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RESEARCH ARTICLE

Superelliptic Affine Lie algebras and orthogonal polynomials

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Abstract
We construct two families of orthogonal polynomials associated with the universal central extensions of the superelliptic Lie algebras. These polynomials satisfy certain fourth-order linear differential equations, and one of the families is a particular collection of associated ultraspherical polynomials. We show that the generating functions of the polynomials satisfy fourth-order linear PDEs. Since these generating functions can be represented by superelliptic integrals, we have examples of linear PDEs of fourth order with explicit solutions without complete integrability.

Contents

Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra and $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ the loop algebra of $\mathfrak{g}$ with commutation relations $[x \otimes f, y \otimes g] = [x, y] \otimes fg$, for $x, y \in \mathfrak{g}$ and $f, g \in \mathbb{C}[t, t^{-1}]$. In the construction of the loop algebra, we may replace the Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$ by any other commutative associative complex algebra R. In particular, if R is the algebra of regular functions over an affine scheme of finite type X, we obtain the Lie algebra of all regular maps from X to $\mathfrak{g}$. If R is the ring of meromorphic functions on a Riemann surface with a fixed number of poles, we obtain a *current Krichever-Novikov algebra*. These algebras where introduced by Krichever and Novikov in their study of string theory in Minkowski space [16], [17] and extensively studied by many authors (cf. [19] and references therein). In particular, the case of the ring of rational functions on the Riemann sphere regular everywhere except a finite number of points is relevant to the study of tensor module structures for affine Lie algebras ([12], [13]). The corresponding Lie algebras generalizing the untwisted affine Kac-Moody Lie algebras are called *N-point algebras*.

Denote by $\hat{\mathcal{G}}$ the universal central extension of $\mathcal{G} = \mathfrak{g} \otimes R$. In [3], Bremner described the center C of the universal central extension $\hat{\mathcal{G}}$ of N-point algebras and determined its dimension. Also, in the case of the ring of regular functions on an elliptic curve with two points removed, the universal cocycle $\hat{\mathcal{G}} \times \hat{\mathcal{G}} \to C$ was computed.

Following these results, Cox and Jurisich [6] described the universal central extension of the 3-point current algebra when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and, furthermore, constructed their explicit realizations by differential operators. In [4], Bremner constructed the universal central extension of the 4-point current algebra.

Date, Jimbo, Kashiwara and Miwa considered the universal central extension of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}, u]$ with $u^2 = (t^2 - b^2)(t^2 - c^2)$, $b \in \mathbb{C} \setminus \{-c, c\}$ in their study of Landau-Lifshitz equation [10] (the so-called *DJKM algebra*). This is an example of a Krichever-Novikov algebra with genus different from zero.

The universal central extension of the DJKM algebras was described in [7]. A realization of these algebras in terms of partial differential operators was constructed in [6] for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and in [8] for a general $\mathfrak{g}$. A study of the universal central extension of the DJKM algebras led to the discovery of new families of orthogonal polynomials in [9], which belong to the families of associated ultraspherical polynomials. These non-classical orthogonal polynomials satisfy differential equations of fourth order. Orthogonal polynomials play an important role in mathematics and physics, particularly in the description of wave phenomena and vibrating systems [18], the theory of random matrix ensembles [15] and other areas. It is curious to have such a connection between the universal central extensions of Krichever-Novikov algebras and associated ultraspherical polynomials. It would be interesting to know in general which orthogonal polynomials correspond to the universal central extensions of Krichever-Novikov algebras aiming to find new families of such polynomials satisfying higher-order differential equations.

Building upon the previous research on the universal central extension of the DJKM algebras, we extend the study to the superelliptic case. The equation of a superelliptic curve is $u^m = p(t)$, where $p(t)$ is a polynomial in t. The DJKM algebras correspond to the case $m = 2$ and $deg\, p(t) = 4$. We generalize the results of [9] for arbitrary superelliptic curve with $m > 2$ and $deg\, p(t) = 4$. Our analysis reveals the existence of two new families of orthogonal polynomials for each particular $m > 2$, which are the solutions of fourth-order differential equations as in the case the DJKM algebras. This is somewhat surprising as one may expect the order of differential equations to grow with m. One constructed family is identified as a family of associated ultraspherical polynomials. The other family is similar but has different initial conditions. As a result, it required an independent proof of the orthogonality.

As a byproduct, we show that certain superelliptic integrals are solutions of elliptic PDEs of fourth-order (cf. Corollaries 2.2 and 2.4). This is of interest on its own since the PDEs we consider are not completely integrable and do not fall into any other class of PDEs with explicit solutions. In the cases when such explicit solutions are known, they are of importance in different areas of mathematics [1] and theoretical physics [11].

1. Superelliptic affine Lie algebras

Let R be a ring of the form $\mathbb{C}[t, t^{-1}, u]$, where $u^m \in \mathbb{C}[t]$ and $m \geq 2$. Thus, R has a basis consisting of t^i, $t^i u$, $t^i u^2 \ldots, t^i u^{m-1}$ for $i \in \mathbb{Z}$. We will assume that $P(t) = \sum_{i=0}^{D} a_i t^i$, where $a_D = 1$ and a_0, a_1 are not both 0 and that $P(t)$ has no multiple roots. The equation $u^m = P(t)$ defines a superelliptic curve. Set R^i for $\mathbb{C}[t, t^{-1}]u^i$. Then $R = R^0 \oplus R^1 \oplus \cdots \oplus R^{m-1}$ is a $\mathbb{Z}/m\mathbb{Z}$-grading.

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Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra. The Lie algebra $\mathcal{G} = \mathfrak{g} \otimes R$ is an example of *superelliptic loop algebras*. The $\mathbb{Z}/m\mathbb{Z}$-grading induces the structure of a $\mathbb{Z}/m\mathbb{Z}$-graded Lie algebra on $\mathcal{G}$ by setting $\mathcal{G}^i = \mathfrak{g} \otimes R^i$ $(i = 0, 1, 2, \ldots, m-1)$.

Let $\hat{\mathcal{G}} = \mathcal{G} \oplus C$ be the universal central extension of $\mathcal{G}$, where C is the center of $\hat{\mathcal{G}}$. By [14], the center C is linearly isomorphic to Ω^1_R/dR, the space of Kähler differentials of R modulo the exact differentials. Our goal is to determine a basis for Ω^1_R/dR.

Let $F = R \otimes R$ be the left R-module with action $f(g \otimes h) = fg \otimes h$ for $f, g, h \in R$. Let K be the submodule generated by the elements $1 \otimes fg - f \otimes g - g \otimes f$. Then $\Omega^1_R = F/K$ is the module of Kähler differentials. We denote the element $f \otimes g + K$ of Ω^1_R by fdg. We define a map $d : R \to \Omega^1_R$ by $d(f) = df = 1 \otimes f + K$, and we denote the coset of fdg modulo dR by $\overline{fdg}$. We have

$$[x \otimes f, y \otimes g] = [xy] \otimes fg + (x, y)\overline{fdg}, \qquad [x \otimes f, \omega] = 0, \qquad [\omega, \omega'] = 0, \tag{1.1}$$

where $x, y \in \mathfrak{g}$, $f, g \in R$ and $\omega, \omega' \in \Omega^1_R/dR$; here, (x, y) denotes the Killing form on $\mathfrak{g}$.

The elements $t^i u^k \otimes t^j u^l$, with $i, j \in \mathbb{Z}$ and $k, l \in \{0, 1, \ldots, m-1\}$ form a basis of $R \otimes R$. The following result is fundamental to the description of the universal central extension for $R = \mathbb{C}[t, t^{-1}, u \mid u^m = P(t)]$.

Theorem 1.1 ([2], Theorem 2.4). *The following elements form a basis of Ω^1_R/dR:*

$$\overline{t^{-1}dt},\ \overline{t^{-1}u^l dt},\ \ldots,\ \overline{t^{-D}u^l dt},$$

where $l \in \{1, 2, \ldots, m-1\}$ and we omit $\overline{t^{-D}u^l dt}$ if $a_0 = 0$.

We have the following relation between the basis elements:

Proposition 1.2 ([7], Lemma 2.0.2). *If $u^m = P(t)$ and $R = [t, t^{-1}, u \mid u^m = P(t)\ ]$, then in Ω^1_R/dR, one has*

$$((m+1)n + im)t^{n+i-1}udt \equiv \sum_{j=0}^{n-1}((m+1)j + mi)a_j t^{i+j-1}udt \mod dR. \tag{1.2}$$

This allows to describe the universal central extension of the superelliptic algebra.

Let us define the sequence of polynomials in $D+3$ parameters $P_{m,n,l}(a_0, a_1, \ldots, a_{D-1}) := P_n$ for $n \geq -D$, $m, l \in \mathbb{Z}_+$ and $a_0, a_1, \ldots, a_{D-1} \in \mathbb{C}$ as follows:

$$(Dl + (1+n)m)P_n = \sum_{k=0}^{D-1}(-a_k(kl + (-D+1+n+k)m))P_{-D+n+k}, \tag{1.3}$$

with initial condition

$$P_{-D} = t^{-D}u^l dt,\ P_{-D+1} = t^{-D+1}u^l dt,\ \ldots,\ P_{-1} = t^{-1}u^l dt. \tag{1.4}$$

Then $P_n = t^n u^l dt$ for all $n \geq 0$.

If $a_0 \neq 0$, we define the sequence of polynomials in $D+3$ parameters $Q_{m,n,l}(a_0, a_1, \ldots, a_{D-1}) := Q_n$ for $n \leq -D-1$, $m, l \in \mathbb{Z}_+$ and $a_0, a_1, \ldots, a_{D-1} \in \mathbb{C}$ as follows:

$$(a_0(1+n)m)Q_n = \sum_{k=1}^{D}(-a_k(kl + (1+n+k)m))Q_{n+k} \tag{1.5}$$

with initial conditions

$$Q_{-D} = t^{-D}u^l dt,\ Q_{-D+1} = t^{-D+1}u^l dt,\ \ldots,\ Q_{-1} = t^{-1}u^l dt. \tag{1.6}$$

Then $Q_n = t^n u^l dt$ for $n \leq -D-1$.

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If $a_0 = 0$, we define the sequence of polynomials in $D+3$ parameters $R_{m,n,l}(a_0, a_1, \dots, a_{D-1}) := R_n$ for $n \le -D-1$, $m, l \in \mathbb{Z}_+$ and $a_0, a_1, \dots, a_{D-1} \in \mathbb{C}$ as follows:

$$a_1(l + (1+n+1)m)R_n = \sum_{k=2}^{D} (-a_k(kl + (1+n+k)m))R_{n+k} \tag{1.7}$$

with initial condition

$$R_{-D+1} = t^{-D+1}u^l dt,\ R_{-D+2} = t^{-D+2}u^l dt,\ \dots,\ R_{-1} = t^{-1}u^l dt. \tag{1.8}$$

We see that $R_n = t^n u^l dt$, $n \le -D-1$.

Set

$$\psi_{i,j} = \begin{cases} P_{i+j-1} \text{ if } i+j \ge -D+1; \\ Q_{i+j-1} \text{ if } i+j \le -D \text{ and } a_0 \ne 0; \\ R_{i+j-1} \text{ if } i+j \le -D \text{ and } a_0 = 0. \end{cases} \tag{1.9}$$

Then $t^i u^l d(t^j) = j\psi_{i,j}$ [[2], Proposition 3.4].

We have the following description of the Lie algebra structure on $\hat{\mathcal{G}}$. Set

$$\omega_0 = \overline{t^{-1}dt} \text{ and } \omega_{ij} = \overline{t^i u^j dt} \text{ for } j \ne 0. \tag{1.10}$$

Theorem 1.3 ([2], Theorem 3.5). *The superelliptic affine Lie algebra $\hat{\mathcal{G}}$ has a $\mathbb{Z}/m\mathbb{Z}$-grading in which*

$$\hat{\mathcal{G}}^0 = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\omega_0, \qquad \hat{\mathcal{G}}^l = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]u^l \bigoplus_{n=1}^{D} \mathbb{C}\omega_{-n,l}.$$

The subalgebra $\hat{\mathcal{G}}^0$ is an untwisted affine Kac-Moody Lie algebra with commutation relations

$$[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j} \oplus \delta_{i+j,0}(x, y) j\omega_0.$$

The commutation relations in $\hat{\mathcal{G}}$ are

$$[x \otimes t^i u^{l_1}, y \otimes t^j u^{l_2}] = [x, y] \otimes (t^{i+j}u^{l_1+l_2}) + (x, y)\left(\frac{jl_1 - il_2}{l_1 + l_2}\right)\omega_{i+j-1, l_1+l_2},$$

if $l_1 + l_2 \le m - 1$. When $l_1 + l_2 > m - 1$,

$$[x \otimes t^i u^{l_1}, y \otimes t^j u^{l_2}] = [x, y] \otimes \left(\sum_{k=0}^{D} a_k t^{i+j+k} u^{l_1+l_2-m}\right) + (x, y)\left(\frac{jl_1 - il_2}{l_1 + l_2}\right)\omega_{i+j-1, l_1+l_2}.$$

The subspace $\hat{\mathcal{G}}^l$ is a $\hat{\mathcal{G}}^0$-module with

$$[x \otimes t^i u^n, y \otimes t^j] = [x, y] \otimes (t^{i+j}u^{n+1}) + (x, y) j\psi_{i,j}. \tag{1.11}$$

In this paper, we will consider the superelliptic affine Lie algebras with the polynomial relation

$$u^m = 1 - 2ct^2 + t^4.$$

In this case, letting $k = i + 3$, the recursion relation (1.2) becomes

$$(4 + m + km)\overline{t^k \mathrm{udt}} = -(k-3)m\overline{t^{-4+k}\mathrm{udt}} + 2c(2 + (k-1)m)\overline{t^{-2+k}\mathrm{udt}}.$$

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Consider a family $P_k := P_k(c)$ of polynomials in c satisfying the recursion relation

$$(km + m + 4)P_k(c) = 2c((k-1)m + 2)P_{k-2}(c) - (k-3)mP_{k-4}(c) \tag{1.12}$$

for $k \geq 0$. Set

$$P(c,z) = \sum_{k=-4}^{\infty} z^{k+4} P_k(c) = \sum_{k=0}^{\infty} z^k P_{k-4}(c).$$

After a straightforward rearrangement of terms, we have

$$\begin{aligned}
0 &= \sum_{k=0}^{\infty}(km+m+4)z^k P_k(c) - (2c)\sum_{k=0}^{\infty}(km-m+2)z^k P_{k-2}(c) \\
&\quad + \sum_{k=0}^{\infty}(k-3)mz^k P_{k-4}(c) \\
&= ((4-3m)z^{-4} + (2c(-2+3m))z^{-2} - 3m)P(c,z) + m\left(z^{-3} - 2cz^{-1} + z\right)\frac{d}{\mathrm{d}z}P(c,z) \\
&\quad + \left(2c(2-3m)z^{-2} + (3m-4)z^{-4}\right)P_{-4}(c) + 2\left(-2c(m-1)z^{-1} + (m-2)z^{-3}\right)P_{-3}(c) \\
&\quad + (m-4)P_{-2}(c)z^{-2} - 4P_{-1}(c)z^{-1}.
\end{aligned}$$

Hence, $P(c,z)$ satisfies the differential equation

$$\begin{aligned}
&\frac{d}{\mathrm{d}z}P(c,z) + \left(\frac{6cmz^2 - 4cz^2 - 3m\left(z^4+1\right) + 4}{mz\left(-2cz^2 + z^4 + 1\right)}\right)P(c,z) \\
&= \frac{(4z^3)P_{-1} + ((-m+4)z^2)P_{-2} + (2\left(2c(m-1)z^2 - m + 2\right)z)P_{-3} + (\left(-2c(2-3m)z^2 - 3m + 4\right))P_{-4}}{m\left(-2cz^3 + z^4 + z\right)}.
\end{aligned} \tag{1.13}$$

It has an integrating factor

$$\begin{aligned}
\mu(c,z) &= \exp\left\{\int_{\cdot}^{z}\left(\frac{6cmw^2 - 4cw^2 - 3m\left(w^4+1\right) + 4}{mw\left(-2cw^2 + w^4 + 1\right)}\right)\mathrm{d}w\right\} \\
&= \frac{1}{z^{3-\frac{4}{m}}(-2cz^2 + z^4 + 1)^{1/m}}.
\end{aligned}$$

Now, we consider different cases depending on the initial conditions.

1.1. Superelliptic case 1

Assume that $P_{-3} = P_{-2} = P_{-1} = 0$ and $P_{-4} = 1$, and denote the generating function in this case by $P_{-4}(c,z)$. Then

$$P_{-4}(c,z) = \sum_{k=-4}^{\infty} P_{-4,k}(c)z^{k+4} = \sum_{k=0}^{\infty} P_{-4,k-4}(c)z^k$$

can be written in terms of a superelliptic integral

$$P_{-4}(c,z) = z^{3-4/m}(1-2cz^2+z^4)^{1/m} \times \lim_{\epsilon\to 0}\left[\int_\epsilon^z \frac{(4-3m+2c(-2+3m)w^2)}{mw^{4-4/m}(1-2cw^2+w^4)^{1+1/m}}\mathrm{d}w + \epsilon^{4/m-3} + 2c(3-2/m)\phi_m(\epsilon)\right], \quad (1.14)$$

where

$$\phi_m(\epsilon) = \begin{cases} \frac{\epsilon^{4/m-1}}{4/m-1} & m \neq 4 \\ \log\epsilon & m = 4. \end{cases}$$

Remark 1.15. The correction term inside of the limit $\epsilon \to 0$ in the formula (1.14) is necessary to ensure finiteness of the limit.

1.2. Superelliptic case 2

Suppose now that $P_{-3} = P_{-4} = P_{-1} = 0$ and $P_{-2} = 1$. Then we arrive to the generating function

$$P_{-2}(c,z) = \sum_{k=-4}^{\infty} P_{-2,k}(c)z^{k+4} = \sum_{k=0}^{\infty} P_{-2,k-4}(c)z^k,$$

which is defined in terms of a superelliptic integral

$$P_{-2}(c,z) = z^{3-4/m}(1-2cz^2+z^4)^{1/m}\lim_{\epsilon\to 0}\left[\int_\epsilon^z \frac{(4-m)}{mw^{2-4/m}(1-2cw^2+w^4)^{1+1/m}}\mathrm{d}w + \epsilon^{4/m-1}\right].$$

1.3. Gegenbauer case 3

If we take $P_{-1}(c) = 1$, and $P_{-2}(c) = P_{-3}(c) = P_{-4}(c) = 0$ and set $P_{-1}(c,z) = \sum_{n\geq 0} P_{-1,n-4}z^n$, then

$$P_{-1}(c,z) = \frac{\left(4z^{3-\frac{4}{m}}\left(-2cz^2+z^4+1\right)^{1/m}\right)}{m}\sum_{n=0}^{\infty}\frac{z^{\frac{4}{m}+2n}Q_n^{\left(1+\frac{1}{m}\right)}(c)}{\frac{4}{m}+2n},$$

where $Q_n^{\left(1+\frac{1}{m}\right)}(c)$ is the n-th Gegenbauer polynomial.

1.4. Gegenbauer case 4

If we take $P_{-3}(c) = 1$, and $P_{-1}(c) = P_{-2}(c) = P_{-4}(c) = 0$ and set $P_{-3}(c,z) = \sum_{n\geq 0} P_{-1,n-4}z^n$, then

$$P_{-3}(c,z) = \frac{\left(z^{3-\frac{4}{m}}\left(-2cz^2+z^4+1\right)^{1/m}\right)}{2}\sum_{n=0}^{\infty} z^{\frac{4}{m}+2n}Q_n^{\left(1+\frac{1}{m}\right)}(c)\left(\frac{m-2}{z^2(m(n-1)+2)} - \frac{2c(m-1)}{mn+2}\right).$$

We will see in the last section that the family of polynomials described in the superelliptic case 1 is an example of associated ultraspherical polynomials.

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2. Orthogonal polynomials in superelliptic cases

2.1. Superelliptic type 1

Let us reindex the polynomials $P_{-4,n}$ as follows:

$$\begin{aligned}
P_{-4}(c,z) &= z^{3-4/m}(1-2cz^2+z^4)^{1/m} \\
&\quad\times \lim_{\epsilon\to 0}\left[\int_\epsilon^z \frac{(4-3m+2c(-2+3m)w^2)}{mw^{4-4/m}(1-2cw^2+w^4)^{1+1/m}}\mathrm{d}w+\epsilon^{4/m-3}+2c(3-2/m)\phi_m(\epsilon)\right] \\
&= \sum_{k=0}^{\infty} P_{-4,k-4}(c)z^k \\
&= 1+\frac{3m}{m+4}z^4+\frac{6cm(m+2)}{(m+4)(3m+4)}z^6+\frac{3m\big(4c^2(m+2)(3m+2)-m(3m+4)\big)}{(m+4)(3m+4)(5m+4)}z^8 \\
&\quad+\frac{12cm(3m+2)\big(2c^2(m+2)(5m+2)-m(5m+8)\big)}{(m+4)(3m+4)(5m+4)(7m+4)}z^{10} \\
&\quad+3m\Bigg(\frac{(16c^4(m+2)(3m+2)(5m+2)(7m+2))}{(m+4)(3m+4)(5m+4)(7m+4)(9m+4)} \\
&\quad+\frac{-12c^2m(3m+2)(5m+2)(7m+12)+5m^2(3m+4)(7m+4)}{(m+4)(3m+4)(5m+4)(7m+4)(9m+4)}\Bigg)z^{12}+O(z^{14}).
\end{aligned}$$

The first nonzero polynomials are

$$\begin{aligned}
P_{-4,0}(c) &= 1, \\
P_{-4,4}(c) &= \frac{3m}{m+4}, \\
P_{-4,6}(c) &= \frac{6cm(m+2)}{(m+4)(3m+4)}, \\
P_{-4,8}(c) &= \frac{3m\big(4c^2(m+2)(3m+2)-m(3m+4)\big)}{(m+4)(3m+4)(5m+4)}, \\
P_{-4,10}(c) &= \frac{12cm(3m+2)\big(2c^2(m+2)(5m+2)-m(5m+8)\big)}{(m+4)(3m+4)(5m+4)(7m+4)},
\end{aligned}$$

and $P_n(c) = P_{-4,n}(c)$ satisfies the recursion

$$0 = (2c((k-1)m+2))P_{k+2}(c)-(km+m+4)P_{k+4}(c)-((k-3)m)P_k(c). \tag{2.1}$$

Our main result is the following theorem which generalizes [[9], Theorem 3.1.1].

Theorem 2.1. *The polynomials $P_n = P_{-4,n}$ satisfy the following fourth-order linear differential equation:*

$$\begin{aligned}
0 &= 16\left(c^2-1\right)^2 m^2 P_n^{(iv)}+160c\left(c^2-1\right)m^2P_n''' \\
&\quad+(8m(m((n-6)n-10)+4(n-6)) \\
&\quad-8c^2\left(m^2(n-10)(n+4)+4m(n-4)+8\right)\Big)P_n'' \\
&\quad-24c\left(m^2(n-6)n+4m(n-4)+8\right)P_n' \\
&\quad+(n-4)n(m(n-6)+4)(m(n-2)+4)P_n.
\end{aligned} \tag{2.2}$$

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Proof. See Appendix. □

Denote $v := \ln z$ and, consequently, $\frac{\partial}{\partial v} = z\frac{\partial}{\partial z}$.

Corollary 2.2. *The generating function $P_{-4} = P_{-4}(c, z)$ satisfies the following linear PDE of fourth order:*

$$0 = m^2\left[4(1-c^2)\frac{\partial^2}{\partial c^2} - 12c\frac{\partial}{\partial c} + \frac{\partial^2}{\partial v^2} + 2(\frac{2}{m} - 3)\frac{\partial}{\partial v} + 4(1 - \frac{2}{m})\right]^2 P_{-4} + 16(m^2 - 8m)\frac{\partial^2 P_{-4}}{\partial c^2} - 16(2-m)^2(c\frac{\partial}{\partial c} + 1)^2 P_{-4}, \tag{2.3}$$

with $(c, z) \in (-1, 1) \times (0, 1)$ and with the following boundary conditions:

$$P_{-4}(c, 0) = 1, \frac{\partial P_{-4}}{\partial z}(c, 0) = \frac{\partial^2 P_{-4}}{\partial z^2}(c, 0) = \frac{\partial^3 P_{-4}}{\partial z^3}(c, 0) = 0. \tag{2.4}$$

Proof. It follows from (1.14) that P_{-4} is an analytic function represented by the series $P_{-4}(c, z) = \sum_{n=0}^{\infty} P_{-4,n}(c)z^n$ (in some ball of small radius). Hence, by Theorem 2.1 and some elementary algebraic calculations, P_{-4} satisfies the linear PDE (2.3) in the ball. By the uniqueness of analytic continuation, it satisfies the PDE (2.3) in the domain $(-1, 1) \times (0, 1)$. The boundary conditions (2.4) immediately follow from the definition of P_{-4}. □

Remark 2.5. Boundary conditions for P_{-4} on other parts of the boundary of the domain $[-1, 1] \times [0, 1]$ can be explicitly calculated using formula (1.14).

2.2. Superelliptic type 2

Now we consider the second family of polynomials $P_{-2,n}$. We have

$$\begin{aligned} P_{-2}(c, z) &= \sum_{k=0}^{\infty} P_{-2,k-4}(c)z^k \\ &= z^{3-\frac{4}{m}}\left(-2cz^2 + z^4 + 1\right)^{1/m} \lim_{\epsilon \to 0}\left[\int_{\epsilon}^{z} \frac{(4-m)w^{\frac{4}{m}-2}}{m\left(-2cw^2 + w^4 + 1\right)^{\frac{1}{m}+1}}\,dw + \epsilon^{4/m-1}\right] \\ &= z^2 - \frac{2c(m-2)}{m+4}z^4 + \frac{z^6\left(m(m+4) - 4c^2(m^2-4)\right)}{(m+4)(3m+4)} \\ &\quad - \frac{4z^8\left(c(m+2)\left(2c^2(m-2)(3m+2) - 3m^2\right)\right)}{(m+4)(3m+4)(5m+4)} + \mathrm{O}(z^9), \end{aligned}$$

and the polynomials $P_{-2,n}(c)$ satisfy the following recursion:

$$0 = 2c(km - m + 2)P_{-2,k+2}(c) - (km + m + 4)P_{-2,k+4}(c) - (km - 3m)P_{-2,k}(c). \tag{2.6}$$

In particular, the first nonzero polynomials of the sequence are

$$P_{-2,2}(c) = 1,$$
$$P_{-2,4}(c) = -\frac{2c(m-2)}{m+4},$$

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$$P_{-2,6}(c) = \frac{m(m+4) - 4c^2\left(m^2-4\right)}{(m+4)(3m+4)},$$
$$P_{-2,8}(c) = -\frac{4c(m+2)\left(2c^2(m-2)(3m+2)-3m^2\right)}{(m+4)(3m+4)(5m+4)}.$$

Theorem 2.3. *The polynomials $P_n = P_{-2,n}(c)$ satisfy the following fourth-order linear differential equation:*

$$\begin{aligned} 0 = {} & 16\left(c^2-1\right)^2 m^2 {P_n}^{(iv)} + 160c\left(c^2-1\right)m^2 P_n''' \\ & + \Big[8m(m(n^2-6n-2)+4(n-8)) \\ & - 8c^2\Big(m^2((n-6)n-32)+4m(n-6)+8\Big)\Big]P_n'' \\ & - 24c\Big(m^2(n-4)(n-2)+4m(n-6)+8\Big)P_n' \\ & + (n^2-4)(mn-8m+4)(mn-4m+4)P_n. \end{aligned} \tag{2.7}$$

Corollary 2.4. *Generating function $P_{-2} = P_{-2}(c,z)$ satisfies the following linear PDE of fourth order:*

$$\begin{aligned} 0 = {} & m^2\left[4(1-c^2)\frac{\partial^2}{\partial c^2} - 12c\frac{\partial}{\partial c} + \frac{\partial^2}{\partial v^2} + 2(\frac{2}{m}-3)\frac{\partial}{\partial v} - 4\right]^2 P_{-2} \\ & + 144m^2\frac{\partial^2 P_{-2}}{\partial c^2} - 16(3m-2)^2(c\frac{\partial}{\partial c}+1)^2 P_{-2}, \end{aligned} \tag{2.8}$$

(where $v = \ln z$, $\frac{\partial}{\partial v} = z\frac{\partial}{\partial z}$) with $(c,z) \in (-1,1)\times(0,1)$ and with the following boundary conditions:

$$\frac{\partial^2 P_{-2}}{\partial z^2}(c,0) = 1, P_{-2}(c,0) = \frac{\partial P_{-2}}{\partial z}(c,0) = \frac{\partial^3 P_{-2}}{\partial z^3}(c,0) = 0. \tag{2.9}$$

Proof. The proof is analogous to the proof of Corollary 2.2. □

Remark 2.10. The linear PDEs (2.3) and (2.8), as long as authors aware, does not seem to fall in any known category of PDEs with explicit solutions.

3. Uniqueness of solutions

Let us check now if the differential equations (2.2) and (2.7) have other polynomial solutions. The argument below was suggested by A. Shyndyapin.

Clearly, if $\{P_n\}$ is a family of polynomials satisfying (2.2) (respectively, (2.7)), then for any choice of nonzero scalars $\lambda_n \in \mathbb{C}$, the family $\{\lambda_n P_n\}$ also satisfies (2.2) (respectively, (2.7)). We will call such families of polynomials equivalent. Also, we will ignore the families of polynomials obtained by zeroing some members of the family.

Consider first the equation (2.2) and assume that

$$Q_n = \sum_{i=0}^{r(n)} \alpha_i(n)c^i$$

is its polynomial solution. We will simply write r for $r(n)$ and α_i for $\alpha_i(n)$. After the substitution in (2.2), we obtain

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$$\sum_{i=0}^{r}((2i-n+4)(2i+n)(m(2i-n+6)-4)(m(2i+n-2)+4))\alpha_i c^i \\ +(-8(i-1)im\Big(m\Big(4i^2-(n-6)n-6\Big)-4(n-6)\Big))\alpha_i c^{i-2} \\ +(16(i-3)(i-2)(i-1)im^2)\alpha_i c^{i-4}=0,$$

which is equivalent to the following system of linear equations:

$$\begin{aligned}&[(2i-n+4)(2i+n)(m(2i-n+6)-4)(m(2i+n-2)+4)]\alpha_i\\&\quad+[-8(i+1)(i+2)m(m(4i(i+4)-(n-6)n+10)-4(n-6))]\alpha_{i+2}\\&\quad+[16(i+1)(i+2)(i+3)(i+4)m^2]]\alpha_{i+4}=0,\quad i=0,1,\dots,r.\end{aligned} \tag{3.1}$$

Here α_s is assumed to be zero if $s>r$. The coefficient matrix of the system is upper triangular with

$$(2i-n+4)(2i+n)(m(2i-n+6)-4)(m(2i+n-2)+4),\ \ i=0,\dots,r$$

on the diagonal. Hence, for each n, the solution of the system is unique and trivial if $(2i-n+4)(m(2i-n+6)-4)\neq 0$ for all $i=0,1,\dots,r$. Suppose $(2i-n+4)(m(2i-n+6)-4)=0$. First consider the case $m>4$. Then we must have $2i-n+4=0$. From (3.1), we get that $Q_{2k+1}=0$ for all $k>0$, and Q_{2k} is defined uniquely up to a scalar for each $k>1$. The degree of Q_{2k} is $\geq k-2$. Now, if $m=4$, then either $i=\frac{n}{2}-2$ or $i=\frac{n+1}{2}-3$. Therefore, Q_{2k} is defined uniquely up to a scalar as a polynomial of degree $\geq k-2$, $Q_1=Q_3=0$ and Q_{2k+1} is defined uniquely up to a scalar as a polynomial of degree $\geq k-2$ for each $k\geq 2$.

Consider now the equation (2.7). Arguing as above, we come a linear system on the coefficients of the polynomial solution Q_n. The matrix of this system is again upper triangular with diagonal entries

$$\Big(4(i+1)^2-n^2\Big)(m(2i-n+8)-4)(m(2i+n-4)+4),\ \ i=0,\dots,r.$$

Assume that $\left(4(i+1)^2-n^2\right)(m(2i-n+8)-4)(m(2i+n-4)+4)=0$. If $m>4$, then $Q_{2k+1}=0$ for all $k>0$, and Q_{2k} is defined uniquely up to a scalar for each $k>1$, while $Q_2=0$. The degree of Q_{2k} is $\geq k-2$. Suppose now that $m=4$ and $n>1$. Then either $i=\frac{n}{2}-1$ or $i=\frac{n+1}{2}-3$. Therefore, Q_{2k} is defined uniquely up to a scalar as a polynomial of degree $\geq k-2$, $Q_3=Q_5=0$, and Q_{2k+1} is defined uniquely up to a scalar as a polynomial of degree $\geq k-2$ for each $k\geq 3$. When $m=4$, $n=1$ and $i=1$, we also get 0. Hence, Q_1 is defined uniquely up to a scalar.

We proved the following theorem.

Theorem 3.1. *For any $m\geq 4$, the differential equations (2.2) and (2.7) have unique polynomial solutions up to equivalence.*

4. Associated ultraspherical polynomials

After shifting the indices back by 4, we obtain that both families of the polynomials $P_{-4,n}(c)$ and $P_{-2,n}(c)$ satisfy the recurrence relation

$$(km+m+4)P_k(c)=2c(km-m+2)P_{k-2}(c)-(k-3)mP_{k-4}(c), \tag{4.1}$$

where $P_k(c)\in\{P_{-4,k-4}(c),P_{-2,k-4}(c)\}$. Note that all odd polynomials are zero. Set $k=2(n+1)$ and $q_s=q_s(c):=P_{2s}(c)$. Then we have

$$2c(2mn+m+2)q_n=(2mn+3m+4)q_{n+1}+m(2n-1)q_{n-1}. \tag{4.2}$$

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For example, we have: $q_0 = 1, q_1 = \frac{6cm(m+2)}{(m+4)(3m+4)}$,

$$q_2 = \frac{3m(3m+2)\left(4c^2(m+2) - 2m\right)}{(m+4)(3m+4)(5m+4)}, \quad q_3 = \frac{12cm(3m+2)(5m+2)\left(2c^2(m+2) - 6m\right)}{(m+4)(3m+4)(5m+4)(7m+4)}.$$

We will show that $\{q_n\}$ is a special case of the associated ultraspherical polynomials.

Let c and ν be two complex numbers. Let $C_n^{(\nu)}(x;c)$ denote the family of *associated ultraspherical polynomials* with initial conditions $C_{-1}^{(\nu)}(x;c) = 0$ and $C_0^{(\nu)}(x;c) = 1$ (cf. [5], p. 729). Then they satisfy the following difference equation:

$$2x(n+\nu+c)C_n^{(\nu)}(x;c) = (n+c+1)C_{n+1}^{(\nu)}(x;c) + (2\nu+n+c-1)C_{n-1}^{(\nu)}(x;c). \tag{4.3}$$

Choosing $c = \frac{1}{2}+\frac{2}{m}$ and $\nu = -\frac{1}{m}$, this equation becomes the recurrence relation (4.2). Hence, we conclude that polynomials $P_{-4,n}$ is a special case of associated ultraspherical polynomials. We immediately have the following:

Corollary 4.1. *Polynomials $P_{-4,n}$ are non-classical orthogonal polynomials.*

5. Orthogonality of $P_{-2,n}$

Polynomials $P_{-2,n}$ satisfy the same recurrence relation as polynomials $P_{-4,n}$ but have different initial conditions. Hence, we do not know immediately whether they are associated ultraspherical polynomials and whether they are orthogonal. We will give an independent proof of the orthogonality of these polynomials. Set $q_s(c) = P_{-2,2s}(c)$. In particular, we have

$$q_1 = 1 \quad q_2 = -\frac{2c(m-2)}{m+4}, \quad q_3 = \frac{m(m+4) - 4c^2\left(m^2-4\right)}{(m+4)(3m+4)}.$$

Now we set $\overline{q}_n := q_{n+1}$, $n \geq -1$. Then polynomials $\overline{q}_n$ have degree n, and (4.2) becomes

$$2(2mn+3m+2)c\overline{q}_n = (2mn+m)\overline{q}_{n-1} + (2mn+5m+4)\overline{q}_{n+1}. \tag{5.1}$$

We recall the following result.

Theorem 5.1 ([9], Theorem 5.0.4). *Let $\{p_n, n \geq 0\}$ be a sequence of polynomials, where p_n has degree n for any n, satisfying the following recursion*

$$\chi p_n = a_{n+1}p_{n+1} + b_n p_n + c_{n-1}p_{n-1}, \qquad n = 0, 1, 2, \ldots \tag{5.2}$$

for some complex numbers a_{n+1}, b_n, c_{n-1}, $n = 0, 1, 2, \ldots$ and $p_{-1} = 0$. Then $\{p_n, n \geq 0\}$ is a sequence of orthogonal polynomials with respect to some (unique) weight measure if and only if $b_n \in \mathbb{R}$ and $c_n = \overline{a}_n + 1 \neq 0$ for all $n \geq 0$.

We apply the theorem in the case when

$$a_{n+1} = \frac{(2mn+5m+4)}{2(2mn+3m+2)}, \quad b_n = 0, \quad c_{n-1} = \frac{(2mn+m)}{2(2mn+3m+2)}.$$

Theorem 5.2. *The polynomials $P_{-2,n}(c)$ are orthogonal with respect to some weight function.*

Proof. It is sufficient to check that there exists a family of orthonormal polynomials f_n and constants λ_n such that $q_n = \lambda_n f_n$ for all n.

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We have the following recursion for f_n:

$$f_n = \frac{(2mn+m)\lambda_{n-1}}{2(2mn+3m+2)c\lambda_n} f_{n-1} + \frac{(2mn+5m+4)\lambda_{n+1}}{2(2mn+3m+2)c\lambda_n} f_{n+1},$$

for $n \geq 1$.

Set

$$A_{n+1} = \frac{(2mn+5m+4)\lambda_{n+1}}{2(2mn+3m+2)\lambda_n} \qquad C_{n-1} = \frac{(2mn+m)\lambda_{n-1}}{2(2mn+3m+2)\lambda_n}.$$

Then $A_n = C_{n-1}$ if and only if

$$\lambda_n^2 = \frac{(2mn+m+2)(2mn+m)}{(2mn+3m+4)(2mn+3m+2)} \lambda_{n-1}^2.$$

Taking $\lambda_0 = 1$, we can find a family of constants λ_n satisfying this relation. □

Appendix

Proof of Theorem 2.1. The proof generalizes the proof of Theorem 3.1.1 in [9] in the case $m = 2$. We start off with the generating function

$$\begin{aligned} P_{-4}(c,z) &= z^{3-4/m}(1-2cz^2+z^4)^{1/m} \\ &\quad \times \lim_{\epsilon\to 0}\left[\int_\epsilon^z \frac{4c\left(w^2\left(\left(\frac{3m}{2}-1\right)w^{\frac{4}{m}-2}\right)\right)-(3m-4)w^{\frac{4}{m}-2}}{mw^2\left(-2cw^2+w^4+1\right)^{\frac{1}{m}+1}}\mathrm{d}w + \epsilon^{4/m-3} + 2c(3-2/m)\phi_m(\epsilon)\right] \\ &= z^{3-4/m}(1-2cz^2+z^4)^{1/m}\left(-\sum_{n=0}^{\infty}\frac{z^{\frac{4}{m}+2n-3}\left((3m-4)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(2n-3)+4}\right. \\ &\quad \left. + \sum_{n=0}^{\infty}\frac{z^{\frac{4}{m}+2n-1}\left(c(6m-4)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(2n-1)+4}\right), \end{aligned}$$

where $Q_n^\lambda(c)$ is the nth-Gegenbauer polynomial. The polynomials $Q_n^\lambda(c)$ satisfy the second-order linear differential equation

$$\left(1-c^2\right)y'' - c(2\lambda+1)y' + ny(2\lambda+n) = 0,$$

where the derivatives are with respect to c. Thus, for $\lambda = 1 + \frac{1}{m}$, we get

$$\left(1-c^2\right)(Q_n^{\left(1+\frac{1}{m}\right)})''(c) - c\left(\frac{2}{m}+3\right)(Q_n^{\left(1+\frac{1}{m}\right)})'(c) + n\left(\frac{2}{m}+n+2\right)Q_n^{\left(1+\frac{1}{m}\right)}(c) = 0. \qquad (5.3)$$

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Rewriting the expansion formula for $P_{-4}(c,z)$, we get

$$z^{-3+4/m}(1-2cz^2+z^4)^{-1/m}P_{-4}(c,z)=\sum_{n=0}^{\infty}\frac{z^{\frac{4}{m}+2n-1}\left(c(6m-4)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(2n-1)+4} -\sum_{n=0}^{\infty}\frac{z^{\frac{4}{m}+2n-3}\left((3m-4)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(2n-3)+4}. \tag{5.4}$$

Now we apply the differential operator $L:=(1-c^2)\frac{d^2}{dc^2}-(c(3+\frac{2}{m}))\frac{d}{dc}$ to the right-hand side and use the identity (formula 4.7.27 in [20])

$$\left(1-c^2\right)\frac{\mathrm{d}c}{\mathrm{d}Q}{}_n^{(\lambda)}(c)=c(2\lambda+n)Q_n^{(\lambda)}(c)-(n+1)Q_{n+1}^{(\lambda)}(c). \tag{5.5}$$

Then we get

$$\begin{aligned}L\left(c(-\frac{4}{m}+6)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)&=\left((1-c^2)\frac{d^2}{dc^2}-(c(3+2/m))\frac{d}{dc}\right)\left(c(-4/m+6)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)\\&=-\frac{2(3m-2)}{m}\left(2m(n+1)Q_{n+1}^{\left(1+\frac{1}{m}\right)}(c)\right.\\&\quad\left.+\,c(n-1)(mn+m+2)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right).\end{aligned}$$

From here, we deduce

$$\begin{aligned}&L\left(\sum_{n=0}^{\infty}\frac{z^{\frac{4}{m}+2n-1}\left(c(6m-4)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(2n-1)+4}\right)\\&=\sum_{n=0}^{\infty}-\frac{z^{\frac{4}{m}+2n-1}\left((2(3m-2))\left(2m(n+1)c_{n+1}^{\left(1+\frac{1}{m}\right)}(c)+c(n-1)(mn+m+2)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)\right)}{m(m(2n-1)+4)}\\&=\sum_{n=0}^{\infty}-\frac{z^{\frac{4}{m}+2n-1}\left((2c(3m-2)(n-1)(mn+m+2))Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(m(2n-1)+4)}\\&\quad+\sum_{n=0}^{\infty}-\frac{z^{\frac{4}{m}+2n-1}\left((4(3m-2)(n+1))c_{n+1}^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(2n-1)+4}.\end{aligned}$$

Since

$$\sum_{n=0}^{\infty}-\frac{z^{\frac{4}{m}+2n-1}\left((2c(3m-2)(n-1)(mn+m+2))Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(m(2n-1)+4)}$$

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$$= \frac{6c(3m-2)(m+4)\int_0^z w^{\frac{4}{m}-2}\left(-2cw^2+w^4+1\right)^{-\frac{1}{m}-1}dw}{4m^2}$$
$$-\frac{c(3m-2)z^{\frac{4}{m}-1}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}}{2m}-\frac{\left(2c(3m-2)z^{4/m}\right)\frac{\mathrm{d}}{\mathrm{d}z}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}}{4m},$$

and

$$\sum_{n=0}^{\infty}-\frac{z^{\frac{4}{m}+2n-1}\left((4(3m-2)(n+1))Q_{n+1}^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(2n-1)+4}$$
$$=-\frac{(2(3m-4)(3m-2))\int_0^z w^{\frac{4}{m}-4}\left(\left(-2cw^2+w^4+1\right)^{-\frac{1}{m}-1}-1\right)dw}{m^2}$$
$$-\frac{(4(3m-2))\left(z^{\frac{4}{m}-3}\left(\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}-1\right)\right)}{2m},$$

we get

$$L\left(\sum_{n=0}^{\infty}\frac{z^{\frac{4}{m}+2n-1}\left(c(6m-4)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(2n-1)+4}\right)$$
$$=-\frac{c(3m-2)z^{\frac{4}{m}-1}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}}{2m}$$
$$-\frac{\left((2c)(3m-2)z^{4/m}\right)\frac{\mathrm{d}}{\mathrm{d}z}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}}{4m}$$
$$+\frac{((2c)(3m-2)(3(m+4)))\int_0^z w^{\frac{4}{m}-2}\left(-2cw^2+w^4+1\right)^{-\frac{1}{m}-1}dw}{4m^2}$$
$$-\frac{(2(3m-4)(3m-2))\int_0^z w^{\frac{4}{m}-4}\left(-2cw^2+w^4+1\right)^{-\frac{1}{m}-1}dw}{m^2}$$
$$+\frac{(4-6m)z^{\frac{4}{m}-3}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}}{m}.$$

In addition, we have

$$L\left(\sum_{n=0}^{\infty}\frac{z^{\frac{4}{m}+2n-3}\left((3m-4)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(2n-3)+4}\right)$$
$$=-\sum_{n=0}^{\infty}\frac{z^{\frac{4}{m}+2n-3}\left((3m-4)\left(n\left(\frac{2}{m}+n+2\right)\right)Q_n^{\left(1+\frac{1}{m}\right)}(c)\right)}{m(2n-3)+4}$$
$$-\sum_{n=0}^{\infty}\left(\frac{7\left(9m^2-24m+16\right)}{4m(2mn-3m+4)}+\frac{(3m-4)n}{2m}+\frac{7(3m-4)}{4m}\right)z^{\frac{4}{m}+2n-3}$$

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$$- \frac{7(3m-4)z^{\frac{4}{m}-3}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}}{4m} - \frac{(3m-4)z^{\frac{4}{m}-2}\frac{\mathrm{d}}{\mathrm{d}z}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}}{4m}$$
$$- \frac{\left(7\left(9m^2-24m+16\right)\right)\int_0^z w^{\frac{4}{m}-4}\left(-2cw^2+w^4+1\right)^{-\frac{1}{m}-1}dw}{4m^2}.$$

Applying the operator L to the left-hand side of (5.4), we get

$$L\left(z^{\frac{4}{m}-3}\left(-2cz^2+z^4+1\right)^{-1/m}P_{-4}(c,z)\right)$$
$$= \frac{z^{\frac{4}{m}-1}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-2}\left(4c^2(2m+1)z^2-2c(3m+2)\left(z^4+1\right)+4(m+1)z^2\right)P_{-4}(c,z)}{m^2}$$
$$+ \frac{z^{\frac{4}{m}-3}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}\left(6c^2mz^2-c(3m+2)\left(z^4+1\right)+4z^2\right)P'_{-4}(c,z)}{m}$$
$$- \left(c^2-1\right)z^{\frac{4}{m}-3}\left(-2cz^2+z^4+1\right)^{-1/m}P''_{n-4}(c,z).$$

Hence, we obtain

$$- \frac{3(m+4)z^{\frac{4}{m}-3}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}}{4m} - \frac{c(3m-2)z^{\frac{4}{m}-1}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}}{2m}$$
$$+ \frac{(m+1)(3m-4)\left(c-z^2\right)z^{\frac{4}{m}-1}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-2}}{m^2}$$
$$- \frac{2c(m+1)(3m-2)\left(c-z^2\right)z^{\frac{m+4}{m}}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-2}}{m^2}$$
$$+ \frac{3(m+4)}{4m}\left(z^{\frac{4}{m}-3}\left(-2cz^2+z^4+1\right)^{-1/m}P_{-4}(c,z)\right)$$
$$= \frac{z^{\frac{4}{m}-1}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-2}\left(4c^2(2m+1)z^2-2c(3m+2)\left(z^4+1\right)+4(m+1)z^2\right)P_{-4}(c,z)}{m^2}$$
$$+ \frac{z^{\frac{4}{m}-3}\left(-2cz^2+z^4+1\right)^{-\frac{1}{m}-1}\left(6c^2mz^2-c(3m+2)\left(z^4+1\right)+4z^2\right)P'_{-4}(c,z)}{m}$$
$$- \left(c^2-1\right)z^{\frac{4}{m}-3}\left(-2cz^2+z^4+1\right)^{-1/m}P''_{n-4}(c,z),$$

which gives us

$$\frac{1}{4}\left(-z^4\left(4c^2(m+2)(3m-2)+(3m+4)(5m-4)\right)+2c\left(9m^2+6m-8\right)z^6\right.$$
$$\left.+4c(3m(m+2)-4)z^2-3m(m+4)\right)$$
$$= -\left(m\left(4c^2z^4-6c\left(z^6+z^2\right)+3z^8+2z^4+3\right)+4z^2\left(-\left(c^2+1\right)z^2+cz^4+c\right)\right.$$
$$\left.+\frac{3}{4}m^2\left(-2cz^2+z^4+1\right)^2\right)P_{-4}(c,z)$$
$$+m\left(-2cz^2+z^4+1\right)\left(6c^2mz^2-c(3m+2)\left(z^4+1\right)+4z^2\right)P'_{-4}(c,z)$$
$$-\left(c^2-1\right)m^2\left(-2cz^2+z^4+1\right)^2P''_{-4}(c,z).$$

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Expanding this out and writing $P_{-4,k}(c)$ as P_k, we will have the following equality:

$$\begin{aligned}0 &= \left(c^2(-(m+2))(3m-2) - \frac{1}{2}m(3m+4) + 4\right)P_{n-4} + (c(3m(m+2)-4))P_{n-2} \\ &\quad - \frac{1}{4}(3m(m+4))P_n + \frac{1}{4}(-3m(m+4))P_{n-8} + (c(3m(m+2)-4))P_{n-6} \\ &\quad - (cm(3m+2))P'_n + \left(4\left(c^2m(3m+1)+m\right)\right)P'_{n-2} \\ &\quad - \left(6cm\left(2c^2m+m+2\right)\right)P'_{n-4} + \left(4\left(c^2m(3m+1)+m\right)\right)P'_{n-6} \\ &\quad + (-cm(3m+2))P'_{n-8} - \left(\left(c^2-1\right)m^2\right)P''_n + \left(4c\left(c^2-1\right)m^2\right)P''_{n-2} \\ &\quad - \left(\left(c^2-1\right)m^2\right)P''_{n-8} + \left(4c\left(c^2-1\right)m^2\right)P''_{n-6} \\ &\quad + \left(2\left(-2c^4+c^2+1\right)m^2\right)P''_{n-4}. \end{aligned} \tag{5.6}$$

We now differentiate twice with respect to c the recursion 2.1:

$$\begin{aligned}0 &= 2((k-1)m+2)P_{k+2} + 2c((k-1)m+2)P'_{k+2} \\ &\quad - (k-3)mP'_k - (km+m+4)P'_{k+4} \end{aligned} \tag{5.7}$$

$$\begin{aligned}0 &= (4(k-1)m+8)P'_{k+2} + 2c((k-1)m+2)P''_{k+2} \\ &\quad - (km+m+4)P''_{k+4} - (k-3)mP''_k. \end{aligned} \tag{5.8}$$

Substituting $k = n-8$ in the last equation, we obtain

$$\begin{aligned}0 &= 4(m(n-9)+2)P'_{n-6} - m(n-11)P''_{n-8} \\ &\quad + 2c(m(n-9)+2)P''_{n-6} - (m(n-7)+4)P''_{n-4}. \end{aligned} \tag{5.9}$$

Multiplying (5.6) by $-11+n$ and adding it to the above equation multiplied by $(1-c^2)m$ gives us

$$\begin{aligned}0 &= 4c\left(c^2-1\right)m^2(n-11)P''_{n-2} - \left(c^2-1\right)m^2(n-11)P''_n \\ &\quad + 4m\left(c^2(2m(n-12)+n-13)+(m+1)(n-9)\right)P'_{n-6} \\ &\quad - 6cm(n-11)\left(2c^2m+m+2\right)P'_{n-4} \\ &\quad + 2c\left(c^2-1\right)m(m(n-13)-2)P''_{n-6} \\ &\quad - \left(c^2-1\right)m\left(m\left(4c^2(n-11)+n-15\right)-4\right)P''_{n-4} \\ &\quad + (n-11)\left(c^2(-(m+2))(3m-2) - \frac{1}{2}m(3m+4)+4\right)P_{n-4} \\ &\quad + 4(n-11)\left(c^2m(3m+1)+m\right)P'_{n-2} - \frac{3}{4}(m+4)m(n-11)P_{n-8} \\ &\quad - \frac{3}{4}(m+4)m(n-11)P_n - c(3m+2)m(n-11)P'_{n-8} \\ &\quad - c(3m+2)m(n-11)P'_n + c(3m(m+2)-4)(n-11)P_{n-6} \\ &\quad + c(3m(m+2)-4)(n-11)P_{n-2}. \end{aligned} \tag{5.10}$$

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Now substitute $k = n - 8$ in (5.7) to get

$$0 = 2(m(n-9)+2)P_{n-6} - m(n-11)P'_{n-8} + 2c(m(n-9)+2)P'_{n-6} - (m(n-7)+4)P'_{n-4}. \tag{5.11}$$

Multiplying this equation by $-c(2+3m)$ and adding it to the previous equation, we obtain

$$\begin{aligned}
0 = {} & 4c\left(c^2-1\right)m^2(n-11)P''_{n-2} - \left(c^2-1\right)m^2(n-11)P''_n \\
& + 2c\left(c^2-1\right)m(m(n-13)-2)P''_{n-6} \\
& + \left(1-c^2\right)m\left(m\left(4c^2(n-11)+n-15\right)-4\right)P''_{n-4} \\
& + (n-11)\left(c^2(-(m+2))(3m-2) - \frac{1}{2}m(3m+4)+4\right)P_{n-4} \\
& + \left(2c^2(m(m(n-21)-14)-4)+4m(m+1)(n-9)\right)P'_{n-6} \\
& + \left(c(3m+2)(m(n-7)+4) - 6cm(n-11)\left(2c^2m+m+2\right)\right)P'_{n-4} \\
& + 4(n-11)\left(c^2m(3m+1)+m\right)P'_{n-2} \\
& - \frac{3}{4}(m+4)m(n-11)P_{n-8} - \frac{3}{4}(m+4)m(n-11)P_n \\
& - c(3m+2)m(n-11)P'_n - c(m(3m(n-7)-2n+42)+4(n-9))P_{n-6} \\
& + c(3m(m+2)-4)(n-11)P_{n-2}.
\end{aligned}$$

Finally, substitute $k = n-8$ in (2.1), multiply it by $-3(4+m)$ and add to the previous equation multiplied by 4:

$$\begin{aligned}
0 = {} & -2c(m(m(9n-69)+8n-18)+8(n-6))P_{n-6} \\
& + \left(-4c^2(m+2)(3m-2)(n-11)+16(m+n-8)+m(4n-3m(n-15))\right)P_{n-4} \\
& + 4c(3m(m+2)-4)(n-11)P_{n-2} - 3m(m+4)(n-11)P_n \\
& + \left(8c^2(m(m(n-21)-14)-4)+16m(m+1)(n-9)\right)P'_{n-6} \\
& + 4\left(c(3m+2)(m(n-7)+4) - 6cm(n-11)\left(2c^2m+m+2\right)\right)P'_{n-4} \\
& + 16(n-11)\left(c^2m(3m+1)+m\right)P'_{n-2} - 4cm(3m+2)(n-11)P'_n \\
& + 8c\left(c^2-1\right)m(m(n-13)-2)P''_{n-6} \\
& + 4\left(1-c^2\right)m\left(m\left(4c^2(n-11)+n-15\right)-4\right)P''_{n-4} \\
& + 16c\left(c^2-1\right)m^2(n-11)P''_{n-2} - 4\left(c^2-1\right)m^2(n-11)P''_n.
\end{aligned}$$

We obtained the equation without terms with index $n-8$.

The next step is to eliminate P_{n-6}. After setting $k = n-6$ in (5.8), (5.7) and in the recursion, we obtain the following equations:

$$0 = 4(m(n-7)+2)P'_{n-4} - m(n-9)P''_{n-6} + 2c(m(n-7)+2)P''_{n-4} - (m(n-5)+4)P''_{n-2}, \tag{5.12}$$

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$$\begin{aligned} 0 &= 2(m(n-7)+2)P_{n-4} - m(n-9)P'_{n-6} \\ &\quad + 2c(m(n-7)+2)P'_{n-4} - (m(n-5)+4)P'_{n-2}, \\ 0 &= -m(n-9)P_{n-6} + 2c(m(n-7)+2)P_{n-4} - (m(n-5)+4)P_{n-2}. \end{aligned} \tag{5.13}$$

Proceeding as in the previous case and using these three equations, we can eliminate all terms with index $n-6$:

$$\begin{aligned} 0 = &\Big(-64c^3(m+1)(2m+1)(m(n-4)+2) - 4cm(m(m(n(3n-104)+629) \\ &+ 2(n-54)n+714) - 24n+184)\Big)P'_{n-4} \\ &- 4\Big(c^2-1\Big)m^3(n-11)(n-9)P''_n \\ &+ \Big(m(n-9)\Big(m^2(29n-179)+36m(n-4)+16(n-4)\Big) \\ &- 16c^2(m+1)(2m+1)(n-4)(m(n-6)+4)\Big)P_{n-4} \\ &+ m\Big((m(n-5)+4)\Big(8c^2\Big(-m(n-21)+\frac{4}{m}+14\Big) - 16(m+1)(n-9)\Big) \\ &+ 16(n-11)(n-9)\Big(c^2m(3m+1)+m\Big)\Big)P'_{n-2} \\ &- 4\Big(c^2-1\Big)m\Big(16c^2(m+1)(2m+1)+m(n-9)(m(n-15)-4)\Big)P''_{n-4} \\ &+ 8c\Big(c^2-1\Big)m(m(m((n-22)n+133)-2n+42)+8)P''_{n-2} \\ &- 3(m+4)m^2(n-11)(n-9)P_n - 4c(3m+2)m^2(n-11)(n-9)P'_n \\ &+ (4cm(3m(m+2)-4)(n-11)(n-9) \\ &+ 2c(m(n-5)+4)(m(m(9n-69)+8n-18)+8(n-6)))P_{n-2}. \end{aligned}$$

The next step is to eliminate terms with index $n-4$. Substituting $k = n-6$ in (5.8), (5.7) and in the recursion, we obtain the following equations:

$$\begin{aligned} 0 &= 4(m(n-5)+2)P'_{n-2} - m(n-7)P''_{n-4} \\ &\quad + 2c(m(n-5)+2)P''_{n-2} - (m(n-3)+4)P''_n, \\ 0 &= 2(m(n-5)+2)P_{n-2} - m(n-7)P'_{n-4} \\ &\quad + 2c(m(n-5)+2)P'_{n-2} - (m(n-3)+4)P'_n, \\ 0 &= -m(n-7)P_{n-4} + 2c(m(n-5)+2)P_{n-2} - (m(n-3)+4)P_n. \end{aligned}$$

Using these equations, we can eliminate terms with index $n-4$:

$$\begin{aligned} 0 = &\, 2c(n-2)(m(n-4)+4)\Big(c^2(m(n-5)+2) - m(n-8)\Big)P_{n-2} \\ &- (n-4)(m(n-6)+4)\Big(c^2(m(n-3)+4) - m(n-9)\Big)P_n \\ &- 4c\Big(c^2(m(n-4)+2)(m(n-3)+4) + m(m(6-(n-7)n) - 6n+46)\Big)P'_n \\ &+ 8c\Big(c^2-1\Big)m\Big(c^2(m(n-5)+2) - m(n-8)\Big)P''_{n-2} \\ &- 4\Big(c^2-1\Big)m\Big(c^2(m(n-3)+4) - m(n-9)\Big)P''_n \end{aligned}$$

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$$+4\Big(2c^4(m(n-5)+2)(m(n-2)+2)+c^2m(m((28-3n)n-47)-8n+52)$$
$$+m^2(n-9)(n-5)\Big)P'_{n-2}. \tag{5.14}$$

Finally, we will eliminate all terms with index $n-2$. Substituting $k=n-2$ in (5.8), (5.7) and in the recursion, we obtain

$$\begin{aligned}0 &= 4(m(n-3)+2)P'_n - m(n-5)P''_{n-2}\\ &\quad +2c(m(n-3)+2)P''_n + (m(-n)+m-4)P''_{n+2},\\ 0 &= 2(m(n-3)+2)P_n - m(n-5)P'_{n-2} + 2c(m(n-3)+2)P'_n\\ &\quad +(m(-n)+m-4)P'_{n+2},\\ 0 &= -m(n-5)P_{n-2} + 2c(m(n-3)+2)P_n + (m(-n)+m-4)P_{n+2}.\end{aligned}$$

These equations substitution $n \to n-2$ lead to the following equation:

$$\begin{aligned}0 &= 2c(n-4)(m(n-6)+4)(-m(n-3)-4)\Big(c^2(m(n-7)+2)-m(n-10)\Big)P_n\\ &\quad +8c\Big(c^2-1\Big)m(-m(n-3)-4)\Big(c^2(m(n-7)+2)-m(n-10)\Big)P''_n\\ &\quad +4c\Big(4c^4(m(n-7)+2)(m(n-5)+2)(m(n-2)+2)\\ &\quad +c^2m\Big(m\Big(-m(n-5)(7(n-11)n+136)-34n^2+404n-1026\Big)-40n+296\Big)\\ &\quad +m^2(m(n(n(3n-56)+303)-454)+2n(5n-78)+554)\Big)P'_{n-2}\\ &\quad +(n-2)(m(n-4)+4)\Big(4c^4(m(n-7)+2)(m(n-5)+2)\\ &\quad +c^2m(-m(n-5)(5n-47)-12(n-9))+m^2(n-11)(n-7)\Big)P_{n-2}\\ &\quad +4\Big(c^2-1\Big)m\Big(4c^4(m(n-7)+2)(m(n-5)+2)\\ &\quad +c^2m(-m(n-5)(5n-47)-12(n-9))+m^2(n-11)(n-7)\Big)P''_{n-2}\\ &\quad +m(-m(n-3)-4)\Bigg(\frac{8c^4(m(n-7)+2)(m(n-4)+2)}{m}\\ &\quad -4c^2(m(n(3n-40)+115)+8n-68)+4m(n-11)(n-7)\Bigg)P'_n.\end{aligned} \tag{5.15}$$

Combining this with (5.14), we obtain

$$\begin{aligned}0 &= 2c(n-2)(m(n-4)+4)\Big(c^2(m(n-5)+2)-m(n-8)\Big)P_{n-2}\\ &\quad -(n-4)(m(n-6)+4)\Big(c^2(m(n-3)+4)-m(n-9)\Big)P_n\\ &\quad -4c\Big(c^2(m(n-4)+2)(m(n-3)+4)+m(m(6-(n-7)n)-6n+46)\Big)P'_n\\ &\quad +8c\Big(c^2-1\Big)m\Big(c^2(m(n-5)+2)-m(n-8)\Big)P''_{n-2}\\ &\quad -4\Big(c^2-1\Big)m\Big(c^2(m(n-3)+4)-m(n-9)\Big)P''_n\\ &\quad +4\Big(2c^4(m(n-5)+2)(m(n-2)+2)+c^2m(m((28-3n)n-47)-8n+52)\\ &\quad +m^2(n-9)(n-5)\Big)P'_{n-2}.\end{aligned} \tag{5.16}$$

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Now we use (5.16) and (5.15) to simplify the coefficient by P''_{n-2}. Combining them, multiplying (5.16) by $-2c(mn-7m+2)$ and adding it to (5.15) gives

$$\begin{aligned}0 =& -4\left(c^2-1\right)m^2(n-11)\left(c^2(m(n-1)+4)-m(n-7)\right)P''_{n-2}\\&+8c\left(c^2-1\right)(3m+2)m^2(n-11)P''_n\\&+4m(n-11)\left(c^2\left(m^2((n-10)n+39)+4m(n-1)+8\right)-m(n-7)(m(n-3)+4)\right)P'_n\\&-4cm(n-11)\left(c^2(m(n-2)+2)(m(n-1)+4)+m(m(16-(n-3)n)-6n+34)\right)P'_{n-2}\\&-m(n-11)(n-2)(m(n-4)+4)\left(c^2(m(n-1)+4)-m(n-7)\right)P_{n-2}\\&+2c(3m+2)m(n-11)(n-4)(m(n-6)+4)P_n.\end{aligned}$$

From now on, we assume that $n>11$. Then the equation simplifies to

$$\begin{aligned}0 =& \left(4m(n-7)(m(n-3)+4)-4c^2\left(m^2((n-10)n+39)+4m(n-1)+8\right)\right)P'_n\\&+(n-2)(m(n-4)+4)\left(c^2(m(n-1)+4)-m(n-7)\right)P_{n-2}\\&+4c\left(c^2(m(n-2)+2)(m(n-1)+4)+m(m(16-(n-3)n)-6n+34)\right)P'_{n-2}\\&+4\left(c^2-1\right)m\left(c^2(m(n-1)+4)-m(n-7)\right)P''_{n-2}\\&-8c\left(c^2-1\right)m(3m+2)P''_n-2c(3m+2)(n-4)(m(n-6)+4)P_n.\end{aligned}\tag{5.17}$$

Multiplying it by $-2c(mn-5m+2)$ and adding it to (5.14) multiplied by $4+mn-m$, we get

$$\begin{aligned}0 =& \left(4c^2\left(m^2((n-6)n+23)+4m(n+1)+8\right)-4m(n-5)(m(n-1)+4)\right)P'_{n-2}\\&+(n-4)(m(n-6)+4)\left(c^2m(n-7)-mn+m-4\right)P_n\\&-4c\left(m\left(c^2(n-7)(m(n-6)+2)-m((n-13)n+24)-2n+38\right)+8\right)P'_n\\&+8c\left(c^2-1\right)m(3m+2)P''_{n-2}\\&+4\left(c^2-1\right)m\left(c^2m(n-7)-mn+m-4\right)P''_n\\&+2c(3m+2)(n-2)(m(n-4)+4)P_{n-2}.\end{aligned}\tag{5.18}$$

Multiplying this equation by $c(-mn+m-4)$ and adding it to (5.17) multiplied by $6m+4$, we get

$$\begin{aligned}0 =& c(n-4)(m(n-6)+4)\left(c^2(m(n-1)+4)-m(n+5)-8\right)P_n\\&+4c\left(m\left(c^2(n-5)(m(n-1)+4)+m(13-(n-6)n)-4n+44\right)+8\right)P'_{n-2}\\&+8\left(c^2-1\right)m(3m+2)P''_{n-2}\\&+4c\left(c^2-1\right)m\left(c^2(m(n-1)+4)-m(n+5)-8\right)P''_n\\&+\Big(-4c^4(m(n-6)+2)(m(n-1)+4)+4c^2(m(n-6)+2)(m(n+5)+8)\\&-8(3m+2)(m(n-3)+4)\Big)P'_n.\\&+2(3m+2)(n-2)(m(n-4)+4)P_{n-2}.\end{aligned}\tag{5.19}$$

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Multiplying it by $-c$ and adding to (5.18), we obtain the following equation:

$$0 = 4\left(c^2 - 1\right)mP_n'' + (n-4)(m(n-6)+4)P_n + 4m(n-5)P_{n-2}' - 4c(m(n-6)+2)P_n'. \tag{5.20}$$

Differentiate the last equation with respect to c to get

$$0 = 4\left(c^2 - 1\right)mP_n''' + (n-6)(m(n-8)+4)P_n' + 4m(n-5)P_{n-2}'' - 4c(m(n-8)+2)P_n''. \tag{5.21}$$

Multiplying it by $-2(c^2-1)(3m+2)$ and adding to (5.19) multiplied by $n-5$ gives us the following equation:

$$\begin{aligned}0 = &-8\left(c^2-1\right)^2 m(3m+2)P_n''' \\ &+4c\left(c^2-1\right)\left(m\left(c^2(n-5)(m(n-1)+4) - m((n-6)n+23) - 4(n-5)\right)+8\right)P_n'' \\ &+c(n-5)(n-4)(m(n-6)+4)\left(c^2(m(n-1)+4) - m(n+5) - 8\right)P_n \\ &+4c(n-5)\left(m\left(c^2(n-5)(m(n-1)+4) + m(13-(n-6)n) - 4n + 44\right)+8\right)P_{n-2}' \\ &+\left((n-5)\left(-4c^4(m(n-6)+2)(m(n-1)+4) + 4c^2(m(n-6)+2)(m(n+5)+8)\right.\right. \\ &-\left.\left. 8(3m+2)(m(n-3)+4)\right) - 2\left(c^2-1\right)(3m+2)(n-6)(m(n-8)+4)\right)P_n' \\ &+2(3m+2)(n-5)(n-2)(m(n-4)+4)P_{n-2}.\end{aligned}$$

Multiplying now (5.20) by $c\left(m\left(c^2(n-5)(m(n-1)+4) + m(13-(n-6)n) - 4n + 44\right)+8\right)$ and adding it to the equation above multiplied by $-m$ gives the following equation:

$$\begin{aligned}0 = &-24c\left(c^2-1\right)m^2P_n'' - 4\left(c^2-1\right)^2 m^2 P_n''' \\ &+\left(c^2(m(3n(m(n-6)+4)-40)+16)\right. \\ &+\left. m(-3m((n-6)n+4) - 12n + 56)\right)P_n' \\ &+m(n-5)(n-2)(m(n-4)+4)P_{n-2} \\ &-c(n-4)(m(n-6)+4)(m(n-2)+2)P_n.\end{aligned}$$

If we differentiate this equation, multiply by -4 and add to (5.20) multiplied by $(n-2)(mn-4m+4)$, we will come to the desired linear differential equation of order 4. This proves the theorem. □

Competing interest. The authors have no competing interests to declare.

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Ethical standards. The research meets all ethical guidelines, including adherence to the legal requirements of the study country.

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