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# ON THE SHORTEST DISTANCE FUNCTION IN CONTINUED FRACTIONS

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**Abstract**

Let $x \in [0,1)$ be an irrational number and let $x = [a_1(x), a_2(x), \ldots]$ be its continued fraction expansion with partial quotients $\{a_n(x) : n \geq 1\}$. Given a natural number $m$ and a vector $(x_1, \ldots, x_m) \in [0,1)^m$, we derive the asymptotic behaviour of the shortest distance function

$$M_{n,m}(x_1, \ldots, x_m) = \max\{k \in \mathbb{N} : a_{i+j}(x_1) = \cdots = a_{i+j}(x_m) \text{ for } j = 1, \ldots, k \text{ and some } i \text{ with } 0 \leq i \leq n-k\},$$

which represents the run-length of the longest block of the same symbol among the first $n$ partial quotients of $(x_1, \ldots, x_m)$. We also calculate the Hausdorff dimension of the level sets and exceptional sets arising from the shortest distance function.

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## 1. Introduction

Let $T : [0,1) \to [0,1)$ be the Gauss map defined by

$$T(0) = 0, \quad T(x) = \frac{1}{x} \pmod 1 \quad \text{for } x \in (0,1).$$

Every irrational number $x \in [0,1)$ can be uniquely expanded into an infinite form

$$x := \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots + \cfrac{1}{a_n + T^n(x)}}}} = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cfrac{1}{\ddots}}}}, \tag{1.1}$$

where $a_n(x) = \lfloor 1/T^{n-1}(x) \rfloor$ are called the partial quotients of $x$. (Here $\lfloor \cdot \rfloor$ denotes the greatest integer less than or equal to a real number and $T^0$ denotes the identity map.)

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For simplicity of notation, we write (1.1) as

$$x = [a_1(x), a_2(x), \ldots, a_n(x) + T^n(x)] = [a_1(x), a_2(x), a_3(x), \ldots].$$

It is clear that the Gauss transformation $T$ acts as the shift map on the continued fraction system. That is, for each $x = [a_1(x), a_2(x), a_3(x), \ldots] \in [0, 1) \cap \mathbb{Q}^c$,

$$T(x) = T([a_1(x), a_2(x), a_3(x), \ldots]) = [a_2(x), a_3(x), \ldots].$$

Gauss observed that $T$ is measure-preserving and ergodic with respect to the Gauss measure $\mu$ defined by

$$d\mu = \frac{1}{\log 2}\frac{1}{x+1}dx.$$

For more information on the continued fraction expansion, see [3].

The metrical theory of continued fractions, which concerns the properties of the partial quotients for almost all $x \in [0, 1)$, is one of the major themes in the study of continued fractions. Wang and Wu [7] considered the metrical properties of the maximal run-length function

$$R_n(x) = \max\{l \in \mathbb{N} : a_{i+1}(x) = \cdots = a_{i+l}(x) \text{ for some } i \text{ with } 0 \le i \le n - l\},$$

which counts the longest run of the same symbol among the first $n$ partial quotients of $x$. They proved that, for $\mu$ almost all $x \in [0, 1)$,

$$\lim_{n\to\infty} \frac{R_n(x)}{\log n} = \frac{1}{2\log((\sqrt{5}+1)/2)}.$$

Song and Zhou [6] gave a more subtle characterisation of the function $R_n(x)$. In this paper, we continue the study by considering the shortest distance function

$$M_{n,m}(x_1, \ldots, x_m) = \max\{k \in \mathbb{N} : a_{i+j}(x_1) = \cdots = a_{i+j}(x_m) \text{ for } j = 1, \ldots, k, \text{ and some } i \text{ with } 0 \le i \le n - k\}.$$

This is motivated by the behaviour of the shortest distance between two orbits,

$$S_{n,2}(x, y) = \min_{i=0,\ldots,n-1} (d(T^i(x), T^i(y)))$$

in the continued fraction system. Shi *et al.* [5] proved that, for $\mu^2$ almost all $(x, y) \in [0, 1) \times [0, 1)$,

$$H_2 \cdot \lim_{n\to\infty} \frac{M_{n,2}(x, y)}{\log n} = \lim_{n\to\infty} \frac{-\log S_{n,2}(x, y)}{\log n},$$

where $H_2$ is the Rényi entropy defined by (1.2). Investigating the shortest distance between two orbits amounts to estimating the longest common substrings between two sequences of partial quotients. In fact, [5] focused on the asymptotics of the length of the longest common substrings in two sequences of partial quotients.

For $n \ge 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, we call

$$I_n(a_1, \ldots, a_n) = \{x \in [0, 1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\}$$

an $n$th cylinder. For $m \geq 2$, we define the generalised Rényi entropy with respect to the Gauss measure $\mu$ by

$$H_m = \lim_{n\to\infty} \frac{-\log \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} \mu(I_n(a_1,\ldots,a_n))^m}{(m-1)n}. \tag{1.2}$$

The existence of the limit (1.2) for the Gauss measure $\mu$ was established in [2].

THEOREM 1.1. *For $\mu^m$-almost all $(x_1,\ldots,x_m) \in [0,1)^m$,*

$$\lim_{n\to\infty} \frac{M_{n,m}(x_1,\ldots,x_m)}{\log n} = \frac{1}{(m-1)H_m}.$$

*Here we use the convention that $1/0 = \infty$ and $1/\infty = 0$.*

It is natural to study the exceptional set in this limit theorem. We define the exceptional set as

$$\widetilde{E} = \Big\{(x_1,\ldots,x_m) \in [0,1)^m : \liminf_{n\to\infty} \frac{M_{n,m}(x_1,\ldots,x_m)}{\log n} < \limsup_{n\to\infty} \frac{M_{n,m}(x_1,\ldots,x_m)}{\log n}\Big\}$$

and the level set as

$$E(\alpha) = \Big\{(x_1,\ldots,x_m) \in [0,1)^m : \lim_{n\to\infty} \frac{M_{n,m}(x_1,\ldots,x_m)}{\log n} = \alpha\Big\}.$$

Throughout the paper, $\dim_H A$ denotes the Hausdorff dimension of the set $A$.

THEOREM 1.2. *For any $\alpha$ with $0 \leq \alpha \leq \infty$, $\dim_H \widetilde{E} = \dim_H E(\alpha) = m$.*

In fact, Theorem 1.2 follows immediately from the following more general result. For any $0 \leq \alpha \leq \beta \leq \infty$, set

$$E(\alpha,\beta) = \Big\{(x_1,\ldots,x_m) \in [0,1)^m : \liminf_{n\to\infty} \frac{M_{n,m}(x_1,\ldots,x_m)}{\log n} = \alpha, \\ \limsup_{n\to\infty} \frac{M_{n,m}(x_1,\ldots,x_m)}{\log n} = \beta\Big\}.$$

THEOREM 1.3. *For any $\alpha,\beta$ with $0 \leq \alpha \leq \beta \leq \infty$, $\dim_H E(\alpha,\beta) = m$.*

## 2. Preliminaries

In this section, we fix some notation and recall some basic properties of continued fraction expansions. A detailed account of continued fractions can be found in Khintchine's book [3].

For any irrational number $x \in [0,1)$ with continued fraction expansion (1.1), we denote by

$$\frac{p_n(x)}{q_n(x)} := [a_1(x),\ldots,a_n(x)]$$

the $n$th convergent of $x$. With the conventions

$$p_{-1}(x) = 1, \quad q_{-1}(x) = 0, \quad p_0(x) = 0, \quad q_0(x) = 1,$$

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we have, for any $n \geq 0$,

$$p_{n+1}(x) = a_{n+1}(x)p_n(x) + p_{n-1}(x), \quad q_{n+1}(x) = a_{n+1}(x)q_n(x) + q_{n-1}(x).$$

Obviously, $q_n(x)$ is determined by the first $n$ partial quotients $a_1(x), \ldots, a_n(x)$. So we also write $q_n(a_1(x), \ldots, a_n(x))$ in place of $q_n(x)$. If no confusion is likely to arise, we write $a_n$ and $q_n$ in place of $a_n(x)$ and $q_n(x)$, respectively.

PROPOSITION 2.1 [3]. *For $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$:*

(1) $q_n \geq 2^{(n-1)/2}$ *and*

$$\prod_{k=1}^{n} a_k \leq q_n \leq \prod_{k=1}^{n}(a_k + 1) \leq 2^n \prod_{k=1}^{n} a_k;$$

(2) *the length of $I_n(a_1, \ldots, a_n)$ satisfies*

$$\frac{1}{2q_n^2} \leq |I_n(a_1, \ldots, a_n)| = \frac{1}{(q_n + q_{n-1})q_n} \leq \frac{1}{q_n^2}.$$

The following $\psi$-mixing property is essential in proving Theorem 1.1.

LEMMA 2.2 [4]. *For any $k \geq 1$, let $\mathbb{B}_1^k = \sigma(a_1, \ldots, a_k)$ and let $\mathbb{B}_k^\infty = \sigma(a_k, a_{k+1}, \ldots)$ denote the $\sigma$-algebras generated by the random variables $(a_1, \ldots, a_k)$ and $(a_k, a_{k+1}, \ldots)$ respectively. Then, for any $E \in \mathbb{B}_1^k$ and $F \in \mathbb{B}_{k+n}^\infty$,*

$$\mu(E \cap F) = \mu(E) \cdot \mu(F)(1 + \theta\rho^n),$$

*where $|\theta| \leq K, \rho < 1$ and $K, \rho$ are positive constants independent of $E, F, n$ and $k$.*

To estimate the measure of a limsup set in a probability space, the following lemma is widely used.

LEMMA 2.3 (Borel–Cantelli lemma). *Let $(\Omega, \mathcal{B}, \nu)$ be a finite measure space and let $\{A_n\}_{n\geq 1}$ be a sequence of measurable sets. Define $A = \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty A_n$. Then*

$$\nu(A) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \nu(A_n) < \infty, \\ \nu(\Omega) & \text{if } \sum_{n=1}^{\infty} \nu(A_n) = \infty \text{ and } \{A_n\}_{n\geq 1} \text{ are pairwise independent.} \end{cases}$$

Let $\mathbb{K} = \{k_n\}_{n\geq 1}$ be a subsequence of $\mathbb{N}$ that is not cofinite. Define a mapping $\phi_{\mathbb{K}} : [0, 1) \cap \mathbb{Q}^c \to [0, 1) \cap \mathbb{Q}^c$ as follows. For each $x = [a_1, a_2, \ldots] \in [0, 1) \cap \mathbb{Q}^c$, put $\phi_{\mathbb{K}}(x) = \overline{x} = [c_1, c_2, \ldots]$, where $[c_1, c_2, \ldots]$ is obtained by eliminating all the terms $a_{k_n}$ from the sequence $a_1, a_2, \ldots$. Let $\{b_n\}_{n\geq 1}$ be a sequence with $b_n \in \mathbb{N}, n \geq 1$. Write

$$E(\mathbb{K}, \{b_n\}) = \{x \in [0, 1) \cap \mathbb{Q}^c : a_{k_n}(x) = b_n \text{ for all } n \geq 1\}.$$

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LEMMA 2.4 [6]. *Assume that* $\{b_n\}_{n\geq 1}$ *is bounded. If the sequence* $\mathbb{K}$ *is of density zero in* $\mathbb{N}$, *that is,*

$$\lim_{n\to\infty}\frac{\sharp\{i\leq n : i\in\mathbb{K}\}}{n}=0,$$

*where* $\sharp$ *denotes the number of elements in a set, then*

$$\dim_H E(\mathbb{K},\{b_n\})=\dim_H \phi_{\mathbb{K}}(E(\mathbb{K},\{b_n\}))=1.$$

We close this section by citing Marstrand's product theorem.

LEMMA 2.5 [1]. *If* $E, F\subset\mathbb{R}^d$ *for some* $d$, *then* $\dim_H(E\times F)\geq\dim_H E+\dim_H F$.

## 3. Proof of Theorem 1.1

Theorem 1.1 can be proved from the following two propositions.

PROPOSITION 3.1. *For* $\mu^m$*-almost all* $(x_1,\ldots,x_m)\in[0,1)^m$,

$$\limsup_{n\to\infty}\frac{M_{n,m}(x_1,\ldots,x_m)}{\log n}\leq\frac{1}{(m-1)H_m}.$$

PROOF. We can assume that $H_m>0$ (the case $H_m=0$ is obvious). Fix $s_1<s_2<(m-1)H_m$. By the definition of the $H_m$,

$$\sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n}\mu(I_n(a_1,\ldots,a_n))^m<\exp\Big\{-\frac{s_1+s_2}{2}n\Big\}\tag{3.1}$$

for sufficiently large $n$. Set $u_n=\lfloor\log n/s_1\rfloor$. Note that, for any $(x_1,\ldots,x_m)\in[0,1)^m$ with $M_{n,m}(x_1,\ldots,x_m)=k$, there exists $i$ with $0\leq i\leq n-k$ such that

$$a_{i+j}(x_1)=\cdots=a_{i+j}(x_m)$$

for $j=1,\ldots,k$. We deduce

$$\begin{aligned}&\mu^m(\{(x_1,\ldots,x_m)\in[0,1)^m : M_{n,m}(x_1,\ldots,x_m)>u_n\})\\ &\quad=\sum_{k=u_n+1}^{\infty}\mu^m(\{(x_1,\ldots,x_m)\in[0,1)^m : M_{n,m}(x_1,\ldots,x_m)=k\})\\ &\quad\leq\sum_{k=u_n+1}^{\infty}\sum_{i=0}^{n-k}\mu^m(\{(x_1,\ldots,x_m)\in[0,1)^m : a_{i+j}(x_1)=\cdots=a_{i+j}(x_m),\ j=1,\ldots,k\}).\end{aligned}$$

By the invariance of $\mu$ under $T$, it follows that

$$\begin{aligned}&\mu^m(\{(x_1,\ldots,x_m)\in[0,1)^m : M_{n,m}(x_1,\ldots,x_m)>u_n\})\\ &\quad\leq n\sum_{k=u_n+1}^{\infty}\mu^m(\{(x_1,\ldots,x_m)\in[0,1)^m : a_j(x_1)=\cdots=a_j(x_m),\ j=1,\ldots,k\})\end{aligned}$$

$$= n \sum_{k=u_n+1}^{\infty} \sum_{(a_1,\dots,a_k)\in\mathbb{N}^k} \mu(I_k(a_1,\dots,a_k))^m$$
$$\le Cn \cdot \exp\Big\{-\frac{s_1+s_2}{2}(u_n+1)\Big\} \qquad \text{(by (3.1))}$$
$$\le Cn^{-(s_2-s_1)/2s_1},$$

where $C = \sum_{k=1}^{\infty} \exp\{-(s_1+s_2)k/2\}$. Choose an infinite subsequence of integers $\{n_k\}_{k\ge1}$, where $n_k = k^L$ and $L \cdot (s_2-s_1)/2s_1 > 1$. Then

$$\sum_{k=1}^{\infty} \mu^m(\{(x_1,\dots,x_m) \in [0,1)^m : M_{n_k,m}(x_1,\dots,x_m) > u_{n_k}\}) < \infty.$$

From the Borel–Cantelli Lemma 2.3, for almost all $(x_1,\dots,x_m) \in [0,1)^m$,

$$M_{n_k,m}(x_1,\dots,x_m) \le u_{n_k}$$

for sufficiently large $k$. Thus,

$$\limsup_{n\to\infty} \frac{M_{n,m}(x_1,\dots,x_m)}{\log n} \le \limsup_{k\to\infty} \frac{M_{n_{k+1},m}(x_1,\dots,x_m)}{\log n_k}$$
$$\le \limsup_{k\to\infty} \frac{M_{n_{k+1},m}(x_1,\dots,x_m)}{\log n_{k+1}} \cdot \limsup_{k\to\infty} \frac{n_{k+1}}{n_k} \le \frac{1}{s_1}.$$

Therefore, by the arbitrariness of $s_1$,

$$\limsup_{n\to\infty} \frac{M_{n,m}(x_1,\dots,x_m)}{\log n} \le \frac{1}{(m-1)H_m}.$$

This completes the proof. □

PROPOSITION 3.2. *For $\mu^m$-almost all $(x_1,\dots,x_m) \in [0,1)^m$,*

$$\liminf_{n\to\infty} \frac{M_{n,m}(x_1,\dots,x_m)}{\log n} \ge \frac{1}{(m-1)H_m}.$$

PROOF. We can assume that $H_m < \infty$ (the case $H_m = \infty$ is obvious). For $1 \le d < n$, set

$$M_{[d,n]}(x_1,\dots,x_m) = \max\{k \in \mathbb{N} : a_{i+j}(x_1) = \cdots = a_{i+j}(x_m) \text{ for } j = 1,\dots,k, \text{ and for some } i \text{ with } d-1 \le i \le n-k\}.$$

We denote $\{(x_1,\dots,x_m) \in [0,1)^m : M_{n,m}(x_1,\dots,x_m) < k\}$ by $\{M_{n,m} < k\}$ for brevity.

For any $s > (m-1)H_m$, by the definition of the $H_m$,

$$\sum_{(a_1,\dots,a_n)\in\mathbb{N}^n} \mu(I_n(a_1,\dots,a_n))^m > \exp\Big\{-\frac{s+(m-1)H_m}{2}n\Big\} \tag{3.2}$$

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for sufficiently large $n$. Let $u_n = \lfloor \log n/s \rfloor$ and $l_n = \lfloor n/u_n^2 \rfloor$. Then

$$\begin{aligned}\{M_{n,m} < u_n\} &\subset \{M_{[iu_n^2+1, iu_n^2+u_n]} < u_n : 0 \le i < l_n\} \\ &\subset \{M_{u_n,m} < u_n\} \cap (\underbrace{T \times \cdots \times T}_{m \text{ times}})^{-u_n^2}\{M_{[iu_n^2+1, i\mu_n^2+u_n]} < u_n : 0 \le i < l_n - 1\}.\end{aligned}$$

By Lemma 2.2, it follows that

$$\begin{aligned}&\mu^m(\{M_{n,m} < u_n\}) \\ &\le \mu^m(\{M_{u_n,m} < u_n\} \cap (\underbrace{T \times \cdots \times T}_{m \text{ times}})^{-u_n^2}\{M_{[iu_n^2+1, i\mu_n^2+u_n]} < u_n, 0 \le i < l_n - 1\}) \\ &\le \mu^m(\{M_{u_n,m} < u_n\})^{l_n}(1 + \theta\rho^{u_n^2 - u_n})^{m \cdot l_n} \\ &\le \Big(1 - \sum_{(a_1,\ldots,a_{u_n}) \in \mathbb{N}^{u_n}} \mu(I_{u_n}(a_1, \ldots, a_{u_n}))^m\Big)^{l_n}(1 + \theta\rho^{u_n^2 - u_n})^{m \cdot l_n} \\ &\le \exp\Big\{-\frac{n}{u_n^2} \cdot n^{-(s+(m-1)H_m)/2s}\Big\} \cdot \exp\Big\{\theta\rho^{u_n^2-u_n}\frac{m \cdot n}{u_n^2}\Big\} \\ &\le M \exp\Big\{-\frac{n^{(s-(m-1)H_m)/2s}}{u_n^2}\Big\},\end{aligned}$$

where the penultimate inequality follows from (3.2) and the two facts $(1 - x) < \exp(-x)$ for $0 < x < 1$ and $\lim_{n\to\infty}(1 + 1/n)^n = e$, and the last inequality follows because $\theta\rho^{u_n^2-u_n} m \cdot n/u_n^2 \to 0$ as $n \to \infty$. Thus,

$$\sum_{n=1}^{\infty} \mu^m(\{M_{n,m} < u_n\}) < \infty.$$

From the Borel–Cantelli Lemma, for $\mu^m$-almost all $(x_1, \ldots, x_m) \in [0, 1)^m$,

$$\liminf_{n\to\infty} \frac{M_{n,m}(x_1, \ldots, x_m)}{\log n} \ge \frac{1}{(m-1)H_m}.$$

This completes the proof of Proposition 3.2 and of Theorem 1.1. □

## 4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. Our strategy is to construct Cantor-like subsets with full Hausdorff dimension. The proof is divided into several cases according to the values of $\alpha$ and $\beta$. We give a detailed proof for the case $0 < \alpha < \beta < \infty$ and a sketch of the proof for the remaining cases.

CASE 1: $0 < \alpha < \beta < \infty$.

Choose two positive integer sequences $\{n_k\}_{k\ge1}$ and $\{s_k\}_{k\ge1}$ such that, for each $k \ge 1$,

$$n_1 = 2, \quad n_{k+1} = \lfloor n_k^{\beta/\alpha} \rfloor, \quad s_k = \lfloor \beta \log n_k \rfloor. \tag{4.1}$$

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We readily check that

$$\lim_{k\to\infty}\frac{s_k}{n_{k+1}-n_k}=0. \tag{4.2}$$

Without loss of generality, we assume that $n_{k+1}-n_k > s_k$ for all $k \geq 1$. Otherwise, we consider only sufficiently large $k$. Put

$$n_{k+1}-n_k = s_k\cdot \iota_k+\theta_k,$$

where

$$\iota_k=\left\lfloor\frac{n_{k+1}-n_k}{s_k}\right\rfloor \quad \text{for } 0\leq\theta_k<s_k.$$

Define a marked set $\mathbb{K}$ of positive integers by

$$\mathbb{K}:=\mathbb{K}(\{n_k\},\{s_k\})=\bigcup_{k\geq1}\{n_k,n_k+1,n_k+2,\dots,n_k+s_k,n_k+2s_k,n_k+3s_k,\dots,n_k+\iota_k s_k\}.$$

Now we define $m$ sequences as follows.

- For $i=1$,

$$a^{(1)}_{n_k}=1,\ a^{(1)}_{n_k+1}=\cdots=a^{(1)}_{n_k+s_k-1}=1,\ a^{(1)}_{n_k+s_k}=a^{(1)}_{n_k+2s_k}=\cdots=a^{(1)}_{n_k+\iota_k s_k}=1.$$

- For $2\leq i\leq m$,

$$a^{(i)}_{n_k}=i,\ a^{(i)}_{n_k+1}=\cdots=a^{(i)}_{n_k+s_k-1}=1,\ a^{(i)}_{n_k+s_k}=a^{(i)}_{n_k+2s_k}=\cdots=a^{(i)}_{n_k+\iota_k s_k}=i.$$

Then, for $i=1,2,\dots,m$, write

$$E(\mathbb{K},\{a^{(i)}_n\}_{n\geq1})=\{x\in[0,1)\cap\mathbb{Q}^c : a_n(x)=a^{(i)}_n \text{ for all } n\in\mathbb{K}\}.$$

Now we prove $\prod_{i=1}^m E(\mathbb{K},\{a^{(i)}_n\}_{n\geq1})\subset E(\alpha,\beta)$. Fix $(x_1,\dots,x_m)\in\prod_{i=1}^m E(\mathbb{K},\{a^{(i)}_n\}_{n\geq1})$ for any $n\geq n_1$ and let $k$ be the integer such that $n_k\leq n<n_{k+1}$. From the construction of the set $\prod_{i=1}^m E(\mathbb{K},\{a^{(i)}_n\}_{n\geq1})$, we see that

$$M_{n,m}(x_1,\dots,x_m)=\begin{cases} s_{k-1}-1 & \text{if } n_k\leq n<n_k+s_{k-1},\\ n-n_k & \text{if } n_k+s_{k-1}\leq n<n_k+s_k,\\ s_k-1 & \text{if } n_k+s_k\leq n<n_{k+1}.\end{cases}$$

Further, by (4.1), we deduce that

$$\begin{aligned}\liminf_{n\to\infty}\frac{M_{n,m}(x_1,\dots,x_m)}{\log n}&=\liminf_{k\to\infty}\min\left\{\frac{M_{n_k+s_{k-1}-1,m}(x_1,\dots,x_m)}{\log(n_k+s_{k-1}-1)},\frac{M_{n_{k+1}-1,m}(x_1,\dots,x_m)}{\log(n_{k+1}-1)}\right\}\\&=\liminf_{k\to\infty}\min\left\{\frac{s_{k-1}-1}{\log(n_k+s_{k-1}-1)},\frac{s_k-1}{\log(n_{k+1}-1)}\right\}\\&=\alpha\end{aligned}$$

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and

$$\limsup_{n\to\infty}\frac{M_{n,m}(x_1,\ldots,x_m)}{\log n}=\limsup_{k\to\infty}\max\left\{\frac{M_{n_k,m}(x_1,\ldots,x_m)}{\log(n_k)},\frac{M_{n_k+s_k-1,m}(x_1,\ldots,x_m)}{\log(n_k+s_k-1)}\right\}$$
$$=\limsup_{k\to\infty}\max\left\{\frac{s_{k-1}-1}{\log(n_k)},\frac{s_k-1}{\log(n_k+s_k-1)}\right\}$$
$$=\beta.$$

Hence, $(x_1,\ldots,x_m)\in E(\alpha,\beta)$.

It remains to prove that the density of $\mathbb{K}\subset\mathbb{N}$ is zero. For $n_k\le n<n_{k+1}$ with some $k\ge 1$:

- if $n_k\le n<n_k+s_k$, then $\sharp\{i\le n: i\in\mathbb{K}\}=\sum_{j=1}^{k-1}(m_j+\iota_j)+n-n_k+1$;
- if $n_k+ls_k\le n<n_k+(l+1)s_k$ for some $l$ with $0<l<\iota_k$, then we see that $\sharp\{i\le n: i\in\mathbb{K}\}=\sum_{j=1}^{k-1}(s_j+\iota_j)+s_k+l$;
- if $n_k+\iota_k s_k\le n<n_{k+1}$, then $\sharp\{i\le n: i\in\mathbb{K}\}=\sum_{j=1}^{k}(s_j+\iota_j)$.

Consequently,

$$\limsup_{n\to\infty}\frac{\sharp\{i\le n: i\in\mathbb{K}\}}{n}\le\limsup_{k\to\infty}\max_{0\le l<\iota_k}\left\{\frac{\sum_{j=1}^{k-1}(s_j+\iota_j)+s_k+l}{n_k+ls_k}\right\}$$
$$\le\limsup_{k\to\infty}\left\{\frac{\sum_{j=1}^{k-1}(s_j+\iota_j)+s_k+\iota_k}{n_k}\right\}$$
$$=0,$$

where the last equality follows by the Stolz–Cesàro theorem and (4.2). By Lemmas 2.4 and 2.5,

$$\dim_H E_{\alpha,\beta}\ge\dim_H\left(\prod_{i=1}^{m}E(\mathbb{K},\{a_n^{(i)}\}_{n\ge1})\right)\ge\sum_{i=1}^{m}\dim_H E(\mathbb{K},\{a_n^{(i)}\}_{n\ge1})=m.$$

Similar arguments apply to the remaining cases. We only give the constructions for the proper sequences $\{n_k\}_{k\ge1}$ and $\{s_k\}_{k\ge1}$.

CASE 2: $0<\alpha=\beta<\infty$. Take $n_k=2^k$ and $s_k=\lfloor\alpha\log n_k\rfloor$ for $k\ge1$.

CASE 3: $\alpha=0<\beta<\infty$. Take $n_k=2^{2^{2^k}}$ and $s_k=\lfloor\beta\log n_k\rfloor$ for $k\ge1$.

CASE 4: $\alpha=0,\beta=\infty$. Take $n_k=2^{2^{2^k}}$ and $s_k=\lfloor k\log n_k\rfloor$ for $k\ge1$.

CASE 5: $0<\alpha<\beta=\infty$. Take $n_k=2^{k!}$ and $s_k=\lfloor\alpha k\log n_k\rfloor$ for $k\ge1$.

CASE 6: $\alpha=\beta=0$. Take $n_k=2^k$ and $s_k=\lfloor\log\log n_k\rfloor$ for $k\ge1$.

CASE 7: $\alpha=\beta=\infty$. Take $n_k=2^k$ and $s_k=\lfloor k\log n_k\rfloor$ for $k\ge1$.

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