β_i is a positive linear combination of noncrossing n-roots. Because no cancellation can occur in the sum, Theorem 5.5 (ii) implies that this is only possible if each β_i is a multiple of β. Collecting terms, we then have $\beta = \lambda\beta_i$. Lemma 5.6 and the assumption that β_i is positive then imply that $\lambda = 1$ and $\beta = \beta_i$, which is a contradiction.

The relation $\leq_{\mathcal{B}}$ in (iii) is clearly a partial order on X. Since X is finite, it contains at least one maximal positive n-root with respect to $\leq$. To prove (iii), it then suffices to show that for every n-root $\gamma \in X$ not equal to θ_C, there is an element $\gamma' \in X$ such that $\gamma <_{\mathcal{B}} \gamma'$. Let $\gamma \in X$ be an n-root not equal to θ_C, so that we can factorize γ as $\gamma_1\gamma'$, where γ' is either an alignment or a nesting. In either case, Theorem 3.10 (vi) implies that there exists a reflection s_α such that $s_\alpha(\gamma')$ is a crossing, and that we have $s_\alpha(\gamma') = \gamma' + \gamma''$, where γ'' is a nesting if γ' is an alignment, or vice versa. We then have

$$s_\alpha(\gamma) = \gamma + \gamma_1\gamma'',$$

where $\gamma_1\gamma''$ is also a positive n-root. If we write the n-root $s_\alpha(\gamma)$ as $s_\alpha(\gamma) = \sum_{\beta\in\mathcal{B}} e_\beta \beta$, then it follows from (i) that $c_\beta \leq e_\beta$ for all $\beta \in \mathcal{B}$. It follows that $\gamma <_{\mathcal{B}} s_\alpha(\gamma)$, proving (iii).

In the situation of (iv), it is immediate that if α_i is a component of γ, then $s_i(\gamma) = -\gamma$. Suppose from now on that this is not the case, and let $A^p N^r$ be the type of γ. Let R be the set of components of γ, let $Q \subseteq R$ be the coplanar quadruple moved by α_i, and let Ψ be the D_4-subsystem of Q. Then the sets $Q, Q' = s_i(Q)$ and $Q'' = \Psi^+ \setminus (Q \cup Q')$ are the three distinct coplanar quadruples partitioning the induced positive system by Proposition 3.8 (ii) and Theorem 3.10. Let Ψ^+_A, Ψ^+_C and Ψ^+_N be the alignment,

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crossing and nesting in Ψ^+, respectively. Then since α_i is a simple root, Proposition 4.7 (i) implies that we must have one of the following two situations:

(1) $Q = \Psi^+_A$, $Q' = \Psi^+_C$, $s_i(\gamma)$ has type $A^{p-1}CN^r$, and $Q'' = \Psi^+_N$;
(2) $Q = \Psi^+_N$, $Q' = \Psi^+_C$, $s_i(\gamma)$ has type A^pCN^{r-1}, and $Q'' = \Psi^+_A$.

Let $\gamma_x = \prod_{\beta\in x}\beta$ for all $x \in \{Q, Q', Q''\}$, and write $\gamma = \gamma_1\gamma_Q$. Then we have $\gamma_{Q'} = \gamma_Q + \gamma_{Q''}$, and thus,

$$s_i(\gamma) = \gamma_1\gamma_{Q'} = \gamma + \gamma_1\gamma_{Q''}$$

in both of the above cases. We have $\alpha_i \in \Psi$ by Proposition 3.8 (ii), and α_i must lie in the induced simple system of Ψ since it is simple. Theorem 3.10 (v) then implies that $\alpha \notin Q'$, and we have $\alpha_i \notin Q$ by assumption, so we have $\alpha \in Q''$. It follows that α divides the n-root $\gamma'' = \gamma_1\gamma_{Q''}$.

It remains to prove that γ'' is noncrossing. We treat case (1) first. If α is any root that is minimal in Q, then α moves the crossing Q' to the nesting Q'' by Theorem 3.10 (vi), so that $\gamma'' = s_\alpha(s_i(\gamma))$. Since $s_i(\gamma)$ has type $A^{p-1}CN^r$, Q' is the unique crossing in $s_i(\gamma)$; therefore, any root that moves a C in $s_i(\gamma)$ to an N must move Q', and it must move Q' to Q'' by Proposition 3.8 (ii). Together with Theorem 3.10 (vi), this further implies that any root moving Q' to Q'' must come from Q, so it follows that α is minimal among roots moving a C in $s_i(\gamma)$ to an N. It follows from Proposition 4.7 (iii) that $\gamma'' = s_\alpha(s_i(\gamma))$ has type $A^{p-1}N^{r+1}$; therefore, γ'' is noncrossing. A similar argument shows that γ'' has type $A^{p+1}N^{r-1}$ in case (2), so γ'' is noncrossing in both cases. □

Remark 5.8.

(i) Since the Weyl group W acts transitively on n-roots in types E_7, E_8 and D_n for n even, the first assertion of Theorem 5.7 (i) is equivalent to the assertion that the noncrossing basis $\mathcal{B}$ is a *sign-coherent basis* of the Macdonald representation $j^\Phi_{nA_1}(\text{sgn})$ in the sense of cluster algebras ([10, Definition 2.2 (i)], [25, Definition 6.12]); that is, with respect to $\mathcal{B}$, every element of W acts on $j^\Phi_{nA_1}(\text{sgn})$ by a matrix where the entries in each column all have the same sign. It would be interesting to know whether these entries (i.e., the coefficients appearing in the expansion of arbitrary n-roots into the noncrossing elements) have any interpretation in terms of categorification.

(ii) There are other constructions of the basis of noncrossing n-roots. For example, one can modify the monomial bases of [23] by specializing the parameter to 1 and twisting by the sign representation, where the monomial basis in turn agrees with a suitable IC basis in the sense of Du [21] by a result of the first named author and Losonczy [29, Theorem 3.6]. However, the n-root approach has the significant advantage that it is relatively easy, given an arbitrary group element w and an arbitrary n-root α, to express $w(\alpha)$ as a linear combination of basis elements. The bases in type D_n may be constructed diagrammatically in terms of perfect matchings, as we explain at the end of Section 6.1. There is also a diagrammatic construction in types E_7 and E_8, as described in [56] and [27], but we do not pursue this here because it is not easy to recover the components of a basis n-root by inspection of the corresponding diagram.

(iii) With some more work, it can be shown that every component of a noncrossing n-root has odd height and, conversely, that every root of odd height occurs as a component of some noncrossing n-root.

5.4. Properties of the nonnesting basis

In this subsection, we show that the nonnesting basis is naturally indexed by a distributive lattice whose unique maximal and minimal elements are given by the maximally crossing and aligned n-roots θ_C and θ_A, respectively. This lattice is induced by the left weak Bruhat order $\leq_L$ of W and is isomorphic to a lattice consisting of certain fully commutative elements. We recall that $\leq_L$ is defined by the condition that $v \leq_L w$ if $w = uv$ for some $u \in W$ such that $\ell(w) = \ell(u) + \ell(v)$ or, equivalently, by the condition that $\ell(wv^{-1}) + \ell(v) = \ell(w)$. An element w in a simply laced Weyl group is fully commutative precisely when no reduced word for w contains a factor of the form $s_i s_j s_i$ [54].

Definition 5.9. Let θ, θ' be two nonnesting positive n-roots. A *nonnesting sequence from θ to θ'* is a (possibly trivial) sequence $(\theta_i) = (\theta_0 = \theta, \theta_1, \ldots, \theta_r = \theta')$ of positive nonnesting n-roots such that for all $1 \leq j \leq r$, there exists a simple root α_{i_j} such that

$$s_{i_j}(\theta_{j-1}) = \theta_j \text{ and } \sigma(\theta_j) > \sigma(\theta_{j-1}). \tag{5.1}$$

If $s_{i_1}, s_{i_2}, \cdots, s_{i_r}$ are simple reflections satisfying the condition in (5.1), we say that $\mathbf{w} = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a (θ, θ')-*word*, and we call the element w expressed by $\mathbf{w}$ a (θ, θ')-*element*. Note that we have $w(\theta') = \theta$.

Remark 5.10. Let θ be a nonnesting positive n-root of type $A^p C^q$ and let α_i be a simple root. The condition that $\sigma(s_i(\theta)) > \sigma(\theta)$ is equivalent to the condition that $B(\sigma(\theta), \alpha_i) < 0$ by Equation (2.1) because $\sigma(s_i(\theta)) = s_i(\sigma(\theta))$. In addition, by Corollary 4.9 (ii) and (iii), the condition that $\sigma(s_i(\theta)) > \sigma(\theta)$ is also equivalent to the condition that $\sigma(s_i(\theta)) = \sigma(\theta) + 2\alpha_i$, or the condition that s_i moves θ to an n-root of type $A^{p-1}C^{q+1}$. It follows that if $(\theta_0, \theta_1, \cdots, \theta_r)$ is a nonnesting sequence, then we have $\lambda(\theta_j) = \lambda(\theta_{j-1}) + 1$ for all $1 \leq j \leq r$. In particular, every nonnesting sequence is a saturated chain with respect to the quasiparabolic order $\leq_Q$.

Remark 5.11. Let θ and θ' be two positive n-roots with $\lambda(\theta') > \lambda(\theta)$, and let $w \in W$ be an element such that $w(\theta') = \theta$. Let $\mathbf{w} = s_{i_1} \cdots s_{i_r}$ be an arbitrary word for w. By the definition of quasiparabolic sets, applying a simple reflection decreases the level by at most 1, so any element taking θ' to θ has length at least $\lambda(\theta') - \lambda(\theta)$. It follows that $r \geq \lambda(\theta') - \lambda(\theta)$. It also follows that if $r = \lambda(\theta') - \lambda(\theta)$, then $\mathbf{w}$ is reduced and successively applying the simple reflections $s_{i_r}, \cdots, s_{i_2}, s_{i_1}$ starting from θ' must reduce the level by 1 at each step. In particular, if $r = \lambda(\theta') - \lambda(\theta)$ and θ' is nonnesting, then it follows from Proposition 4.7 (i) that each of these simple reflections is a CA move, so that conversely the sequence $\theta_{\mathbf{w}} := (\theta_0, \cdots, \theta_r)$ defined by $\theta_0 = \theta, \theta_j = s_{i_j}(\theta_{j-1})$ for $1 \leq j \leq r$ must be a nonnesting sequence by Remark 5.10.

Proposition 5.12. *Let $W_I \subset W$ be the parabolic subgroup of Proposition 5.2 (i), and let θ be a nonnesting positive n-root of type $A^p C^q$.*

(i) *If θ is the maximally crossing element θ_C, then we have $B(\sigma(\theta), \alpha_i) \geq 0$ for every simple root α_i. Otherwise, there exists a simple root α_i such that $B(\sigma(\theta), \alpha_i) < 0$.*
(ii) *There exists a nonnesting sequence from θ to θ_C, and we have $ht(\sigma(\theta)) = ht(\sigma(\theta_C)) - 2p$.*
(iii) *Every (θ, θ_C)-word is reduced. Every (θ, θ_C)-element is fully commutative and has length p. Every shortest element taking θ_C to θ has length p.*
(iv) *There is a unique (θ, θ_C)-element w. It is the unique shortest element in the coset wW_I and is also the unique shortest element in W taking θ_C to θ.*
(v) *There exists a nonnesting sequence from the maximally aligned element θ_A to θ_C that includes θ.*

Proof. The first assertion of (i) follows from Proposition 5.1 (iii) and the fact that $\sigma(\theta_C) = \sigma(\theta_N)$ by Proposition 5.2. Let V be the reflection representation of W and let

$$D = \{v \in V : B(v, \alpha_i) \geq 0 \text{ for all simple roots } \alpha_i\}.$$

The set D is a fundamental domain for the action of W on V by [33, Theorem 1.12 (a)], and we have $\theta_C \in D$. If θ is a nonnesting n-root different from θ_C, then θ and θ_C are conjugate under the action of W by Lemma 3.2, and therefore so are $\sigma(\theta)$ and $\sigma(\theta_C)$. It follows that $\theta \notin D$; therefore, we have $B(\sigma(\theta), \alpha_i) < 0$ for some simple root α_i.

We prove (ii) by induction on p. If $p = 0$, then $\theta = \theta_C$ by Proposition 5.2 (iii) and the conclusion of (ii) holds trivially. If $p > 0$, then $\theta \neq \theta_C$ and there exists a simple root α_i with $B(\sigma(\theta), \alpha_i) < 0$ by (i). The simple reflection s_i satisfies the condition (5.1), adds 2 to the height of the sum, and sends θ to an n-root of type $A^{p-1}C^{q+1}$ by Remark 5.10, so (ii) follows by induction.

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Let $\mathbf{w} = s_{i_1} \cdots s_{i_r}$ be a (θ, θ_C)-word expressing a (θ, θ_C) element w. Then w takes θ_C to θ, and we have $r = \lambda(\theta_C) - \lambda(\theta) = p$ by by Remark 5.10. Remark 5.11 then implies that $\mathbf{w}$ is reduced. It follows that $\ell(w) = r = p$. Since an element taking θ_C to θ has length at least $\lambda(\theta_C) - \lambda(\theta) = p$ by Remark 5.11 and w is such a shortest element, it also follows that every shortest element taking θ_C to θ has length p.

To prove (iii), it remains to show that $\mathbf{w}$ cannot contain a factor of the form $s_{\alpha_i} s_{\alpha_j} s_{\alpha_i}$. By Remark 5.10 and direct computation, such a factor would imply the existence of a subsequence $(\theta_a, \theta_{a+1}, \theta_{a+2}, \theta_{a+3})$ such that

$$s_{\alpha_i}(\theta_{a+2}) = s_{\alpha_i}(\theta_a + 2\alpha_i + 2\alpha_j) = \theta_a + 2\alpha_i + 2\alpha_j = \theta_{a+2}.$$

This contradicts the fact that $s_{\alpha_i}(\theta_{a+2}) = \theta_{a+3}$, which completes the proof of (iii).

Let w be a shortest element taking θ_C to θ. Then w has length $p = \lambda(\theta_C) - \lambda(\theta)$ by (iii). Let $\mathbf{w} = s_{i_1} \ldots s_{i_p}$ be a reduced word for w. Remark 5.11 implies that if we start from θ_C and apply $s_{i_p}, \cdots, s_{i_2}, s_{i_1}$ successively, each simple reflection must be a CA move. In particular, s_{i_p} must perform a CA move on θ_C, so we have

$$s_{i_p}(\sigma(\theta_C)) = \sigma(s_{i_p}(\theta_C)) = \sigma(\theta_C) - 2\alpha_{i_p}$$

by Corollary 4.9 (ii). This implies that $B(\sigma(\theta_N), \alpha_{i_p}) = B(\sigma(\theta_C), \alpha_{i_p}) > 0$, so it follows from Proposition 5.1 (iii) that $\alpha_{i_p} = \alpha_x$ where α_x is the unique Coxeter generator of W not in I. In other words, every reduced word for w ends in s_{α_x}. It follows from [33, Proposition 1.10 (c)] that w is the unique shortest element in wW_I.

If w' is another shortest element taking θ_C to θ, then $w'w^{-1}(\theta_C) = \theta_C$, so $w'w^{-1} \in W_I$ by Proposition 5.1 (iv). It follows that the cosets wW_I and $w'W_I$ are equal; therefore, we have $w = w'$ because w and w' are both the unique shortest element in the common coset $wW_I = w'W_I$ by the last paragraph. This proves the uniqueness of the shortest element taking θ_C to θ. Part (iii) says that each (θ, θ_C)-element is such a shortest element, and (iv) now follows.

Finally, to prove (v), we recall from Proposition 4.11 (i) that θ_A is the unique minimal element of the set X. It follows from [43, Theorem 2.8] that there exists an element $u \in W$ such that $u(\theta_A) = \theta$ and $\ell(u) = \lambda(\theta) - \lambda(\theta_A) = q$. Let v be the unique (θ, θ_C)-element, let $s_{i_1} \cdots s_{i_p}$ be a reduced word for v, and let $s'_{i_1} \cdots s'_{i_q}$ be a reduced word for u^{-1}. Then $u^{-1}v$ takes θ_C to θ_A and has length at most $p + q = M = \lambda(\theta_C) - \lambda(\theta_A)$, so $u^{-1}v$ must be the unique (θ_A, θ_C)-element by (ii) and (iii), and the word $\mathbf{w} = s'_{i_1} \cdots s'_{i_q} s_{i_1} \cdots s_{i_p}$ must be a reduced word for $u^{-1}v$. Remark 5.11 now implies that starting from θ_A and applying $s'_{i_1}, \cdots, s'_{i_q}, s_{i_1}, \cdots, s_{i_p}$ successively yields a nonnesting sequence $\theta_{\mathbf{w}}$ from θ_A to θ_C that reaches $\theta = v\theta_C$ after the first q steps, and (v) follows. □

Theorem 5.13. *Let W be a Weyl group of rank n of types E_7, E_8 or D_n for n even. Let M be the number of coplanar quadruples in a maximal orthogonal set, and let W_I be the parabolic subgroup of Proposition 5.2 (i).*

(i) *There is a unique element $w_N \in W$ of minimal length such that $w_N(\theta_C) = \theta_A$. The element w_N is fully commutative and has length $\ell(w_N) = M$, and is the unique element of minimal length in the coset $w_N W_I$.*

(ii) *The set*

$$L = \{v(\theta_C) : v \leq_L w_N\}$$

is a complete, irredundantly described set of nonnesting positive n-roots. The set L has the structure of a distributive lattice, induced by the weak Bruhat order $\leq_L$.

(iii) *If γ_1 and γ_2 are positive n-roots satisfying $ht(\sigma(\gamma_1)) - ht(\sigma(\gamma_2)) = 2M$, and w is an element expressed by a word $\mathbf{w}$ of length M satisfying $w(\gamma_1) = \pm\gamma_2$, then $\mathbf{w}$ is reduced, and we must have $\gamma_1 = \theta_C$, $\gamma_2 = \theta_A$, $w = w_N$ and $w(\gamma_1) = \gamma_2$.*

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Proof. Part (i) follows from Proposition 5.12 in the case where $\theta_0 = \theta_A$.

Let $w_N \in W$ be as in (i) and let $v \leq_L w_N$. We may complete a reduced word $\mathbf{v}$ for v to a reduced word of the form $\mathbf{w} = \mathbf{u} \cdot \mathbf{v}$ for w_N. Remark 5.11 implies that $\mathbf{w}$ gives rise to a nonnesting sequence $\theta_{\mathbf{w}}$ from θ_A to θ_C that passes $v(w_N)$. It follows that the elements of L are indeed all nonnesting positive n-roots. Conversely, for every nonnesting positive root θ, it follows from Proposition 5.12 (v) and its proof that $\theta = v(\theta_C)$ for some element $v \leq_L w_N$, so the list L is complete. Finally, if $v \leq_L w_N$ and $v' \leq_L w_N$ are elements such that $v(\theta_C) = v'(\theta_C)$, then $v'v^{-1}$ stabilizes θ_C and hence $\sigma(\theta_C)$, so we have $v'v \in W_I$ and $v'W_I = vW_I$. Since w_N is the shortest element in $w_N W_I$, the elements v' and v must be the shortest elements in vW_I and $v'W_I$ as well, which implies that $v = v'$ as well. It follows that the list L irredundantly describes the positive nonnesting n-roots, proving the first statement of (ii).

By [54, Theorem 3.2], the fact that w_N is fully commutative implies that the poset $\{x : x \leq_L w\}$ is a distributive lattice. This completes the proof of (ii).

Suppose that the conditions of (iii) hold, and let $s_{i_1} s_{i_2} \cdots s_{i_M}$ be a reduced expression for w. Since $\mathrm{ht}(\sigma(\gamma_1)) - \mathrm{ht}(\sigma(\gamma_2)) = 2M$, as we start from θ_C and successively apply the simple reflection $s_{i_M}, \ldots, s_{i_2}, s_{i_1}$, the application of each simple reflection s_{i_j} must subtract 2 from the height of the sum and change a C to an A by Corollary 4.9 (ii). It is therefore not possible at any stage for a simple reflection to negate a component of an n-root, which implies that we have $w(\gamma_1) = \gamma_2$. Each simple reflection s_{i_j} also causes no change in the number of nestings by Proposition 4.7 (i), so the fact that $\mathrm{ht}(\sigma(\gamma_1)) - \mathrm{ht}(\sigma(\gamma_2)) = 2M$ implies that γ_1 and γ_2 have types C^M and A^M, respectively, so we have $\gamma_1 = \theta_C$ and $\gamma_2 = \theta_A$. We then have $w = w_N$ by (i), which completes the proof of (iii). □

Definition 5.14. We call the element w_N from Theorem 5.13 (i.e., the unique element of minimal length that sends θ_C to θ_A) the *nonnesting element* of W.

In Section 6, we will compute the nonnesting element explicitly with the help of Theorem 5.13 (iii).

5.5. *Sum equivalence*

We say that two positive n-roots β and γ of W are *sum equivalent*, or *σ-equivalent*, if $\sigma(\beta) = \sigma(\gamma)$. If C and C' are two σ-equivalence classes, then we write $C \leq_\sigma C'$ if $\sigma(\beta) \leq \sigma(\gamma)$ for any $\beta \in C$ and $\gamma \in C'$ in the usual order $\leq$ on roots (Section 2.2). The goal of this subsection is to show that the σ-equivalence classes of X are highly compatible with the quasiparabolic order $\leq_Q$ and the feature-avoiding n-roots.

Proposition 5.15. *Let $\mathcal{B}$ be the set of nonnesting positive n-roots of W.*

(i) *If $\beta, \beta' \in \mathcal{B}$ are nonnesting positive n-roots with $\sigma(\beta) = \sigma(\beta')$, then we have $\beta = \beta'$.*
(ii) *Each positive n-root γ is σ-equivalent to a unique nonnesting n-root $f(\gamma)$ and a unique noncrossing n-root $g(\gamma)$. We have $f(\gamma) \leq_Q \gamma$, and*

$$\gamma = f(\gamma) + \sum_{\beta \in \mathcal{B} : \sigma(\beta) < \sigma(\gamma)} \lambda_{\beta,\gamma} \beta$$

for suitable integers $\lambda_{\beta,\gamma}$.
(iii) *Every σ-equivalence class contains a unique nonnesting n-root, β_1, and a unique noncrossing n-root, β_2. The σ-equivalence class containing β_1 and β_2 is equal to the interval*

$$[\beta_1, \beta_2]_Q = \{\gamma \in X : \beta_1 \leq_Q \gamma \leq_Q \beta_2\}$$

in the quasiparabolic set X.
(iv) *The set of alignment-free positive n-roots is a σ-equivalence class and is equal to the interval $[\theta_C, \theta_N]_Q$ in the quasiparabolic set X. It is the unique maximal σ-equivalence class with respect to the partial order $\leq_\sigma$.*

Proof. Suppose that β and β' are nonnesting positive n-roots with $\sigma(\beta') = \sigma(\beta)$. It follows from Proposition 5.12 (ii) that β and β' have the same number of alignments – namely, the number $p =$

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$(\mathrm{ht}(\sigma(\theta_C)) - \mathrm{ht}(\sigma(\beta)))/2$. If $p = 0$, then we have $\beta = \theta_C = \beta'$ by Proposition 5.2 (iii). If $p > 0$, then neither β nor β' equals θ_C, so there is a simple root α_i satisfying $B(\sigma(\beta), \alpha_i) = B(\sigma(\beta'), \alpha_i) < 0$ by Proposition 5.12 (i). By Remark 5.10, both $s_i(\beta)$ and $s_i(\beta')$ are nonnesting positive n-roots with $p - 1$ alignments, and we have

$$\sigma(s_i(\beta)) = s_i(\sigma(\beta)) = \sigma(\beta) + 2\alpha_i = \sigma(\beta') + 2\alpha_i = s_i(\sigma(\beta')) = \sigma(s_i(\beta')),$$

so (i) follows by induction on p.

Let γ be a positive n-root, and let $\leq_N$ be the nesting order defined in Section 5.2. If γ contains no nesting, we can simply take $f(\gamma) = \gamma$. Otherwise, we can factorize $\gamma = \gamma'\gamma_N$ where γ_N is a nesting. By the second paragraph of Section 5.2, we can write $\gamma = \gamma'\gamma_C - \gamma'\gamma_A$ where we have (a) $\gamma'\gamma_C \leq_N \gamma$, because $\sigma(\gamma'\gamma_C) = \sigma(\gamma)$ and $\gamma'\gamma_C <_Q \gamma$, and (b) $\gamma'\gamma_A \leq_N \gamma$, because $\sigma(\gamma'\gamma_A) < \sigma(\gamma)$. Taking $f(\gamma) = f(\gamma'\gamma_C)$ proves the existence of $f(\gamma)$ and the required expression for γ by induction on the order $\leq_N$. The uniqueness of $f(\gamma)$ follows from (i). We can use a similar induction using the crossing order $\leq_C$ to show that γ is σ-equivalent to a noncrossing n-root $g(\gamma)$ such that $\gamma \leq_Q g(\gamma)$, and this completes the proof of (ii).

It follows from (ii) that every σ-equivalence class contains a unique nonnesting n-root, and that the number of σ-equivalence classes equals the number of nonnesting n-roots. The latter number is the dimension of the Macdonald representation and also the number of noncrossing roots by Theorem 5.5 (i) and (ii). Since each σ-equivalence class contains at least one noncrossing n-root by (ii), it follows that each σ-equivalence class must contain exactly one nonnesting element and exactly one noncrossing element. This proves the first sentence of (iii).

Let E be a σ-equivalence class with unique nonnesting element β_1 and unique noncrossing element β_2. Then we have $[\beta_1, \beta_2]_Q \subseteq E$ by Corollary 4.9 (i). Conversely, if γ is an n-root in E, then (ii) and its proof imply that we may find a nonnesting n-root $f(\gamma) \in E$ and a noncrossing n-root $g(\gamma) \in E$ such that $f(\gamma) \leq_Q \gamma \leq_Q g(\gamma)$. We must have $f(\gamma) = \beta_1$ and $g(\gamma_2) = \beta_2$ by the uniqueness of the nonnesting and noncrossing elements in E; therefore, we have $\gamma \in [\beta_1, \beta_2]_Q$. It follows that $E = [\beta_1, \beta_2]_Q$.

For every nonnesting root β not equal to θ_C, there is a nontrivial nonnesting sequence from β to θ_C by by Proposition 5.12 (ii), so $\sigma(\beta) < \sigma(\theta_C)$ by Definition 5.9. Part (iv) now follows from (iii) and Proposition 5.2 (ii)–(iii). □

Theorem 5.16. *Let W be a Weyl group of type E_7, E_8 or D_n for n even. If $\mathcal{B}$ is any set of σ-equivalence class representatives, then $\mathcal{B}$ is a basis for the Macdonald representation $j_{nA_1}^{\Phi}(\mathrm{sgn})$. Furthermore, if we order each such basis $\mathcal{B} = \{\beta_1, \cdots, \beta_k\}$ in a way compatible with the order $\leq_\sigma$ (i.e., in such a way that $i < j$ whenever $\beta_i <_\sigma \beta_j$), then the change of basis matrix between any two such bases is unitriangular with integer entries. In particular, this is true for the change of basis matrix between the nonnesting basis and the noncrossing basis.*

Proof. By Proposition 5.15 (ii), each element $\gamma \in \mathcal{B}$ is the sum of the nonnesting element γ' that is σ-equivalent to γ and a $\mathbb{Z}$-linear combination of nonnesting elements with strictly lower sums. The nonnesting elements form a basis for $j_{nA_1}^{\Phi}(\mathrm{sgn})$ by Theorem 5.5 (i), from which it follows that the set $\mathcal{B}$ is also a basis, and that the change of basis from $\mathcal{B}$ to the nonnesting basis is unitriangular with integer entries. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two such bases, then the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ is unitriangular with integer entries because it is the product of the matrix changing $\mathcal{B}_1$ to $\mathcal{B}'$ with the inverse of the matrix changing $\mathcal{B}_2$ to $\mathcal{B}'$, both of which are unitriangular with integer entries. Finally, the last assertion follows because Proposition 5.15 (iii) implies that both the nonnesting and noncrossing bases are sets of σ-equivalence class representatives. □

Remark 5.17. Recall that the *Möbius function*, μ, of a partially ordered set P is defined to satisfy $\mu(x, x) = 1$, $\mu(x, y) = 0$ if $x \not\leq y$, and

$$\sum_{z : x \leq z \leq y} \mu(x, z) = 0$$

if $x < y$. A poset is *Eulerian* if we have $\mu(x,y) = (-1)^{\lambda(y)-\lambda(x)}$ whenever $x \le y$. It can be shown that if $x, y \in X$ correspond to a nonnesting and noncrossing element, respectively, then the interval $I = [x, y]$ corresponds to a σ-equivalence class if and only if I is Eulerian.

Remark 5.18. Reading [44] defines a *poset congruence* to be an equivalence relation on a poset X such that

(i) each equivalence class is an interval;
(ii) the projection mapping $x \in X$ to the maximal element in its equivalence class is order preserving; and
(iii) the projection mapping $x \in X$ to the minimal element in its equivalence class is order preserving.

It can be shown using [58, Proposition 42] that, in type D_n, the equivalence relation induced on X by σ is a poset congruence. It can also be shown (by direct computational verification, for example) that the same is true in types E_7 and E_8.

6. Examples

In this section, we give type-specific details about the n-roots in types D_n for n even, E_7, and E_8. In all types, we explicitly describe the maximally aligned, crossing and nesting n-roots θ_A, θ_C and θ_N. We find the nonnesting element w_N (Definition 5.14), and we use w_N and Theorem 5.13 (ii) to deduce the dimension of the Macdonald representation $j^{\Phi}_{nA_1}(\text{sgn})$. We also discuss type-specific properties of the set X_I of alignment-free positive n-roots for all types. In addition, we explain precise connections between the Macdonald representation $j^{\Phi}_{nA_1}(\text{sgn})$ of type D_{2k} and a Specht module of the symmetric group S_{2k} (Proposition 6.2),

We note that by Lemma 3.1 and Remark 3.11, the noncrossing and nonnesting positive n-roots of type D_{2k} can be easily recovered from the well-studied noncrossing and nonnesting perfect matchings of $[2k]$. More generally, in all types, the nonnesting positive n-roots can be computed efficiently via the elements the elements θ_C and w_N by Theorem 5.13 (ii), and it is possible to construct the noncrossing n-roots using Fan's construction of monomial cells in [23]. In the notation of [23], the maximally aligned n-root θ_A can be identified with the element $b_1 b_3 \cdots b_{2k-1}$ in type D_{2k}, with $b_2 b_4 b_6 b_7$ in type E_7 (with the labelling of Figure 1 (d)), and with $b_2 b_3 b_5 b_7$ in type E_8 (with the labelling of Figure 1 (e)). In types E_7 and E_8, it is also possible to use a computer program to find all noncrossing and nonnesting n-roots by generating all the (finitely many) positive n-roots and then removing all n-roots where a crossing or nesting can be found. For these reasons, and to save space, we have chosen not to list the noncrossing and nonnesting bases in type E_7 or E_8 in this paper (although the complete lists are available upon request).

6.1. *Type* D_{2k}

If W has type D_n for an even integer $n = 2k$, then the positive n-roots can be naturally identified with the perfect matchings of $[n]$, as explained in Lemma 3.1 (ii). Under this identification, the actions of W on the n-roots and on the matching agree, and the alignments, crossings and nestings in the n-roots correspond to the alignments, crossings and nestings in the matchings in the obvious way by Remark 3.11. We also recall from Section 2.3 and Remark 3.7 that the reflection $r = s_\alpha \in W$ acts as the transposition (ij) on the n-roots for each root $\alpha = \varepsilon_i \pm \varepsilon_j$ of W, so that the action of W factors through the homomorphism $\phi : W \longrightarrow S_{2k}$ of Equation (2.2) to induce an action of $S_{2k} = W(A_{2k-1})$ on the n-roots, giving the Macdonald representation $j^{\Phi}_{nA_1}(\text{sgn})$ the structure of an S_{2k}-module (where the elements of S_{2k} permute the indices of the terms ε_i^2). The above facts will allow us to connect the theory of n-roots in type D_n to some widely studied type-A objects and results.

Recall that the number of coplanar quadruples in each positive n-root is $M = \binom{k}{2}$, the number of pairs of 2-blocks, by Corollary 3.19.

Let ν_A, ν_C and ν_N be the positive n-roots corresponding to the matchings $\{12, 34, \cdots, (n-1)n\}, \{1(k+1), 2(k+2), \cdots, k(2k)\}$ and $\{1n, 2(n-1), \cdots, k(k+1)\}$, respectively. Every pair of

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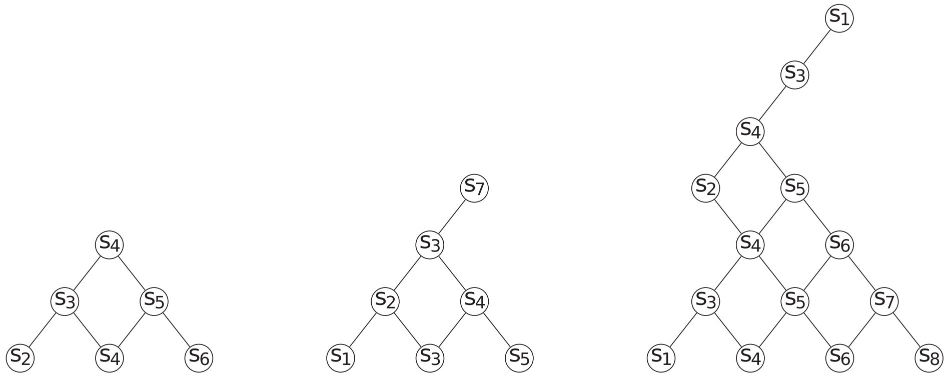

Figure 2. *The heaps of the nonnesting elements of types D_8, E_7, and E_8.*

2-blocks in the first matching forms an alignment, so the matching contains $\binom{k}{2} = M$ alignments. It then follows from Proposition 4.11 (i) that ν_A is the unique maximally aligned n-root θ_A in the set X. Similar arguments show that $\nu_C = \theta_C$ and $\nu_N = \theta_N$ by Proposition 5.2 (iii) and Proposition 4.11 (ii), respectively. Note that we have $\sigma(\theta_N) = 2\sum_{i=1}^{k} \varepsilon_i$.

Let w be the element expressed by the word

$$\mathbf{w} = \mathbf{w}_{2,k-2}\mathbf{w}_{3,k-3}\cdots\mathbf{w}_{k,0},$$

where $\mathbf{w}_{i,j} := s_i s_{i+2} s_{i+4} \cdots s_{i+2j}$. For example, in type D_8, we have $w = (s_2s_4s_6)(s_3s_5)(s_4)$, and the heap of w is shown in Figure 2 (a). The word $\mathbf{w}$ has M letters, and it is straightforward to verify that $w(\theta_C) = \theta_A$, so it follows from Theorem 5.13 (i) that w is the fully commutative nonnesting element w_N and $\mathbf{w}$ is a reduced word for it.

Since $w_N = w$ is fully commutative, the elements in the set $\{v \in W : v \leq_L w_N\}$ are in bijection with the order filters of the heap poset of w_N. (See [54, Section 2.2] for the definition of the heap poset; an order filter of a poset P is a subset of P such that $y \in I$ whenever the conditions $y \in P$, $x \in I$, and $x \leq y$ hold.) These filters are in canonical correspondence with Dyck paths of order k, (i.e. staircase walks from $(0,0)$ to (k,k) that lie strictly below (but may touch) the diagonal $y = x$). It is well known [53, Theorem 1.5.1 (vi)] that the number of such paths is the k-th Catalan number, $C_k = \frac{1}{k+1}\binom{2k}{k}$. Theorem 5.13 (ii) and Theorem 5.5 imply that the number of nonnesting positive n-roots of W is given by C_k, as are the number of noncrossing positive n-roots and the dimension of the Macdonald representation $j_{nA_1}^{\Phi}(\mathrm{sgn})$.

The level function λ in type D_{2k} has a combinatorial interpretation that is natural in the context of combinatorial game theory [36]. The matching corresponding to an n-root β can be identified with a Steiner system $S(1,2,2k)$ (i.e., a collection of 2-blocks of $[2k]$ with the property that any singleton lies in a unique 2-block). The level $\lambda(\beta)$ then counts the number of 2-element subsets E of $[2k]$ with the property that the matching corresponding to β contains no 2-blocks of the form $(E \setminus \{j\}) \cup \{i\}$ where $i \leq j$ and $j \in E$ (in particular, the matching cannot contain E). With some more work, it can be shown that each crossing gives rise to one such subset E, and each nesting gives rise to two such subsets. This gives a combinatorial interpretation of the formula $\lambda(x) = C(x) + 2N(x)$, and also explains the appearance of the product of odd quantum integers in [36, Equation (4.2)]. In addition, the quantity $C(m) + 2N(m)$ associated to each matching m appears as '$C(m) + 2U(m)$' in the context of octabasis Laguerre polynomials in [52, Sections 4 and 5], as the weight '$\omega(m)$' in the context of Gaussian q-distributions in [20, Theorem 4], and as '$cov(m) - cro(m)$' in the context of q-Bessel numbers in [15, Section 4].

The poset structure on the set X_I in type D_{2k} coincides with a familiar one.

Proposition 6.1. *Suppose W has type D_{2k}. Then as a poset, the interval $X_I = [\theta_C, \theta_N]$ in the quasi-parabolic set X is canonically isomorphic to the symmetric group S_k under the (strong) Bruhat order via the map $\varphi : S_k \longrightarrow X_I$ sending each element $\tau \in S_n$ to the n-root*

$$\varphi(\tau) = \prod_{i=1}^{k} (\varepsilon_i^2 - \varepsilon_{\tau(i)+k}^2). \tag{6.1}$$

Under this bijection, we have

$$\lambda(\varphi(\tau)) = M + \ell(\tau)$$

for every $\tau \in S_k$, where M is the number of coplanar quadruples in each n-root and ℓ denotes Coxeter length.

Proof. By Proposition 5.2 (ii), the set X_I is the σ-equivalence class of the n-root θ_N. We noted earlier that $\sigma(\theta_N) = 2\sum_{i=1}^{k} \varepsilon_i$, which implies that n-roots in X_I are precisely the positive n-roots whose components are all of the form $\varepsilon_i \pm \varepsilon_j$, where $1 \leq i \leq k < j \leq 2k$. These are precisely the n-roots listed in the theorem, so the map φ is surjective. It is clear that φ is also injective, so φ is a bijection.

The Bruhat order on S_k is generated by relations of the form $\tau < r\tau$ where we have $\tau \in S_k$ and r is a reflection $r = (\tau(i), \tau(j)) \in S_k$ for some $i < j$ such that $\tau(i) < \tau(j)$ [33, Section 5.9, Example 2]. In this case, the quadruple $\{\varepsilon_i \pm \varepsilon_{\tau(i)+k}, \varepsilon_j \pm \varepsilon_{\tau(j)+k}\}$ contained in $\varphi(\tau)$ is a crossing and is moved to the nesting $\{\varepsilon_i \pm \varepsilon_{\tau(j)+k}, \varepsilon_j \pm \varepsilon_{\tau(i)+k}\}$, so we have $\lambda(\varphi(\tau)) < \lambda(r\varphi(\tau))$ in X_I by Proposition 4.7 (iii). Conversely, if we have $\lambda(\varphi(\tau)) < \lambda(r\varphi(\tau))$ in X_I for some $\tau \in S_k$ and some reflection $r \in W$, then since $\varphi(\tau)$ has no alignments, r must move a crossing in $\varphi(\tau)$ to a nesting by Proposition 4.7 (iii). The crossing moved must be of the form $\{\varepsilon_i \pm \varepsilon_{\tau(i)+k}, \varepsilon_j \pm \varepsilon_{\tau(j)+k}\}$ for some $i, j \in [k]$ such that $i < j$ and $\tau(i) < \tau(j)$, and the only possibilities for r are (ij) and $(\tau(i) + k, \tau(j) + k)$. In either case, we have $r\varphi(\tau) = \varphi(r'\tau)$ for the reflection $r' = (\tau(i), \tau(j)) \in S_k$, so that we have

$$\varphi^{-1}(\varphi(\tau)) = \tau < r'\tau = \varphi^{-1}(r\varphi(\tau)),$$

where $<$ denotes the Bruhat order in S_k. It now follows that φ is a poset isomorphism.

To prove the last assertion, we note that each inversion of a permutation $\tau \in S_k$ corresponds to a nesting in the corresponding alignment-free n-root $\gamma = \varphi(\tau)$, and we recall that $\ell(\tau)$ equals the number of inversions in τ. It follows that $N(\gamma) = \ell(\tau)$; therefore, we have

$$\lambda(\gamma) = C(\gamma) + 2N(\gamma) = (C(\gamma) + N(\gamma)) + N(\gamma) = M + N(\gamma) = M + \ell(\tau). \qquad \square$$

We now discuss the structure of the space $j_{nA_1}^{D_n}(\text{sgn})$ underlying the Macdonald representation as an S_{2k}-module. As a vector space, $j_{nA_1}^{D_n}(\text{sgn})$ is isomorphic to the free vector space on the noncrossing perfect matchings of $[n] = [2k]$, which is denoted by $V(n, k, 0)$ in the work of Rhoades [47]. Furthermore, given a simple reflection $s_i = (i, i+1) \in S_n$ and a noncrossing perfect matching m corresponding to an n-root γ, the reflection s_i acts on m in one of the following ways:

(1) if $i(i+1)$ is a 2-block in m, then the n-root γ contains $\varepsilon_i^2 - \varepsilon_{i+1}^2$ as a factor, so $s_i(m) = -m$;
(2) if $i(i+1)$ if not a 2-block in m, then m contains two blocks ia and $(i+1)b$ which either form an alignment (if $a < i < i+1 < b$) or a nesting (if $b < a < i$ or $i+1 < b < a$). In all cases, we have $s_i(m) = m''$ where m'' is the matching $(m \setminus \{ia, (i+1)b\}) \cup \{(i+1)a, ib\}$. Here, the blocks $(i+1)a$ and ib form a crossing, and the Ptolemy relation $\gamma_C = \gamma_N + \gamma_A$ from Theorem 3.10 (vi) implies that

$$s_i(m) = m'' = m + m', \tag{6.2}$$

where m' is the perfect matching $m' = (m \setminus \{ia, (i+1)b\}) \cup \{i(i+1), ab\}$. Here, the blocks $i(i+1)$ and ab form the nesting in the Ptolemy relation if $\{ia, (i+1)b\}$ is an alignment and form the

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alignment in the Ptolemy relation if $\{ia, (i+1)b\}$ is a nesting. The matching m' is noncrossing by Theorem 5.7 (iv).

It follows from the above analysis that the action of S_{2k} on $j_{nA_1}^{D_n}(\mathrm{sgn})$ agrees with the action of S_{2k} on the space $V(n,k,0)$ defined by Rhoades. The precise formula in Equation (6.2) appears in the work of Kim [38, Equation (1.3)]. By [47, Proposition 5.2], as an S_{2k} module $V(n,k,0)$ is isomorphic to the Specht module $S^{(k,k)}$ corresponding to the 2-row partition (k,k), so we may summarize our discussion as follows:

Proposition 6.2. *If W has type D_n for $n = 2k$ even, then the W-action on the Macdonald representation $j_{nA_1}^{D_n}(\mathrm{sgn})$ factors through the map ϕ defined by Equation (2.2) to induce an S_n-module structure on $j_{nA_1}^{D_n}(\mathrm{sgn})$. The resulting S_n-module is isomorphic to the Specht module $S^{(k,k)} \cong V(n,k,0)$. In particular, it is irreducible.*

Remark 6.3. The nonnesting and noncrossing bases for the S_n-module $j_{nA_1}^{D_n}(\mathrm{sgn}) \cong S^{(k,k)}$ are also studied extensively in the works of Russell–Tymoczko [48], Im–Zhu [35], Hwang–Jang–Oh [34] and Heard–Kujawa [32]. In these papers, the noncrossing basis is called the *web basis*, and the nonnesting basis can be naturally identified with the *standard basis* (or the *polytabloid* or *Specht* basis) as explained in [35, Lemma 3.1] and [34, Section 1]. Under this identification, the isomorphism of [48, Theorem 2.2] associates each nonnesting perfect matching with the unique noncrossing matching in the same σ-equivalence class, and Theorem 5.5 of [48] follows from Theorem 5.16 as a special case. The restriction of the quasiparabolic order to the noncrossing basis gives rise to the *web graph* of [48, Section 2.3], which therefore has the structure of a distributive lattice by Theorem 5.13 (ii). Our definition of the nesting number (Definition 4.2 (i)) agrees with the nesting number of [48] when restricted to noncrossing n-roots, and is inspired by [48]. It also follows from [34, Corollary 4.2] that if W has type D_{2k} and we expand the maximally crossing n-root θ_C as a linear combination of the noncrossing basis, $\theta_C = \sum \lambda_\beta \beta$, then the sum $\sum \lambda_\beta$ of the nonnegative integers λ_β is given by the number E_{k+1} in the family (1,1,1,2,5,16,272, ...) of *Euler numbers*, which are characterized by the equation

$$\sec(x) + \tan(x) = \sum_{i=0}^{\infty} E_i \frac{x^i}{i!}.$$

Coefficients in the expansion of the maximally crossing $2k$-root θ_C into the noncrossing basis have a combinatorial interpretation in terms of the so-called web permutations in S_k by [34, Theorem 1.2].

6.2. Type E_7

Suppose W has type E_7. We define ν_A to be the positive 7-root with the following components:

$$\alpha_2,\ \alpha_4,\ \alpha_6,\ \alpha_7,\ \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_7,$$

$$\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7,\ 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7.$$

We define ν_C to be the positive 7-root with the following components:

$$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7, \quad \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_7, \quad \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7$$

$$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_7, \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \quad \alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7.$$

We define ν_N to be the positive 7-root with the following components:

$$\alpha_7, \quad \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_7, \quad \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_7, \quad \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$$

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7, \quad \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7.$$

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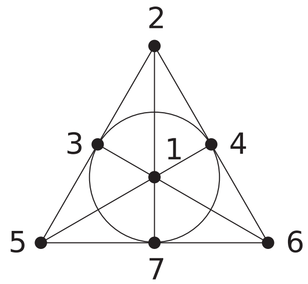
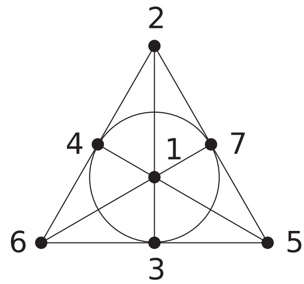


(a) L_C



(b) L_N

Figure 3. *The inequivalent labellings of the Fano plane corresponding to θ_C and θ_N.*

Finally, we define the element $w \in W$ to be the element expressed by the word

$$\mathbf{w} = (s_1 s_3 s_5)(s_2 s_4)(s_3)(s_7).$$

The heap of w is shown in Figure 2 (b).

Proposition 6.4. *If W has type E_7, then the 7-roots ν_A, ν_C and ν_N given above are respectively the maximally aligned, maximally crossing and maximally nesting 7-roots of W. The element w is the nonnesting element w_N, and $\mathbf{w}$ a reduced word for it. The Macdonald representation $j_{7A_1}^{E_7}(\mathrm{sgn})$ has dimension* 15.

Proof. Recall from Corollary 3.19 that the number of coplanar quadruples in any 7-root is $M = 7$. Direct verification shows that $\mathrm{ht}(\sigma(\nu_C)) - \mathrm{ht}(\sigma(\nu_A)) = 49 - 35 = 2M$ and that $w(\nu_C) = \nu_A$, which implies the assertions about ν_A, ν_C, $\mathbf{w}$ and w by Theorem 5.13 (iii). The dimension of the Macdonald representation $j_{7A_1}^{E_7}(\mathrm{sgn})$ equals the cardinality of the set $\{v \in W : v \leq_L w\}$ by Theorem 5.5 (i) and Theorem 5.13 (ii). As explained in Section 6.1, this set is in bijection with the order filters of the heap of the fully commutative w_N, and direct computation shows that this heap has 15 filters, so the dimension of $j_{7A_1}^{E_7}(\mathrm{sgn})$ is 15.

It remains to show that ν_N is the maximally aligned n-root θ_A. By inspection, the heights of the components of ν_N are 1, 5, 7, 9, 9, 9, 9 when listed in increasing order. The sum of the first three terms of this sequence is bigger than the largest term, so ν_N cannot contain any alignments by Proposition 3.13 (ii). If ν_N contains a crossing, it follows from Proposition 3.13 (iv) that the crossing cannot contain any component of height 1, and that the crossing can contain at most one component of height 9. It then follows from the listed heights that ν_N contains no crossing either. It follows that all the M coplanar quadruples in ν_N are nestings, so that $\nu_N = \theta_N$ by Proposition 4.11 (ii). □

The set X_I of alignment-free positive 7-roots in type E_7 is intimately related to the combinatorics of the *Fano plane* (Figure 3), the finite projective plane of order 2 over the field $\mathbb{F}_2$ with two elements. We recall that any two points in the Fano share a unique line that contains them both, so that the vertex labellings of the Fano points using the labels $1, 2, .., 7$ correspond precisely to the Steiner triple systems $S(2, 3, 7)$ via the bijection that associates each line in the Fano plane with the triple of labels for the vertices in that line. For example, the labellings shown in Figure 3 (a) and (b) correspond respectively to the Steiner systems L_C and L_N from Proposition 6.5. It is well known that the automorphism group of the Fano plane is the simple group $GL(3, 2)$ of order 168, so that the number of inequivalent vertex labellings is $7!/168 = 30$.

Proposition 6.5. *If W has type E_7, then every component of every 7-root $\gamma \in X_I$ has the form*

$$\eta_{abc} = \left(\sum_{i=0}^{7} \varepsilon_i\right) - 2(\varepsilon_0 + \varepsilon_a + \varepsilon_b + \varepsilon_c)$$

for a 3-element subset $abc := \{a, b, c\}$ of the set [7], and the map φ sending each 7-root $\gamma \in X_I$ to the set

$$L_\gamma = \{abc : \eta_{abc} \mid \gamma\}$$

gives a canonical bijection from X_I to the 30 inequivalent labellings of the Fano plane. Under this bijection, the minimal element θ_C of X_I corresponds to the labelling

$$L_C = L_{\theta_C} = \{136, 145, 127, 235, 246, 347, 567\},$$

and the maximal element θ_N corresponds to the labelling

$$L_N = L_{\theta_N} = \{123, 145, 246, 257, 347, 356, 167\}.$$

Proof. In the Fano coordinates, the components of the maximally crossing and maximally nesting n-roots θ_C and θ_N are given by the rows of the following matrices M_C and M_N, respectively, where each '+' stands for 1 and each '−' stands for −1 for brevity.

$$M_C = \begin{bmatrix} - & - & + & - & + & + & - & + \\ - & - & + & + & - & - & + & + \\ - & - & - & + & + & + & + & - \\ - & + & - & - & + & - & + & + \\ - & + & - & + & - & + & - & + \\ - & + & + & - & - & + & + & - \\ - & + & + & + & + & - & - & - \end{bmatrix}, \quad M_N = \begin{bmatrix} - & - & - & - & + & + & + & + \\ - & - & + & + & - & - & + & + \\ - & + & - & + & - & + & - & + \\ - & + & - & + & + & - & + & - \\ - & + & + & - & - & + & + & - \\ - & + & + & - & + & - & - & + \\ - & - & + & + & + & + & - & - \end{bmatrix} \tag{6.3}$$

By inspection, the components of θ_C have the properties (a) each of them is a root of the form η_{abc} for some triple $abc \subseteq [7]$, and (b) the triples corresponding to components form a Steiner triple system.

By Proposition 6.4, the rightmost generator appearing in w_N is s_7, which implies that $I = S\backslash\{s_7\}$. It follows from Section 2.3 that W_I is a Weyl group of type A_6, isomorphic to S_7, and that W_I acts on X_I by permuting the Fano coordinates. Since all elements of X_I are conjugate to θ_C under the action of W_I by Proposition 5.2 (ii) and (iii), it now follows from the previous paragraph that for every 7-root $\gamma \in X_I$, the components of γ satisfy the properties (a) and (b) satisfied by the components of θ_C. This implies that the map φ takes each element to a Steiner triple system (and thus one of the 30 inequivalent labellings of the Fano plane). The map φ is clearly injective, and it is surjective because all Steiner triple systems are isomorphic via the permutation action of S_7 by Remark 3.16. This proves the first sentence of the proposition. The second sentence holds by inspection of the matrices M_C and M_N. □

Remark 6.6. The labellings canonically corresponding to X_I have the following additional properties.

(i) The triples ijk in the labelling L_C corresponding to the 7-root θ_C appear in [55, Section IV] as the triples indexing the 'globally invariant linear forms' $\pm x_i \pm x_j \pm x_k$ of type E_7.
(ii) The labelling L_N corresponding to θ_N is the unique labelling with the property that if the digits are written in binary, then the third digit of each triple is the bitwise exclusive or (XOR) of the other two.
(iii) Recall from Remark 5.3 that X_I naturally splits into two equal-sized components that are interchanged by the action of a reflection in W_I. As discussed in [50], any two distinct labellings in the same component have precisely one triple in common.
(iv) The level $\lambda(\gamma)$ of each 7-root γ in X_I equals $(14-d)$, where d is the number of 3-element subsets E of the set [7] with the property that the labelling L_γ contains no blocks of the form $(E \setminus \{j\}) \cup \{i\}$ where $i \leq j$ and $j \in E$. This fact can be verified computationally, and is similar to the interpretation of the level function in type D_{2k} via Steiner systems $S(1, 2, 2k)$ given in Section 6.1.

Remark 6.7. The noncrossing basis in type E_7 is illustrated by the diagram labelled $\mathfrak{M}_6$ in [14, Appendix], where each rectangle can be identified with a noncrossing basis element β. A label of i on a rectangle indicates that $\alpha_i|\beta$, i.e., that α_i is a component of β. The edges connecting rectangles refer

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to *star operations* in the sense of [37], which can be interpreted directly in terms of n-roots as follows. If i and j are adjacent vertices of the Dynkin diagram, then two noncrossing basis elements β and β' such that $\alpha_i|\beta$ and $\alpha_j|\beta'$ are joined by an edge if we have $s_i s_j(\beta) = \beta'$ or, equivalently, $s_j s_i(\beta') = \beta$. (A similar construction appears in [30, Lemma 2.8].) Note that the Dynkin diagram of type E_7 in [14] differs from the Dynkin diagram of type E_7 shown in Figure 1 (d) in the labelling of vertices, but it can be obtained by removing the vertex '8' and its incident edge from the Dynkin diagram of type E_8 shown in Figure 1 (e).

Remark 6.8. Ren–Sam–Schrader–Sturmfels [46, Theorem 4.1] give an 'utterly explicit' basis for the 15-dimensional Macdonald representation in type E_7 that is natural in the context of the Göpel variety in algebraic geometry. The elements of the nonnesting basis and the noncrossing basis in type E_7 all factorize into linear factors in $\mathrm{Sym}(V^*)$ by construction, but not all the basis elements of [46, Theorem 4.1] do, even after extending scalars to $\mathbb{C}$. It follows that the basis of [46] is not the same as either the noncrossing basis or the nonnesting basis, even after applying a change of basis of V^*.

6.3. Type E_8

Suppose W has type E_8. We define v_A to be the positive 8-root with the following components:

$$\alpha_2,\ \alpha_3,\ \alpha_5,\ \alpha_7,\ \alpha_2+\alpha_3+2\alpha_4+\alpha_5,$$

$$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7,\ 2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7,$$

$$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8.$$

We define v_C to be the positive 8-root with the following components:

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8,\quad \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7,$$

$$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,\quad \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6,$$

$$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7,\quad \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8,$$

$$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+3\alpha_6+2\alpha_7+\alpha_8,\quad \alpha_1+3\alpha_2+3\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8.$$

We define v_N to be the positive 8-root with the following components:

$$\alpha_1,\quad \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5,\quad \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7,$$

$$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8,\quad \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7+\alpha_8,$$

$$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8,\quad \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8,$$

$$\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7.$$

Finally, we define the element $w \in W$ to be the element expressed by the word

$$\mathbf{w} = (s_1s_4s_6s_8)(s_3s_5s_7)(s_4s_6)(s_2s_5)(s_4)(s_3)(s_1).$$

The heap of w is shown in Figure 2 (c).

Proposition 6.9. *If W has type E_8, then the* 8*-roots ν_A, ν_C and ν_N given above are respectively the maximally aligned, maximally crossing and maximally nesting* 8*-roots of W. The element w is the nonnesting element w_N, and* $\mathbf{w}$ *a reduced word for it. The Macdonald representation $j_{8A_1}^{E_8}(\mathrm{sgn})$ has dimension* 50*.*

Proof. The statements can be proved using the same strategy used in the proof of Proposition 6.4 except for the following changes in numerical details. The number M of coplanar quadruples in an n-root is now 14, and the number of order filters in the heap of the nonnesting element w_N is 50. The components of ν_N have heights $1, 7, 11, 13, 15, 15, 15$ and 15, which implies that ν_N has no alignments by Proposition 3.13 (ii). Furthermore, if ν_N had a crossing, then Proposition 3.13 (iv) implies that the only possibility would be for the crossing to contain roots of heights 7, 11, 13, and 15, but this is not possible either because $11 + 13 > 7 + 15$. □

Remark 6.10.

(i) Schmidt [51, Lemma 3.4] gives an explicit partition of the 120 positive-negative pairs of roots in type E_8 into 15 sets of size 8. The components of θ_A, θ_N and θ_C appear in Schmidt's list as V_1, V_{14} and V_{15}, respectively.

(ii) In standard coordinates, the components of the maximally crossing and maximally nesting 8-roots $\theta_C = \nu_C$ and $\theta_N = \nu_N$ are given by the rows of the following matrices M'_C and M'_N, respectively, where each '+' stands for 1 and each '−' stands for −1 for brevity.

$$M'_C = \begin{bmatrix} + & + & - & - & - & - & + & + \\ + & - & + & - & - & + & - & + \\ + & - & - & + & + & - & - & + \\ - & + & + & - & + & - & - & + \\ - & + & - & + & - & + & - & + \\ - & - & + & + & - & - & + & + \\ - & - & - & - & + & + & + & + \\ + & + & + & + & + & + & + & + \end{bmatrix}, \quad M'_N = \begin{bmatrix} + & - & - & - & - & - & - & + \\ - & + & + & + & - & - & - & + \\ - & + & - & - & + & + & - & + \\ - & - & + & - & + & - & + & + \\ - & - & - & + & - & + & + & + \\ + & + & + & - & - & + & + & + \\ + & + & - & + & + & - & + & + \\ + & - & + & + & + & + & - & + \end{bmatrix}$$

All rows in M'_C other than the bottom row contain four '+' and four '−', and the 14 quadruples recording the column numbers of the positive and negative entries in these rows form a Steiner quadruple system. Furthermore, these 14 quadruples are precisely the ones indexing the 'globally invariant linear forms' of type E_8 in [55, Section V].

(iii) The matrix M'_C above is a *Hadamard matrix*, meaning a matrix with entries in $\{+1, -1\}$ that has orthogonal rows (and, therefore, orthogonal columns). By rearranging the rows, the matrix can be expressed more simply as the Kronecker product $H \otimes H \otimes H$, where

$$H = \begin{bmatrix} +1 & +1 \\ -1 & +1 \end{bmatrix}.$$

Remark 6.11. The noncrossing basis in type E_8 is illustrated by the diagram labelled $\mathfrak{M}_{50}$ in [13, Appendix].

The set X_I of alignment-free positive 8-roots in type E_8 can be used to give a convenient realization of the graph $\bar{\Gamma}_1$ studied by Schmidt in [51]. The graph $\bar{\Gamma}_1$, which is the complement of another graph Γ_1, has the property that it is quantum isomorphic (in the sense of [3]) but not isomorphic to the orthogonality graph G_{E_8} of the roots of type E_8. The vertices of G_{E_8} are the 120 positive roots of type E_8, and two roots are adjacent in G_{E_8} if and only if they are orthogonal.

To realize the graph $\bar{\Gamma}_1$ via X_I, recall from Remark 5.3 that X_I naturally splits into two equal-sized components, X_I^e and X_I^o, which consist of all the elements in X_I with even levels and odd levels, respectively. The components each have $240/2 = 120$ elements since $|X_I| = 240$ by Proposition 7.1 (ii).

Definition 6.12. We define Γ to be the following graph: the vertex set is X_I^e, the set of all alignment-free positive 8-roots of even parabolic level, and two vertices are adjacent if and only if they have no common components.

The next four paragraphs recall Schmidt's construction of the graph Γ_1 and explain why Γ is isomorphic to $\bar{\Gamma}_1$. (The isomorphism will also hold if we replace X_I^e with X_I^o in Definition 6.12.)

Start with the folded halved 8-cube, where the vertices are the 64 pairs of the form $\{x, \mathbf{1} + x\}$ for all length-8 binary strings $x \in \mathbb{F}_2^8$ with an even number of 1s (where $\mathbf{1}$ is the string with all entries equal to 1). We can naturally identify these vertices with the 64 positive roots of the form $\alpha = (\sum_{i=1}^{7} \pm\varepsilon_i) + \varepsilon_8$, via the bijection sending α to the pair $\{x_\alpha, \mathbf{1} + x_\alpha\}$ where x_α is the string $(x_i)_{i=1}^{8}$ such that $x_i = 1$ if and only if ε_i appears with coefficient -1 in α for all $i \in [8]$. The 64 positive roots of the form $(\sum_{i=1}^{7} \pm\varepsilon_i) + \varepsilon_8$ are precisely the positive roots of x-height 1 in the sense of Remark 5.3, so they are also precisely the roots that can appear as a component of an 8-root in X_I by Remark 5.3.

By definition, two vertices $\{x, \mathbf{1} + x\}$ and $\{y, \mathbf{1} + y\}$ in the folded halved cube are adjacent if and only if x and y differ in 2 or 6 entries. It follows that in the complement of the folded halved cube, two distinct vertices $\{x, \mathbf{1} + x\}$ and $\{y, \mathbf{1} + y\}$ are adjacent if and only if x and y differ in 4 positions. This complement is denoted $VO_6^+(2)$. The condition that x and y differ in 4 entries holds if and only if the positive roots corresponding to $\{x, \mathbf{1} + x\}$ and $\{y, \mathbf{1} + y\}$ are orthogonal; therefore, each clique of size 8 in $VO_6^+(2)$ corresponds to an 8-root in X_I.

Schmidt defines the vertex set of Γ_1 to be any orbit of cliques of size 8 under the group $\mathbb{Z}_2^6 \rtimes A_8$ in $VO_6^+(2)$, where A_8 is the alternating subgroup of S_8. There are two such orbits, both of size 120, and the choice of the orbit does not matter, so we may assume that the orbit contains a clique corresponding to an 8-root with even level, that is, to a vertex, β, of Γ. The vertices of Γ_1 then match precisely the vertices of Γ for the reasons sketched below. We have $I = S \setminus \{s_1\}$ by Proposition 6.9; therefore, we have $W_I \cong W(D_7)$. The action of $W_I = W(D_7)$ on X_I can be extended to an action of a larger subgroup $G \leq W(E_8)$, generated by W_I together with the reflection s_θ corresponding to the highest root $\theta = 2(\varepsilon_7 + \varepsilon_8)$. We have $G \cong W(D_8) \cong N \rtimes S_8$, where $N \cong \mathbb{Z}_2^7$ is the elementary abelian group of order 2^7. By considerations involving x-heights (in the sense of Remark 5.3), each reflection in G changes every 8-root in X_I, and when it does so, it changes the parabolic level by an odd number because it moves a C to an N or vice versa. It follows that the commutator subgroup $G' \cong \mathbb{Z}_2^7 \rtimes A_8$ of G acts on X_I with X_I^e and X_I^o as its orbits. This action induces a transitive action of $G'/Z(G) \cong \mathbb{Z}_2^6 \rtimes A_8$ on Γ that matches the action of $\mathbb{Z}_2^6 \rtimes A_8$ on Γ_1.

Two vertices in Γ_1 are defined to be adjacent if and only if they are cliques that intersect in exactly two elements from $VO_6^+(2)$. This occurs if and only if their corresponding 8-roots have two components in common. With some more work, or using computation, one can show that two distinct 8-roots whose levels have the same parity either have disjoint components or have exactly two components in common. It follows that Γ is isomorphic to $\bar{\Gamma}_1$. To summarize, we have the following result.

Remark 6.13. The graph Γ from Definition 6.12 (i.e., the graph whose vertices are the alignment-free positive 8-roots of even parabolic level and where two vertices are adjacent if and only if they have no common component) is isomorphic to the graph $\bar{\Gamma}_1$ from [51]. As a consequence, the graph Γ is quantum isomorphic but not isomorphic to the orthogonality graph G_{E_8} of the E_8 root system.

The graphs Γ_{E_8} and $\bar{\Gamma}_1$ are known to be strongly regular graphs with parameters $(120, 63, 30, 36)$. It follows that Γ is also such a graph. Mathon and Street [40, Table 2.2] mention that the graphs Γ_{E_8} and $\bar{\Gamma}_1$ each have 2025 8-cliques. The 8-cliques of Γ_{E_8} are the positive 8-roots of E_8, and the 8-cliques of $\Gamma \cong \bar{\Gamma}_1$ are classified by Fitz in [24, Theorem 7.6]. Finally, we note that, as [51] points out, there are other constructions of Γ_1 in the literature [8, 40]. However, the construction in terms of 8-roots has the advantages of being concise, and being clearly related to the E_8 root system.

Remark 6.14. The group $\mathrm{Aut}(\Gamma_1)$ acts as a permutation group of rank 4 on Γ_1 [8]. It follows that if two n-roots $x, y \in X_I$ both have even or odd levels, then x and y can be in one of four relative positions.

These can be shown to be the following (where N is the elementary abelian group of order 2^7 mentioned above):

(i) $x = y$;
(ii) x and y have precisely two common components;
(iii) x and y have disjoint components, and $y = n.x$ for some $n \in N$;
(iv) x and y have disjoint components, and $y \neq n.x$ for any $n \in N$.

The situations in (iii) and (iv) correspond to the edges in the graph Γ, and they show that the edges of Γ naturally split into two types. This is not the case for the graph G_{E_8}: the automorphism group of G_{E_8} has rank 3, and two vertices can only be in three relative positions: equality, adjacency and non-adjacency.

7. Concluding remarks

7.1. Poincaré polynomials

Rains and Vazirani [43, Section 8] define the *Poincaré series* of a quasiparabolic set $\mathcal{X}$ to be $PS(q) = \sum_{x\in\mathcal{X}} q^{\lambda(x)}$. They point out that in many cases, the Poincaré series factorizes in a very simple way, and the factors are often *quantum integers*

$$[d]_q := \frac{q^d - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{d-1}.$$

These quantum integers often behave as if they were the degrees of polynomial invariants of a Coxeter group, and in some cases, the integers can be interpreted in terms of degrees of invariants in characteristic 2 (see [43, Section 8, Example 9.4]).

Proposition 7.1. *Let W be a Weyl group of type E_7, E_8 or D_n for n even.*

(i) *For the quasiparabolic set X for W consisting of all positive n-roots of W, equipped with the level function such that $\lambda(\theta_A) = 0$, we have*

$$PS_X(q) = \begin{cases} \prod_{i=2}^{k}[2i-1]_q & \text{if } W \text{ is of type } D_{2k}, \\ [3]_q[5]_q[9]_q & \text{if } W \text{ is of type } E_7, \\ [3]_q[5]_q[9]_q[15]_q & \text{if } W \text{ is of type } E_8. \end{cases}$$

(ii) *For the quasiparabolic set $X_I \subseteq X$ for W_I consisting of the alignment-free positive n-roots of W (with its level function inherited from X), we have*

$$PS_{X_I}(q) = \begin{cases} q^M \prod_{i=2}^{k}[i]_q & \text{if } W \text{ is of type } D_{2k}, \\ q^M [2]_q[3]_q[5]_q & \text{if } W \text{ is of type } E_7, \\ q^M [2]_q[3]_q[5]_q[8]_q & \text{if } W \text{ is of type } E_8, \end{cases}$$

where M is the level $\lambda(\theta_C)$ of the minimal element of X_I (which is also the number of coplanar quadruples in each n-root). In particular, each of the factors $[i]_q$ of $PS_{X_I}(q)$ corresponds to a factor $[2i-1]_q$ of $PS_X(q)$ in (i).

Proof. We have verified both (i) and (ii) in types E_7 and E_8 computationally. (We do not have conceptual proofs at the moment.) For type D_n, part (i) follows from [52, Equation (5.4)], or [20, Theorem 4] or [15, Corollary 3.3], after we identify X with the perfect matchings of the set $[n]$ as usual. Finally, part (ii) for type D_n follows from Proposition 6.1 and the well-known form $\prod_{i=2}^{k}[i]_q$ for the Poincaré series $\sum_{\tau\in S_k} q^{\ell(\tau)}$ of the symmetric group S_k. □

Remark 7.2. The exponents 3, 5, 9 and 3, 5, 9, 15 that respectively appear in the Poincaré series $PS_X(q)$ of types E_7 and E_8 show up as the degrees of generators in the cohomology modulo 2 of compact exceptional Lie groups [2], and as the codimensions of generators of Chow rings associated to linear algebraic groups in characteristic 2 [42, Section 4].

Remark 7.3. Recall from Proposition 5.15 that the set X_I is the top σ-equivalence class with respect to the order $\leq_\sigma$. It turns out that for every σ-equivalence class C in X, the polynomial $PS_C(q) = \sum_{x \in C} q^{\lambda(x)}$ has the form

$$PS_C(q) = \prod_{d \in D} q^{d-1}[d]_q \tag{7.1}$$

for some set of nonnegative integers D. This is particularly remarkable because in general, there is no obvious way to turn a σ-equivalence class into a W-set for a suitable Weyl group W. In type D_n, the integers from D have an interpretation in terms of rook placements [58, Theorem 1]. Summing over all σ-equivalence classes gives rise to the expression for $PS_X(q)$ that appears in the abstract of [6]. In types E_7 and E_8, we verified Equation (7.1) by computation.

7.2. *Coxeter elements*

Let $d_1, d_2, \ldots, d_r$ be the numbers that appear in the factorization $\prod_{i=1}^r [d_i]_q$ of the Poincare series $PS_X(q)$ in Proposition 7.1 (i). It follows easily from the definitions that $\prod_{i=1}^r d_i$ is the number of positive n-roots, and that $\sum_{i=1}^r (d_i - 1) = 2M$, where M is the number of coplanar quadruples in an n-root. It also turns out that the largest integer d_r in each case (which is $n-1$ in type D_n, is 9 in type E_7, and is 15 in type E_8) is equal to $h/2$, where h is the Coxeter number.

We recall that, by definition, a Coxeter element is a product of all the simple reflections in some order, and the Coxeter number is the order of any Coxeter element c. All such elements are conjugate and therefore have the same order. It turns out that $c^{h/2}$ acts as -1 in the reflection representation, and therefore acts trivially on the set X. If C is the cyclic group of order $h/2$ generated by the action of c on the positive n-roots, then it can be shown that the nonidentity elements of C act without fixed points on the positive n-roots. The factor of $[h/2]_q$ in $PS_X(q)$ then implies that the triple $(X, PS_X(q), C)$ satisfies the cyclic sieving phenomenon of Reiner, Stanton and White [45]: the number of fixed points of c^d is equal to $PS_X(e^{2\pi i d/m})$, where $m = h/2$.

It is possible, by choosing a suitable Coxeter element c and n-root β, to find an orbit of n-roots

$$O = \{c^d(\beta) : 0 \leq d < h/2\}$$

that contains every positive root exactly once as one of its components. This can be achieved in type D_n by taking $\beta = \theta_N$ and $c = s_1 s_2 \cdots s_{n-1} s_n$. We also verified that such an orbit can also be found in types E_7 and E_8, although it is necessary to make a different choice of β. The existence of such an orbit O in type E_8 is related to the Kochen–Specker theorem in quantum mechanics [57].

7.3. *Feature-avoidance via quasiparabolic structure*

It can be shown that each of the three types of feature-avoiding n-roots in the set X can be characterized using only the quasiparabolic structure of X, without reference to the combinatorics of n-roots. Specifically, the following holds for all n-roots $x \in X$:

(i) x is alignment-free if and only if there does not exist a reflection r such that $\lambda(r(x)) - \lambda(x)$ is a strictly positive even number;
(ii) x is noncrossing if and only if there is a sequence

$$x <_Q r_1(x) <_Q r_2 r_1(x) <_Q \cdots <_Q x_1,$$

where x_1 is the unique maximal element of X and the level increases by 2 at each step;

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(iii) x is nonnesting if and only if there does not exist a reflection r such that $\lambda(r(x)) - \lambda(x)$ is a strictly negative even number.

In addition, Remark 5.17 shows that the σ-equivalence classes can be characterized as the Eulerian intervals between nonnesting and noncrossing elements. It may be interesting to use these characterizations to extend the notions of feature-avoiding elements and σ-equivalence to more general quasiparabolic sets.

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