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The C^1 Closing Lemma, including Hamiltonians

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Abstract. An Axiom of Lift for classes of dynamical systems is formulated. It is shown to imply the Closing Lemma. The Lift Axiom is then verified for dynamical systems ranging from C^1 diffeomorphisms to C^1 Hamiltonian vector fields.

1. *Introduction*

If a dynamical system has a recurrent trajectory γ, then one may ask: does there exist a nearby system with a periodic trajectory near γ? This is the *Closing Problem for* γ. For example,

$$\dot{x} = 1, \qquad \dot{y} = \alpha, \qquad \alpha \text{ irrational},$$

generates a flow on the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ each of whose trajectories is everywhere recurrent. A slight change of the differential equation (take $\dot{x} = 1$, $\dot{y} = \beta$, where β is rational and nearly equal to α) makes all trajectories periodic, i.e. it closes up all the recurrence. Depending on the type of perturbation permitted, the Closing Problem is trivial, solved, or open. See §§ 5–9 for details and its solution. See [**20**] for a conceptual outline of our proof.

Solving the Closing Problem is interesting in its own right, but more because it implies generically that a dynamical system already has its periodic trajectories dense in its set of recurrent trajectories. (This is Axiom Ab of Smale [**21**].) To be more precise, let M be a smooth manifold, and consider the three spaces of dynamical systems:

$$\mathscr{X}^rM = \text{the } C^r \text{ tangent vector fields on } M;$$
$$\mathscr{F}^rM = \text{the } C^r \text{ flows on } M;$$
$$\mathscr{D}^rM = \text{the } C^r \text{ diffeomorphisms of } M \text{ to itself.}$$

Each space has a natural C^r topology, which we detail below, making it a *Baire space* – one in which every G_δ-subset is dense. (Recall that a G_δ-subset of a topological space X is a countable intersection of open-dense subsets of X.) A property enjoyed by all elements of a G_δ-subset of a Baire space is a *generic property* – 'most' elements possess the property.

General density theorem (GDT) for C^1 diffeomorphisms. If M is compact, then the generic $f \in \mathscr{D}^1 M$ satisfies

$$\mathrm{Per}\,(f) \text{ is dense in } \Omega(f),$$

where Per (f) *is the set of periodic points of f and $\Omega(f)$ is the set of non-wandering points of f.*

Recall that p is *periodic* for f if $f^n p = p$ for some $n \neq 0$; p is *non-wandering* for f if, for each neighbourhood U of p, $f^n(U) \cap U \neq \varnothing$ for some $n \neq 0$. (f^n denotes $f \circ \cdots \circ f$, n times.) The set of non-wandering points of f is closed, f-invariant, and contains all periodic points, all recurrent points, all α- and ω-limit points.

The corresponding genericity result holds for $\mathscr{F}^1$, $\mathscr{X}^1$. See § 11. See also § 11 for the case of non-compact M.

If M has more structure, say a volume or a symplectic structure, then one may examine the subspaces of $\mathscr{X}^r$, $\mathscr{F}^r$, $\mathscr{D}^r$ whose elements preserve that structure. In many such cases we prove again that generically – in the subspace – the periodic trajectories are dense in the non-wandering set. Since the subspaces are closed, nowhere dense, genericity results in the overlying space are no help in the subspace (see § 8). More special yet is the space of Hamiltonian systems $\mathscr{X}_H$ on a symplectic manifold. $\mathscr{X}_H$ is Baire. Our results verify a:

Conjecture of Poincaré [**14**, p. 82]. *C^1 generically in $\mathscr{X}_H$, the periodic trajectories are dense in the compact energy surfaces.* See § 9 and (11.4).

When $r \geq 2$ very little is known about the Closing Problem. For instance, it is not known on T^2 whether every recurrent trajectory of a smooth differential equation can be closed up by a C^2 small change of the equation. The results of Peixoto [**12**], however, do imply the GDT in $\mathscr{X}^r(M^2)$, $\mathscr{F}^r(M^2)$, where $1 \leq r \leq \infty$ and M^2 is compact, orientable. For general non-orientable M^2 and all higher dimensional manifolds it remains unknown whether the generic C^2 differential equation has its periodic trajectories dense in its non-wandering set. Likewise, the C^r Conjecture of Poincaré, $r \geq 2$, remains open.

One negative result for $r \geq 2$ is that some kinds of 'double closing' are C^2 impossible but C^1 possible [**18**]. See [**11**, problems 21–23] for further open problems and discussion.

We owe thanks to all our colleagues, but especially we are grateful to the Instituto de Matemática Pura e Aplicada for the final push to publish this work.

The C^r topology

Let V, W be C^∞ finite dimensional manifolds and let $C^r(V, W)$ be the set of C^r maps $f: V \to W$. We topologize $C^r(V, W)$ as follows.

Case 1. V is compact and $0 \leq r < \infty$. Fix $f \in C^r(V, W)$ and choose a pair of finite C^∞ chart coverings such that for each α, v_α extends to a C^∞ chart defined on a neighbourhood of $\bar{V}_\alpha$ and $f(\bar{V}_\alpha) \subset W_\beta$ for some $\beta = \beta(\alpha)$. For $\varepsilon > 0$, call $\mathscr{U}_\varepsilon(f)$

$$\begin{array}{ccccccc} V_\alpha & \hookrightarrow & V & \xrightarrow{f} & W & \hookleftarrow & W_\beta \\ \Big\downarrow{\scriptstyle v_\alpha} & & & & & & \Big\downarrow{\scriptstyle w_\beta} \\ \mathbb{R}^v & & & & & & \mathbb{R}^w \end{array}$$

$$\alpha = 1, \ldots, N \qquad \beta = 1, \ldots, L$$

the set of all $g \in C^r(V, W)$ such that, for all $\alpha = 1, \ldots, N$ and $\beta = \beta(\alpha)$,

$$g(V_\alpha) \subset W_\beta,$$
$$\|D^k(w_\beta g v_\alpha^{-1} - w_\beta f v_\alpha^{-1})_x\|_k < \varepsilon,$$

as x ranges over $v_\alpha(V_\alpha)$ and $0 \le k \le r$. By D^k we denote the kth Frechet derivative of a map between linear spaces and by $\| \ \|_k$ the norm on k-linear maps arising from the Euclidean norms on $\mathbb{R}^v$, $\mathbb{R}^w$. A second choice of coverings, say $\{V'_\alpha\}$ and $\{W'_\beta\}$, produces neighbourhoods $\mathcal{U}'_\varepsilon(f)$ comparable to the $\mathcal{U}_\varepsilon(f)$ in the sense that each $\mathcal{U}_\varepsilon(f)$ contains a $\mathcal{U}'_\delta(f)$ and vice versa. A C^r neighbourhood of f is any subset of $C^r(V, W)$ containing some $\mathcal{U}_\varepsilon(f)$ and a set in $C^r(V, W)$ is C^r-open if it is a neighbourhood of each of its points. This is the natural C^r topology on $C^r(V, W)$. See [**7**, chapter 2].

PROPOSITION. *$C^r(V, W)$ is completely metrizable and is hence a Baire space.*

Proof. [**7**, p. 62].

Case 2. V is paracompact and $0 \le r < \infty$. Then

$$V = V_1 \cup V_2 \cup \cdots,$$

where each V_n is compact and smooth but no sequence $p_n \in V_n$ converges in V. If $\mathcal{U}_n$ is a C^r neighbourhood of $f \mid V_n$ in $C^r(V_n, W)$, then

$$\{g \in C^r(V, W) \colon g \mid V_n \in \mathcal{U}_n\}$$

is a C^r *Whitney neighbourhood* of f in $C^r(V, W)$. Again, different choices of $\{V_n\}$, $\{\mathcal{U}_n\}$ produce comparable families of neighbourhoods of f, so we again unambiguously topologize $C^r(V, W)$. ('Fine' or 'strong' are other names for this topology.)

PROPOSITION. *$C^r(V, W)$ is Baire.*

Proof. [**7**, p. 62].

Remark. Topologically, $C^r(V, W)$ is horrible.

Case 3. $r = \infty$. The C^∞ topology on $C^\infty(V, W)$ is the smallest one making every inclusion $C^\infty(V, W) \hookrightarrow C^r(V, W)$ continuous, $0 \le r < \infty$.

PROPOSITION. *$C^\infty(V, W)$ is Baire.*

Proof. [**7**, p. 62].

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The C^r topology on $\mathscr{D}^r$

Think of $\mathscr{D}^r$ as a subset of $C^r(M, M)$. As such it inherits its natural C^r topology.

PROPOSITION. *If M is compact and boundaryless, then $\mathscr{D}^r$ is an open subset of $C^r(M, M)$, $1 \le r \le \infty$, and is hence Baire. Indeed, $\mathscr{D}^r$ is a Baire space, no matter what.*

Proof. [**7**, p. 37 and p. 62].

The C^r topology on $\mathscr{X}^r$

Think of $\mathscr{X}^r$ as a subset c. $C^r(M, TM)$ where TM is the tangent bundle of M. As such, it inherits its natural C^r ιopology.

PROPOSITION. *If M is compact and $0 \le r < \infty$, then $\mathscr{X}^r$ is Banachable and hence Baire; if $r = \infty$ then $\mathscr{X}^r$ is Frechetable and hence Baire. Even if M is not compact, $\mathscr{X}^r$ is Baire.*

Proof. [**7**, p. 62] since $\mathscr{X}^r$ is closed in $C^r(M, TM)$.

The topology on $\mathscr{F}^r$

A C^r flow is a C^r map $\varphi : \mathbb{R} \times M \to M$ such that

$$t \mapsto \varphi(t, \cdot) \qquad \mathbb{R} \to \mathscr{D}^r$$

is a homomorphism. For any $a < b$ we can take the restriction

$$\rho : \begin{array}{l} \mathscr{F}^r \to C^r([a, b] \times M, M) \\ \varphi \mapsto \varphi \,|\, [a, b] \times M. \end{array}$$

The C^r topology on $\mathscr{F}^r$ is the smallest one making ρ continuous. It is independent of a, b. Note that we do *not* put the C^r Whitney topology on $\mathscr{F}^r$ as a subset of $C^r(\mathbb{R} \times M, M)$, because that would permit almost no perturbations.

PROPOSITION. *$\mathscr{F}^r$ is Baire.*

Proof. [**7**, p. 62] since $\rho(\mathscr{F}^r)$ is closed in $C^r([a, b] \times M, M)$.

The relation between $\mathscr{F}^r$ and $\mathscr{X}^r$

Each C^r vector field X integrates to a C^r flow φ

$$i : \begin{array}{c} X \mapsto \varphi \\ \mathscr{X}^r \to \mathscr{F}^r \end{array}$$

when $1 \le r \le \infty$. This i is continuous, but except for $r = \infty$ it is not surjective. Results of Dorroh [**4**] and Hart [**5**] show i has a dense range. The differentiation operator $d : \mathscr{F}^r \to \mathscr{X}^{r-1}$, $1 \le r < \infty$, produces a non-linear subspace of $\mathscr{X}^{r-1}$, which has little to do with $\mathscr{F}^r$ from our point of view.

The relation between $\mathscr{F}^r$ and $\mathscr{D}^r$

The time-t-map of a C^r flow is a C^r diffeomorphism, but there are many C^r diffeomorphisms which are not part of a flow, even if they are isotopic to the identity. See the work of Palis [**10**]. On the other hand, any C^r diffeomorphism f

of M naturally 'suspends' to a C^r flow on a manifold of one higher dimension, $M\times I/f$. See [**21**]. In § 7A, remark 3, we see how to suspend f in such a way that its flow is generated by a C^r vector field.

2. *Lift*

Here we make precise the perturbation property of C^1 dynamical systems indicated in the local trajectory figures (figures 1 and 2).

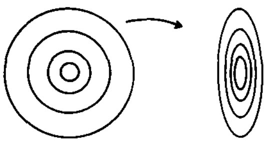

FIGURE 1*a*. Diffeo before lift.

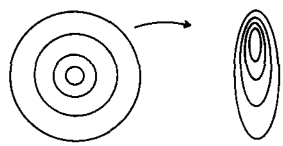

FIGURE 1*b*. Diffeo after lift.

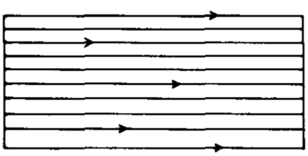

FIGURE 2*a*. Flow before lift.

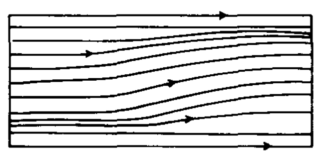

FIGURE 2*b*. Flow after lift.

In § 5 we prove that Lift$\Rightarrow$Closing and in §§ 6–8 we verify the Lift Axiom in the most interesting cases.

To keep track of uniformities we assume M is compact, has a C^∞ Riemann structure, and the associated exponential map exp *embeds each unit ball into M,*

$$\exp_p : T_pM(1)\to M.$$

By $M_p(r)$ we denote the r-ball at p, $\exp_p(T_pM(r))$, $0\le r\le 1$.

Now suppose $\mathcal{S}$ is some subset of $\mathcal{D}^1$. For example, $\mathcal{S}$ could be all of $\mathcal{D}^1$ or it could be the symplectic diffeomorphisms of M if M has a symplectic structure.

Definition. $\mathcal{S}$ satisfies the *Lift Axiom* if, for each $f\in\mathcal{S}$ and each C^1 neighbourhood $\mathcal{N}$ of f in $\mathcal{S}$, there exists $\varepsilon>0$ such that, whenever $v\in T_pM(1)$, we have a perturbation g of the identity satisfying $g\circ f\in\mathcal{N}$ and

(L1) $g(p)=\exp(\varepsilon v)$,

(L2) $\operatorname{supp}(g)\subset M_p(|v|)$.

In addition, we require some 'flexibility' of $\mathcal{S}$:

(L3) If $g_1,\dots,g_n$ are several such perturbations having disjoint supports then $g_1\circ\cdots\circ g_n\circ f\in\mathcal{S}$.

(The support of a diffeomorphism is the closure of the set where it differs from the identity.) If $\mathcal{S}$ is closed under composition (L3) is automatic. What the Lift Axiom means is:

Inside $\mathcal{S}$, one can lift points p in prescribed directions v with results proportional to the support radius.

Next we formulate the Lift Axiom for subsets of $\mathcal{F}^1$ and $\mathcal{X}^1$. To do so, we use *Poincaré maps*. Let $\varphi \in \mathcal{F}^1$, say $\varphi = \varphi_t(x)$. Call $\dot{\varphi}$ the continuous vector field generating φ

$$\dot{\varphi}(p) = \frac{d}{dt}\bigg|_{t=0} \varphi_t(p).$$

By van Kampen's Uniqueness Theorem [**6**, p. 35], the only trajectories of $\dot{\varphi}$ are the φ-curves themselves. Suppose $\dot{\varphi}(p) \neq 0$. Let

$$\hat{\Pi}_p = (\text{span } \dot{\varphi}(p))^{\perp} = \text{the hyperplane in } T_pM \text{ perpendicular to } \dot{\varphi}(p),$$

$$\Pi_p = \exp(\hat{\Pi}_p(1)) \quad \text{and} \quad \Pi_p(\delta) = \exp(\hat{\Pi}_p(\delta)),$$

where $\hat{\Pi}_p(\delta)$ is the δ-ball in $\hat{\Pi}_p$. Then Π_p is a smooth $(m-1)$-disk in M, transverse to φ at p. Call

$$p' = \varphi_1(p), \qquad \Pi'_p = \Pi_{p'}.$$

The flow φ uniquely defines the *canonical Poincaré map* $f: \Pi_p(\delta) \to \Pi'_p$ provided that δ is small enough. Due to singularities, δ may tend to 0 as p varies. Indeed,

PROPOSITION. *With φ, Π_p, Π'_p as above, there is a $\delta > 0$ such that*:

(*a*) *There is a unique continuous choice of $t = t(y)$ with $t(p) = 1$ and $\varphi_{t(y)}(y) \in \Pi'_p$ for all $y \in \Pi_p(\delta)$. This map $f(y) = \varphi_{t(y)}(y)$ is a C^1 embedding whose image is interior to Π'_p.*

(*b*) *If $\varphi_t p \neq p$ for $0 < t \leq 1$ then $(t, y) \mapsto \varphi_t y$ C^1-embeds*

$$\{(t, y) \in \mathbb{R} \times \Pi_p(\delta) : 0 \leq t \leq t(y)\}.$$

(*c*) *If $\mathcal{V}$ is a sufficiently small neighbourhood of φ in $\mathcal{F}^1$ then there is a unique continuous $t = t(y, \psi)$ defined for $y \in \Pi_p(\delta)$ and $\psi \in \mathcal{V}$ with $t(y, \varphi) = t(y)$ as in (a) and $\psi_{t(y,\psi)}(y) \in \Pi'_p$. This map $f_\psi(y) = \psi_{t(y,\psi)}(y)$ is a C^1 embedding whose image is interior to Π'_p.*

(*d*) *$\psi \mapsto f_\psi$ is a continuous map $\mathcal{V} \to \mathrm{Emb}^1(\Pi_p(\delta), \Pi'_p)$.*

Proof. Standard by transversality theory; see also figure 3; see § 7A for the converse perturbation question.

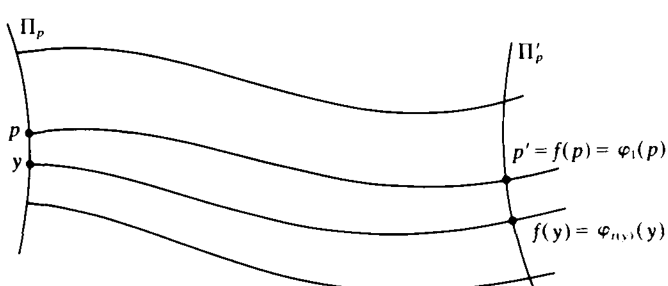

FIGURE 3. The Poincaré map and flowbox F.

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Definition. The image

$$\{\varphi_t y : y \in \Pi_p(\delta) \text{ and } 0 \le t \le t(y)\}$$

is the (unit time length) *φ-flowbox* and is denoted by $F = F_p(\delta)$.

Now suppose $\varphi \in \mathscr{S} \subset \mathscr{F}^1$ and $S \subset M$ is open.

Definition. $\mathscr{S}$ satisfies the *Lift Axiom at φ on S* if for each C^1 neighbourhood $\mathscr{N}$ of φ in $\mathscr{S}$, there exist $\varepsilon > 0$ and a continuous function $\delta : S \to (0, 1)$ such that whenever $v \in \hat{\Pi}_p(\delta(p))$ we have a perturbation ψ of φ in $\mathscr{N}$ satisfying:

(L′1) The Poincaré map f_ψ of ψ is defined via (c) above and $f_\varphi^{-1} \circ f_\psi(p) = \exp(\varepsilon v)$.

(L′2) $\operatorname{supp}(\dot{\psi} - \dot{\varphi}) \subset F_p(|v|)$.

(L′3) If several such perturbations of φ are made in disjoint flowboxes then their union-perturbation belongs to $\mathscr{S}$.

Note that (L′1) says: in the flowbox coordinates we can push p in the v-direction. The function δ probably tends to 0 at ∂S. In § 6 we verify that $\mathscr{S} = \mathscr{F}^1$ satisfies the Lift Axiom at all $\varphi \in \mathscr{F}^1$ on $S = M - \mathscr{P}$ where $\mathscr{P} = \{p : \varphi_t p = p \text{ for some } t, 0 < t \le 1\}$.

Finally, suppose $X \in \mathscr{S} \subset \mathscr{X}^1$ and $S \subset M$ is open.

Definition. $\mathscr{S}$ satisfies the *Lift Axiom at X on S* provided that $i(\mathscr{S})$ satisfies the $\mathscr{F}^1$ Lift Axiom at $i(X)$ on S using the $i(\mathscr{X}^1)$ topology. That is, given a C^1 neighbourhood $\mathscr{N}$ of X in $\mathscr{S}$, then there exist $\varepsilon > 0$ and a continuous function $\delta : S \to (0, 1)$ such that, whenever $v \in \hat{\Pi}_p(\delta(p))$, we have a perturbation Y of X in $\mathscr{N}$ whose flow ψ satisfies (L′1), (L′2), (L′3), φ being the X-flow. If $\mathscr{S}$ is closed under addition, (L′3) is automatic. In § 7 we verify that $\mathscr{S} = \mathscr{X}^1$ satisfies the Lift Axiom at all $X \in \mathscr{X}^1$ on $S = M - \mathscr{P}$, $\mathscr{P}$ as above.

In § 9, we give a modification of the Lift Axiom for Hamiltonian vector fields. In this case it is possible to push only along the energy surface. We then verify the axiom and show how it implies the Hamiltonian Closing Lemma.

3. *Linear algebra*

In this section we do the linear algebra necessary for our Closing Lemma proof. The techniques used are irrelevant to the other sections of this paper, so the impatient reader need only absorb the content of (3.1), (3.2), (3.3). A more general version of this section occurs in [**17**].

As a standing hypothesis, let $T : \mathbb{R}^m \to H$ be a monomorphism, H a Euclidean space and $V \subset \mathbb{R}^m$ a subspace. By T^{-1} we mean the inverse of T with domain $T(\mathbb{R}^m)$.

Definition. The *bolicity* of T is the ratio of the maximum stretch to the minimum stretch

$$\operatorname{bol}(T) = \frac{\|T\|}{\mathbf{m}(T)},$$

where

$$\mathbf{m}(T) = \min_{|x|=1} |Tx| = \text{the } \textit{minimum norm} \text{ of } T,$$

is also called the *conorm*. Clearly, $\mathbf{m}(T) = \|T^{-1}\|^{-1}$,

$$1 = \|TT^{-1}\| \le \|T\|\|T^{-1}\| = \mathrm{bol}\,(T),$$

and

$$\mathrm{bol}\,(T) = 1 \quad \text{iff} \quad T \text{ is conformal}$$

(preserves $|x \cdot y|/|x||y|$).

Definition. The *V-altitude* of T, $A\colon V \to H$, is defined by commutativity of

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "T"] & H \arrow[d, "\pi"] \\ V \arrow[u, hook] \arrow[r, "A"] & T(V^\perp)^\perp \end{tikzcd}$$

where $\pi : H \to (TV^\perp)^\perp$ is the orthogonal projection (see figure 4).

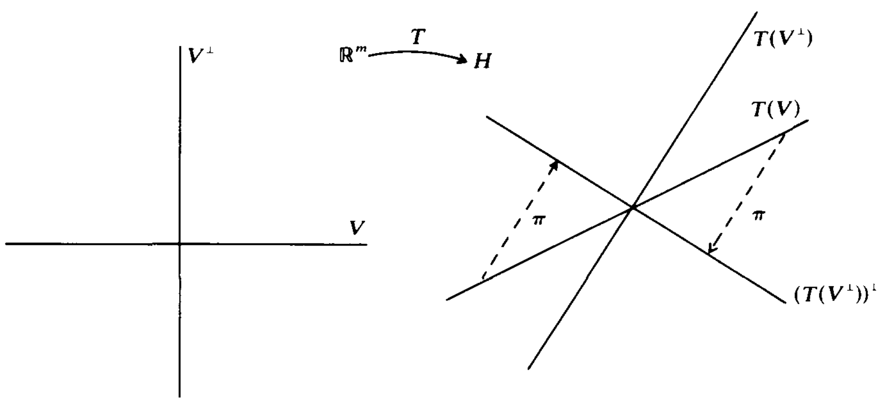

FIGURE 4. The V-altitude $A = \pi \circ T/V$.

Definition. The *V-co-altitude of T*, $A^\perp : V^\perp \to H$, makes

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "T"] & H \arrow[d, "\rho = \text{orthogonal projection}"] \\ V^\perp \arrow[u, hook] \arrow[r, "A^\perp"] & (TV)^\perp \end{tikzcd}$$

commute. That is, $A^\perp$ is the $V^\perp$-altitude of T.

Definition. The *hyperbolicity of T respecting V* is

$$\mathrm{hyp}_V\,(T) = \frac{\mathbf{m}(A^\perp)}{\|A\|}.$$

For $V =$ the x-axis in $\mathbb{R}^2$, this is the ratio of the two altitudes of the parallelogram, $T(I^2)$ (see figure 5). If T preserves the orthogonality, $TV \perp TV^\perp$, then the hyperbolicity is the minimum stretch on $V^\perp$ divided by the maximum stretch on V.

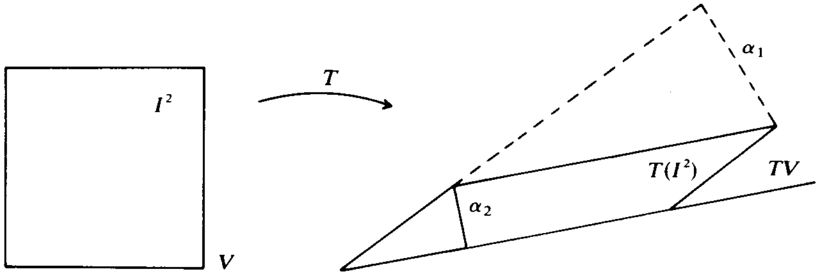

FIGURE 5. $\mathrm{hyp}_{x\text{-axis}}(T) = \alpha_1/\alpha_2$.

Definition. The *V-orthogonalization of T* is $S : \mathbb{R}^m \to H$ with

$$S = A \oplus (T \mid V^\perp) \quad \text{resp.} \quad V \oplus V^\perp.$$

Thus, $SV \perp S(V^\perp)$. By $\oplus$ we mean orthogonal direct sum.

(3.1) PROPOSITION. *If S is the V-orthogonalization of T, then*

$$\|T^{-1}S - \mathrm{Id}\| \le 1/\mathrm{hyp}\,(T),$$
$$\mathrm{hyp}\,(T) \ge 1 \Rightarrow \mathbf{m}(S) = \mathbf{m}(A),$$
$$\left(1 + \frac{1}{\mathrm{hyp}\,(T)}\right)\mathbf{m}(T) \ge \mathbf{m}(S) \ge \left(1 - \frac{1}{\mathrm{hyp}\,(T)}\right)\mathbf{m}(T),$$

where hyp *denotes* hyp_V *and* Id *is the identity on* $\mathbb{R}^m$.

(3.2) *Selection Theorem. Let T_t be a sequence of monomorphisms $\mathbb{R}^m \to H$. Then there exists an orthogonal splitting $\mathbb{R}^m = V^1 \oplus \cdots \oplus V^L$ and a subsequence $\{T_{t_k}\}$ such that the V^i-altitudes*

$$A_k^i \colon V^i \xrightarrow{T_{t_k}} H \xrightarrow{\pi_k^i} [T_{t_k}((V^i)^\perp)]^\perp$$

satisfy

$$\mathrm{bol}\,(A_k^i) \text{ is bounded as } k \to \infty; \tag{1}$$

$$\mathbf{m}(A_k^i)/\|A_k^{i+1}\| \to \infty \quad as \quad k \to \infty \quad (1 \le i < L). \tag{2}$$

(3.3) ADDENDUM. *Let $h, M > 0$ be given perhaps depending on $\{T_{t_k}\}, \oplus V^i$. Then there exist dilations $\Delta^i : V^i \to V^i$ and intervals $I^1 < I^2 < \cdots < I^L$ in $\{t_k\}$, each having length M, and*

$$\mathrm{hyp}_{V^i}\,(T_{t_k} \circ \Delta) > h \tag{3}$$

for all $t_k \in I^i$, where

$$\Delta = \begin{pmatrix} \Delta^1 & 0 & \cdots & 0 \\ 0 & \Delta^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta^L \end{pmatrix} \textit{resp. } V^1 \oplus V^2 \oplus \cdots \oplus V^L.$$

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Remarks. (1) That one set of numbers is $<$ another means that each number in the first is $<$ each in the second. (3) means that each subspace V^i gets its chance, as the t_k increase through I^i, to be a subspace of huge hyperbolicity.

(2) $A_k^i \circ \Delta^i$: $V^i \to H$ is the V^i-altitude of $T_{t_k} \circ \Delta$.

(3) For $x \in V^i$, $|(T_{t_k} \circ \Delta)^{-1} \circ A_k^i \circ \Delta^i(x) - x| < 1/\text{hyp}\,(T_{t_k} \circ \Delta)$. So when we push in the $A_k^i\Delta(x)$ direction in the proof in § 5 this is approximately the $T_{t_k}\Delta(x)$ direction. Next let $\mathcal{B}$ be a bound on the bolicities bol (A_k^i), $i = 1, \ldots, L$, $k = 1, 2, 3, \ldots$. Then for $x \in V^i$ with $|x| \leq 1$,

$$|A_k^i \Delta x| \leq \mathcal{B}\mathbf{m}(A_k^i\Delta|V^i) \leq \mathcal{B}(1 + 1/\text{hyp}\,(T_{t_k}\Delta))\mathbf{m}(T_{t_k}\Delta)$$

so this vector has a magnitude bounded by the minimum norm of $T_{t_k}\Delta$. These are the two properties of altitudes that are most important in § 5. Note the second property does not hold for $|T_{t_k}\Delta x|$, i.e. it is not bounded by a quantity involving $\mathbf{m}(T_{t_k}\Delta)$. This is the reason we need to introduce altitudes.

Definition. Let $F, F' \subset \mathbb{R}^m$ be linear subspaces. Let $\pi: \mathbb{R}^m \to F$, $\pi': \mathbb{R}^m \to F'$ be orthogonal projections. The *angle* from F to F' is

$$\measuredangle(F, F') = \sup\{\measuredangle(f, \pi' f) : f \in F\}.$$

Here $\measuredangle(x, 0) = \pi/2$. Let $\sin(F, F') = \sin \measuredangle(F, F')$, etc.

(3.4) Lemma. *With π, π' as above, assume $\pi F' = F$, $\pi' F = F'$. Then*

$$\measuredangle(F, F') = \measuredangle(F', F).$$

Proof. $\measuredangle(\pi'\pi f', \pi f') \leq \measuredangle(f', \pi f')$ so

$$\measuredangle(\mathrm{F}, \mathrm{F}') = \sup\{\measuredangle(\pi f', \pi'\pi f) : f' \in F'\} \leq \sup\{\measuredangle(f', \pi f') : f' \in F'\} = \measuredangle(F', F).$$

By symmetry

$$\measuredangle(F, F') = \measuredangle(F', F).$$

and so they are equal. Q.E.D.

Proof of Proposition 3.1. Denote the orthogonal projections by

$$\begin{tikzcd} & H \arrow[dl, "\pi"'] \arrow[dr, "\pi^\perp"] & \\ (TV^\perp)^\perp \arrow[rr] & & TV^\perp \end{tikzcd} \qquad \begin{tikzcd} & H \arrow[dl, "\rho"'] \arrow[dr, "\rho^\perp"] & \\ (TV)^\perp \arrow[rr] & & TV \end{tikzcd}$$

$\ker(\pi) = TV^\perp \perp \ker(\pi^\perp)$ $\qquad$ $\ker(\rho) = TV \perp \ker(\rho^\perp)$

Thus

$$\pi + \pi^\perp = \rho + \rho^\perp = \mathrm{Id}_H$$

and

$$A = \pi T | V, \qquad A^\perp = \rho T | V^\perp.$$

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To estimate $\|T^{-1}S-\mathrm{Id}\|$ we first note that $T^{-1}S-\mathrm{Id}$ vanishes on $V^\perp$ since S is T on $V^\perp$. On the other hand, for $x\in V$, we have

$$T^{-1}Sx-x=T^{-1}Ax-T^{-1}Tx=T^{-1}(\pi TT^{-1}-\mathrm{Id}_H)Tx$$
$$=-T^{-1}\pi^\perp Tx.$$

But $\pi^\perp Tx$, being in $TV^\perp$, is carried by T^{-1} into $V^\perp$, so we can apply the conorm estimate to $A^\perp$ on $V^\perp$:

$$\mathbf{m}(A^\perp)|T^{-1}\pi^\perp Tx|\le|A^\perp T^{-1}\pi^\perp Tx|=|\rho\pi^\perp Tx|.$$

But $\rho\pi^\perp=-\rho\pi$ on TV since $\rho\circ(\pi+\pi^\perp)$ on TV is ρ on TV which is zero. Thus, x being in V, the last term is

$$|\rho Ax|\le\|A\|\,|x|.$$

Dividing through by $\mathbf{m}(A^\perp)$ gives

$$|T^{-1}Sx-x|\le\frac{\|A\|\,|x|}{\mathbf{m}(A^\perp)}=\frac{|x|}{\mathrm{hyp}\,(T)},$$

demonstrating the first claim of the proposition.

Clearly,

$$\mathbf{m}(S\mid V^\perp)=\mathbf{m}(T\mid V^\perp)\ge\mathbf{m}(A^\perp)=\|A\|\,\mathrm{hyp}\,(T)\ge\mathbf{m}(A)=\mathbf{m}(S\mid V)$$

since $\mathrm{hyp}\,(T)\ge1$. Since S preserves the orthogonality $V\oplus V^\perp$, its minimum norm is the smaller of the two minimum norms $\mathbf{m}(S\mid V)$, $\mathbf{m}(S\mid V^\perp)$, which we have just seen to be $\mathbf{m}(S\mid V)$. This proves the second claim.

Call $h=\mathrm{hyp}\,(T)$. Then

$$\|T^{-1}S-\mathrm{Id}\|\le1/h$$

implies

$$(1+h^{-1})U^m\supset T^{-1}SU^m\supset(1-h^{-1})U^m,$$

where U^m is the unit ball in $\mathbb{R}^m$, and

$$(1+h^{-1})TU^m\supset SU^m\supset(1-h^{-1})TU^m$$

which implies

$$(1+h^{-1})\mathbf{m}(T)\ge\mathbf{m}(S)\ge(1-h^{-1})\mathbf{m}(T).\qquad\text{Q.E.D.}$$

Proof of the Selection Theorem. It is notationally simpler and involves no loss of generality to reverse the indices i and prove (1), (2′) and (3) for $I^1>I^2>\cdots>I^L$, where (2′) is as follows:

$$\frac{\|A_k^i\|}{\mathbf{m}(A_k^{i+1})}\to0,\quad\text{as }k\to\infty.\tag{2′}$$

(We have stated the theorem in the form most easily used, not proved.)

Each T_t factors as $T_t=O_tP_t$, where O_t is orthogonal and P_t is positive definite symmetric [**8**, p. 102]. The orthogonal part does not affect the properties in the theorem, so we can assume that $T_t=P_t$ is a positive definite symmetric automorphism of $\mathbb{R}^m$.

Let $0 < \lambda_t^1 \leq \cdots \leq \lambda_t^m$ be the eigenvalues of T_t. Take a subsequence t_k so that ratios of eigenvalues converge:

$$\lim_{k\to\infty} \frac{\lambda_{t_k}^i}{\lambda_{t_k}^j} = \alpha_{ij} \qquad 0 \leq \alpha_{ij} \leq \infty.$$

By an abuse of notation we shall write k for the subsequence t_k, i.e. T_k for T_{t_k}, λ_k^i for $\lambda_{t_k}^i$, etc.

Two eigenvalues are said to be *equivalent* if $\alpha_{ij} \neq 0, \infty$. Split $\mathbb{R}^m$ into subspaces of equivalent eigenvectors for each k

$$\mathbb{R}^m = E_k^1 \oplus \cdots \oplus E_k^K.$$

The subspaces are orthogonal because the T_t are symmetric. dim (E_k^i) is independent of k. Since the Grassmannian is compact, there is a finer subsequence (unrelabelled) such that E_k^i converges to a limit plane E^i as $k \to \infty$. Thus

$$E_k^1 \oplus \cdots \oplus E_k^K \xrightarrow[k]{} E^1 \oplus \cdots \oplus E^K.$$

Let

$$V_k = E_k^1 \qquad V_k^\perp = E_k^2 \oplus \cdots \oplus E_k^K \qquad V_k \xrightarrow[k]{} V$$

$$V = E^1 \qquad V^\perp = E^2 \oplus \cdots \oplus E^K \qquad V_k^\perp \xrightarrow[k]{} V^\perp$$

$$\varphi_k = \measuredangle(V_k, V) = \measuredangle(V_k^\perp, V^\perp).$$

$$\begin{tikzcd} & T_k V \arrow[dr, "\pi_k"] & \\ V \arrow[ur, "T_k"] \arrow[rr, "A_k"] & & (T_k V^\perp)^\perp \end{tikzcd} \qquad \begin{tikzcd} & T_k V^\perp \arrow[dr, "\rho_k"] & \\ V^\perp \arrow[ur, "T_k"] \arrow[rr, "A_k"] & & (T_k V)^\perp \end{tikzcd}$$

In Lemma 3.5, below, we show (combine (i) and (ii)) that

$$\text{bol}\,(A_k) = \frac{\|A_k\|}{\mathbf{m}(A_k)} \leq \frac{\|T_k \mid V_k\|}{\mathbf{m}(T_k \mid V_k) \cos^2(\varphi_k)} \tag{4}$$

which stays bounded as $k \to \infty$, since $V_k = E_k^1$ is a space of equivalent eigenvectors for T_k and $\varphi_k \to 0$. By the same lemma (combine (i) and (iii))

$$\frac{\|A_k\|}{\mathbf{m}(A_k^\perp)} \leq \frac{\|T_k \mid V_k\|}{\cos(\varphi_k)\|T_k \mid V_k\|} (h_k^{-2} + \text{bol}\,(T_k \mid V_k)^2 \tan^2 \varphi_k)^{1/2}, \tag{5}$$

where $h_k = \text{hyp}_{V_k}(T_k)$. As $k \to \infty$, $h_k \to \infty$, $\varphi_k \to 0$, $\cos(\varphi_k) \to 1$, $\tan(\varphi_k) \to 0$, and bol $(T_k \mid V_k)$ is bounded; so

$$\frac{\|A_k\|}{\mathbf{m}(A_k^\perp)} \to 0. \tag{6}$$

(6) will show $V^1 = V$ satisfies the theorem. Unfortunately, the other E^i will not work as V^k, but we can use induction. The difficulty is that the spaces E^i are not eigenspaces, but are only limits of eigenspaces as $k \to \infty$. See [**17**] for an example showing that the $E^2, \ldots, E^K$ do not work.

$V^\perp$ has dimension lower than m, so apply the Selection Theorem to $\{A_k^\perp\}$. (Note that here it is convenient to assume that T_t is a monomorphism, not an isomorphism, since $A_k^\perp : V^\perp \to H$ is bound to be only a monomorphism.) Thus, there is a splitting

$$V^\perp = V^2 \oplus \cdots \oplus V^L$$

and a subsequence (unrelabelled) of $\{A_k^\perp\}$ so that the altitudes C_k^i of $A_k^\perp$ satisfy (1), (2′). The C_k^i are related to the A_k^i, respecting $V^1 \oplus V^2 \oplus \cdots \oplus V^L$, via the commutativity of

$$\begin{tikzcd}
 & \mathbb{R}^m \arrow[r, "T_k"] & \mathbb{R}^m \arrow[r, "\pi_k"] \arrow[dd, "\rho_k"] & {[T_k((V^i)^\perp)]^\perp} \arrow[dd, hook] \\
V^i \arrow[ur, hook] \arrow[dr, hook] \arrow[urrr, bend left, "A_k"] \arrow[drrr, bend right, "C_k^i"'] & & & \\
 & V^\perp \arrow[uu, hook] \arrow[r, "A_k^\perp"] & (T_k V^\perp)^\perp \arrow[r, "\omega_k"] & {[A_k^\perp(\bigoplus_{2 \le i \ne j} V^i)]^\perp}
\end{tikzcd}$$

where ρ_k, π_k^j, ω_k^j are orthogonal projections, $j \geq 2$. That there is an inclusion for the rightmost map is true because $A_k^\perp V^i$ is just $T_k V^i$ made orthogonal to $T_k V^1$ and so

$$A_k^\perp V^i \subset T_k V^1 \oplus T_k V^i, \qquad i = 2, \ldots, L.$$

Thus,

$$A_k^\perp\Big(\bigoplus_{2 \le i \ne j} V^i\Big) \subset T_k((V^i)^\perp)$$

and the reverse is true of their orthogonal complements. (The idea here is that if a vector z lies outside two spaces $X \subset Y$ of H and if one (orthogonally) projects z to z' making $z' \perp Y$ then one obtains the same answer by first projecting z to z'' making $z'' \perp X$, and then projecting z'' to z''' making $z''' \perp Y$; $z' = z'''$.)

Thus $C_k^j = A_k^j$, $j \geq 2$, and so (1), (2′) hold for $j \geq 2$. For $j = 1$, $A_k^1 = A_k^\perp$ so (1) holds. As for (2′) when $j = 1$, we have

$$\frac{\|A_k^1\|}{\mathbf{m}(A_k^2)} = \frac{\|A_k^1\|}{\mathbf{m}(A_k^\perp)} \frac{\mathbf{m}(A_k^\perp)}{\mathbf{m}(A_k^2)}.$$

The first factor tends to zero by (6). The second factor is ≤ 2 when hyp $A_k^\perp \geq 2$ by Proposition 3.1 applied to $T = A_k^\perp$. Note that either $K = 2$ and $A_k^\perp = A_k^2$ or else hyp $(A_k^\perp) \to \infty$. This proves that (2′) holds for $j = 1$ also, and completes the proof of the Selection Theorem, except for the inequalities (4), (5) which we now verify.

(3.5) LEMMA. *Let H be a second subspace of $\mathbb{R}^m$ having the same dimension as V and call $\varphi = \measuredangle(V, H)$. Suppose $T : \mathbb{R}^m \to \mathbb{R}^m$ leaves $H \oplus H^\perp$ invariant*

$$T = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \textit{ resp. } H \oplus H^\perp$$

and is pseudohyperbolic in the sense that $\mathbf{m}(Q) \geq \|P\|$. Then the altitudes A, $A^\perp$ of T respecting V, $V^\perp$ obey

(i) $\|A\| \leq \dfrac{\|P\|}{\cos(\varphi)}$,

(ii) $\mathbf{m}(A) \geq \mathbf{m}(P) \cos(\varphi)$,

(iii) $\mathbf{m}(A^\perp) \geq \dfrac{\|P\|}{[h^{-2} + \operatorname{bol}(P)^2 \tan^2(\varphi)]^{1/2}}$,

where $h = \operatorname{hyp}_H(T)$.

Remark. In the Selection Theorem $H = V_k$, so the invariance hypothesis is satisfied by T_k, as is the pseudohyperbolicity.

Proof. (i) Let $\sigma: V \to H$ be the projection parallel to $V^\perp$ and let $\tau: V^\perp \to H^\perp$ be the projection parallel to V (see figure 6).

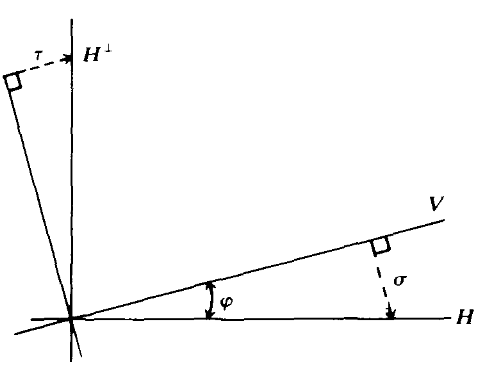

FIGURE 6. $\mathbb{R}^m = H \oplus H^\perp = V \oplus V^\perp$, $\varphi = \measuredangle(V, H)$.

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Observe that

$$A = \pi P \sigma. \tag{7}$$

For take $v \in V$. Then $P \oplus Q = T$ resp. $H \oplus H^\perp$ implies

$$\pi P(\sigma v) = \pi T(\sigma v).$$

But $\sigma(v) = v + w$ for some $w \in V^\perp$ and $\pi T(w) = 0$. Hence

$$Av = \pi Tv = \pi T(\sigma v) = \pi P \sigma v,$$

proving (7). From (7) we have

$$\|A\| \leq \|\pi P\| \|\sigma\| \leq \|P\| \|\sigma\| \leq \|P\| / \cos \varphi$$

almost by the definition of cosine.

(ii) Using (7) again,

$$\mathbf{m}(A) \geq \mathbf{m}(\pi \,|\, H)\mathbf{m}(P)\mathbf{m}(\sigma) \geq \mathbf{m}(\pi \,|\, H)\mathbf{m}(P).$$

Since $\mathbf{m}(Q) \geq \|P\|$, $\varphi = \measuredangle(V, H) \geq \measuredangle(H^\perp, TV^\perp)$. Thus,

$$\mathbf{m}(\pi \,|\, H)\mathbf{m}(P) = \cos\,(H, TV^\perp)^\perp \mathbf{m}(P) \geq \cos\,(\varphi)\mathbf{m}(P)$$

proving (ii).

(iii) As with A, we have $A^\perp = \rho Q \tau$ and so

$$\mathbf{m}(A^\perp) \geq \mathbf{m}(\rho Q) = \operatorname{dist}\,(TS, TV),$$

where S is the unit sphere in $H^\perp$ and 'dist' refers to the minimum distance. Let R be the automorphism of $\mathbb{R}^m$ given by

$$R = \begin{pmatrix} \mathbf{m}(P)P^{-1} & 0 \\ 0 & \mathbf{m}(Q)Q^{-1} \end{pmatrix} \text{ resp. } H \oplus H^\perp.$$

Thus $\|R\| = 1$ and RP, RQ are conformal automorphisms of H, $H^\perp$ with norms $\mathbf{m}(P)$, $\mathbf{m}(Q)$. Let C_φ be the cone of angle φ around H:

$$C_\varphi = \{x \in \mathbb{R}^m : \measuredangle(x, H) \leq \varphi\}.$$

Thus $V \subset C_\varphi$ and

$$\operatorname{dist}\,(TS, TV) \geq \operatorname{dist}\,(TS, TC_\varphi) \geq \operatorname{dist}\,(RT(S), RT(C_\varphi))$$

the latter, since $\|R\| \leq 1$. This distance equals $\mathbf{m}(RQ)\cos(\psi)$, where ψ is the angle between $H^\perp$ and the lines from $RT(S)$ perpendicular to $\partial(RT(C_\varphi))$ (see figure 7).

Using the conformal nature of RP, RQ, this angle ψ is independent of which point in $RT(S)$ we start at. Since $\mathbf{m}(RQ) = \mathbf{m}(Q)$, this gives

$$\mathbf{m}(A^\perp) \geq \mathbf{m}(Q) \cos \psi.$$

Clearly, $RT(C_\varphi) = C_\psi$. Thus ψ can be calculated by choosing any $x \in H$, $y \in H^\perp$ such that $x + y \in \partial C_\varphi$. Then

$$\tan(\psi) = \frac{|RTy|}{|RTx|} = \frac{\mathbf{m}(Q)|y|}{\mathbf{m}(P)|x|}$$
$$= \frac{\mathbf{m}(Q)}{\|P\|} \frac{\|P\|}{\mathbf{m}(P)} \frac{\sin \varphi}{\cos \varphi} = \operatorname{hyp}_H(T) \operatorname{bol}(P) \tan \varphi.$$

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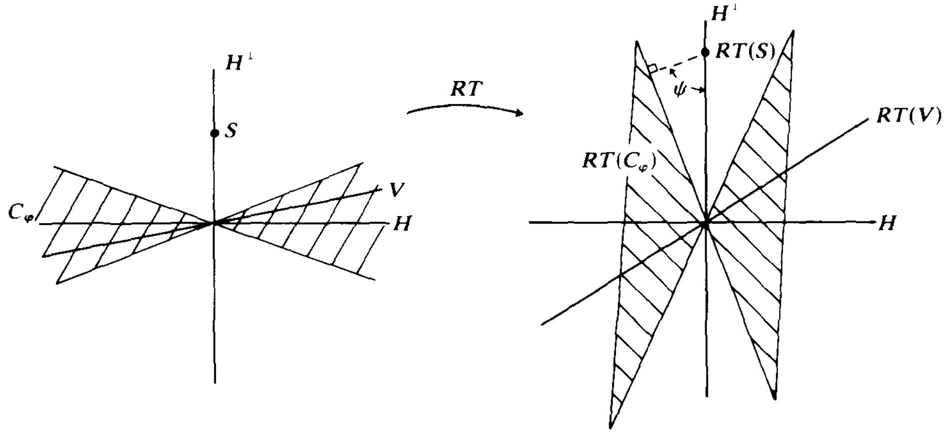

FIGURE 7. Cones mapped by RT.

Therefore

$$\begin{aligned}\mathbf{m}(A^\perp) &\geq \mathbf{m}(Q)\cos(\tan^{-1}(h\ \mathrm{bol}\,(P)\tan(\varphi)))\\ &= \mathbf{m}(Q)\frac{1}{[1+h^2\,\mathrm{bol}\,(P)^2\tan^2\varphi]^{1/2}}\\ &= \frac{\|P\|}{[h^{-2}+\mathrm{bol}\,(P)^2\tan^2\varphi]^{1/2}}\end{aligned}$$

proving (iii). Q.E.D.

Proof of addendum 3.3. Since the assertion is vacuous for $L=1$, suppose $L\geq 2$ and we have proved it for $L-1$. Recall that the altitudes A_k^j, $j\geq 2$ are also the altitudes of $A_k^\perp : V^\perp \to H$, where $V=V^1$ and

$$V^\perp = V^2 \oplus \cdots \oplus V^L.$$

By the induction assumption applied to $\{A_k^\perp\}$ there exist dilations Δ^i of V^i, $i\geq 2$, intervals $I^L < \cdots < I^2$ in $\{t_k\}$ of length M, and

$$\mathrm{hyp}_{V^i}\,(A_k^\perp \Delta^\perp) > h \tag{8}$$

for all $t_k \in I^i$. By $\Delta^\perp$ we mean diag $(\Delta^2, \ldots, \Delta^L)$. We introduce the notation

$${}^iV = V^1\oplus\cdots\oplus\widehat{V^i}\oplus\cdots\oplus V^L, \qquad {}^i(V^\perp) = V^2\oplus\cdots\oplus\widehat{V^i}\oplus\cdots\oplus V^L,$$
$${}^i\Delta = \mathrm{diag}\,(\Delta^1,\ldots,\widehat{\Delta^i},\ldots,\Delta^L), \qquad {}^i(\Delta^\perp) = \mathrm{diag}\,(\Delta^2,\ldots,\widehat{\Delta^i},\ldots,\Delta^L),$$
$${}^iA_k = \text{the } {}^iV\text{-altitude of } T_{t_k}, \qquad {}^iC_k = \text{the } {}^i(V^\perp)\text{-altitude of } A_k^\perp,$$

where Δ^1 will be chosen soon. Recall that C_k^j is the V^j-altitude of $A_k^\perp$ and that it equals the V^j-altitude of T_{t_k}, $j\geq 2$. A similar situation holds here. Omitting the subscripts k, we have a commutative diagram (see below), where ${}^i\pi$, ρ, ${}^i\omega$ are orthogonal projections. That there is an orthogonal projection for the rightmost arrow is because, as we saw before,

$$TV^1 \oplus A^\perp V^j = T(V^1 \oplus V^j) \supset T\check{V}^j$$

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$$\begin{tikzcd}
{}^jV \arrow[r, hook] \arrow[rrr, bend left, "{}^jA"] & \mathbb{R}^m \arrow[r, "T"] & \mathbb{R}^m \arrow[r, "{}^j\pi"] \arrow[d, "\rho"] & (TV^j)^\perp \arrow[d] \\
{}^j(V^\perp) \arrow[r, hook] \arrow[u, hook] \arrow[rrr, bend right, "{}^jC"'] & V^\perp \arrow[r, "A^\perp"] \arrow[u, hook] & (TV^1)^\perp \arrow[r, "{}^j\omega"] & (A^\perp V^j)^\perp \cap (TV^1)^\perp
\end{tikzcd}$$

and, since $TV^1 \perp A^\perp V^j$, when we take orthogonal complements we get

$$(TV^1)^\perp \cap (A^\perp V^j)^\perp = (T(V^1 \oplus V^j))^\perp \subset (TV^j)^\perp.$$

Thus, ${}^jA$ restricts to ${}^jC$, just as $A^j = C^j$ and so, if we make Δ^1 an enormous dilation, we get

$$\mathbf{m}({}^jA_k \circ {}^j\Delta) = \mathbf{m}({}^jC_k \circ {}^j(\Delta^\perp))$$

for all $t_k \in I^2 \cup \cdots \cup I^L$. Since the Δ^j's factor through all altitudes, this gives

$$\begin{aligned}\text{hyp}_{V^j}\,(T_{t_k} \circ \Delta) &= \frac{\mathbf{m}({}^jA_k \circ {}^j\Delta)}{\|A_k^j \circ \Delta^j\|} = \frac{\mathbf{m}({}^jC_k \circ {}^j(\Delta^\perp))}{\|C_k^j \circ \Delta^j\|} \\ &= \text{hyp}_{V^j}\,(A_k^\perp \circ \Delta) > h\end{aligned}$$

by (8), provided $j \geq 2$ and $t_k \in I^j$. For $j \geq 2$ this proves (3).

To find I^1 we use (6) which states

$$\frac{\|A_k\|}{\mathbf{m}(A_k^\perp)} \to 0, \quad \text{as} \quad k \to \infty.$$

Note $A_k = A_k^1$ and $A_k^\perp = {}^1A_k$. Thus

$$\text{hyp}_{V^1}\,(T_{t_k} \circ \Delta) = \frac{\mathbf{m}({}^1A_k \circ {}^1\Delta)}{\|A_k^1 \circ \Delta^1\|} = \frac{\mathbf{m}(A_k^\perp \circ \Delta^\perp)}{\|A_k \circ \Delta^1\|} \geq \frac{\mathbf{m}(A_k^\perp)\mathbf{m}(\Delta^\perp)}{\|A_k\|\|\Delta^1\|} > h$$

for all $k \geq$ some k_1. To complete the proof, let I^1 be any interval in $\{t_k\}$ beyond t_{k_1} having length M. Q.E.D.

4. *Boxes: the Fundamental Lemma*

In this section we handle the problem of 'intermediate intersections'.

(4.1) *Fundamental Lemma. Let p^* be a non-wandering point for $f \in \mathscr{D}^1$, $\{e^k\}$ an orthonormal basis for $T_{p^*}M$, and let $\xi^1, \ldots, \xi^m > 0$ be given. Then there exist $\lambda^1, \ldots, \lambda^m > 0$ and $r \in T_{p^*}M$ such that, for some $p, q \in \exp_{p^*}(T_{p^*}M)$,*

(i) *$f^s p = q$, for some $s > 0$,*
(ii) *$p, q \in B(a_1 I^m)$, $a_1 = \sqrt{\tfrac{3}{4}}$,*

(iii) $f^t p \notin BI^m$, $0 < t < s$,
(iv) $\lambda^j/\lambda^k \le 2m^{1/2}\xi^j/\xi^k$, for all j, k,

where B is the box $\exp_{p^*}(r + \Lambda | I^m)$, *$I^m$ is the unit cube in $T_{p^*}M$ respecting $\{e^k\}$, and* $\Lambda = \text{diag}(\lambda^1, \ldots, \lambda^m)$ *respecting $\{e^k\}$. Moreover, $|r|$ and $\|\Lambda\|$ may be made arbitrarily small.*

Remark. (iv) says that up to a dimensional constant we can specify the proportions of the box B to be $\xi^1 : \xi^2 : \cdots : \xi^m$.

Proof. Define the partial order $\prec$ on a neighbourhood of 0 in $T_{p^*}M$ by

$$x \prec y \quad \text{iff} \quad x \ne y \quad \text{and} \quad f^s \exp_{p^*}(x) = \exp_{p^*}(y) \qquad \text{for some } s > 0.$$

First we prove:

(v) For some arbitrarily small $x, y \in T_{p^*}M$ with $x \prec y$,

$$x \prec z \prec y \quad \text{implies} \quad z \notin \xi(x, y),$$

where

$$\xi(x, y) = \{z \in T_{p^*}M : d(x, z) \le (\tfrac{3}{4})^{1/2} d(x, y) \text{ or } d(z, y) \le (\tfrac{3}{4})^{1/2} d(x, y)\}$$

$$d(x, y) = \left(\sum_{j=1}^{m}\left(\frac{x^j - y^j}{\xi^j}\right)^2\right)^{1/2}, \quad x = \sum x^j e^j, \quad y = \sum y^j e^j.$$

Since p^* is non-wandering, there exist arbitrarily small x_0, y_0 with $x_0 \prec y_0$. Between x_0 and y_0 at most finitely many points z may intervene, $x_0 \prec z \prec y_0$. If one of these lies in $\xi(x_0, y_0)$ let

$$z \text{ replace } y_0 \quad \text{if} \quad d(z, x_0) \le d(z, y_0),$$
$$z \text{ replace } x_0 \quad \text{if} \quad d(z, y_0) \le d(z, x_0).$$

The pair so formed, (x_1, y_1), has $x_1 \prec y_1$ and either $x_1 = x_0$ or $y_1 = y_0$. Proceed as with (x_0, y_0), generating a sequence (x_n, y_n). All the x_n, y_n lie between x_0 and y_0, so the process ends at some (x_k, y_k) with $x_k \prec y_k$ and no intermediate points z in $\xi(x_k, y_k)$. Clearly,

$$d(x_n, y_n) \le (\tfrac{3}{4})^{n/2} d(x_0, y_0)$$

and either

$$x_n = x_{n-1} \quad \text{or} \quad y_n = y_{n-1}, \qquad 1 \le n \le k.$$

Therefore

$$d(x_k, x_0) \le \sum_{n=1}^{k} d(x_n, x_{n-1}) < \left[\sum_{1}^{\infty} (\tfrac{3}{4})^{n/2}\right] d(x_0, y_0).$$

This proves that x_k, y_k are arbitrarily small, completing the proof of (v). The particular form of d has been irrelevant, so far.

Let (x, y) be as in (v). Set

$$r = \tfrac{1}{2}(x + y), \qquad p = \exp_{p^*}(x), \qquad q = \exp_{p^*}(y),$$

and

$$\lambda^j = \xi^j(\tfrac{1}{3}d_j(x, y)^2 + d(x, y)^2/8m)^{1/2}$$

$$d_j(x, y)^2 = \left(\frac{x^j - y^j}{\xi^j}\right)^2.$$

Thus $d^2 = \sum d_j^2$ and (i) holds. We shall see that these λ^j make (ii), (iv) hold and

$$r + \Lambda I^m \subset \xi(x, y).$$

This makes (iii) also true. Proof of (iv) is just the calculation

$$\left(\frac{\lambda^j}{\lambda^k}\right)^2 \le \left(\frac{\xi^j}{\xi^k}\right)^2 \frac{(\frac{1}{3} + 1/8m)d(x, y)^2}{d(x, y)^2/8m} = \left(\frac{\xi^j}{\xi^k}\right)^2 \left(\frac{8m}{3} + 1\right) < 4m\left(\frac{\xi^j}{\xi^k}\right)^2.$$

Likewise to prove $x \in r + \Lambda a_1 I^m$ we compute

$$(x^j - r^j)^2 = \tfrac{1}{4}(x^j - y^j)^2 = \tfrac{3}{4}(\xi^j)^2[\tfrac{1}{3}d_j(x, y)^2] < \tfrac{3}{4}(\lambda^j)^2.$$

Similarly for y. This proves (ii). If $u \in \Lambda I^m$ then

$$\begin{aligned} d(r + u, x)^2 + d(r + u, y)^2 &= \sum_{j=1}^m \left(\frac{u^j + \frac{1}{2}(y^j - x^j)}{\xi^j}\right)^2 + \sum_{j=1}^m \left(\frac{u^j + \frac{1}{2}(x^j - y^j)}{\xi^j}\right)^2 \\ &= \sum_{j=1}^m \frac{2(u^j)^2 + \frac{1}{2}(x^j - y^j)^2}{(\xi^j)^2} \\ &\le \sum_{j=1}^m \frac{2(\xi^j)^2(\frac{1}{3}d_j(x, y)^2 + d(x, y)^2/8m)}{(\xi^j)^2} + \tfrac{1}{2}\sum_{j=1}^m d_j(x, y)^2 \\ &= (\tfrac{2}{3} + \tfrac{1}{4} + \tfrac{1}{2})d(x, y)^2 < 2(\tfrac{3}{4})d(x, y)^2. \end{aligned}$$

Hence either

$$d(r + u, x)^2 < \tfrac{3}{4}d(x, y)^2$$

or

$$d(r + u, y)^2 < \tfrac{3}{4}d(x, y)^2.$$

That is, $r + \Lambda I^m \subset \xi(x, y)$, proving (iii), and completing the proof of the Fundamental Lemma. Q.E.D.

Remark. For $\varphi \in \mathscr{F}^1$, the Fundamental Lemma is valid with the substitutions: $m - 1$ for m, $\hat{\Pi}_{p^*} = (\text{span } \dot{\varphi}(p^*))^\perp$. The proof is the same.

5. *Lift Implies Closing*

Let us divide our attention among $\mathscr{D}^1$, $\mathscr{F}^1$, $\mathscr{X}^1$ where our manifold M is compact.

(5.1) *Closing Lemma for C^1 diffeomorphisms. Let $\mathscr{S} \subset \mathscr{D}^1$ obey the Lift Axiom. If $f \in \mathscr{S}$, $\mathscr{N}$ is a neighbourhood of f in $\mathscr{S}$, p^* is a non-wandering point of f, and U is a neighbourhood of p^* in M, then there exists $gf \in \mathscr{N}$ having a periodic point in U.*

(5.2) *Closing Lemma for C^1 flows. If $p^* \in \Omega(\varphi)$ for some $\varphi \in \mathscr{S} \subset \mathscr{F}^1$ and if S is some neighbourhood of $\mathscr{O}_+(p^*)$, then the Lift Axiom for $\mathscr{S}$ at φ on S implies that there exists $\psi \in \mathscr{S}$ arbitrarily near φ in $\mathscr{F}^1$ with a periodic orbit passing arbitrarily near p^*.*

(5.3) *Closing Lemma for C^1 tangent vector fields. Substitute X, $\mathscr{X}^1$, Y for φ, $\mathscr{F}^1$, ψ in* (5.2).

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Remark. The C^1 Closing Lemma for non-compact M is treated in § 11.

Proof of (5.1). We may replace $\mathcal{N}$ by a neighbourhood $\mathcal{N}'$ of f such that, if $g_1 f, \ldots, g_k f \in \mathcal{N}'$ and supp $g_1, \ldots$, supp g_k are disjoint, then $g_1 \circ \cdots \circ g_k \circ f \in \mathcal{N}'$. For instance, put a C^1 metric on $\mathcal{D}^1$ and let $\mathcal{N}'$ be a ball around f contained in $\mathcal{N}$. Then use (L3) from the Lift Axiom. Note that $\mathcal{N}'$ is independent of k. Thus, we may and do assume $\mathcal{N} = \mathcal{N}'$.

Our first step is to analyse the sequence of isomorphisms $T_t = T_{p^*} f^t$

$$T_t : T_{p^*} M \to T_{f^t p^*} M \hookrightarrow H, \qquad t = 1, 2, 3, \ldots,$$

where M is embedded in some Euclidean space H. We select a sequence $\{t_k\}_{k=1}^{\infty}$ and a splitting $\bigoplus_{i=1}^{L} V^i = T_{p^*} M$ according to the Selection Theorem 3.2.

Let $\mathcal{B}$ be a uniform bound for the altitude bolicities of $\{T_{t_k}\}_{k=1}^{\infty}$. Let $\varepsilon = \varepsilon(f, \mathcal{N})$ be given by the Lift Axiom. Choose and fix, once and for all, an integer

$$N \geq 40\mathcal{B}/\varepsilon. \tag{1}$$

The secret of our proof is that N will never be changed. Also $\bigoplus V^i$, $\mathcal{B}$, and ε are fixed.

By Addendum 3.3, given any h, we can find intervals $I^1 < I^2 < \cdots < I^L$ in $\{t_k\}_1^{\infty}$ and an automorphism Δ of $T_{p^*} M \circlearrowleft$ such that

$$I^i \text{ consists of } N \dim V^i \text{ integers from } \{t_k\}_1^{\infty}; \tag{2i}$$

$$\Delta \text{ preserves } V^1 \oplus \cdots \oplus V^L \text{ and is conformal on each } V^i; \tag{2ii}$$

$$\operatorname{hyp}_{V^i} (T_{t_k} \circ \Delta) \geq h \quad \text{for} \quad t_k \in I^i. \tag{2iii}$$

Once such a set $\mathcal{Q} = \{I^1, \ldots, I^L\}$ is chosen, we call

$$f_n = f^{t_{k(n)}},$$

where $t_{k(n)}$ is the n'th integer in $I^1 \cup \cdots \cup I^L$. There are mN of these integers n and it is these f_n we care most about. We shall use the Lift Axiom at some y_n near $f_n(p^*)$, producing perturbations g_n, $n = 1, \ldots, mN$, with $g_n f \in \mathcal{N}$. Good choice of $\mathcal{Q}$, Δ, y_n, and the lift directions will imply

$$\text{the supports of the } g_n \text{ are disjoint, } 1 \leq n \leq mN; \tag{3i}$$

for some p, q near p^*, $f^s p \approx q$, $s > T$ (recall T was defined by $f_{mN} = f^T$);
the f-orbit from $f^T p$ to q is the same as the gf-orbit; and $(gf)^T q = f^T p$. (3ii)

Here $g = g_1 \circ \cdots \circ g_{mN}$. From (3i), (L3) in the Lift Axiom, and the fact that $g_n f \in \mathcal{N}$, it follows that $gf \in \mathcal{N}$. By (3ii) it follows that q is gf-periodic because its orbit is

$$q, (gf)q, \ldots, (gf)^{T-1} q, (gf)^T q = f^T p, f^{T+1} p, \ldots, f^s p = q.$$

Thus (3) gives the Closing Lemma for $\mathcal{S} \subset \mathcal{D}^1$.

Selecting $\mathcal{Q}$, Δ, etc to make (3) hold is complicated; four preliminary constructions are required. The idea is to calculate the effect on f, in the limit, of mN perturbations whose supports tend to

$$\{f_1(p^*), \ldots, f_{mN}(p^*)\}$$

while $h(\mathcal{Q}, \Delta) \to \infty$.

(a) Some standard boxes

Let $\{e^k\}$ be an orthonormal basis of $T_{p^*}M$ subordinate to $V^1 \oplus \cdots \oplus V^L$. Thus,

$$\{e^1, \ldots, e^{\dim V^1}\}$$

is a basis of V^1, etc. Call

$$a_1 = (\tfrac{3}{4})^{1/2}$$

and let

$$a_1 < a_2 < a_3 < 1$$

be evenly spaced:

$$a = 1 - a_3 = a_3 - a_2 = a_2 - a_1 = \tfrac{1}{3}(1 - (\tfrac{3}{4})^{1/2}).$$

Consider the closed unit cube

$$\bar{I}^m = \left\{\sum_1^m x^i e^i : |x^i| \le 1\right\}$$

in $T_{p^*}M$ respecting the basis $\{e^k\}$ and consider the boxes

$$a_1\bar{I}^m \subset a_2\bar{I}^m \subset a_3\bar{I}^m \subset \bar{I}^m.$$

(b) The index set Ω

Let Ω_1 be the set of $\omega_1 = (\mathcal{Q}, \Delta)$ such that (2i, ii) hold. Let Ω be the set of all (ω_1, ω_2) such that $\omega_1 = (\mathcal{Q}, \Delta) \in \Omega_1$ and $\omega_2 = (p, q, r, s, \Lambda)$ satisfies

$$\left.\begin{array}{l} \Lambda : T_{p^*}M \circlearrowleft \text{ preserves } V^1 \oplus \cdots \oplus V^L; \\ \text{bol}\,(\Lambda^{-1}\Delta) \le 2m^{1/2}; \\ r \in T_{p^*}M; \\ p, q \in B(a_1\bar{I}^m), \qquad f^s(p) = q; \\ f^t(p) \notin B\bar{I}^m \quad \text{for all } t,\ 0 < t < s. \end{array}\right\} \tag{4}$$

Here B is the box defined by the commutativity of

$$\begin{tikzcd} T_{p^*}M \arrow[r, "r+\Lambda"] & T_{p^*}M \arrow[d, "\exp"] \\ \bar{I}^m \arrow[u, hook] \arrow[r, "B"] & M \end{tikzcd}$$

We measure the *smallness* of ω_2 by

$$\text{sm}\,(\omega_2) = |r| + \|\Lambda\|.$$

Given $\omega_1 = (\mathcal{Q}, \Delta) \in \Omega_1$, the Fundamental Lemma 4.1 applied to f, Δ, at p^* gives

$$\omega_2 = (p, q, r, s, \Lambda)$$

satisfying (4) with sm (ω_2) arbitrarily small. That is, for any $\omega_1 \in \Omega_1$,

$$\inf\{\text{sm}\,(\omega_2) : (\omega_1, \omega_2) \in \Omega\} = 0.$$

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Similarly, by (3.3) there are $\omega_1 \in \Omega$, with $h(\omega_1) \overset{\text{def}}{=}$ the largest value of h making (2iii) true arbitrarily large. The pair $h(\ )$, sm$(\ )$ *bi-orders the index set* $\Omega = \{(\omega_1, \omega_2)\}$. A subset $\Omega' \subset \Omega$ is called *ascending* if

$$\sup\{h(\omega_1) : (\omega_1, \omega_2) \in \Omega'\} = \infty$$

and if, for each fixed $\omega_1 \in \Omega_1$ having some $(\omega_1, \omega_2) \in \Omega'$,

$$\inf\{\mathrm{sm}\,(\omega_2) : (\omega_1, \omega_2) \in \Omega'\} = 0.$$

Thus, Ω itself is ascending and so is any *tail* of Ω – a set of the form

$$\{(\omega_1, \omega_2) \in \Omega : h(\omega_1) \geq h_0, \mathrm{sm}\,(\omega_2) \leq s_0\}.$$

Such subsets occur naturally for iterated limits.

Definition. If $\Omega' \subset \Omega$ is ascending, X is a metric space with metric d, and $f : \Omega' \to X$ then

$$\lim_{\Omega'} f = z$$

means: given $\nu > 0$ there exist

$$h_0 = h_0(\nu) > 0 \quad \text{and} \quad s_0 = s_0(\nu, \omega_1) > 0$$

such that

$$\left.\begin{array}{l} (\omega_1, \omega_2) \in \Omega' \\ h(\omega_1) \geq h_0 \\ \mathrm{sm}\,(\omega_2) \leq s_0 \end{array}\right\} \Rightarrow d(f(\omega_1, \omega_2), z) \leq \nu.$$

This convergence is weaker than the iterated limit

$$\lim_{h \to \infty} \lim_{\mathrm{sm} \to 0} f(\omega) = z$$

since $\lim_{\mathrm{sm}\to 0} f(\omega)$ need not exist for any fixed ω_1. But if $\lim_{\mathrm{sm}\to 0} f(\omega)$ is interpreted to be the Hausdorff lim sup (= the set of accumulation points) then the two notions of convergence coincide – at least if X is locally compact. This convergence is weaker than the joint limit

$$\lim_{\substack{\mathrm{sm} \to 0 \\ h \to \infty}} f(\omega) = z$$

because s_0 can depend on ω_1.

(c) *Trial perturbations*

Fix $\omega = (\mathcal{Q}, \Delta, p, q, r, s, \Lambda) \in \Omega$. Henceforth n will run from 1 to mN. Let

$$J_n = T_{p^*} f_n : T_{p^*} M \to T_{p_n^*} M, \qquad p_n^* = f_n(p^*),$$

where, as above, $f_n = f^{t_{k(n)}}$ and $k(n)$ is the nth integer in $I^1 \cup \cdots \cup I^L$. Let S_n = the V^i-orthogonalization of J_n where $t_{k(n)} \in I^i$. As n increases from 1 to mN, S_n is at first the V^1-orthogonalization of J_n for a while, then ..., and finally is the V^L-orthogonalization of J_n for a while. It stays the V^i-orthogonalization while n stays in I^i.

Arrange the $T_{p^*} M$ basis in blocks:

$$\underbrace{e^1, \ldots, e^1}_{N}, \underbrace{e^2, \ldots, e^2}_{N}, \ldots, \underbrace{e^m, \ldots, e^m}_{N}.$$

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Each e_n is some e^k; in fact, $k-1$ is the integer part of $(n-1)N$. Call

$$v_n^* = \frac{a}{2\mathcal{B}} S_n \Lambda(e_n) \in T_{p_n^*}M.$$

Recall that a is a numerical constant and $\mathcal{B}$ bounds all altitude bolicities. This v_n^* is approximately our lift direction at the nth step.

Let $\rho_1, \ldots, \rho_{mN}$ be independent real parameters with $-1 \le \rho_n \le 1$. Call

$$\rho = (\rho_1, \ldots, \rho_{mN})$$

and inductively define

$$\left.\begin{aligned} &y_0 = q, \qquad \text{identify} = f_0, \quad v_0 = 0, \quad F_n = f_n \circ f_{n-1}^{-1} \\ &y_n = F_n \circ g(y_{n-1}; \rho_{n-1} v_{n-1}) \\ &v_n = \text{par}_n\, v_n^* \end{aligned}\right\} \tag{5}$$

where par_n is parallel translation of $T_{p_n^*}M$ to $T_{y_n}M$ along the minimal geodesic and $g(\quad; \rho v)$ is the perturbation supplied by the Lift Axiom with lift direction $\rho v \in T_y M$. Since v_n^* is small, so is v_n.

What (5) means is: start with q, jump to $y_1 = F_1(q) = f_1(q)$, consider the direction $\rho_1 v_1$ at y_1, lift y_1 in that direction to say y_1', jump from y_1' to y_2 by F_2, lift y_2 in the $\rho_2 v_2$-direction, jump ahead by F_3, etc. If all the ρ_n's are zero, these y_n^ρ are just the q_n, so everything is happening near $p_1^*, \ldots, p_{mN}^*$. Replacing Ω by a tail, if necessary, we can assume that (5) makes y_n, v_n well-defined and makes

$$B, fB, \ldots, f^T B \text{ disjoint,}$$

since T is fixed and B is small when $\|\Lambda\|$ is small. We call g_n the perturbation $g(\quad; v_n)$ and denote the dependence of y_n, v_n, g_n on $\rho_1, \ldots, \rho_{n-1}$ by writing

$$y_n = y_n^\rho, \quad v_n = v_n^\rho, \quad g_n = g_n^\rho.$$

These g_n's are the *trial perturbations*. The eventual lift direction making q gf-periodic is

$$\rho_1 v_1, \ldots, \rho_{mN} v_{mN}$$

for the proper choice of ρ. Observe that y_n depends continuously on ρ. Clearly, y_1^ρ is independent of ρ. Assuming y_n^ρ depends continuously on ρ, it is clear that

$$v_n^\rho = \text{par}_n\,(v_n^*)$$

depends continuously on ρ and therefore that

$$\begin{aligned} y_{n+1}^\rho &= F_{n+1} \circ g_n(y_n^\rho; \rho_n v_n^\rho) \\ &= F_{n+1} \exp_{y_n}(\varepsilon \rho_n v_n^\rho) \end{aligned}$$

also does. This uses (L1) which is weaker than continuity of g respecting v, a property we do not need.

Remark. If

$$\text{supp}\, g_n^\rho \subset f_n B \qquad [B = \exp(r + \Lambda I^m)] \tag{6}$$

then $(gf)^T$ factors as

$$(gf)^T = g_{mN}^\rho \circ F_{mN} \circ g_{mN-1}^\rho \circ F_{mN-1} \circ \cdots \circ F_3 \circ g_2^\rho \circ F_2 \circ g_1^\rho \circ F_1 \tag{7}$$

since the $B, \ldots, f^T B$ are disjoint and contain the supports of g^ρ. What must be established is that

$$\rho_1, \ldots, \rho_{mN} \text{ exist} \begin{cases} \text{large enough so } (gf)^T q = f^T p, \\ \text{small enough so (6) holds.} \end{cases} \tag{8}$$

(d) Pseudo-orbits and limit configurations

Our strategy is to deal with the composed map in (7)

$$\chi^\rho = g^\rho_{mN} \circ F_{mN} \circ \cdots \circ g^\rho_1 \circ F_1$$

even though, due to failure of (6), it may not equal $(gf)^T$. As $h(\omega_1) \to \infty$ and $\mathrm{sm}\,(\omega_2) \to 0$ we obtain certain limit estimates on χ^ρ which enable us to prove (8).

The pseudo-orbit of q by χ^ρ is

$$q, q^\rho_1, q^\rho_2, \ldots, q^\rho_{mN},$$

where

$$q^\rho_n = g^\rho_n(y^\rho_n) = g^\rho_n F_n(q^\rho_{n-1}).$$

The purpose of (6) is to prove that this pseudo-orbit is part of a true orbit; i.e. that

$$q^\rho_n = (gf)^{t_{k(n)}}(q), \qquad 1 \le n \le mN.$$

Let $\omega = (\mathcal{Q}, \Delta; p, q, r, s, \Lambda) \in \Omega$ be given. When $(\mathcal{Q}, \Delta)$ is fixed and $|r| + \|\Lambda\|$ becomes small we want to keep track of q^ρ so we enlarge the picture near p^*_n to a standard size, using the box chart B_n

$$\hat{q}^\rho_n = B_n^{-1} q^\rho_n, \quad B_n = f_n \circ \exp_{p^*} \circ (r + \Lambda).$$

Replacing Ω by a tail, if necessary, we can assume that $\hat{q}^\rho_n$ is well-defined. For even when $|r| + \|\Lambda\|$ is small, $\exp_{p^*} \circ (r + \Lambda)$ is defined on a huge ball in $T_{p^*}M$ and f_n sends its image to a neighbourhood of p^*_n having fixed size. Note that $\mathcal{Q}$ being fixed means that $p^*_1, \ldots, p^*_{mN}$ and $f_1, \ldots, f_{mN}$ are fixed.

The quantity $\hat{q}^\rho_n - \hat{q}^\rho_{n-1}$ is called the *proportional lift in the box* B_n. As $h(\omega) \to \infty$ and $\mathrm{sm}\,(\omega_2) \to 0$ the points $\hat{q}^\rho_0, \ldots, \hat{q}^\rho_{mN}$ assume a certain limit configuration depending on ρ. We proceed to prove the key equation

$$\lim_\Omega (\hat{q}^\rho_n - \hat{q}^\rho_{n-1}) = \frac{a\varepsilon\rho_n}{2\mathcal{B}} e_n, \qquad 1 \le n \le mN, \tag{9}$$

which determines the limit configuration of $\hat{q}^\rho_1, \ldots, \hat{q}^\rho_{mN}$ in terms of $\lim_\Omega \hat{q}^\rho_0$.

Since

$$\hat{q}^\rho_{n-1} = B_n^{-1}(y^\rho_n),$$

we can estimate the proportional lift in B_n by the C^1 Mean Value Theorem. We obtain

$$\begin{aligned} \hat{q}^\rho_n - \hat{q}^\rho_{n-1} &= B_n^{-1}(q^\rho_n) - B_n^{-1}(y^\rho_n) = B_n^{-1} \exp_{y^\rho_n}(\varepsilon\rho_n v_n) - B_n^{-1} \exp_{y^\rho_n}(0) \\ &= \int_0^1 D(B_n^{-1} \exp_{y^\rho_n})_{t\varepsilon\rho_n v_n} dt(\varepsilon\rho_n v_n). \end{aligned}$$

Substituting

$$v_n = \mathrm{par}_n\left(\frac{a}{2\mathcal{B}} S_n \Lambda(e_n)\right)$$

and using the Chain Rule, this integral becomes

$$\left(\frac{\varepsilon a \rho_n}{2\mathscr{B}}\right)\int_0^1 \Lambda^{-1} D(\exp_{p^*}^{-1} \circ f_n^{-1} \circ \exp_{y_n^\rho})_{t\varepsilon\rho_n v_n} \circ \mathrm{par}_n \circ (S_n\Lambda)(e_n)dt.$$

As $|r|+\|\Lambda\|\to 0$, Λ^{-1} blows up so some caution is required. However, the composition $D(\quad)\circ \mathrm{par}_n$ uniformly approaches J_n^{-1} as

$$\mathrm{sm}\,(\omega_2)=|r|+\|\Lambda\|\to 0$$

while $\omega_1=(\mathscr{Q},\Delta)$ stays fixed. Writing

$$j_n = J_n - D(\quad)\circ \mathrm{par}_n$$

gives the proportional lift in B_n as

$$\left(\frac{\varepsilon a \rho_n}{2\mathscr{B}}\right)\Lambda^{-1}J_n^{-1}S_n\Lambda(e_n)+\frac{\varepsilon a\rho_n}{2\mathscr{B}}\int_0^1 \Lambda^{-1} j_n S_n\Lambda(e_n)\,dt.$$

Using (3.1), (2iii), and (4) we have

$$\|(J_n\Lambda)^{-1}(S_n\Lambda)-\mathrm{Id}\|\le \frac{1}{\mathrm{hyp}_{V^i}\,(J_n\Lambda)}\le\frac{\mathrm{bol}\,(\Delta^{-1}\Lambda)}{\mathrm{hyp}_{V^i}\,(J_n\Delta)}\le\frac{2m^{1/2}}{h},$$

where $h=h(\omega_1)$. Thus

$$\left|(\hat{q}_n^\rho-\hat{q}_{n-1}^\rho)-\frac{a\varepsilon\rho_n}{2\mathscr{B}}e_n\right|\le\frac{\varepsilon a|\rho_n|}{2\mathscr{B}}\left[\frac{2m^{1/2}}{h}+\mathrm{bol}\,(\Lambda)\|j_n\|\|S_n\|\right].$$

Now $\lim_\Omega$ means we can fix ω_1 with h enormous and then let $|r|+\|\Lambda\|\to 0$; the former puts a uniform bound on bol (Λ) by (4) and $S_1,\ldots,S_{mN}$ are fixed for fixed ω_1. Thus, $\|j_n\|\to 0$ dominates and the r.h.s. above is arbitrarily small, proving (9).

Since $\hat{q}_0^\rho = B^{-1}q\in a_1\bar{I}^m$ which is compact, there is an ascending $\Omega'\subset\Omega$ such that

$$Q=\lim_{\Omega'}\hat{q}_0^\rho$$

exists, $Q\in a_1\bar{I}^m$.

By (9) this completely determines the limit position of the other $\hat{q}_n^\rho$, $1\le n\le mN$. In the same way, we find the limit position of supp g_n^ρ – namely

$$\limsup_{\Omega'} B_n^{-1}(\mathrm{supp}\; g_n^\rho)\subset \lim_{\Omega'}\hat{q}_{n-1}^\rho + a\bar{I}^m. \tag{10}$$

First we calculate that

$$|v_n|\le a\mathbf{m}(J_n\Lambda).$$

Parallel translation along geodesics is orthogonal so

$$|v_n|=|v_n^*|,\qquad v_n^*=aS_n\Lambda e_n/2\mathscr{B}.$$

By Proposition 3.1,

$$2\mathbf{m}(J_n\Lambda)\ge\mathbf{m}(S_n\Lambda\,|\,V^i),\quad e_n\in V^i,$$

if $h\ge 2$. Thus

$$|v_n|=|v_n^*|=\frac{a}{2\mathscr{B}}|S_n\Lambda e_n|\le\frac{a}{2\mathscr{B}}\|S_n\Lambda|\,V^i\|$$
$$\le\frac{a}{2}\mathbf{m}(S_n\Lambda\,|\,V^i)\le a\mathbf{m}(J_n\Lambda).$$

By (L2),

$$\operatorname{supp} g_n^\rho \subset \exp_{y_n^\rho} U_n^\rho,$$

where $U_n^\rho = T_{y_n^\rho}M(|\rho_n v_n|)$. Thus

$$B_n^{-1}(\operatorname{supp} g_n^\rho) \subset B_n^{-1}(\exp_{y_n^\rho} T_{y_n^\rho}M(a\mathbf{m}(J_n\Lambda))).$$

But the map $B_n^{-1} \circ \exp_{y_n^\rho}$ sends the origin of $T_{y_n^\rho}M$ onto $\hat{q}_{n-1}^\rho$ and has derivative

$$\Lambda^{-1}D_u(\exp_{p^*}^{-1} \circ f_n^{-1} \circ \exp_{y_n^\rho}), \qquad u \in T_{y_n^\rho}M.$$

When $|u| \le a\mathbf{m}(J_n\Lambda)$, D_u tends uniformly to J_n^{-1} as sm $(\omega_2) \to 0$, and using the C^1 Mean Value Theorem as above, we see that the B_n^{-1}-image of supp g_n^ρ is contained in the ball at $\hat{q}_{n-1}^\rho$ of radius

$$\le \|(\Lambda^{-1}J_n^{-1})a\mathbf{m}(J_n\Lambda)\| + \|\Lambda^{-1}\|\|j_n\|a\|J_n\|\|\Lambda\|,$$

where j_n is as above. The second term tends to 0 as sm $(\omega_2) \to \infty$, since bol (Λ) stays bounded. The first term is a. Since the unit ball is contained in the unit cube, this proves (10).

Let us calculate $\lim_{\Omega'} q_n^\rho$ explicitly. For convenience, equate the so far independent parameters $\rho_1, \ldots, \rho_{mN}$ by blocks of length N; i.e. for $\sigma \in \bar{I}^m$ put

$$\rho = \rho(\sigma) = (\underbrace{\sigma^1, \ldots, \sigma^1}_{N}, \underbrace{\sigma^2, \ldots, \sigma^2}_{N}, \underbrace{\sigma^m, \ldots, \sigma^m}_{N}).$$

From (9) we calculate

$$\lim_{\Omega'} \hat{q}_n^{\rho(\sigma)} = Q + \ell_n\sigma$$

where $\ell_n : T_{p^*}M \to T_{p^*}M$ is a diagonal matrix with (l, l)th entry

$$\ell_n^{ll} = \begin{cases} 0, & n \le (l-1)N; \\ [n-(l-1)N]\varepsilon a/\mathscr{B}, & (l-1)N < n \le lN; \\ N\varepsilon a/\mathscr{B}, & n > lN. \end{cases}$$

Thus $\ell_{mM}^{ll} = N\varepsilon a/\mathscr{B}$, for all l, $1 \le l \le m$. Call b this common value

$$\ell_{mN} = \begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{bmatrix}.$$

By (1)

$$b \ge a_1 + a_2 = \tfrac{1}{3} + \tfrac{5}{6}\sqrt{3}.$$

The arithmetic involving $\sqrt{\tfrac{3}{4}}$, 40, etc. is left to the reader.

We call $\mathscr{A}$ the set of acceptable parameter values

$$\mathscr{A} = \{\sigma \in \bar{I}^m : Q + b\sigma \in a_2\bar{I}^m\}.$$

Since $b \ge a_1 + a_2$ and $Q \in a_1\bar{I}^m$, the (affine) map $\sigma \mapsto Q + b\sigma$ is a homeomorphism of $\mathscr{A}$ onto $a_2\bar{I}^m$. Also we observe

$$\sigma \in \mathscr{A} \Rightarrow Q + \ell_n\sigma = \lim_{\Omega'} \hat{q}_n^\rho(\sigma) \in a_2\bar{I}^m, \qquad 1 \le n \le mN; \tag{11}$$

$$\sigma \in \mathscr{A} \Rightarrow \limsup_{\Omega'} \hat{B}_n^{-1}(\operatorname{supp} g_n^{\rho(\sigma)}) \subset a_3\bar{I}^m, \qquad 1 \le n \le mN. \tag{12}$$

For ℓ_n, ℓ_{mN} are diagonal matrices with

$$\ell_{mN}^{ll} \geq \ell_n^{ll} \geq 0$$

and so Q, $Q+\ell_{mN}\sigma$, $Q+\ell_n\sigma$ are three points Q, Q', Q'' of $\mathbb{R}^m$ such that the kth coordinate of Q'' lies between the kth coordinates of Q and of Q', $1 \leq k \leq m$. Any rectangular box containing Q, Q' must contain every such Q''. This proves (11), while (12) is a direct consequence of (10), (11). This completes our analysis of the limit configurations.

Remark. It is absolutely necessary to deal with boxes (as opposed to 'ellipsoids' = images of the unit Euclidean ball U^m, $\exp \circ (r+\Lambda)U^m$) in order to get (11). Except for this, we could replace $\bar{I}^m$ by U^m throughout, and, in Lemma 4.1, make $\Lambda \equiv \Delta$.

Finally, we are ready to say how $\mathcal{Q}$, Δ, v_n should be picked to verify (3).

We consider for each $\omega \in \Omega'$ the map

$$Q_n^\omega : \mathcal{A} \to T_{p^*}M$$
$$\sigma \mapsto \hat{q}_n^{\rho(\sigma)}.$$

We know that Q_n^ω converges uniformly to the affine map $\sigma \mapsto Q + \ell_n\sigma$ as $h(\omega_1) \to \infty$ and sm $(\omega_2) \to 0$. Thus

$$Q_{mN}^\omega(\mathcal{A}) \supset a_1 \bar{I}^m \tag{13}$$

for all ω is in some tail of Ω'. For

$$\sigma \mapsto Q + \ell_{mN}\sigma = Q + b\sigma$$

is a homeomorphism of $\mathcal{A}$ onto $a_2\bar{I}^m$ and any continuous map (Q_{mN}^ω for instance) near a homeomorphism of one m-cell to another has almost the same image. (We know that $Q_n^\omega(\sigma)$ is a continuous function of σ because y_n^ρ depends continuously on ρ and so does $g_n^\rho(y_n^\rho) = q_n^\rho$.) In particular, $\sigma \in \mathcal{A}$ can be found so that

$$\hat{q}_{mN}^{\rho(\sigma)} = B^{-1}p,$$

since $p \in B(a_1\bar{I}^m)$. Let such a σ be called $\sigma(\omega)$

$$\hat{q}_{mN}^{\rho(\sigma(\omega))} = B^{-1}(p).$$

This proves the first half of (8): for each ω in some tail of Ω' we have a $\sigma(\omega) \in \mathcal{A}$ such that the pseudo-orbit

$$q, q_1^\rho, q_2^\rho, \ldots, q_{mN}^\rho, \quad \rho = \rho(\sigma(\omega)),$$

starts at q and ends at f^Tp.

From (12) and convergence we see that

$$B_n^{-1}(\text{supp } g_n^{\rho(\sigma(\omega))}) \subset \bar{I}^m$$

for all ω in some tail of Ω', since $\sigma(\omega) \in \mathcal{A}$. Multiplying through by B_n gives (6), the second half of (8):

$$\text{supp } g_n^{\rho(\sigma(\omega))} \subset B_n.$$

Let Ω_0 be the intersection of all tails of Ω' so far discussed; choose and fix any $\omega_0 \in \Omega_0$,

$$\omega_0 = (\mathcal{Q}, \Delta, p, q, r, s, \Lambda);$$

set $\rho = \rho(\sigma(\omega_0))$ and

$$g_n = g_n^{\rho}.$$

Then $(8) + (4) \Rightarrow (3)$. As we observed earlier, (3) gives the Closing Lemma for $\mathscr{S} \subset \mathscr{D}^1$.

Q.E.D.

Proof of theorem 5.2. We apply the diffeomorphism proof to the Poincaré maps of our flow φ along the φ-orbit through p^* as follows. We may assume that p^* is not in the closure of Per (φ) – otherwise we need make no perturbation of φ at all. At the points $p^*, \varphi_1 p^*, \varphi_2 p^*, \ldots$ we examine the canonical Poincaré maps

$$\Pi_{p^*} \to \Pi_{\varphi_n(p^*)}$$

and select from the sequence of their tangent maps a subsequence and a splitting

$$T_{p^*}(\Pi_{p^*}) = V^1 \oplus \cdots \oplus V^L$$

by (3.2). We fix ε and N as above. Using (3.1), (3.2), (3.3) we look at the bi-ordered set

$$\Omega = \{\mathscr{Q}, \Delta, p, q, r, s, \Lambda\}$$

as before. For $(\mathscr{Q}, \Delta)$ satisfying the flow version of (4), we determine the $(m-1)N$ points p_n^* as in the $\mathscr{D}$-case. Then we construct disjoint flowboxes F_n starting at p_n^*, $1 \le n \le (m-1)N$. The continuous function δ from the Lift Axiom for φ is positive on F_n, since p^* is not in the closure of Per (φ). Hence $\delta | F_n \ge$ some $\delta_0 > 0$. We choose a tail of Ω with $\|\Lambda\|$ so that

$$|v_n^*| = \frac{a\varepsilon}{2\mathscr{B}} |S_n \Lambda e_n| < \delta_0, \quad 1 \le n \le (m-1)N.$$

The rest of the construction is the same as in the $\mathscr{D}$-case except that we lift points gradually as they travel across the flowbox instead of lifting them after they land on M.

Q.E.D.

Proof of theorem 5.3. There is no formal difference from that of (5.2), the flow case.

The difference between fields and flows is in the verification of the Lift Axiom. For fields it is harder.

6. *$\mathscr{D}^1$ and $\mathscr{F}^1$ have lift*

(6.1) THEOREM. *$\mathscr{D}^1$ satisfies the Lift Axiom.*

(6.2) COROLLARY. *The C^1 Closing Lemma holds for any $f \in \mathscr{D}^1$.*

Proof. Apply (5.1).

Proof of (6.1). This is easy: we glue standard lifts into M. Let $\alpha : \mathbb{R}^m \to [0, 1]$ be a C^∞ bump-function which equals 1 near 0, has support in $U^m \equiv$ the unit ball of $\mathbb{R}^m$, and has C^1-size ≤ 2. When $0 \ne v \in T_pM$ and $\varepsilon > 0$ define

$$\hat{g}_\varepsilon(\ \ ; v) : T_pM \to T_pM$$
$$y \mapsto y + \varepsilon\alpha(y/|v|)v = \hat{g}_\varepsilon(y; v).$$

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If $\varepsilon<\frac{1}{2}$ and $v\in T_pM$ with $0<|v|\leq 1$ then $\hat{g}_\varepsilon(\ \ ;v)$ is a diffeomorphism of T_pM onto itself. For the C^1-distance from $\hat{g}_\varepsilon$ to the identity map is

$$|\hat{g}_\varepsilon-\mathrm{Id}|_{C^1}\leq\left|\varepsilon(D\alpha)_{y/|v|}(\ \ )\frac{v}{|v|}\right|\leq 2\varepsilon \tag{1}$$

and any map of $\mathbb{R}^m$ whose C^1 distance to the identity is <1 is a diffeomorphism. Define g by the commutativity of

$$\begin{tikzcd} T_pM \arrow[r, "\hat{g}_\varepsilon(\ \ ;v)"] \arrow[d, "\exp_p"] & T_pM \arrow[d, "\exp_p"] \\ M \arrow[r, "g_\varepsilon(\ \ ;v)"] & M \end{tikzcd}$$

Set $g_\varepsilon=$ identity off $M_p(1)$. By construction

(L1) $g_\varepsilon(p;v)=\exp_p(\varepsilon v)$,

(L2) $\operatorname{supp}(g_\varepsilon(\ \ ;v))\subset M_p(|v|)$

and (L3) is automatic since $\mathscr{D}^1$ is closed under composition.

It remains to show that given $f\in\mathscr{D}^1$ and $\mathscr{N}$ a neighbourhood of f in $\mathscr{D}$

there exists $\varepsilon>0$ such that $g_\varepsilon f\in\mathscr{N}$.

The ε is not allowed to depend on v, $|v|\leq 1$. Since composition is C^1-continuous, it suffices to show that g_ε converges C^1-uniformly to the identity as $\varepsilon\to 0$; but in questions of C^1 convergence, the particular choice of exp never plays a significant role, so this follows from (1). Q.E.D.

(6.3) THEOREM. *$\mathscr{F}^1$ satisfies the Lift Axiom.*

(6.4) COROLLARY. *The C^1 Closing Lemma holds for any $\varphi\in\mathscr{F}^1$.*

Proof. Apply (5.2).

Remark. The proof of (6.3) is quite similar to (6.1) if the given flow has no fixed points, for instance if it is the suspension of a diffeomorphism. But, in the general case, more care must be taken because we wish to work in flowbox coordinates even though no finite number of them covers M and flowboxes may be badly distorted near fixed points. We use the following topological result.

(6.5) LEMMA. *Let U be the interior of a compact m-ball $\bar{U}$ in $\mathbb{R}^m$ and let $f:\bar{U}\to\mathbb{R}^m$ be a continuous map which is locally injective on U. Then*

$$f|\partial U \text{ is injective} \Rightarrow f \text{ embeds } \bar{U}.$$

Question. Does (6.5) remain true for all open, bounded, connected $U\subset\mathbb{R}^m$?

Proof. Invariance of domain implies $f|U$ is open so fU is an open subset of $\mathbb{R}^m$.

Its closure $\overline{fU}$ equals $f(\bar{U})$ since f is continuous, so fU is bounded. If $q \in \partial(fU)$ then $q \notin fU$ since fU is open. Hence

$$\partial(fU) \subset f(\partial U). \tag{i}$$

Since $f|\partial U$ is injective $f(\partial U)$ is a topological sphere and it separates $\mathbb{R}^m$ into two components. fU has no points in the unbounded component; for such a point could be connected to ∞ by an arc γ missing $f(\partial U)$, but γ would leave the bounded set fU somewhere, and that would only occur at a boundary point of $\partial(fU)$, contradicting (i). Hence

$$fU \subset \mathbb{R}^m - \text{outside } (f(\partial(U)). \tag{ii}$$

No $p \in U$ can be sent to $f(p) \in f(\partial U)$ because some neighbourhood V of p would thereby be sent onto a neighbourhood of $f(p)$ in $\mathbb{R}^m$ and $f(\partial u)$ being a topological sphere, there are points of its outside arbitrarily near $f(p)$, contradicting (ii). Hence

$$fU \cap f(\partial U) = \varnothing. \tag{iii}$$

On the other hand, if $fp \in f(\partial U)$ there are points of fU arbitrarily near it but points outside $f(\partial U)$ also near it. Hence $f(\partial U) \subset \partial(fU)$ so with (i) this shows

$$f(\partial U) = \partial(fU). \tag{iv}$$

In particular, fU is a topological open m-ball in $\mathbb{R}^m$. But any locally injective map of one open m-ball to another which extends continuously to a map carrying the boundary of one to the boundary of the other is globally injective, for it is easily seen to be a covering map and U is simply connected. Q.E.D.

Remark. There do exist locally injective maps of one open ball onto another which are not globally injective, for example, figure 8. Of course, boundaries are not preserved.

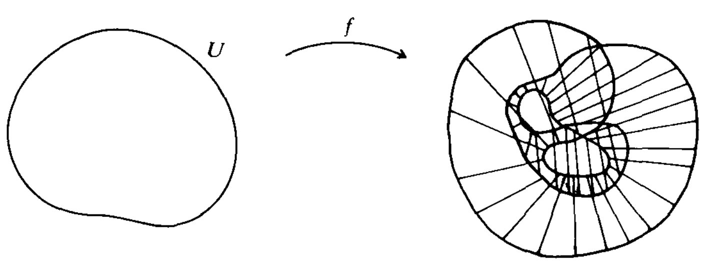

FIGURE 8. A locally injective map of one ball onto another.

Proof of (6.3). Let $\varphi \in \mathscr{F}^1$ and $\mathcal{N}$ a neighbourhood of φ in $\mathscr{F}^1$ be given. Call $\dot{\varphi} = X$. At each $p \in M_\varphi$ consider the natural flowbox chart at p

$$\begin{array}{ccc} T_pM \supset \hat{U}_p^1 = \{tX(p) + y : 0 \le t \le 1 \text{ and } y \in \Pi_p(1)\} \\ \downarrow F_p^1 \qquad\qquad \downarrow \\ M \qquad\qquad \varphi_t(\exp y) \end{array}$$

As above, $\hat{\Pi}_p(\delta)$ is the δ-ball in the orthogonal complement to span $X(p)$.

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We claim that there is a continuous function $\delta : M_\varphi \to (0, 1)$ such that, if we call

$$\hat{U}_p = \{tX(p) + y \in \hat{U}_p^1 : |y| \le \delta(p)\}$$
$$F_p = F_p^1 \mid \hat{U}_p,$$

then each F_p is an embedding and, for $u \in U_p$,

$$\mathbf{m}(T_u F_p) \text{ and } \|T_u F_p\| \text{ are uniformly bounded away from 0 and } \infty. \tag{2}$$

These thin flowboxes F_p will be used to construct lift for φ. When $M_\varphi = M$, (2) is easy because M is compact, but when φ has fixed points (2) is not immediate. First take u varying only along the segments $[0, X(p)]$ in TM. These segments are the central axes of the flowboxes. Then

$$T_u F_p = (T_p \varphi_t) \circ \iota, \quad u \in [0, X(p)], \tag{3}$$

where $\iota : T_u(T_p M) \to T_p M$ is the canonical identification. (3) is clear because, for $u = tX(p)$, $0 \le t \le 1$, we have

$$T_u F_p^0 |\iota^{-1} \hat{\Pi}_p = T_p \varphi_t \circ (T_0 \exp_p) | \iota^{-1} \hat{\Pi}$$

and both $(T_u F_p^0) \circ \iota^{-1}$, $T_p \varphi_t$ have the same effect on $X(p)$: they send it onto $X(\varphi_t p)$. (Recall that any exponential map has the property that $T_p(\exp_p) : T_p M \to T_p M$ is the identity.) From (3) we see that (2) holds when everything is restricted to these segments $[0, X(p)]$, for $\{T_p \varphi_t\}$ is a family of isomorphisms continuously indexed by the compact set $[0, 1] \times M$. Since TM_φ is locally compact, it is then easy to find a function δ on M_φ which tends to zero so rapidly at ∂M_φ that (2) holds for the thin flowboxes F_p as claimed.

Again, the particular choice of exp is irrelevant to C^1-convergence questions and it entails no loss of generality to assume that $M = \mathbb{R}^m$, exp is the inclusion, and $F_p \subset U^m$. Interpreted in $\mathbb{R}^m$, (2) says that the derivative matrices of the flowbox charts F_p form a compact subset of the invertible matrices. The latter are open among all matrices, so there is a $\nu > 0$ such that

$$\|A - (Df_p)_u\| < \nu \text{ for some } u \in U_p \Rightarrow A \text{ is invertible.} \tag{4}$$

Note that ν is independent of $p \in U^m$.

Let $g_\varepsilon(\ \ ; v)$ be the standard lift constructed in the proof of (6.1), but restricted to Π_p. Thus $g_\varepsilon(\ \ , v)$ is a diffeomorphism of Π_p onto itself and

$$\left.\begin{array}{ll} g_\varepsilon(0; v) = v, & \operatorname{supp}(g_\varepsilon(\ \ ; v)) \subset \Pi_p(|v|), \\ |g_\varepsilon(\ \ ; v) - \mathrm{Id}|_{C^1} < 2\varepsilon. & \end{array}\right\} \tag{5}$$

We suspend g_ε as follows. Let β be a C^∞bump-function on $\mathbb{R}$ which is 0 near 0 and 1 near 1. Set

$$\begin{array}{c} \hat{U}_p = \{tX(p) + y : 0 \le t \le 1 \text{ and } y \in \Pi_p(\delta(p))\} \\ \Big\downarrow{\scriptstyle F_p^{\varepsilon, v}} \\ (1 - \beta(t))\varphi_t(y) + \beta(t)\varphi_t(\bar{y}) \qquad \bar{y} = g_\varepsilon(y; v). \end{array}$$

Addition occurs in $\mathbb{R}^m = M$. When $\varepsilon = 0$, $F_p^{\varepsilon,v}$ becomes the standard flowbox chart F_p. Call

$$\Phi = \sup \{\|(D\phi_t)_p\|: 0 \le t \le 1, p \in M\}.$$

We claim that for some continuous small $\rho_0 : M_\varphi \to (0, 1)$

$$\|(DF_p^{\varepsilon,v})_u - (DF_p)_u\| < 2\varepsilon\Phi \tag{6}$$

for all $u \in \hat{U}_p$, all $p \in M_\varphi$, and all $v \in \Pi_p(\rho_0(p))$. We calculate

$$\frac{\partial F_p^{\varepsilon,v}(tX(p)+y)}{\partial t} = \beta'(t)(\varphi_t\bar{y} - \varphi_t y) + X(\varphi_t y) + \beta(t)(X(\varphi_t\bar{y}) - X(\varphi_t y))$$

$$\frac{\partial F_p^{\varepsilon,v}(tX(p)+y)}{\partial y} = (1-\beta(t))(D\varphi_t)_y + \beta(t)(D\varphi_t)_{\bar{y}}(Dg)_y$$

$$\bar{y} = g_\varepsilon(y; v), \quad g = g_\varepsilon(\quad; v)$$

so that, for all $u = tX(p) + y \in \hat{U}_p$,

$$\begin{aligned}\|(DF_p^{\varepsilon,v})_u - (DF_p)_u\| &\le |\beta'(t)\|\varphi_t y - \varphi_t\bar{y}| + \beta(t)|X(\varphi_t\bar{y}) - X(\varphi_t y)| \\ &\quad + \beta(t)\|(D\varphi_t)_{\bar{y}}(Dg)_y - (D\varphi_t)_y\| \\ &\le |\beta|_1\{|\varphi_t y - \varphi_t\bar{y}| + |X(\varphi_t y) - X(\varphi_t\bar{y})| + \|(D\varphi_t)_y - (D\varphi_t)_{\bar{y}}\|\} \\ &\quad + \|(D\varphi_t)_y\|\|D(g)_y - I\|.\end{aligned}$$

Now $|y - \bar{y}| \le 2\rho_0$ while $(D\varphi_t)_y$ and $X(x)$ are uniformly continuous for $0 \le t \le 1$ and $x, y \in U^m$. Thus

$$\|DF^{\varepsilon,v} - DF\| < 2\varepsilon\Phi$$

provided $\rho \le$ some constant ρ_0. Choose

$$\rho_0(p) = \min(\delta(p), \rho_0)$$

and (6) follows. Choose $\varepsilon_0 \le \nu/2\Phi$. By (6), (4) and the IFT, $F_p^{\varepsilon,v}$ is locally injective. On $\partial\hat{U}_p$, $F_p^{\varepsilon,v}$ agrees with F_p except on

$$\{tX(p) + y : t = 1\}.$$

Here, however, $F_p^{\varepsilon,v}$ sends

$$X(p) + y \quad \text{to} \quad \varphi_1(\bar{y}) = \varphi_1(g_\varepsilon(y; v))$$

which is also injective; thus, $F_p^{\varepsilon,v}$ is injective on $\partial\hat{U}_p$. By (6.5), $F_p^{\varepsilon,v}$ embeds $\hat{U}_p$ and, since

$$F_p^{\varepsilon,v}(\partial\hat{U}_p) = F_p(\partial\hat{U}_p),$$

we see that $F_p^{\varepsilon,v}$ and F_p have the same image flowbox: call it

$$U_p = F_p(\hat{U}_p) = F_p^{\varepsilon,v}(\hat{U}_p).$$

Since $F_p^{\varepsilon,v}$ is an embedding, it defines a C^1 flow ψ on U_p whose trajectories are those of φ outside U_p and while on U_p they are

$$(t, g) \mapsto (1-\beta(t))\varphi_t y + \beta(t)\varphi_t(\bar{y}), \qquad \bar{y} = g_\varepsilon(y; v),$$

for $0 \le t \le 1$ and $y \in \Pi_p$. That is, ψ in the flowbox is the $F_p^{\varepsilon,v}$-conjugate of the translation flow χ on $\hat{U}_p$

$$\chi_s : tX(p) + y \mapsto (t+s)X(p) + y$$

while φ is the F_p-conjugate of χ. Now ψ and φ are C^1-conjugate to the same flow by conjugacies which, according to (6), are arbitrarily C^1-close to each other, and which, by (2), are C^1 bounded. Hence $\psi \in \mathcal{N}$, the given neighbourhood of φ.

The ψ trajectory through y strikes Π'_p exactly where the φ-trajectory through $\bar{y} = g_\varepsilon(y; v)$ does. For $\psi_1(y) = \varphi_1(\bar{y})$ and the bump function $\beta(t)$ is constantly equal to 1 near $t = 1$. Thus

$$g_\varepsilon(\quad; v) = (\text{Poincaré map of } \varphi)^{-1} \circ (\text{Poincaré map of } \psi)$$

so ψ lifts p as it should to verify (L'1). The flows of φ and ψ differ only in U_p so (L'2) is satisfied; and (L'3) is automatic for $\mathcal{F}^1$. Q.E.D.

7. $\mathcal{X}^1$ *has lift*

This section corrects the proof of Closing Lemma appearing in [**15**]. First let us point out what was wrong. Flowboxes are used as charts on M. If the given vector field is C^1 then these flowboxes are just C^1 and so perturbing vector fields constructed in them may fail to be C^1 – let alone C^1 small. (See [**15**, p. 668 lines 14–17], [**16**, p. 1013] where $\partial^2 x^i/\partial u^j\, \partial u^k$ may fail to exist in (4.5).) On the other hand, when the original field is C^2 then so are the flowboxes and the perturbations constructed are C^1 and C^1 small. See (6.5), (6.6). Why should anyone spend time making this theorem work also for C^1 fields? Because in the proof of the General Density Theorem (see § 11 and [**16**]) one must close up a non-wandering orbit of a given C^1 (non-C^2) vector field.

(7.1) THEOREM. *$\mathcal{X}^1$ satisfies the Lift Axiom.*

(7.2) COROLLARY. *The C^1 Closing Lemma holds for any $X \in \mathcal{X}^1$.*

Proof. Apply (5.3).

Proof of (7.1). Since the Lift Axiom is local it again entails no loss of generality to replace M by $\mathbb{R}^m$ and to assume that all flowboxes are contained inside the unit ball U^m. Let $\mathcal{N}$ be the given neighbourhood of X in $\mathcal{X}^1$. There is a $\mu > 0$ such that, if $Y \in \mathcal{X}^1$, supp $(Y) \subset U^m$ and

$$\|(DY)_x - (DX)_x\| < \mu \tag{1}$$

for all $x \in U^m$, then $Y \in \mathcal{N}$.

We use the same 'suspension' perturbations in $\mathcal{X}^1$ as we did in $\mathcal{F}^1$ (see § 6). Thus, we have a flow ψ which differs from the X-flow φ only in a flowbox U_p and whose trajectories there are

$$\psi_t(y) = (1 - \beta(t))\varphi_t y + \beta(t)\varphi_t \bar{y},$$
$$\bar{y} = g_\varepsilon(y; v).$$

In § 6 we showed that this standard lift on Π_p, $g_\varepsilon(\quad; v)$, suspends to the flow ψ, and ψ verifies the Lift Axiom. This required $0 \le \varepsilon \le \varepsilon_0$ and $|v| \le \rho_0(p)$, where $\rho_0 \colon M_\varphi \to (0, 1)$ is continuous and small, while $\varepsilon_0 > 0$ is a constant independent of p. It remains to show that $Y = \dot{\psi}$ is C^1 and satisfies (1) when $\varepsilon \le \varepsilon_0$, $\rho \le \rho_0$ are small enough.

Fix a constant K,

$$K \geq \sup_{p \in M} |X(p)| + \sup_{p \in M} \|(DX)_p\| + \sup_{\substack{p \in M \\ |t| \leq 1}} \|(D\varphi_t)_p\|.$$

By (6.1) we can find $\rho_1 \leq \rho_0$ and $\varepsilon_1 < \varepsilon_0$ small enough that

$$\sup_{\substack{p \in M_\varphi \\ |t| \leq 1 \\ y \in \Pi_p(\rho_1(p))}} \|(D\psi_t)_y - (D\varphi_t)_y\| \leq \mu/20(K+1)^2.$$

Recall from § 6 that $\|Dg - I\| < 2\varepsilon$. Also recall that $F_p^{\varepsilon,v} : \hat{U}_p \to U_p$ is the ψ-flowbox chart and that

$$(F_p^{\varepsilon,v})^{-1}(x) = \tau(x)X(p) + y(x) \in \hat{U}_p$$

makes $\tau(x)X(p)$ and $y(x)$ C^1 functions of x having uniformly bounded first derivatives: say bounded by B. The point $y(x)$ is where the ψ-trajectory through x strikes Π_p, and $\tau(x)$ is how much time it takes $y(x)$ to reach x along it. Thus

$$\psi_t(x) = (1 - \beta(\tau + t))\varphi_{\tau+t}(y) + \beta(\tau + t)\varphi_{\tau+t}(\bar{y}),$$

where $\tau = \tau(x)$, $y = y(x)$, $\bar{y} = g(y(x))$. Differentiating this with respect to t gives

$$\begin{aligned} Y(x) &= \frac{d}{dt}\bigg|_{t=0} \psi_t(x) \\ &= \beta'(\tau)\{\varphi_\tau(\bar{y}) - \varphi_\tau(y)\} + X(\varphi_\tau y) + \beta(\tau)\{X(\varphi_\tau \bar{y}) - X(\varphi_\tau y)\}, \end{aligned}$$

where τ, y, $\bar{y}$ are as above. $Y(x)$ is C^1 because all the functions in its expression are C^1. Moreover,

$$(DY)_x - (DX)_x = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}),$$

where

$$\begin{aligned}
(\mathrm{I}) &= \beta''(\tau)(D\tau)[\varphi_\tau \bar{y} - \varphi_\tau y], \\
(\mathrm{II}) &= \beta'(\tau)\{X(\varphi_\tau \bar{y})(D\tau) - X(\varphi_\tau y)(D\tau) + (D\varphi_\tau)_{\bar{y}}(D\bar{y}) - (D\varphi_\tau)_y(Dy)\}, \\
(\mathrm{III}) &= D(X(\varphi_\tau y) - X)_x, \\
(\mathrm{IV}) &= \beta'(\tau)(D\tau)[X(\varphi_\tau \bar{y}) - X(\varphi_\tau y)], \\
(\mathrm{V}) &= \beta(\tau)((DX)_{\varphi_\tau \bar{y}}\{X(\varphi_\tau \bar{y})(D\tau) + (D\varphi_\tau)_{\bar{y}}(D\bar{y})\} \\
&\qquad - (DX)_{\varphi_\tau y}\{X(\varphi_\tau y)(D\tau) + (D\varphi_\tau)_y(Dy)\}).
\end{aligned}$$

All these quantities depend on v and ε. Note that

$$\|D\tau\| \leq B/|X(p)|$$

is unbounded if φ has fixed points, so some care is necessary. However, the bound $B/|X(p)|$ is continuous on M_φ. If $\rho_{\mathrm{I}}(p)$ is a small enough continuous function on M_φ, then uniform continuity of φ gives

$$|\mathrm{I}| < \mu/10 \quad \text{whenever} \quad |v| \leq \rho_{\mathrm{I}}(p),$$

since $|y - \bar{y}| \leq 2\rho_{\mathrm{I}}(p)$. Similarly, $\rho_{\mathrm{IV}}(p)$ can be found small enough that

$$|\mathrm{IV}| < \mu/10 \quad \text{whenever} \quad |v| \leq \rho_{\mathrm{IV}}(p).$$

To estimate (II) note that $D\bar{y}=(Dg)(Dy)$. Thus

$$|\mathrm{II}|\le|\beta'|\{|X(\varphi_\tau\bar{y})-X(\varphi_\tau)|\,\|D_\tau\|+\|(D\varphi_\tau)_{\bar{y}}\|\,\|D\bar{y}-Dy\|+\|(D\varphi_\tau)_{\bar{y}}-(D\varphi_\tau)_y\|\,\|Dy\|\}$$
$$\le|\beta'|\{|X(\varphi_\tau\bar{y})-X(\varphi_\tau y)|\,\|D\tau\|+K\|Dg-\mathrm{Id}\|B+\|(D\varphi_\tau)_{\bar{y}}-(D\varphi_\tau)_y\|B\}$$

which gives

$$|\mathrm{II}|\le\mu/20+2KB\varepsilon+\mu/20$$

by requiring $|v|\le\rho_{\mathrm{II}}(p)$.

Estimating (V) is similar:

$$|\mathrm{V}|\le\|(DX)_{\varphi_\tau\bar{y}}-(DX)_{\varphi_\tau y}\|\,\|\{X(\varphi_\tau\bar{y})(D\tau)+(D\varphi_\tau)_{\bar{y}}(D\bar{y})\}\|$$
$$+\|(DX)_{\varphi_\tau y}\|\,|\{X(\varphi_\tau\bar{y})(D\tau)-X(\varphi_\tau y)(D\tau)+(D\varphi_\tau)_{\bar{y}}[(D\bar{y})-(Dy)]$$
$$+[(D\varphi_\tau)_{\bar{y}}-(D\varphi_\tau)_y](Dy)\}|$$
$$\le\|(DX)_{\varphi_\tau\bar{y}}-(DX)_{\varphi_\tau y}\|\{K\|D\tau\|+K(1+\varepsilon)B\}$$
$$+K\{|X(\varphi_\tau\bar{y})-X(\varphi_\tau y)|\,\|D\tau\|+2KB\varepsilon+\|(D\varphi_\tau)_{\bar{y}}-(D\varphi_\tau)_y\|B\},$$

which gives

$$|\mathrm{V}|\le\mu/30+\mu/30+2K^2B\varepsilon+\mu/30$$

provided $|v|\leqq\rho_{\mathrm{V}}(p)$.

To estimate (III) we derive $y(x)$:

$$(Dy)_x=D(\psi_{-\tau(x)}(x))=(D\psi_{-\tau})_x-Y(y)D\tau$$
$$=D(\psi_{-\tau})_x-X(y)(D\tau),$$

since $X=Y$ on Π. Thus

$$D(X(\varphi_\tau y))_x=(DX)_{\varphi_\tau y}\{X(\varphi_\tau y)(D\tau)+(D\varphi_\tau)_y(Dy)\}$$
$$=(DX)_{\varphi_\tau y}\{X(\varphi_\tau y)(D\tau)+(D\varphi_\tau)_y(D\psi_{-\tau})_x-(D\varphi_\tau)_yX(y)(D\tau)\}$$
$$=(DX)_{\varphi_\tau y}(D\varphi_\tau)_y(D\psi_{-\tau})_x,$$

since $(D\varphi_\tau)_yX(y)=X(\varphi_\tau y)$. This gives

$$|\mathrm{III}|=\|(DX(\varphi_\tau y))_x-(DX)_x\|$$
$$\le\|(DX)_{\varphi_\tau y}-(DX)_x\|\,\|(D\varphi_\tau)_y(D\psi_{-\tau})_x\|$$
$$+\|(DX)_x\|\,\|(D\varphi_\tau)_y[(D\varphi_{-\tau})_x-(D\psi_{-\tau})_x]\|$$
$$\le\|(DX)_{\varphi_\tau y}-(DX)_x\|KB+K^2\mu/20(K+1)^2$$

which gives

$$|\mathrm{III}|\le\mu/20+\mu/20$$

provided $|v|\le\rho_{\mathrm{III}}(p)$. Taking

$$\rho(p)=\min\,(\rho_0(p),\rho_{\mathrm{I}}(p),\dots,\rho_{\mathrm{V}}(p))$$

gives

$$\|(DY)_x-(DX)_x\|\le\mu/2+2KB\varepsilon$$

whenever $p\in M_\varphi$, $|v|\le\rho(p)$, and $\varepsilon\le\varepsilon_0$. For $\varepsilon<\mu/4KB$, (1) is verified and the proof of (7.1) is complete. Q.E.D.

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7A. *Smooth suspension*

The preceding construction of lift grew out of an alternate proof of the C^1 Closing Lemma for $\mathscr{X}^1$ via a smoothing process which we explain here.

(7.3) *Smoothing Theorem. If $X \in \mathscr{X}^r$ has a non-compact orbit through p and if $\mathscr{U}$ is a neighbourhood of X in $\mathscr{X}^r$, $r \geq 1$, while U is a neighbourhood of $\mathcal{O}(p)$ in M then there is $\tilde{X} \in \mathscr{U}$ and a conjugacy $h : M \circlearrowleft$ of the X-flow to the $\tilde{X}$-flow such that*

(a) $\tilde{X}$ is C^∞ on some neighbourhood of $h(\mathcal{O}(p))$;

(b) $h(p) = p$ and $h \equiv$ identity off U.

Moreover, h is C^r and C^r approximates the identity.

Remark 1. The point of (7.3) is: X can be smoothed semi-locally without losing the non-wandering nature of p. Global smoothing of X can radically change the X-flow, for instance the Denjoy flow.

Remark 2. The construction in (7.1) demonstrates the:

C^r-Perturbation Principle. Every C^r-small perturbation of a Poincaré map – as a map – arises from a C^r small perturbation of the generating vector field, $1 \leq r \leq \infty$.

In fact, in (7.1) we estimate the C^1 size of the vector field perturbation in terms of the C^1 size of the given Poincaré map perturbation. Although we worked locally (in $U^m \subset \mathbb{R}^m$), this suspension construction is also global. If Σ and Σ' are hypersurfaces in M and $X \in \mathscr{X}^r$ defines a Poincaré map $f : \Sigma \to \Sigma'$ satisfying

$$\Sigma \ni y \mapsto f(y) = \varphi_{t(y)}(y) \in \Sigma', \qquad \varphi_s y \notin \Sigma \cup \Sigma' \quad \text{for} \quad 0 < s < t(y),$$

where φ is the X-flow, and, if $X \pitchfork (\Sigma \cup \Sigma')$, then any C^r small perturbation of f as a map arises from a C^r small perturbation of X. To see this, embed M in some $\mathbb{R}^k$ and let $\mu : N \to M$ be a C^∞ tubular neighbourhood retraction. Let $\beta : \mathbb{R} \times \Sigma \to [0, 1]$ be a C^∞ bump function such that $\beta \equiv 0$ near $0 \times \Sigma$ and $\beta = 1$ near $\{(t(y), y) : y \in \Sigma\}$. Set

$$\Psi^g(t, y) = \mu((1 - \beta(t, y))\varphi_t y + \beta(t, y)\varphi_t \bar{y}), \qquad \bar{y} = f^{-1}g(y).$$

Addition occurs in $\mathbb{R}^k$ and μ brings the answer back to M. It is easy to see that Ψ^g defines a C^r flow ψ^g whose Poincaré map is g, that $\dot{\psi}^g$ is C^r, and that $\dot{\psi}^g$ C^r-approximates X. To check *how well* $\dot{\psi}^g$ C^r-approximates X in terms of g is messy as was seen in (7.1).

Remark 3. In Smale's exposition [**21**] the reader may be puzzled about the apparent loss of differentiability in the suspension construction. If $f : \Sigma \to \Sigma$ is a C^r diffeomorphism, $M_f = \Sigma \times I / f$ is naturally a C^r manifold and so its tangent vector fields are naturally only C^{r-1}. By general theory, there is a C^∞ structure on M_f, say $\tilde{M}_f$, and susp $(f) = X$ is C^r on $\tilde{M}_f$. If g C^r-approximates f, then $\tilde{M}_g \neq \tilde{M}_f$ although they are diffeomorphic. It is not clear from the general smoothing theory that the field on $\tilde{M}_f$ corresponding to susp $(g) = Y$ C^r-approximates X (although C^{r-1} is easy). In remark 2 this problem is solved: take $M = \tilde{M}_f$, $\Sigma = \Sigma'$. Then C^r small perturbations of f give C^r small perturbations of susp (f) as Smale asserts.

Proof of (7.3). We use the C^r Perturbation Principle of Remark 2 and an $\varepsilon/2^n$-argument to squeeze local perturbations down the entire trajectory of p. For simplicity, we work only with the positive semi-orbit $\mathcal{O}_+(p)$. Since $\mathcal{O}_+(p)$ is non-compact, there is some $w \in \omega(p) - \mathcal{O}_+(p)$. We choose a flowbox F_0 for x of time length >1 which contains p in its interior, $w \notin F_0$. We divide F_0 into two subflowboxes

$$G_0 \cup H_0 = F_0,$$

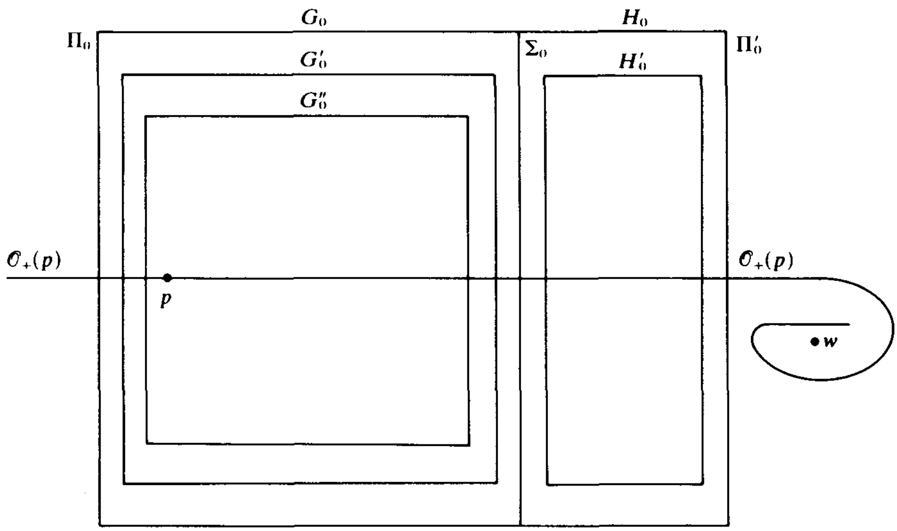

FIGURE 9. F_0 divided into subflowboxes.

as shown in figure 9, and construct subflowboxes

$$G_0 \supset G_0' \supset G_0'', \qquad H_0 \supset H_0'$$

as shown also. We make G_0'' have time length ≥ 1. Using C^∞ convolution approximation and C^∞ bump functions, we can find a field Y_0 on G_0 such that

$$Y_0 \text{ is } C^\infty \text{ on a neighbourhood of } G_0'', \tag{a}$$

$$Y_0 \equiv X \text{ on } G_0 - G_0'. \tag{b}$$

The Y_0-Poincaré map from Π_0 to Σ_0 C^r approximates that of X. By the C^r Perturbation Principle we can find a C^r field Z_0 which equals X off H_0' and such that the field

$$X_0 = \begin{cases} Y_0 \text{ on } G_0 \\ Z_0 \text{ on } H_0 \\ X \text{ off } F_0 \end{cases}$$

satisfies

$$|X - X_0|_{C^r} < \varepsilon/2, \tag{c}$$

$$\text{the } X_0\text{-Poincaré map } \Pi_0 \to \Pi_0' \text{ equals that of } X. \tag{d}$$

(d) is the 'mirror image property' of Wilson [**22**]. Call the X_0-flow $\varphi^{(0)}$. From (c), (d) it follows that there is a C^r conjugacy $h_0 : M \circlearrowleft$ of φ to $\varphi^{(0)}$ and

$$d_{C^r}(h_0, \mathrm{Id}) < \varepsilon/2, \qquad \operatorname{supp}(h_0) \subset F_0. \tag{e}$$

We can easily arrange that $h_0(p) = p$.

Now we replace p by $\varphi_1^{(0)}(p) = p_1$ and repeat the construction on an X_0-flowbox F_1. We make sure that w and p do not belong to F_1 and that F_1 is long enough so that its right end face Π_1' misses F_0. Since $w \notin F_0$ this is possible. See figure 10.

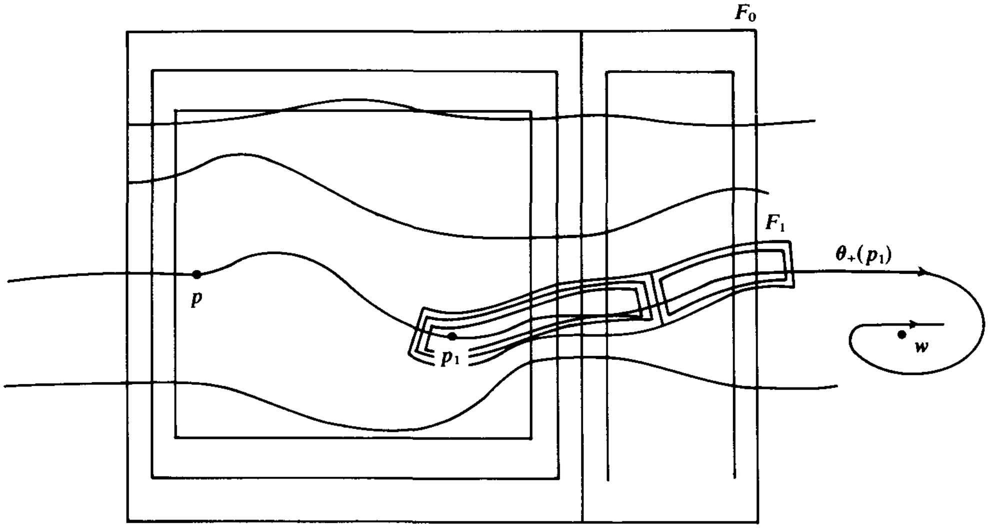

FIGURE 10. F_1 meeting F_0, with X_0-flow curves shown.

We then smooth X_0 in the left part of F_1, G_1'', and cancel the effect in the right part, H_1'. Such smoothing affects X_0 as little as we want on G_0''; in particular, the field remains C^∞ on G_0''. In this way we obtain a sequence of fields X_n, flowboxes F_n, and conjugacies h_n with

$$X_n \text{ is } C^\infty \text{ on a neighbourhood of } G_n''; \qquad (a_n)$$

$$|X_n - X_{n-1}|_{C^r} < \varepsilon/2^{n+1}; \qquad (c_n)$$

$$d_{C^r}(h_n \circ \cdots \circ h_0, h_{n-1} \circ \cdots \circ h_0) < \varepsilon/2^{n+1}, \quad \operatorname{supp}(h_n) \subset F_n; \qquad (e_n)$$

$$|X_n - X_{n-1}|_{C^\infty} < \varepsilon/2^{n+1} \quad \text{on} \quad G_0'' \cup \cdots \cup G_{n+1}''. \qquad (f)$$

h_n conjugates X_{n-1} to X_n. The flowboxes F_n never contain w and only F_0 contains p. Thus,

$$h_n \circ \cdots \circ h_0(p) = p.$$

The fields X_n and composed conjugacies $h_n \circ \cdots \circ h_0$ converge to limits $\tilde{X}$ and $\tilde{h}$ in $\mathscr{X}^r$ and $\mathrm{Diff}^r(M)$, since these spaces are complete. Clearly,

$$|\tilde{X} - X|_{C^r} \le \varepsilon \quad \text{and} \quad d_{C^r}(\tilde{h}, \mathrm{Id}) \le \varepsilon.$$

It is easy to check that $\tilde{h}$ conjugates X to $\tilde{X}$. By (f), $\tilde{X}$ is C^∞ on $\bigcup_{n=0}^\infty G_n''$ which is a neighbourhood of $h(\varphi_y(p))$, completing the proof of (7.3). Q.E.D.

Remark. Recently David Hart has shown that any C^r flow is C^r conjugate to a C^r flow generated by a C^r tangent vector field [**5**]. He is also able to prove the C^r Perturbation Principle somewhat differently. This provides an alternative way of deducing (7.1) from (6.1).

8. *Symplectic diffeomorphisms, volume-preserving diffeomorphisms and volume-preserving vector fields satisfy the Lift Axiom*

First some notation is given and then the cases are treated one at a time. Let $\Omega_i^r(M)$ be all the C^r i-forms on M. The exterior derivative is denoted by

$$d:\Omega_i^r(M)\to\Omega_{i+1}^{r-1}(M).$$

For a vector field X and differential form $\rho\in\Omega_i(M)$, the interior product of X with ρ is denoted by $X \lrcorner \rho$, where

$$(X \lrcorner \rho)(v_1,\ldots,v_{i-1})=\rho(X,v_1,\ldots,v_{i-1}).$$

The Lie derivative of ρ by X is denoted by $L_X\rho$. If φ_t is the flow of X, then

$$(L_X\rho)(x)=\frac{d}{dt}(\varphi_t^*\rho)(x)|_{t=0}.$$

It follows easily that, if $L_X\rho\equiv 0$, then $\rho=\varphi_t^*\rho$ for all t. Also there is the following formula to calculate the Lie derivative:

$$L_X\rho=d(X \lrcorner \rho)+X \lrcorner d\rho.$$

See [**1**, chapter 3], [**8**], and [**19**] for further discussion of definitions and concepts.

If ω is a form on M, let

$$\mathscr{D}_\omega^r=\{f\in\mathscr{D}^r: f^*\omega=\omega\},$$
$$\mathscr{X}_\omega^r=\{X\in\mathscr{X}^r: L_X\omega=0\},$$

and

$$\mathscr{F}_\omega^r=\{\varphi\in\mathscr{F}^r:\varphi_t^*\omega=\omega \text{ for all } t\}.$$

The cases we are interested in are when ω is a volume element or a non-degenerate two-form (symplectic).

Assume ω is a C^∞ two form on M. A map $\bar\omega$ is induced from the tangent bundle to the cotangent bundle, $\bar\omega: TM\to T^*M$, given by

$$\bar\omega(v)=v \lrcorner \omega.$$

The two-form is called *non-degenerate* if the induced map

$$\bar\omega_p;T_pM\to T_p^*M$$

is an isomorphism for each p. The map $\bar\omega$ induces a map from vector fields to one-forms,

$$\bar\omega:\mathscr{X}^r(M)\to\Omega_1^r(M),$$

which is an isomorphism if ω is non-degenerate. If ω is a closed ($d\omega=0$) non-degenerate two-form on M, then (M,ω) is called a symplectic manifold. It follows that the dimension of M is even, say $2m$.

The diffeomorphisms on M which preserve ω are called *symplectic diffeomorphisms* and are denoted by $\mathscr{D}_\omega^r$, as noted above. The vector fields which preserve ω are called *locally Hamiltonian vector fields* or *symplectic vector fields* and are denoted by $\mathscr{X}_\omega^r$. Since

$$0=L_X\omega=d(X \lrcorner \omega)+X \lrcorner d\omega=d(X \lrcorner \omega),$$

locally Hamiltonian vector fields are ones for which $\bar{\omega}(X) = X \lrcorner \omega$ is closed. If $\bar{\omega}(X)$ is exact, i.e. $\bar{\omega}(X) = dH$ for some function H, then X is called a *Hamiltonian vector field*. Let $\mathscr{X}^1_H$ be the set of Hamiltonian vector fields.

Similarly, if μ is an m-form on an m-dimensional manifold, there is induced a map from vector fields to $(m-1)$-forms

$$\bar{\mu} : \mathscr{X}^r \to \Omega^r_{m-1},$$

defined by

$$\bar{\mu}(X) = X \lrcorner \mu.$$

If μ is a volume element (non-degenerate at all points), then $\bar{\mu}$ is an isomorphism. Since μ is an m-form, $d\mu = 0$. Therefore,

$$L_X\mu = d\bar{\mu}(X) + X \lrcorner d\mu = d\bar{\mu}(X),$$

and X preserves μ if and only if $\bar{\mu}(X)$ is closed. Such vector fields are called *volume preserving* or *divergence free* and the set is denoted by $\mathscr{X}_\mu$. The set of volume-preserving diffeomorphisms is denoted by $\mathscr{D}^r_\mu$.

8(*a*) *Symplectic diffeomorphisms*

Assume (M, ω) is a C^∞ symplectic manifold, where ω is a non-degenerate two-form and $\phi \in \mathscr{D}^1_\omega$ is a symplectic diffeomorphism with $\mathscr{N}$ a neighbourhood of ϕ in $\mathscr{D}^1_\omega$. See § 8 for definitions and notation. Note that, if M is compact, all points are non-wandering. Therefore for a generic C^1 set the periodic points are dense (see 11.1).

The symplectic manifold can be covered by a finite number of C^∞ symplectic coordinate charts, i.e. coordinate charts in terms of which the two-form ω is

$$\sum_{j=1}^{m} dx^j \wedge dx^{j+m}$$

or the matrix of ω in the coordinates is

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix. Darboux's theorems show that such symplectic coordinate charts exist. As in the earlier cases, all the estimates are made in one of these coordinate charts, so we may assume that $M = \mathbb{R}^{2m}$, $T_pM = \mathbb{R}^{2m}$, and exp is the usual inclusion. By translation we can take $p = 0$. Let $\mu > 0$ be a constant such that, if $\phi \circ g \in \mathscr{D}^1_\omega$, the support of g is in the coordinate chart, and $\|g - \mathrm{Id}\|_1 < \mu$ in terms of the C^1 topology induced by the coordinate chart, then $\phi \circ g \in \mathscr{N}$. Let $\gamma : \mathbb{R}^{2m} \to \mathbb{R}$ be a C^∞ bump function with support in the unit ball, supp $\gamma \subset U^{2m}$, and $\gamma(x) = 1$ for $x \in \frac{1}{2}U^{2m}$. Let $\varepsilon > 0$.

There are two standard ways to construct symplectic diffeomorphisms, as the time-one map of a Hamiltonian vector field $\bar{\omega}(dH)$ or from a generating function. We use the former. To define the one-form dH, start by defining

$$\lambda : T_pM(\delta) \to \Omega_1,$$

$$\lambda_v = \bar{\omega}(\varepsilon v) = \varepsilon v^t J.$$

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This one-form λ_v is independent of x, so $d\lambda_v = 0$. The vector field $\tilde{\omega}^{-1}(\lambda_v) = \varepsilon v$ is Hamiltonian and points in the right direction but it does not have the correct support. To cut down the support and keep the one-form exact, it suffices to multiply the function giving λ_v by a bump function. Let

$$\begin{aligned} f(x; v) &= \int_0^1 \lambda_v(x)\, dt \\ &= \varepsilon v^t J x \end{aligned}$$

be the function associated to λ_v by the Poincaré lemma, so

$$\lambda_v = d(f(\quad; v))$$

because $d\lambda_v = 0$. Define

$$H(x; v) = f(x; v)\gamma(x|v|^{-1})$$

and let $X(\quad; v)$ be the Hamiltonian vector field associated to H by ω. Thus

$$X(x; v)^t J = DH(\quad; v).$$

Let $g(x; v, t)$ be the flow of $X(\quad; v)$ through x evaluated at time t and

$$\begin{aligned} &g: M \times T_pM(\delta) \to M, \\ &g(x; v) = g(x; v, 1), \end{aligned}$$

where δ is taken small enough so that δU^{2m} is contained in the coordinate chart. It follows that $g(\quad; v): M \to M$ is a C^∞ symplectic diffeomorphism with

$$\operatorname{supp} g(\quad; v) \subset |v| U^{2m}.$$

To show the C^1 size of $g(\quad; v) - \mathrm{Id}$ is less than μ, it suffices to show that the C^2 size of $H(\quad; v)$ is less than μ. Fix v and write

$$H(x; v) = H(x) \quad \text{and} \quad f(x; v) = f(x) = \varepsilon v^t J x.$$

Because $\operatorname{supp} f \subset |v| U^{2m}$, we can assume that $|x| \le |v|$. Then

$$|f(x)| \le \varepsilon |v||x| \le \varepsilon |v|^2$$

and $Df_x = \varepsilon v^t J$ so

$$\|Df_x\| \le \varepsilon |v| \quad \text{and} \quad D^2 f_x = 0.$$

Thus, letting $y = x/|v|$,

$$\begin{aligned} \|DH_x\| &\le |D\gamma_y\| |v|^{-1} |f(x)| + |\gamma(y)| \|Df_x\| \\ &\le \|\gamma\|_1 |v|^{-1} \varepsilon |v|^2 + \varepsilon |v| \\ &\le 2\varepsilon |v| \|\gamma\|_1 \\ &< \mu \end{aligned}$$

for ε sufficiently small. Next,

$$\begin{aligned} \|D^2 H_x\| &\le \|D^2\gamma_y\| |v|^{-2} |f(x)| + 2\|D\gamma_y\| |v|^{-1} \|Df_x\| + |\gamma(y)| \|D^2 f_x\| \\ &\le \|\gamma\|_2 |v|^{-2} \varepsilon |v|^2 + 2\|\gamma\|_1 |v|^{-1} \varepsilon |v| + 0 \\ &\le 3\varepsilon \|\gamma\|_2 \\ &\le \mu \end{aligned}$$

for ε sufficiently small.

To show $g(0; v) = \varepsilon v$, notice that the bound

$$|X(x)| = \|DH_x\| \leq 2\varepsilon |v| \|\gamma\|_1 \leq |v|/2$$

for ε sufficiently small implies that, for $0 \leq t \leq 1$,

$$g(0; v, t) \in |v| 2^{-1} U^{2m} \quad \text{and} \quad \gamma(|v|^{-1} g(0; v, t)) = 1.$$

Then

$$\frac{d}{dt} g(0; v, t) = X \circ g(0, v, t) = \bar{\omega}^{-1} \lambda_v = \varepsilon v$$

so

$$g(0; v) = g(0; v, 1) = \varepsilon v.$$

8(b) *Volume-preserving diffeomorphisms*

Let μ be a non-degenerate m-form on M, i.e. a volume element. Again, if M is compact, all points are non-wandering for $\phi \in \mathscr{D}^1_\mu$. Therefore for a generic C^1 set of ϕ the periodic points are dense in M. (See § 8 for more definitions and notation.)

Let $\mathcal{N}$ be a neighbourhood of ϕ in $\mathscr{D}^1_\mu$. The manifold can be covered by a finite number of C^∞ coordinate charts in which μ is given by $dx^1 \wedge \cdots \wedge dx^m$ (see **[9]**). Let $\gamma : \mathbb{R}^m \to \mathbb{R}$ be a bump function as in § 8(a), and $\varepsilon > 0$, $\delta > 0$. By taking a coordinate chart we can take $p = 0$. Define $\lambda : T_pM(\delta) \to \Omega_{m-1}$ by

$$\lambda_v = \bar{\mu}(\varepsilon v).$$

Let $f(\quad; v)$ be the $(m-2)$-form associated to λ_v by the Poincaré operator such that

$$df(\quad; v) = \lambda_v.$$

Let

$$H(x; v) = \gamma(|v|^{-1} x) f(x, v),$$

and

$$X(\quad; v) = \bar{\mu}^{-1}(dH(\quad; v)),$$

as before. The estimates are exactly as for symplectic diffeomorphisms.

Remark. The above discussion assumes $m \geq 2$. For $m = 1$, i.e. $M = S^1$, the Closing Lemma is trivial even though the Lift Axiom is false. Also, the General Density Theorem (11.1) is false for $\mathscr{D}^1_\mu(S^1)$, since irrational rotations are generic.

8(c) *Volume-preserving vector fields*

Let (M, μ) be a C^∞ manifold with C^∞ volume element μ. Assume $m = \dim M \geq 3$. See § 8 for definitions and notations. Remember, in particular, that X is volume preserving if and only if $\bar{\mu}(X)$ is closed.

Let $\varphi \in \mathscr{X}^1_\mu$ and $\mathcal{N}$ be a neighbourhood of φ in $\mathscr{X}^1_\mu$. Cover M by finitely many coordinate charts $\mathbf{x}$ such that for each $p \in M_\varphi$

$$\dot{\varphi}(p) \text{ is transverse to the plane } x^1 = \text{constant}$$

in some charts $\mathbf{x}$, say $\mathbf{x} = \mathbf{x}(\ ;p)$. If q is in the chart $\mathbf{x}(\ ;p)$ let $\mathcal{F}^p_q$ be the plane $x^1 = \text{constant}$ through q. Since $\varphi_t(p)$ is a uniformly C^2 function of t, there is $T > 0$ independent of $p \in M$ such that the curve $t \mapsto \varphi_t(p)$, $-T \leq t \leq T$ is transverse to the foliation of planes $\mathcal{F}^p$.

Let $\beta : \mathbb{R} \to [0, 1]$ be a C^∞ bump function which is $\equiv 0$ near 0 and $\equiv 1$ near T. The curve

$$\psi_t(p) = (1 - \beta(t))\varphi_t(p) + \beta(t)\varphi_t(p + \varepsilon v)$$

is also transverse to the foliation $\mathcal{F}^p$ if $\varepsilon \leq 1$ and if $|v| \leq \delta(p)$, where $\delta : M_\varphi \to (0, 1)$ is a small continuous function. Addition is carried out in the $\mathbf{x}$-chart.

In this case and the Hamiltonian vector field case we only show that we can prescribe this one trajectory and not all 'volume preserving perturbations' of the Poincaré map as is done for the set of all vector fields. Because we want to apply the Whitney extension operator along $\psi_t(p)$, we introduce the $\mathbf{z}$-coordinates as follows. Let $\pi : \mathcal{F}^p_q \to \mathbb{R}^{m-1}$ be the linear projection in the $\mathbf{x}(\ ;p)$ chart. On a neighbourhood of p in this chart, put

$$z^1(q) = \text{signed arc length of } \psi_t p \text{ from } p \text{ to } \mathcal{F}^p_q \text{ if } |t| \leq T$$
$$(z^2(q), \ldots, z^m(q)) = \pi(q - \psi_t(p)) \qquad \text{if } \psi_t p \in \mathcal{F}^p_q \text{ and } |t| \leq T.$$

Thus the $\mathbf{z}$-chart carries the curve $\psi_t(p)$ isometrically to the z^1-axis. The $\mathbf{z}$-charts are C^2 and uniformly C^2 bounded when $\varepsilon \leq 1$ and $|v| \leq \delta(p)$. Besides, as $\varepsilon \to 0$, $\mathbf{z}(\ ;p, \varepsilon, v)$ uniformly C^2 converges to $\mathbf{z}(\ ;p, 0, 0)$. For ψ_t uniformly C^2-converges to φ_t as $\varepsilon \to 0$, arc length is universally C^∞, and so is linear projection.

Because we want $\psi_t(p)$ as a trajectory of a volume-preserving vector field, we want a C^1 closed $(m-1)$-form λ such that

$$\lambda(z_1, 0) = \bar{\mu}(\dot{\psi}(z_1, 0) - \dot{\varphi}(z_1, 0)).$$

To ensure λ is closed, we represent it as $d\eta$, where η is a $C^2(m-2)$-form. Care must be taken to ensure that η is C^2, so we use the Whitney Extension Theorem to extend η (not λ) and then let $\lambda = d\eta$.

At points $(z_1, 0)$,

$$\lambda(z_1, 0) = \sum_i \lambda_i(z_1, 0)\, dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge dz^m,$$

where the $\lambda_i(z_1, 0)$ are C^1 functions of z_1. If

$$\eta(z) = \sum_{j<k} \eta_{jk}(z)\, dz^1 \wedge \cdots \wedge \widehat{dz^j} \wedge \cdots \wedge \widehat{dz^k} \wedge \cdots \wedge dz^m$$

then

$$d\eta = \sum_i \left(\sum_j \frac{\partial \eta_{ij}}{\partial z^j} (-1)^s \right) dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge dz^m,$$

where $\eta_{ii}=0$, $\eta_{ij}=\eta_{ji}$, and $s=j-1$ if $j<i$ and $s=j-2$ if $j>i$. We can specify that

$$\eta_{ij}(z_1,0)\equiv 0;$$

$$\frac{\partial \eta_{ij}}{\partial z^1}(z_1,0)\equiv 0 \quad \text{for all } i, j \text{ and } z;$$

$$-\frac{\partial \eta_{23}}{\partial z^3}(z_1,0)=\lambda_2(z_1,0);$$

$$\frac{\partial \eta_{2j}}{\partial z^j}(z_1,0)=0 \quad \text{if } j\neq 3;$$

$$-\frac{\partial \eta_{i2}}{\partial z^2}(z_1,0)=\lambda_i(z_1,0) \quad \text{if } i\geq 3;$$

$$\frac{\partial \eta_{12}}{\partial z^2}(z_1,0)=\lambda_1(z_1,0);$$

$$\frac{\partial \eta_{ij}}{\partial z^j}(z_1,0)=0 \quad \text{if } i\neq 2, j\geq 3.$$

For the second derivatives,

$$\frac{\partial}{\partial z^1}\frac{\partial}{\partial z^k}\eta_{ij}(z_1,0) \quad \text{and} \quad \frac{\partial}{\partial z^k}\frac{\partial}{\partial z^1}\eta_{ij}(z_1,0) \text{ are determined;}$$

$$\frac{\partial}{\partial z^n}\frac{\partial}{\partial z^k}\eta_{ij}(z_1,0)\equiv 0, \quad n>1 \text{ and } k>1.$$

By the Whitney Extension Theorem [**2**, Appendix A] there are C^2 extensions of the η_{ij} with these derivatives on $\{(z_1,0)\}$.

To localize the support of η, let $\gamma:\mathbb{R}^{m-1}\to[0,1]$ and $\alpha:\mathbb{R}\to[0,1]$ be C^∞ bump functions with $\operatorname{supp}\gamma=\frac{3}{4}U^{m-1}$, $\gamma(x)=1$ for $x\in\frac{1}{2}U^{m-1}$, $\operatorname{supp}\alpha\subset\text{interior }[0,T]$, and $\alpha(x)=1$ for $x\in\operatorname{supp}\beta'$. For v fixed, let

$$\sigma(z)=\alpha(z_1)\gamma(z_2|v|^{-1},\ldots,z_m|v|^{-1}).$$

As used below in the estimates

$$|\sigma(z)|\leq 1;$$
$$\|D\sigma_z\|=O(|v|^{-1});$$
$$\|D^2\sigma_z\|=O(|v|^{-2}).$$

Let

$$\lambda=d(\sigma\eta).$$

Then λ agrees with its values specified along $\psi_t(p)$, because $\gamma(z)\equiv 1$ near $\psi_t(p)$ and $d\eta=0$ whenever $\alpha(z_1)\neq 1$ so

$$\begin{aligned}\lambda(z_1,0)&=d(\sigma\eta)\\&=d\alpha\wedge\eta+\alpha\wedge d\eta\\&=d\eta.\end{aligned}$$

Therefore

$$\dot{\psi}(z_1, 0) = \bar{\mu}^{-1}(\lambda(z_1, 0)) + \dot{\varphi}(z_1, 0)$$

and $\psi_t(p)$ is a trajectory of the volume-preserving vector field $\dot{\phi} + \bar{\mu}^{-1}(\lambda)$. The support of λ is as needed by the use of the bump function σ.

To get C^1 bounds on λ, we need C^2 bounds on the extended η. First, we get bounds of η along $\{(z_1, 0)\}$. By the definition of $\psi_t(p)$, it follows directly that

$$|\dot{\psi}(z_1, 0) - \dot{\varphi}(z_1, 0) = O(\varepsilon|v|).$$

Also the operator $\bar{\mu}$ is bounded so, for $z = (z_1, 0)$,

$$\begin{aligned}\|D\eta_{ijz}\| \le |\lambda_z| &= |\bar{\mu}(\dot{\psi}(z) - \dot{\varphi}(z))| \\ &= O(\varepsilon|v|).\end{aligned}$$

Clearly,

$$|\eta_{ij}(z_1, 0)| = 0.$$

Lastly, for $z = (z_1, 0)$,

$$\begin{aligned}\|D^2\eta_{ijz}\| &= \left\|\frac{\partial}{\partial z^1} D\eta_{ijz}\right\| \\ &\le \left\|\frac{\partial}{\partial z^1}\bar{\mu}(\dot{\psi}(z_1, 0) - \dot{\varphi}(z_1, 0))\right\| \\ &\le \|\partial\bar{\mu}/\partial z\| \|\dot{\psi}(z_1, 0) - \dot{\varphi}(z_1, 0)\| + \|\bar{\mu}\| \left|\frac{\partial\dot{\psi}(z_1, 0)}{\partial z^1} - \frac{\partial\dot{\varphi}(z_1, 0)}{\partial z^1}\right|.\end{aligned}$$

Since the chart $\mathbf{z}(\ \ ; p, \varepsilon, v)$ converges uniformly in the C^2 sense to $\mathbf{z}(\ \ ; p, 0, 0)$, we see that the $\mathbf{z}(\ \ ; p, \varepsilon, v)$-representation of the vector field $X = \dot{\varphi}$ converges C^1-uniformly to the $\mathbf{z}(\ \ ; p, 0, 0)$-representation of $\dot{\varphi}$. But $\dot{\psi}(z_1, 0) \equiv \dot{\varphi}(z_1, 0)$ if $\varepsilon = 0$. Thus

$$\left|\frac{\partial\dot{\psi}(z_1, 0)}{\partial z^1} - \frac{\partial\dot{\varphi}(z_1, 0)}{\partial z^1}\right| = o(\varepsilon^0).$$

Since $\|\partial\bar{\mu}/\partial z\|$ and $\|\bar{\mu}\|$ are uniformly bounded, while

$$|\dot{\psi}(z_1, 0) - \dot{\varphi}(z_1, 0)| = O(\varepsilon|v|)$$

we see that

$$\|D^2\eta_{ij}\| = o(\varepsilon^0)$$

Because the Whitney extension operator is a continuous function of the derivatives prescribed on $\{(z_1, 0)\}$, the extended functions η_{ij} satisfy

$$\|D^2\eta_{ijz}\| = o(\varepsilon^0).$$

By the estimates of $\|D\eta_z\|$ on $z = (z_1, 0)$ and the mean value theorem, for $z \in \operatorname{supp} \sigma$

$$\|D\eta_{ijz}\| = |v| o(\varepsilon^0)$$

and

$$|\eta_{ij}(z)| = |v|^2 o(\varepsilon^0).$$

Then

$$
\begin{aligned}
|\lambda(z)| &\le \sum_{ij} \|D(\sigma\eta_{ij})_z\| \\
&\le \sum_{ij} |\sigma(z)|\,\|D\eta_{ijz}\| + \|D\sigma_z\|\,|\eta_{ij}(z)| \\
&= |v|\,o(\varepsilon^0) + O(|v|^{-1})|v|^2 o(\varepsilon^0) \\
&= |v|\,o(\varepsilon^0).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|D\lambda_z\| &\le \sum_{ij} \|D^2(\sigma\eta_{ij})_z\| \\
&\le \sum_{ij} |\sigma(z)|\,\|D^2\eta_{ij}\| + 2\|D\sigma_z\|\,\|D\eta_{ijz}\| \\
&\quad + \|D^2\sigma_z\|\,|\eta_{ij}(z)| \\
&\le o(\varepsilon^0) + 2O(|v|^{-1})|v|\,o(\varepsilon^0) + O(|v|^{-2})|v|^2 o(\varepsilon^0) \\
&= o(\varepsilon^0).
\end{aligned}
$$

Therefore, for ε sufficiently small (uniform in p), $\dot{\varphi} + \bar{\mu}^{-1}(\lambda) \in \mathcal{N}$.

8(*d*) *Symplectic and volume-preserving flows*

We were unable to prove the closing lemma for C^1 flows which preserve either a symplectic two-form or a volume element. The problem is that $\mathcal{F}^1_\omega$ is not a linear space so it is not clear how to perturb the Poincaré map and keep a C^1 flow which preserves the form. See [**5**]. Even in dimension two we were unable to close up area-preserving C^1 flows, although it seems likely that there is special proof for this special case.

9. *Closing Lemma for Hamiltonian vector fields*

9(*a*) *Lift Axiom for Hamiltonian vector fields*

Let (M, ω) be a symplectic manifold. First, we remark that the flow of a Hamiltonian vector field preserves the level sets of the function that induces it,

$$\dot{\varphi}_H \lrcorner\, \omega = dH.$$

Thus, when we push q to p, we never have to push in the direction of increasing energy. In fact, local perturbations do not change the level sets of H at the ends of the flow box so we can only push along the level sets. More precisely, we have the following Lift Axiom for Hamiltonian vector fields, $\mathcal{X}^1_H$. See § 8(*a*) for definitions.

Lift Axiom for $\mathcal{X}^1_H$. *Let* $\dot{\varphi} \in \mathcal{X}^1_H$ *with energy function* H *and* $\mathcal{N}$ *be a neighbourhood of* $\dot{\varphi}$. *Then there exist a constant*

$$\varepsilon = \varepsilon(\dot{\varphi}, \mathcal{N}), \quad 0 < \varepsilon \le 1,$$

and a continuous function

$$\delta : M_\varphi \to (0,1), \quad \delta = \delta(\dot\varphi_H, \mathcal{N}, p),$$

such that, whenever $p \in M_\varphi$,

$$v \in \hat\Pi(\delta(p)) \cap \exp_p^{-1}(H^{-1}(h)),$$

where $h = H(p)$, *we have a perturbation* $\dot\psi$ *of* $\dot\varphi$ *in* $\mathcal{N}$ *satisfying* (L′1), (L′2), *and* (L′3).

Note we do not push in the $\partial/\partial H$ direction and that (L′3) is automatic because $\mathscr{X}^1_H$ is a linear space.

9(*b*) *Lift implies Closing for Hamiltonian vector fields*

The main change in the proof from the usual flow case is the need to restrict attention to the energy surface when analysing the sequence of linear maps and considering the boxes which contain the supports of the perturbations.

Let $h = H(p^*)$ be the energy of p^* and Π_x be the transversal at x as before. Consider the Poincaré maps restricted to the energy surface $H^{-1}(h)$,

$$f^t : \Pi_{p^*} \cap H^{-1}(h) \to \Pi_{\varphi_t(p^*)} \cap H^{-1}(h).$$

Applying the selection theorem to the sequence of the tangent maps $T_t = T_{p^*}f^t$ gives a sequence $\{t_k\}$ of $\{2, 4, 6, \dots\}$ and a splitting

$$T_{p^*}(\Pi_{p^*} \cap H^{-1}(h)) = V^1 \oplus \cdots \oplus V^L.$$

Notice that that splitting only spans the tangent space to the energy surface and that the shear in the $\partial/\partial H$ direction is ignored. Proceed as before but now Δ, J_n, and S_n are all restricted to the energy surface. Applying Lemma 4.1 produces

$$\omega_2 = (p, q, r, s, \Lambda)$$

satisfying (5.4) with

$$h' = H(p) = H(q) = H(r),$$

$$\mathrm{pr} : \Pi_{p^*} \to \Pi_{p^*} \cap H^{-1}(h')$$

the projection into the energy surface of p and q, and

$$B = \mathrm{pr} \,.\, \exp_{p^*}(r + \Lambda)$$

taking its values in this energy surface.

Because the boxes which are controlled are restricted to the energy surface, the proof that the supports of the perturbations are disjoint must be augmented. The supports of the perturbations restricted to the energy surface are contained in the appropriate boxes which are disjoint as before,

$$\mathrm{supp}_n^\rho \cap H^{-1}(h') \cap \Pi_{p_n^*} \subset f_n B(I^{2m-2}).$$

Since the gradient of H is non-zero at the points in question, $H^{-1}(h')$ cannot accumulate on itself, so the total supports are disjoint,

$$\mathrm{supp}\, g_n^\rho \cap \mathrm{supp}\, g_k^\rho = \varnothing.$$

Also, as before, there is a ρ such that

$$\mathrm{pr}\, f_T^{(\rho)}(q) = \mathrm{pr}\, f_T(p).$$

Since

$$H(f_T(p)) = H(f_T^{(\rho)}(q)),$$

it follows that

$$f_T(p) = f_T^{(\rho)}(q).$$

Again $f_T(p)$ flows back to q outside the supports of the perturbations so q is on a closed orbit.

9(*c*) *Proof of the Lift Axiom for Hamiltonian vector fields*

The verification for Hamiltonian vector fields is similar to volume-preserving vector fields so we indicate the changes. Let (M, ω) be a symplectic manifold and $\dot{\varphi} \in \mathscr{X}_H^1$ be given with energy function $H : M^{2n} \to \mathbb{R}$, so $\bar{\omega}(\dot{\varphi}) = dH$. Assume $2m \geq 4$. Let $\mathcal{N}$ be a neighbourhood of $X = \dot{\varphi} \in \mathscr{X}_H^1$. Cover M by finitely many C^∞ coordinate charts $\mathbf{x}$ such that for each $p \in M_\varphi$

$$\dot{\varphi}(p) \text{ is transverse to the plane } x^1 = \text{constant}$$

$$\frac{\partial H}{\partial x^2} > 0$$

in some chart $\mathbf{x}$, say $\mathbf{x} = \mathbf{x}(\ \ ; p)$. Again call $\mathscr{F}_q^p$ the plane $x^1 = \text{constant}$ through q in the $\mathbf{x}(\ \ ; p)$-chart. Let $\mathbf{y} = \mathbf{y}(\ \ ; p)$ be the chart at p in which $y^i = x^i$ except that for $i = 2$, $y^i = H$. This requires infinitely many charts $\mathbf{y} = \mathbf{y}(\ \ ; p)$ but they are uniformly C^2, and, as before there is a $T > 0$ independent of $p \in M_\varphi$, and a continuous function $\delta : M_\varphi \to (0, 1)$ such that the curve

$$\psi_t(p) = (1 - \beta(t))\varphi_t(p) + \beta(t)\varphi_t(p + \varepsilon v)$$

is transverse to the foliation $\mathscr{F}^p$ if $\varepsilon \leq 1$. Addition is carried out in the $\mathbf{y}$-chart, β is as above, and so

$$H(\psi_t p) = H(p).$$

Let $\tilde{\pi} : \mathscr{F}_q^p \to \mathbb{R}^{m-1}$ be the linear projection in the $\mathbf{y}$-chart. As in § 8*c*, define $\mathbf{z}$-coordinates on a neighbourhood of p by

$$z^1(q) = \text{signed arc length of } \psi_t(p) \text{ from } \mathscr{F}_p^p \text{ to } \mathscr{F}_q^p$$

$$(z^2(q), \ldots, z^m(q)) = \tilde{\pi}(q - \psi_t(p)) \text{ if } \psi_t(p) \in \mathscr{F}_q^p.$$

To make $\psi_t(p)$ the trajectory of a Hamiltonian vector field, we want a C^2 function η such that for $z = (z_1, 0)$

$$\begin{aligned} D\eta_z &= \bar{\omega}(\dot{\psi}(z_1, 0)) - DH_z \\ &= \bar{\omega}(\dot{\psi}(z_1, 0)) - \dot{\varphi}(z_1, 0). \end{aligned}$$

Since $d\eta$ is a C^1 function of z_1, we can use the Whitney Extension Theorem to extend η as before. This time, the second partials

$$\frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} \eta(z_1, 0)$$

are determined when either $i = 1$ or $j = 1$ and we can set

$$\frac{\partial}{\partial z^i}\frac{\partial}{\partial z^j}\eta(z_1, 0) = 0, \quad \text{when } i \neq 1 \text{ and } j \neq 1;$$

$$\eta(0, 0) = 0.$$

To see that $\eta(z_1, 0) = 0$, let ν_t be the flow of the Hamiltonian vector field associated to $H + \eta$. Then $\nu_t(p) = \psi_t(p)$ on this one trajectory, so

$$H(p) = H(p) + \eta(p),$$

$$H(\nu_t(p)) + \eta(\nu_t(p)) = H(p) + \eta(\nu_t(p))_x$$

because the Hamiltonian $H + \eta$ is preserved by its flow and $\psi_t(p)$ lies on the energy surface $H^{-1}(H(p))$. Thus

$$\eta'(\psi_t(p)) = 0$$

as claimed.

Let $\alpha(z_1)$, $\gamma(z_2, \ldots, z_{2n})$, and

$$\sigma(z) = \alpha(z_1)\gamma(z_2|v|^{-1}, \ldots, z_{2n}|v|^{-1})$$

be as before and

$$K(z) = \sigma(z)\eta(z).$$

The C^2 bounds on K are exactly as before to show that $H + K$ is in the neighbourhood $\mathcal{N}$ for $\varepsilon > 0$ sufficiently small. Q.E.D.

10. *Are closed geodesics dense?*

The geodesic flow on a manifold is a flow on TM or T^*M. On T^*M there is a canonical two-form and the geodesic flow corresponds to the Hamiltonian flow of a function that is quadratic on each fibre (the kinetic energy function). A C^2 Riemannian metric corresponds to a C^2 function on T^*M which corresponds to a C^1vector field. Thus it seems reasonable to ask if the closing lemma applies.

The problem is that the equations are second-order differential equations. What this means is that, if the vector field is changed at any point of a fibre T^*_xM, it must be changed on the whole fibre. Thus the perturbation cannot be localized in T^*M (or TM) but only in M.

To take account of this difficulty, one might try to modify the Fundamental Lemma (4.1). In T^*M all points are non-wandering (if M is compact) and a residual set of points is recurrent. If M has dimension ≥ 3 then generically a residual set of points is recurrent with trajectories passing through a fibre in T^*M at most one time. Thus one might try to find $(2m-2)N$ flowboxes of the form T^*_UM, where $U \subset M$ is such that the trajectory only passes through each flowbox once before returning close to $p^* \in T^*M$. If this were possible, then the perturbation could be localized in the T^*_UM and the proof probably would go through.

Even the C^0 Geodesic Closing Lemma is unproved. Imagine the manifold embedded in $\mathbb{R}^3$ and a recurring geodesic γ on M. Now γ returns near its starting point with nearly its starting direction. Try to put dents or bumps on the manifold to warp γ, causing it to close up smoothly.

11. *The General Density Theorem*

Here we show that generically Axiom Ab of Smale, holds, see § 1, [**16**], and [**21**].

(11.1) *General Density Theorem. Let* $\mathcal{S} = \mathcal{D}^1$, $\mathcal{F}^1$, $\mathcal{X}^1$, $\mathcal{D}^1_\mu(M^m)$, $\mathcal{F}^1_\mu(M^{m+1})$; $\mathcal{X}^1_\mu(M^{m+1})$, $\mathcal{D}^1_\omega(M^{m+2})$, $\mathcal{F}^1_\omega(M^{m+2})$, $\mathcal{X}^1_\omega(M^{m+2})$, *or* $\mathcal{X}^1_H(M^{m+2})$ *where M is compact and* $m \geq 2$. *Then the generic* $S \in \mathcal{S}$ *satisfies*

$$\bar{\Gamma}(S) = \Omega(S), \tag{1}$$

where $\Gamma(S)$ *is the set of periodic trajectories of S,* $\bar{\Gamma}(S)$ *is its closure, and* $\Omega(S)$ *is the set of its non-wandering points. In the conservative cases, this implies* $\bar{\Gamma} = M$.

'Periodic' includes period zero: fixed points of flows. To handle periodic trajectories that cannot be destroyed by small perturbations, we make the following two-step definition.

Definition. Let $\mathcal{S}$ be a class of dynamical systems. If γ is a periodic trajectory of $S \in \mathcal{S}$ then γ is *persistent* relative to $\mathcal{S}$ provided that each S' near S in $\mathcal{S}$ has a periodic trajectory γ' near γ; and γ is *permanent* relative to $\mathcal{S}$ provided that each S' near S in $\mathcal{S}$ has a persistent periodic trajectory γ' near γ.

Note that the topologically stable fixed point 0 of $x \mapsto x - x^3$ is persistent but not permanent.

(11.2) *Stabilization Proposition. Let* γ *be a periodic trajectory of S. If* $\mathcal{S}$ *is one of the classes of dynamical systems in* (11.1), *then there exists* S' *near S in* $\mathcal{S}$ *with a permanent periodic trajectory* γ' *near* γ. *Besides,* $S' \equiv S$ *off a small neighbourhood of* γ.

Proof. For the function spaces listed above, the proof of the Kupka–Smale Theorem, [**2**], [**13**], and [**19**], gives this stabilization γ' of γ. Q.E.D.

Proof of (11.1). Let $\Gamma_{\text{perm}}(S)$ denote the permanent periodic trajectories of S and let $\bar{\Gamma}_{\text{perm}}(S)$ be its closure. The compact subsets of M form a natural metric space $K(M)$ with the Hausdorff metric [**3**, p. 112]. The map

$$\bar{\Gamma}_{\text{perm}}: \begin{array}{l} \mathcal{S} \to K(M) \\ S \mapsto \bar{\Gamma}_{\text{perm}}(S) \end{array}$$

is lower semi-continuous: its values can explode but not implode by permanence. Semi-continuity implies continuity at a residual subset $\mathfrak{S}$ of $\mathcal{S}$, [**3**, p. 114].

If $S \in \mathfrak{S}$ then we claim

$$\bar{\Gamma}_{\text{perm}}(S) = \Omega(S) \tag{2}$$

which implies (1). We know $\bar{\Gamma}_{\text{perm}}$ is continuous at S. Suppose (2) is false, say

$$p^* \in \Omega(S) - \bar{\Gamma}_{\text{perm}}(S).$$

By the C^1 Closing Lemma for $\mathcal{S}$, we can find $S' \doteq S$ in $\mathcal{S}$ with a periodic trajectory γ' almost through p^*. By (11.2) we find $S'' \doteq S'$ in $\mathcal{S}$ with a permanent $\gamma'' \doteq \gamma'$. This defies continuity of $\bar{\Gamma}_{\text{perm}}$ at S, for we produced S'' in $\mathcal{S}$ arbitrarily near S with a permanent periodic trajectory γ'' passing near p^*, and p^* is not near $\bar{\Gamma}_{\text{perm}}(S)$.

Q.E.D.

The case of non-compact M

The natural topology for a space of dynamical systems on a non-compact manifold M is the Whitney topology. See § 1 and [**7**, Ch. 2]. In this topology, the perturbations must grow smaller near infinity.

The proof of the Closing Lemma given in § 5 works equally well when M is non-compact, provided that the point p^* to be closed lies in

$$\Omega_c = \{p \in \Omega : \alpha(p) \cup \omega(p) \neq \emptyset\},$$

where $\alpha(p)$ and $\omega(p)$ refer to the α- and ω-limit sets of the orbit through p. For suppose $p^* \in \Omega_c(\varphi)$ and $\omega(p^*) \neq \emptyset$. Infinitely many points

$$\phi_n(p^*), \quad n = n_1, n_2, n_3, \ldots \to \infty,$$

lie in a compact subset of M, and, in the proof of the Closing Lemma in § 5, we can choose to analyse only the subsequence of the derivatives,

$$(D\phi_n)_{p^*}, \quad n = n_1, n_2, \ldots .$$

The N disjoint lift-perturbations of § 5 are constructed near N of these points and so the perturbation which produces the periodic orbit passing near p^* has support in a fixed compact subset of M. There, the C^1 Whitney topology becomes the usual C^1 topology and the rest of the proof of the Closing Lemma reverts to the compact case. Using this non-compact Closing Lemma, we get:

(11.3) *General Density Theorem for non-compact manifolds. For any of the function spaces $\mathscr{S}$ in* (11.1) *on a non-compact manifold M, the generic $S \in \mathscr{S}$ satisfies*

$$\bar{\Gamma}(S) \supset \Omega_c(S). \tag{3}$$

(11.4) COROLLARY. *General Density Theorem for Hamiltonians. The generic C^1 Hamiltonian vector field on a manifold W has its set of periodic trajectories dense in the union U of the compact energy surfaces. In particular, the generic C^2 proper function $H : T^*M \to \mathbb{R}$ has a Hamiltonian flow whose periodic trajectories are dense in T^*M.*

Proof. U is open in W and is contained in Ω_c, so (11.3) with

$$\mathscr{S} = \mathscr{X}^1_H(M^{m+2})$$

yields the first assertion in (11.4), if $\dim(W) \geq 4$. But if $\dim(W) = 2$ then the compact level surfaces have dimension one and are periodic trajectories or saddle connections. The latter are discrete in U, so the former are dense in U.

The proper functions are open in $C^2(T^*M, \mathbb{R})$ and their level surfaces are compact. To each H corresponds a C^1 Hamiltonian field on T^*M, X_H, and, up to a constant, X_H determines H. Hence, residuality in $\mathscr{X}^1_H$ pulls up to residuality in $C^2_{\text{prop}}(T^*M, \mathbb{R})$. Q.E.D.

Proof of (11.3). Although not metrizable, the spaces $\mathscr{S}$ of (11.1) have the Baire property when M is non-compact – residual sets are dense. Let $K_1, K_2, \ldots,$ be a sequence of compact subsets of M such that

$$\bigcup_{i=1}^{\infty} \operatorname{Int}(K_i) = M.$$

Call

$$\bar{\Gamma}_i(S) = \text{Closure}\,(\bar{\Gamma}_{\text{perm}}(S) \cap \operatorname{Int}(K_i))$$

for $S \in \mathcal{S}$. This map $\bar{\Gamma}_i : \mathcal{S} \to K(M)$ is lower semi-continuous: under perturbation of S, $\bar{\Gamma}_{\text{perm}}(S) \cap \operatorname{Int}(K_i)$ cannot implode. (Note that $\bar{\Gamma}_{\text{perm}}(S) \cap K_i$ could implode if entire permanent periodic orbits lie in $M - \operatorname{Int}(K_i)$. They could slide off K_i. This is why we intersect with $\operatorname{Int}(K_i)$, not K_i.)

Let $\mathfrak{S}_i$ be the set of continuity points of $\bar{\Gamma}_i$ and let $\mathfrak{S} = \bigcap_{i=1}^{\infty} \mathfrak{S}_i$. Since $\mathcal{S}$ has the Baire property, $\mathfrak{S}$ is residual and dense. If $S \in \mathfrak{S}$ we claim

$$\bigcup \bar{\Gamma}_i(S) \supset \Omega_c(S). \tag{4}$$

Suppose (4) is false and take

$$p^* \in \Omega_c(S) - \bigcup_i \bar{\Gamma}_i(S)$$

for some $S \in \mathfrak{S}$. Choose i with $p^* \in \operatorname{Int}(K_i)$. By the Closing Lemma and (11.2) there is a permanent periodic point p' of some $S' \in \mathcal{S}$ and $S' \doteq S$, $p' \doteq p^*$. Since $p^* \in \operatorname{Int}(K_i)$, $p' \in \operatorname{Int}(K_i)$ also; i.e. $p' \in \bar{\Gamma}_i(S')$, contradicting continuity of $\bar{\Gamma}_i$ at S. This verifies (4) which implies (3). Q.E.D.

Added in proof. Using a generic Fubini argument (11.4) can be strengthened to

(11.5) THEOREM. *The generic compact energy surface of the generic C^1 Hamilton vector field contains a dense set of periodic trajectories.*

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