J. Aust. Math. Soc. **80** (2006), 375–382

THE DOUBLE COVER RELATIVE TO A CONVEX DOMAIN AND THE RELATIVE ISOPERIMETRIC INEQUALITY

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(Received 18 March 2004; revised 16 February 2005)

Communicated by K. Wysocki

Abstract

We prove that a domain Ω in the exterior of a convex domain C in a four-dimensional simply connected Riemannian manifold of nonpositive sectional curvature satisfies the relative isoperimetric inequality $64\pi^2 \operatorname{Vol}(\Omega)^3 \leq \operatorname{Vol}(\partial\Omega \sim \partial C)^4$. Equality holds if and only if Ω is an Euclidean half ball and $\partial\Omega \sim \partial C$ is a hemisphere.

2000 *Mathematics subject classification*: primary 49Q20, 58E35.

1. Introduction

The classical isoperimetric inequality states that if Ω is a domain in $\mathbb{R}^n$ then

$$n^n \omega_n \operatorname{Vol}(\Omega)^{n-1} \leq \operatorname{Vol}(\partial\Omega)^n, \tag{1}$$

where ω_n represents the volume of a unit ball in $\mathbb{R}^n$. Here equality holds if and only if Ω is a ball. One natural way to extend this optimal inequality is the following. Let $\mathbb{H}$ be a half-space $\{(x_1, \ldots, x_n) : x_n \geq 0\}$ in $\mathbb{R}^n$ and let Ω be a domain in $\mathbb{H}$ with $\partial\Omega \cap \partial\mathbb{H} \neq \phi$. If we define $\widetilde{\Omega} = \{(x_1, \ldots, x_{n-1}, -x_n) : (x_1, \ldots, x_n) \in \Omega\}$, then it follows from (1) that

$$n^n \omega_n \operatorname{Vol}(\Omega \cup \widetilde{\Omega})^{n-1} \leq \operatorname{Vol}(\partial(\Omega \cup \widetilde{\Omega}))^n.$$

Dividing this inequality by 2^n yields

$$\frac{1}{2} n^n \omega_n \operatorname{Vol}(\Omega)^{n-1} \leq \operatorname{Vol}(\partial\Omega \sim \partial\mathbb{H})^n.$$

Supported in part by KRF-2003-015-C00056 through RIBS-SNU.

https://doi.org/10.1017/S1446788700014087 Published online by Cambridge University Press

Motivated by this, one can ask the following question. Given a convex domain $C \subset \mathbb{R}^n$ and a domain Ω in $\mathbb{R}^n \sim C$ with $\partial\Omega \cap \partial C \neq \phi$, does Ω satisfy the relative isoperimetric inequality

$$\frac{1}{2} n^n \omega_n \operatorname{Vol}(\Omega)^{n-1} \leq \operatorname{Vol}(\partial\Omega \sim \partial C)^n, \tag{2}$$

with equality holding if and only if Ω is a half-ball and $\partial\Omega \sim \partial C$ is a hemisphere?

In [1] Aubin conjectured that (1) should hold for a domain Ω in an n-dimensional simply connected Riemannian manifold M^n of nonpositive sectional curvature. This conjecture is still open except for the dimensions $n = 2, 3, 4$; these cases were proved by Weil [10], Kleiner [9], and Croke [7], respectively.

Extending Aubin's conjecture, one can ask the following. Does (2) hold for a simply connected Riemannian manifold M^n of nonpositive sectional curvature, C a convex domain in M, and Ω a domain in $M \sim C$? Does equality hold if and only if Ω is a Euclidean half ball?

One can easily prove (2) in a two-dimensional M by considering the convex hull of Ω. Recently, the relative isoperimetric inequality in M^3 was proved in [6]. In this paper we prove the inequality in M^4. However, in dimensions higher than four, the problem is still open. In Euclidean space $\mathbb{R}^n$, there are some partial results [8, 4] and, recently, a general result [5].

The key idea of this paper in the proof of (2) is that the concavity of $M \sim C$ conforms naturally to the negativity of the curvature of M. We employ Croke's method [7] in this paper.

We would like to thank Rick Schoen for inviting us to have a valuable sabbatical year at Stanford University in 2002-2003.

2. Double cover of Ω relative to C

Let M be an n-dimensional Riemannian manifold and SM the unit sphere bundle of M. A geodesic flow Φ_t on M satisfies

$$\gamma_v(t) = \pi \circ \Phi_t(v) \quad \text{and} \quad \gamma_v'(t) = \Phi_t(v)$$

where γ_v denotes the geodesic with initial point $\pi(v)$ and initial velocity vector v, and π is the projection from SM onto M. Note that Φ_t takes SM to itself. Liouville proved that Φ_t preserves the canonical measure on SM, the local product of the Lebesgue measure on the unit tangent spheres with the Riemannian measure on M. From this theorem one obtains Santalo's formula as follows.

Let $\Omega \subset M$ be a relatively compact domain. For $v \in SM$, we set

$$l(v) = \sup\{\tau : \gamma_v(t) \in \Omega, \quad \forall t \in (0, \tau)\},$$

that is, $\gamma_v(l(v))$ will be the first point on the geodesic to hit $\partial\Omega$. Denote by ν the inward unit normal vector field along $\partial\Omega$, and let $S^+\partial\Omega$ denote the set of inward pointing unit vectors along $\partial\Omega$, that is,

$$S^+\partial\Omega = \left\{u \in S\overline{\Omega}|_{\partial\Omega} : \langle u, \nu_{\pi(u)}\rangle > 0\right\}.$$

The measure du on $S^+\partial\Omega$ is the local product of the canonical measure on unit tangent hemispheres with the Riemannian measure on $\partial\Omega$.

Since the measure dv on SM is invariant with respect to the geodesic flow Φ_t, integration on $S\Omega$ can be performed by summing up the one-dimensional integrals along all geodesics in Ω starting from $\partial\Omega$. This is the gist of Santalo's formula

$$\int_{S\Omega} f(v)\,dv = \int_{S^+\partial\Omega} \langle u, \nu_{\pi(u)}\rangle\,du \int_0^{l(u)} f(\Phi_t u)\,dt.$$

For a proof, see [3, pages 231–232].

A characteristic of the relative isoperimetric inequality is that it does not count the volume of $\partial\Omega \cap \partial C$. In other words, $\partial\Omega \cap \partial C$ is not considered to be part of the boundary of Ω. This motivates us to consider the *gluing* of Ω with itself along $\partial\Omega\cap\partial C$. More precisely, let Ω_1 and Ω_2 be two replicas of Ω, let $\approx$ be the equivalence relation which identifies the two points of $\partial\Omega_1$ and $\partial\Omega_2$ that correspond to a point of $\partial\Omega\cap\partial C$, and define $\Omega^* = \Omega_1 \cup \Omega_2/\approx$. Let us call Ω^* the *double cover of* Ω *relative to* C. Obviously, Ω^* is a smooth manifold if ∂C is smooth. Its boundary $\partial\Omega^*$ is the double cover of $\partial\Omega \sim \partial C$.

Although the metric of Ω^* is smooth away from ∂C, it is just continuous on $\partial\Omega\cap\partial C$. Being a Riemannian manifold, Ω^* has geodesics. When a geodesic of Ω^* moves from Ω_1 to Ω_2, or the other way around, it bounces off C at $\partial\Omega \cap \partial C$ just as a light ray is reflected by a mirror. Given a point p off ∂C and $v \in M_p$ there exists a unique geodesic γ_v starting from p in the direction of v. However, if p is in $\partial\Omega \cap \partial C$ and v is tangent to ∂C then there are three geodesics γ_v on Ω^* since there are two identical geodesics γ_v on Ω_1 and Ω_2, and the third is the geodesic of ∂C in v direction.

Nonuniqueness of geodesics is due to the nonsmoothness of the metric of Ω^* along ∂C. Since the metric is only continuous, the Christoffel symbols Γ^i_{jk} are discontinuous at $p \in \partial C$ and so the sectional curvature can be infinite at p if ∂C is strictly convex. Still, the Jacobi field J is well defined. J is smooth away from ∂C and continuous along ∂C. Because of nonuniqueness of geodesic, the geodesic flow Φ_t on Ω^* along a geodesic path γ is not well defined when γ is tangent to ∂C. However, it is well defined and smooth almost everywhere. In particular, it is not difficult to see that Φ_t is measure preserving along γ when γ is transversal to ∂C. This is because even though the metric of Ω^* is not smooth at $p \in \gamma \cap \partial C$, Φ_t is measure preserving both up to p and after p. Therefore we still have Santalo's formula on the C^0 Riemannian manifold Ω^*

$$\int_{S\Omega^*} f(v)\,dv = \int_{S^+\partial\Omega^*} u_\nu\,du \int_0^{l(u)} f(\Phi_t u)\,dt,$$

where $u_\nu := \langle u, \nu_{\pi(u)}\rangle$. Hence letting $f(v) \equiv 1$ gives the following.

LEMMA 2.1. $\mathrm{Vol}(\Omega^*) = (1/n\omega_n)\int_{S^+\partial\Omega^*} l(u)u_\nu\,du$.

Recall that Ω^* is a double cover of Ω and $\partial\Omega^*$ is a double cover of $\partial\Omega \sim \partial C$. Therefore the relative isoperimetric inequality (2) for $\Omega \subset M$ will follow if we can prove the classical isoperimetric inequality for Ω^*,

$$n^n\omega_n\,\mathrm{Vol}(\Omega^*)^{n-1} \le \mathrm{Vol}(\partial\Omega^*)^n. \tag{3}$$

For the following lemma let us write $\mathrm{ant}\,u := -\gamma_u'(l(u))$. See [7] for its proof.

LEMMA 2.2. *For an integrable function g on $S^+\partial\Omega^*$,*

$$\int_{S^+\partial\Omega^*} g(u)u_\nu\,du = \int_{S^+\partial\Omega^*} g(\mathrm{ant}\ u)u_\nu\,du.$$

3. Concavity vs negativity of curvature

Suppose that M is a 2-dimensional Riemannian manifold, C is a convex domain in M, and $D \subset M$ is a domain in the exterior of C. Then the Gaussian curvature of D^* along $\partial D \cap \partial C$ can be $-\infty$. For example, let $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $D = \{(x, y) : 1 < x^2 + y^2 < 2\}$. Then the integral of the Gaussian curvature of D^* along $\partial D \cap \partial C$ equals -4π. This follows from the Gauss-Bonnet theorem applied to the annulus D^*.

Thus the concavity of D along $\partial D \cap \partial C$ implies the negativity of curvature on $D^* \cap \partial C$. However, this is not the case for a domain Ω in M^n, $n \ge 3$. The sectional curvature of Ω^* along the section of ∂C is even positive. However, if M is simply connected and nonpositively curved, Ω^* still enjoys properties of a negatively curved manifold: (i) the volume of a geodesic ball in Ω^* grows as in a negatively curved manifold; and (ii) two rays emanating from a point never intersect each other. First we need the following.

LEMMA 3.1. *Suppose that M is simply connected and nonpositively curved, $C \subset M$ is a convex domain and a domain $\Omega \subset M$ lies in the exterior of C. Suppose that $\sigma : [0, l] \to \Omega^*$ is a geodesic segment passing through $\partial\Omega \cap \partial C$ at $\sigma(a)$ transversally. Then $\sigma(t) \notin \partial\Omega \cap \partial C$ for any $t \neq a$.*

PROOF. Suppose that $\sigma \subset \Omega$ hits $\partial\Omega \cap \partial C$ when $t = b$. Since M is simply connected and nonpositively curved, σ is the unique geodesic from $\sigma(a)$ to $\sigma(b)$. By the convexity of C, $\sigma(a, b)$ lies in C, which is a contradiction. □

Lemma 3.1 implies that a geodesic which moves from Ω_1 to Ω_2 transversally crossing $\partial\Omega \cap \partial C$ never comes back to Ω_1. This partially proves property (ii) mentioned above.

Let dx be the volume form of M^n, du_p the volume form of the unit sphere in M_p, and (u, r) the polar coordinates about $p \in M$. Then $dx = h(u, r)du_p dr$ for some positive function $h(u, r)$. If M has nonpositive sectional curvature then $h(u, r) \geq r^{n-1}$ with equality if and only if the sectional curvatures of all sections containing γ_u' are 0 (see [2, Section 11.10]).

LEMMA 3.2. (a) *Let M, C, Ω be as in Lemma 3.1. Then $h(u, r) \geq r^{n-1}$ on Ω^*, with equality for every p if and only if Ω^* is flat.*
(b) *Two rays in Ω^* emanating from a point and transversal to $\partial\Omega \cap \partial C$ never intersect each other.*

PROOF. (a) We have only to consider the case when the geodesic realizing r hits ∂C transversally. Fix $p \in \Omega^*$ and let S be a 2-dimensional surface in Ω^* consisting of geodesics emanating from p. Let $J(t)$ be the Jacobi field along a geodesic $\gamma(t)$ from $p = \gamma(0)$ with $J(0) = 0$, $|J'(0)| = 1$, and $J'(0) \perp \gamma'(0)$. J satisfies the Jacobi equation

$$J'' + R(\gamma', J)\gamma' = 0, \tag{4}$$

where R is the Riemann curvature tensor of S. However, this equation is not well defined because $R = -\infty$ when γ hits ∂C. So let us consider J' instead of J''. Equation (4) implies that $|J'(t)|$ is nondecreasing as a function of t away from ∂C. When γ hits ∂C, $|J'|$ is discontinuous. The point is that $|J'|$ jumps up on ∂C. This is where the convexity of C plays a key role. Hence $|J'|$ can be said to be nondecreasing everywhere. Therefore

$$|J'(t)| \geq |J'(0)| = 1,$$

and hence

$$|J(t)| \geq t. \tag{5}$$

This inequality implies that the exponential map $\exp : \Omega_p^* \to \Omega^*$ is length increasing (nondecreasing, to be precise). Now let us show that exp is volume increasing. Suppose that $d\exp(u_i) = v_i$, $i = 1, \ldots, n-1$, and that v_i are orthogonal to each

https://doi.org/10.1017/S1446788700014087 Published online by Cambridge University Press

other and $v_i \perp \gamma'$. Let U and V be $(n-1)$-dimensional parallelepipeds generated by u_i and v_i, respectively. Then

$$\mathrm{Vol}(U) \leq \prod_{1\leq i\leq n-1} |u_i| \leq \prod_{1\leq i\leq n-1} |v_i| = \mathrm{Vol}(V).$$

Hence exp is volume increasing and it follows that $h(u, r) \geq r^{n-1}$.

If equality holds at every p, then $\mathrm{Vol}(U) = \mathrm{Vol}(V)$ and so $|u_i| = |v_i|$ and u_i are pairwise orthogonal. Thus exp is an isometry and Ω^* is flat.

(b) We see from (5) that exp has nonsingular differential and hence it is a local diffeomorphism. Therefore the exponential map is one-to-one. □

LEMMA 3.3.
$$\int_{S^+\partial\Omega^*} \frac{l(u)^{n-1}}{(\mathrm{ant}\, u)_\nu}\, du \leq \mathrm{Vol}(\partial\Omega^*)^2$$
with equality if and only if Ω is flat and convex.

PROOF. Let dA be the volume form of $\partial\Omega^*$. If we denote $B = \exp\{tu : t = l(u)\}$, then $B \subset \partial\Omega^*$ and $dA|_B = h(u, l(u))/(\mathrm{ant}\, u)_\nu\, du_p$. Write $S_p^+\partial\Omega^* = \pi^{-1}\{p\}$ for $\pi : S^+\partial\Omega^* \to \partial\Omega^*$. Then the map $\varphi : S_p^+\partial\Omega^* \to \partial\Omega^*$ defined by $\varphi(u) = \exp(l(u)u)$ is a one-to-one map by Lemma 3.2 (b). This is another place where the convexity of C is critically used. Therefore we have

$$\int_{S_p^+\partial\Omega^*} \frac{h(u, l(u))}{(\mathrm{ant}\, u)_\nu}\, du_p = \mathrm{Vol}(B) \leq \mathrm{Vol}(\partial\Omega^*).$$

Note that $\mathrm{Vol}(B) = \mathrm{Vol}(\partial\Omega^*)$ if and only if $\partial\Omega^*$ is star-shaped from p. Integrating over $p \in \partial\Omega^*$ yields
$$\int_{S^+\partial\Omega^*} \frac{h(u, l(u))}{(\mathrm{ant}\, u)_\nu}\, du \leq \mathrm{Vol}(\partial\Omega^*)^2$$
with equality if and only if Ω is convex. Thus Lemma 3.2 (a) completes the proof. □

See [7] for the proof of the following.

LEMMA 3.4.

$$\int_{S^+\partial\Omega^*} (\mathrm{ant}\, u)_\nu{}^{1/(n-2)} u_\nu{}^{(n-1)/(n-2)}\, du \leq \alpha_n \mathrm{Vol}(\partial\Omega^*)$$

where

$$\alpha_n = (n-1)\omega_{n-1} \int_0^{\pi/2} \cos^{n/(n-2)} t \sin^{n-2} t\, dt.$$

Equality holds if and only if $(\mathrm{ant}\, u)_\nu = u_\nu$ *everywhere.*

https://doi.org/10.1017/S1446788700014087 Published online by Cambridge University Press

4. Theorem

We are now ready to prove the relative isoperimetric inequality for $\Omega \subset M \sim C$.

THEOREM. *Let M be a four-dimensional simply connected Riemannian manifold of nonpositive sectional curvature. If $C \subset M$ is a convex domain and Ω is a domain in $M \sim C$, then Ω satisfies*

$$64\,\pi^2\,\mathrm{Vol}(\Omega)^3 \le \mathrm{Vol}(\partial\Omega \sim \partial C)^4.$$

Equality holds if and only if Ω is a Euclidean half-ball and $\partial\Omega \sim \partial C$ is a hemisphere.

PROOF. We will first prove the classical isoperimetric inequality (3) for Ω^*.

$$\begin{aligned}
\mathrm{Vol}(\Omega^*) &= \frac{1}{2\pi^2}\int_{S^+\partial\Omega^*} l(u)u_\nu\,du \quad \text{(Lemma 2.1)}\\
&= \frac{1}{2\pi^2}\int_{S^+\partial\Omega^*} \frac{l(u)}{(\mathrm{ant}\,u)_\nu^{1/3}}\,(\mathrm{ant}\,u)_\nu^{1/3}u_\nu\,du\\
&\le \frac{1}{2\pi^2}\left(\int_{S^+\partial\Omega^*}\frac{l(u)^3}{(\mathrm{ant}\,u)_\nu}\,du\right)^{1/3}\left(\int_{S^+\partial\Omega^*}(\mathrm{ant}\,u)_\nu^{1/2}u_\nu^{3/2}\,du\right)^{2/3} \quad \text{(Hölder)}\\
&\le \frac{1}{2\pi^2}\,\mathrm{Vol}(\partial\Omega^*)^{2/3}\left(\frac{\pi^2}{4}\right)^{2/3}\mathrm{Vol}(\partial\Omega^*)^{2/3}. \quad \text{(Lemmas 3.3–3.4)}
\end{aligned}$$

Therefore

$$128\,\pi^2\,\mathrm{Vol}(\Omega^*)^3 \le \mathrm{Vol}(\partial\Omega^*)^4.$$

Dividing this inequality by 2^4 gives the desired relative isoperimetric inequality for Ω.

Equality holds only if we have equalities in Lemmas 3.3–3.4 as well as in the Hölder inequality. Hence equality holds only if Ω is flat and convex, $(\mathrm{ant}\,u)_\nu = u_\nu$, and $l(u) = d\,u_\nu$ for some constant $d > 0$. Therefore Ω^* is an Euclidean ball of diameter d. □

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