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COMMUTATOR THEORY ON HILBERT SPACE

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1. Introduction. Commutator theory has its origins in constructive quantum field theory. It was initially developed by Glimm and Jaffe [**7**] as a method to establish self-adjointness of quantum fields and model Hamiltonians. But it has subsequently proved useful for a variety of other problems in field theory [**17**] [**15**] [**8**] [**3**], quantum mechanics [**5**], and Lie group theory [**6**]. Despite all these detailed applications no attempt appears to have been made to systematically develop the theory although reviews have been given in [**22**] and [**9**]. The primary aim of the present paper is to partially correct this situation. The secondary aim is to apply the theory to the analysis of first and second order partial differential operators associated with a Lie group.

The basic ideas of commutator theory and perturbation theory are very similar. One attempts to derive information about a complex system by comparison with a simpler reference system. But the nature of the comparison differs greatly from one theory to the other. Perturbation theory applies when the difference between the two systems is small but commutator theory only requires the complex system to be relatively smooth with respect to the reference system. No small parameters enter in the latter theory and hence it could be referred to as singular perturbation theory [**10**], but this latter term is used in many different contexts [**20**].

In order to give a more precise comparison suppose H is a given self-adjoint operator on a Hilbert space $\mathfrak{h}$, with domain $D(H)$ and C^∞-elements $\mathfrak{h}_\infty = \cap_{n\geqq 1} D(H^n)$, and further suppose that K is a symmetric operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$. Then the simplest theorem of Kato-Rellich perturbation theory [**12**] [**22**] states that K is essentially self-adjoint whenever

$$\|(K - H)a\| \leqq k\|Ha\| + l\|a\|, \quad a \in \mathfrak{h}_\infty,$$

for some $k \in [0, 1\rangle$ and $l \geqq 0$, i.e., the difference between K and H is small in comparison with H. In contrast the Glimm-Jaffe commutator theorem states that K is essentially self-adjoint whenever $K\mathfrak{h}_\infty \subseteq D(H)$ and the commutator $(\text{ad } H)(K) = HK - KH$ satisfies

$$(1.1) \qquad \|(\text{ad } H)(K)a\| \leqq k_1\|Ha\| + l_1\|a\|, \quad a \in \mathfrak{h}_\infty,$$

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for some $k_1, l_1 \geqq 0$. In fact most versions of the commutator theorem also require $H \geqq 0$ but this is not necessary [**4**]. If $H \geqq 0$ essential self-adjointness of K even follows from the weaker commutator bound

$$|(a, (\text{ad } H)(K)b)| \leqq k_1'\|H^{1/2}a\| \cdot \|H^{1/2}b\| + l_1'\|a\| \cdot \|b\|,$$

$$a, b \in \mathfrak{h}_\infty.$$

The next refinement of perturbation theory occurs if K is positive, or lower semi-bounded. Then self-adjointness properties can be obtained from weaker perturbative estimates. One of the principal results of Section 2 establishes that analogous refinements arise in commutator theory. If $K \geqq 0$ then it is essentially self-adjoint whenever $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_2$ and

$$(1.2) \qquad \|(\text{ad } H)^2(K)a\| \leqq k_2\|H^2a\| + l_2\|a\|, \quad a \in \mathfrak{h}_\infty.$$

If, in addition, $H \geqq 0$ then it suffices that the form bound

$$|(a, (\text{ad } H)^2(K)b)| \leqq k_2'\|Ha\| \cdot \|Hb\| + l_2'\|a\| \cdot \|b\|, \quad a, b \in \mathfrak{h}_\infty$$

is valid. A partial result of this kind was given earlier by Driessler and Summers [**4**].

Thus, in summary, there are four basic self-adjointness results of commutator theory classified by stability properties, expressed by positivity of H or K, and smoothness properties, expressed by commutator estimates. In rough terms, the stronger the stability the weaker the smoothness required for self-adjointness. In each case one can also derive invariance properties of the unitary group $S_t = \exp\{it\bar{K}\}$ or, if $K \geqq 0$, the contraction semigroup $T_t = \exp\{-t\bar{K}\}$. For example, if (1.1) is satisfied then $\mathfrak{h}_\alpha = D(|H|^\alpha)$ is invariant under S for all $\alpha \in [0, 1]$ and the restriction of S to $\mathfrak{h}_\alpha$ is continuous in the graph norm. Alternatively, if $K \geqq 0$ and (1.2) is satisfied then $\mathfrak{h}_\alpha$ is invariant under T for all $\alpha \in [0, 2]$, and the restriction of T to $\mathfrak{h}_\alpha$ is continuous in the graph norm. Details are described in Section 2. Earlier results on invariance occur in [**8**] [**3**] [**6**] [**15**] and continuity properties in [**16**].

A simple illustration of the self-adjointness results is given by taking $\mathfrak{h} = L^2(\mathbf{R})$ and $H = id/dx$ the self-adjoint generator of the unitary group of translations. Then if A denotes the operator of multiplication by a real-valued function with bounded second derivative it follows that the first-order differential operator $K = HA + AH$ is a symmetric operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$. Moreover condition (1.1) is satisfied, with $k_1 = 2\|A'\|_\infty$ and $l_1 = \|A''\|_\infty$, and hence K is essentially self-adjoint. Alternatively if A is positive with a bounded third derivative and B is positive with a bounded second derivative then the second-order differential operator $K = HAH + B$ is a positive symmetric operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ which satisfies (1.2). Therefore K is self-adjoint. These examples illustrate two important points. First the differentiability, i.e., the smoothness, of the coefficients is necessary for the boundedness of the commutators (ad H)(K), (ad $H)^2(K)$, on $\mathfrak{h}_\infty$. Second the order of the bound, i.e., the order in H, reflects the order of the differential operator. Since the $O(H)$ bound cannot be relaxed in the Glimm-Jaffe theorem (see [**22**] Example 1 in Section X.5) this theorem is restricted to first-order differential operators, unless one makes a different choice of H. In contrast the theorem for $K \geqq 0$ applies to second-order differential operators because the commutator bound is $O(H^2)$. Again this order cannot be increased (see [**4**] Section 4).

It is natural to try and extend the foregoing discussion of differential operators on $L^2(\mathbf{R})$ to partial differential operators on $L^2(\mathbf{R}^d)$. This then leads to the consideration of operators associated with a unitary representation of $\mathbf{R}^d$ or a general Lie group G. Commutator theorems for such operators are derived in Section 3. They generalize results of Poulsen [**21**] for operators commuting with the group action. The extension of the results of Section 2 is quite straightforward since the differential structure of the representation is completely determined by a Laplace operator. This Laplacian then plays the role of the self-adjoint operator H. We note that many of the quantum-mechanical applications of commutator theory have been based on the choice of H as the Hamiltonian of the harmonic oscillator, acting on $L^2(\mathbf{R}^d)$. But this operator is exactly the Laplacian of the Heisenberg group, acting in the Schrödinger representation on $L^2(\mathbf{R}^d)$. Hence these examples are naturally described by the results of Section 3.

In Sections 4 and 5 we establish commutation theorems. Basically we prove that if K_1 and K_2 are two symmetric operators each satisfying the criteria of one of the commutator theorems of Section 2, and if K_1 formally commutes with K_2, then the self-adjoint extensions $\bar{K}_1$ and $\bar{K}_2$ commute.

Finally, in Section 6, we elaborate on the application of our results to general partial differential operators associated with a Lie group.

Although this paper only considers commutator theory on Hilbert space we note that the Glimm-Jaffe theorem, as described above, is also valid on Banach space [**1**] [**23**]. It would be of great interest to find a Banach space analogue of the double commutator theorem.

2. Commutator theorems. In this section we develop the theory of commutator theorems on Hilbert space. There are three classes of theorem, for dissipative, symmetric, and positive, operators, and there are also three types of result, generator properties, invariance, and boundedness and continuity properties. In each successive class the results are more detailed.

Throughout the section H denotes a self-adjoint operator on the Hilbert space $\mathfrak{h}$ with domain $D(H)$. For each $\alpha \geqq 0$ we set

$$\mathfrak{h}_\alpha = D((I + H^2)^{\alpha/2})$$

equipped with the norm

$$\|a\|_\alpha = \|(I + H^2)^{\alpha/2}a\|.$$

In particular $\mathfrak{h} = \mathfrak{h}_0$ and $\|\cdot\| = \|\cdot\|_0$. Next for $\alpha < 0$ we define $\mathfrak{h}_\alpha$ to be the completion of $\mathfrak{h}$ with respect to the norm

$$a \in \mathfrak{h} \to \|a\|_\alpha = \|(I + H^2)^{\alpha/2}a\|.$$

It follows that each $\mathfrak{h}_\alpha$, $\alpha \in \mathbf{R}$, is a Hilbert space, the operators $(I + H^2)^{\beta/2}$ define unitary maps from $\mathfrak{h}_\alpha$ into $\mathfrak{h}_{\alpha-\beta}$, and the dual $\mathfrak{h}_\alpha^*$ of $\mathfrak{h}_\alpha$ is identifiable as $\mathfrak{h}_{-\alpha}$. Moreover $\mathfrak{h}_\alpha \supseteq \mathfrak{h}_\beta$ if $\alpha \leqq \beta$. Finally we define $\mathfrak{h}_\infty$ by

$$\mathfrak{h}_\infty = \bigcap_\alpha \mathfrak{h}_\alpha.$$

The space $\mathfrak{h}_\infty$ is a Fréchet space with respect to the topology defined by the family of norms $\{\|\cdot\|_\alpha;\ \alpha \in \mathbf{R}\}$.

If $H \geqq 0$ it is also convenient to introduce a different family of norms

$$\|a\|'_\alpha = \|(I + H)^\alpha a\|, \quad a \in \mathfrak{h}_\infty$$

on the spaces $\mathfrak{h}_\alpha$. It is easy to see that $\|\cdot\|'_\alpha$ is equivalent to $\|\cdot\|_\alpha$. Moreover $(I + H)^\beta$ defines a map from $\mathfrak{h}_\alpha$ into $\mathfrak{h}_{\alpha-\beta}$ which is unitary with respect to the norm $\|\cdot\|'_\alpha$.

a. *Dissipative operators.* The operator K on the Hilbert space $\mathfrak{h}$ is defined to be dissipative if

$$\mathrm{Re}(a, Ka) \geqq 0, \quad a \in D(K),$$

or, equivalently, if

$$\|(I + \epsilon K)a\| \geqq \|a\|, \quad a \in D(K),$$

for all $\epsilon > 0$. Each densely defined dissipative operator is closable, and its closure is also dissipative.

Now we consider dissipative operators K from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ and their commutators $(\mathrm{ad}\ H)(K)$ with H. These commutators are not necessarily defined as operators but they can be interpreted as sequilinear forms over $\mathfrak{h}_\infty \times \mathfrak{h}_\infty$, and we will consistently make this interpretation. Thus, with a slight abuse of notation, we define the sequilinear form $(\mathrm{ad}\ H)(K)$ by

$$a, b \in \mathfrak{h}_\infty \times \mathfrak{h}_\infty \mapsto (a, (\mathrm{ad}\ H)(K)b) = (Ha, Kb) - (a, KHb).$$

THEOREM 2.1. *Let K be a dissipative operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ and suppose that*

$$|(a, (\mathrm{ad}\ H)(K)b)| \leqq k_\alpha \|a\|_\alpha \cdot \|b\|_{1-\alpha}, \quad a, b \in \mathfrak{h}_\infty, \tag{2.1}$$

for some $\alpha \in [0, 1\rangle$ and a $k_\alpha \geqq 0$.

It follows that the closure $\bar{K}$ of K generates a strongly continuous contraction semigroup.

Proof. It follows from the Hille-Yosida theorem that $\bar{K}$ generates a contraction semigroup if, and only if, the range $R(I + \epsilon K)$ of $I + \epsilon K$ is dense in $\mathfrak{h}$ for small $\epsilon > 0$. Now assume this is false. Then for each $\epsilon > 0$ there is a non-zero $a \in \mathfrak{h}$ such that

$$(a, (I + \epsilon K)b) = 0, \quad b \in \mathfrak{h}_\infty.$$

Next introduce the self-adjoint contraction semigroup $S = \{S_t\}_{t \geqq 0}$ where

$$S_t = \exp\{-tH^2\}.$$

Then $S_t\mathfrak{h} \subseteq \mathfrak{h}_\infty$, by spectral theory. Therefore

$$(a, (I + \epsilon K)S_{2t}a) = 0$$

for all $t > 0$. Consequently

$$\begin{aligned}(2.2) \qquad \|S_t a\|^2 &= -\epsilon \operatorname{Re}(a, KS_{2t}a)\\ &= -\epsilon \operatorname{Re}(S_t a, KS_t a) + \epsilon \operatorname{Re}(a, (\operatorname{ad} S_t)(K)S_t a)\\ &\leqq \epsilon \operatorname{Re}(a, (\operatorname{ad} S_t)(K)S_t a).\end{aligned}$$

Now consider the everywhere defined operators

$$C_t = (\operatorname{ad} S_t)(K)S_t, \quad t > 0.$$

We will prove that these operators are bounded and

$$c = \sup\{\|C_t\|;\ 0 < t < 1\} < +\infty,$$

Then from (2.1) one deduces that

$$\|a\|^2 = \lim_{t \to 0+} \|S_t a\|^2 \leqq \epsilon c\|a\|^2$$

which gives an inconsistency if $\epsilon c < 1$. Thus it remains to establish the boundedness properties of the C_t.

Let $c, d \in \mathfrak{h}_\infty$ then

$$\begin{aligned}(2.3) \qquad (c, (\operatorname{ad} S_t)(K)S_t d) &= -t \int_0^1 ds (S_{ts}c, (\operatorname{ad} H^2)(K)S_{t(2-s)}d)\\ &= -t \int_0^1 ds\{ (HS_{ts}c, (\operatorname{ad} H)(K)S_{t(2-s)}d)\\ &\qquad + (S_{ts}c, (\operatorname{ad} H)(K)HS_{t(2-s)}d) \}.\end{aligned}$$

Therefore

$$|(c, (\operatorname{ad} S_t)(K)S_t d)| \leqq k_\alpha t \int_0^1 ds\{ \|S_{ts}c\|_{1+\alpha} \cdot \|S_{t(2-s)}d\|_{1-\alpha} + \|S_{ts}c\|_\alpha \cdot \|S_{t(2-s)}d\|_{2-\alpha}\}.$$

Now for $0 < s \leqq 1$, and $\beta \geqq 0$, one has

$$\|S_s c\|_\beta = \|(I + H^2)^{\beta/2} e^{-sH^2} c\| \leqq c_\beta s^{-\beta/2} \|c\|$$

where

$$c_\beta = \sup_{\lambda \geqq 0} (1 + \lambda)^{\beta/2} e^{-\lambda},$$

by spectral theory. Consequently

$$|(c, (\text{ad } S_t)(K) S_t d)| \leqq k_\alpha \|c\| \cdot \|d\| \{c_{1+\alpha} c_{1-\alpha} I_{1+\alpha} + c_\alpha c_{2-\alpha} I_\alpha\}$$

for $0 < t < 1$ where

$$I_\alpha = \int_0^1 ds s^{-\alpha/2} (2 - s)^{-1+\alpha/2}.$$

But since $0 \leqq \alpha < 1$ both integrals I_α, and $I_{1+\alpha}$, are finite. Thus C_t is bounded and $\|C_t\|$ is uniformly bounded for $0 < t < 1$. This establishes that $\bar{K}$ is a generator.

Remark 2.2. This proof follows an argument of Nelson [**17**] who proved a similar result under the additional hypothesis $H \geqq 0$.

Next we consider properties of the semigroup generated by $\bar{K}$, in particular boundedness, continuity, and invariance properties, which can be derived if $\alpha = 0$ in (2.1).

THEOREM 2.3. *Let K be a dissipative operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$. Suppose*

(2.4) $\quad |(a, (\text{ad } H)(K)b)| \leqq k\|a\| \cdot \|b\|_1, \quad a, b \in \mathfrak{h}_\infty$

Let $T_t = \exp\{-t\bar{K}\}$ denote the contraction semigroup generated by $\bar{K}$.
It follows that $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_1$ and

1. $T_t \mathfrak{h}_1 \subseteq \mathfrak{h}_1, \quad t \geqq 0,$
2. $\|T_t a\|_1 \leqq e^{tk} \|a\|_1, \quad a \in \mathfrak{h}_1$
3. $T|_{\mathfrak{h}_1}$ *is* $\|\cdot\|_1$*-continuous.*

Proof. By definition

$$|(Ha, Kb)| \leqq |(a, KHb)| + |(a, (\text{ad } H)(K)b)|$$
$$\leqq \|a\| \{ \|KHb\| + k\|b\|_1 \}, \quad a, b \in \mathfrak{h}_\infty$$

and hence $Kb \in D(H) = \mathfrak{h}_1$.

Next we prove that $K + kI$ is $\|\cdot\|_1$-dissipative. For this first note that $\|a\|_1 = \|(I + iH)a\|$. Then

$$\|(I + \epsilon(K + kI))a\|_1 = \|(I + iH)(I + \epsilon(K + kI))a\|$$
$$\geqq \|(I + \epsilon(K + kI)(I + iH)a\|$$
$$- \epsilon\|(\text{ad } H)(K)a\|$$

$$\geqq (1 + \epsilon k)\|a\|_1 - \epsilon k\|a\|_1 = \|a\|_1$$

where we have used dissipativity of K, and (2.4).

Finally we prove that $R(I + \epsilon(K + kI))$ is $\|\cdot\|_1$-dense in $\mathfrak{h}_1$ for small $\epsilon > 0$. Then it follows from the Hille-Yosida theorem that the $\|\cdot\|_1$-closure of $K + kI$ generates a $\|\cdot\|_1$-continuous, $\|\cdot\|_1$-contractive, semigroup on $\mathfrak{h}_1$. The stated properties of T are an immediate consequence. We use an argument devised in [**1**]. Suppose there is an $f \in \mathfrak{h}_1^*$ such that

(2.5) $\quad f((I + \epsilon(K + kI))a) = 0, \quad a \in \mathfrak{h}_\infty.$

Now $R = (I + iH)^{-1}$ is a bounded map from $\mathfrak{h}$ into $\mathfrak{h}_1$ and hence the adjoint R^* defines a bounded map from $\mathfrak{h}_1^*$ into $\mathfrak{h}$. But to prove $f = 0$ it suffices to prove that $R^*f = 0$, because the range of R^* equals $\mathfrak{h}$. It follows from (2.5), however, that

$$\begin{aligned}(R^*f, (I + \epsilon K)a) &= -\epsilon k(R^*f, a) + \epsilon f((\text{ad } R)(K)a)\\ &= -\epsilon k(R^*f, a) + i\epsilon(R^*f, (\text{ad } H)(K)Ra)\end{aligned}$$

where we have used $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_1$ to define $(\text{ad } H)(K)$ on $\mathfrak{h}_\infty$. Thus

$$|(R^*f, (I + \epsilon K)a| \leqq 2\epsilon k\|R^*f\| \cdot \|(I + \epsilon K)a\|.$$

Now $R(I + \epsilon K)$ is dense in $\mathfrak{h}$ and consequently

$$\|R^*f\| \leqq 2\epsilon k\|R^*f\|.$$

Thus if $\epsilon < 1/2k$ one must have $R^*f = 0$.

b. *Symmetric operators.* If K is a symmetric operator on $\mathfrak{h}$ then $\pm iK$ are both dissipative. Hence Theorem 2.1 gives criteria for $\pm iK$ to generate contraction semigroups. But $\pm iK$ generate such semigroups if, and only if, $\bar{K}$ is self-adjoint. Thus Theorem 2.1 immediately yields a result on essential self-adjointness of a symmetric operator. But this result can be strengthened by symmetry.

OBSERVATION 2.4. *Let s be a symmetric sesquilinear form over $\mathfrak{h}_\infty \times \mathfrak{h}_\infty$ and suppose*

$$|s(a, b)| \leqq c\|a\|_\alpha \cdot \|b\|_\beta, \quad a, b \in \mathfrak{h}_\infty,$$

for some $\alpha, \beta \in \mathbf{R}$. Then

$$|s(a, b)| \leqq c\|a\|_{\alpha+\gamma} \cdot \|b\|_{\beta-\gamma}, \quad a, b \in \mathfrak{h}_\infty,$$

for all $\gamma \in [0, \beta - \alpha]$ if $\beta \geqq \alpha$, or $\gamma \in [\beta - \alpha, 0]$ if $\beta \leqq \alpha$.

This is established in two steps. First by symmetry one has

$$|s(a, b)| \leqq c\|a\|_\beta \cdot \|b\|_\alpha$$

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and then by interpolation theory (see, for example, **[22]** appendix to Section IX.4) the bound extends to all pairs of intermediate indices with sum $\alpha + \beta$.

The observation immediately implies that if K is symmetric then Theorem 2.1 can be extended to the value $\alpha = 1$, by symmetry. It also implies that if one has a family of conditions such as

$$|s(a, b)| \leqq c\|a\|_\alpha \cdot \|b\|_{n-\alpha}, \quad a, b \in \mathfrak{h}_\infty,$$

with $\alpha \in [0, n]$ then the weakest condition is for $\alpha = n/2$ and the strongest is for $\alpha = 0$, or $\alpha = n$. The foregoing discussion of dissipative operators was based on commutation conditions with index $n = 1$ and we next discuss generalizations for symmetric operators. Subsequently we show that similar results can be obtained for positive symmetric operators from commutation conditions with index $n = 2$.

First we consider the weak conditions with index $n = 1$ then Theorems 2.1 and 2.3 can be generalized if $H \geqq 0$. Second we consider the strong conditions with index $n = 1$ and show that then the generalized invariance properties can be derived without positivity of H.

THEOREM 2.5. *Let K be a symmetric operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ and suppose that*

(2.6) $\quad |(a, (\text{ad } H)(K)b)| \leqq k\|a\|_{1/2} \cdot \|b\|_{1/2}, \quad a, b \in \mathfrak{h}_\infty$

for some $k \geqq 0$.

It follows that K is essentially self-adjoint.

If, in addition, $H \geqq 0$ then $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_{1/2}$. Moreover, for each $\alpha \in [0, 1/2]$ the unitary group $V_t = \exp\{it\bar{K}\}$ has the following properties.

1. $V_t\mathfrak{h}_\alpha = \mathfrak{h}_\alpha, \quad t \in \mathbf{R}$,
2. $\|V_t a\|'_\alpha \leqq e^{|t|\alpha k/2}\|a\|'_\alpha, \quad t \in \mathbf{R}, a \in \mathfrak{h}_\alpha$,
3. $V|_{\mathfrak{h}_\alpha}$ *is* $\|\cdot\|'_\alpha$*-continuous.*

Proof. The essential self-adjointness follows from Theorem 2.1, with $\alpha = 1/2$, applied to $\pm iK$. Now we consider the additional properties with $H \geqq 0$. Recall that if $H \geqq 0$ the norms $\|\cdot\|'_\alpha$ are defined by

$$\|a\|'_\alpha = \|(I + H)^\alpha a\|, \quad a \in \mathfrak{h}_\alpha.$$

It then follows that $\|a\|_\alpha \leqq \|a\|'_\alpha$, for $\alpha \geqq 0$.

To prove $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_{1/2}$ we observe that if $a, b \in \mathfrak{h}_\infty$ then

$$\begin{aligned} |((I + H)^{1/2}a, Kb)| &\leqq |((I + H)^{-1/2}a, (\text{ad } H)(K)b)| \\ &\quad + |((I + H)^{-1/2}a, K(I + H)b)| \\ &\leqq k\|a\| \cdot \|b\|'_{1/2} + \|a\| \cdot \|K(I + H)b\|. \end{aligned}$$

But since $\mathfrak{h}_\infty$ is a core for $(I + H)^{1/2}$ it follows that

$$Kb \in D((I + H)^{1/2}).$$

In order to establish the properties of the unitary group V it is necessary to have an estimate for commutators of fractional powers.

LEMMA 2.6. *Let K be a closable operator from* $\mathfrak{h}_\infty$ *into* $\mathfrak{h}$. *Suppose* $H \geqq 0$ *and*

(2.7) $$|(a, (\mathrm{ad}\, H)(K)b)| \leqq k\|a\|'_{1/2} \cdot \|b\|'_{1/2}, \quad a, b \in \mathfrak{h}_\infty.$$

Then, for each $\alpha \in [0, 1]$,

(2.8) $$|(a, (\mathrm{ad}\, (I + H)^\alpha)(K)b)| \leqq \alpha k\|a\|'_{\alpha/2} \cdot \|b\|'_{\alpha/2}, \quad a, b \in \mathfrak{h}_\infty.$$

Proof. Set $L = I + H$. We use the integral representation

$$L^\alpha = c_\alpha \int_0^\infty d\lambda\lambda^{-\alpha}L(I + \lambda L)^{-1}$$

where

$$c_\alpha^{-1} = \int_0^\infty d\lambda\lambda^{-\alpha}(1 + \lambda)^{-1}.$$

Then

$$(a, (\mathrm{ad}\, L^\alpha)(K)b)$$
$$= c_\alpha \int_0^\infty d\lambda\lambda^{-\alpha}((I + \lambda L)^{-1}a, (\mathrm{ad}\, H)(K)(I + \lambda L)^{-1}b).$$

Consequently

$$|(a, (\mathrm{ad}\, L^\alpha)(K)b)|$$
$$\leqq kc_\alpha \int^\infty d\lambda\lambda^{-\alpha}\|L^{1/2}(I + \lambda L)^{-1}a\| \cdot \|L^{1/2}(I + \lambda L)^{-1}b\|.$$

Therefore by the Cauchy-Schwarz inequality one obtains

$$|(a, (\mathrm{ad}\, L^\alpha)(K)b)| \leqq kM_\alpha(a)M_\alpha(b)$$

where

$$M_\alpha(a)^2 = c_\alpha \int_0^\infty d\lambda\lambda^{-\alpha}\|L^{1/2}(I + \lambda L)^{-1}a\|^2.$$

But if E denotes the spectral measure of H then

$$M_\alpha(a)^2 = c_\alpha \int_1^\infty d(a, E(x)a)x \int_0^\infty d\lambda\lambda^{-\alpha}(1 + \lambda x)^{-2}$$
$$= c_\alpha \int_1^\infty d(a, E(x)a)x^\alpha \int_0^\infty d\rho\rho^{-\alpha}(1 + \rho)^{-2}$$
$$= \alpha \int_1^\infty d(a, E(x)a)x^\alpha = \alpha(\|\alpha\|'_{\alpha/2})^2.$$

This gives the desired bounds.

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We note in passing that a slight adaptation of this argument gives bounds

$$|(a, (\mathrm{ad}(I + H)^{\alpha})(K)b)| \leqq k_{\alpha,\beta}\|a\|_{\beta} \cdot \|b\|_{\alpha-\beta}, \quad a, b \in \mathfrak{h}_{\infty}$$

with $k_{\alpha,\beta} < +\infty$ whenever $\beta, \alpha - \beta \in [0, 1/2\rangle$.

Now we are prepared to prove the invariance properties in Theorem 2.5.

First note that since K is symmetric it is closable, and hence closable as an operator from the Fréchet space $\mathfrak{h}_{\infty}$ into the Hilbert space $\mathfrak{h}$. Therefore, by the closed graph theorem, there is an integer $p \geqq 1$ and a $c_p \geqq 0$ such that

$$\|Ka\| \leqq c_p\|a\|'_p = c_p\|(I + H)^p a\|, \quad a \in \mathfrak{h}_{\infty}. \tag{2.9}$$

Consequently for each $\epsilon > 0$ there is a $c_p(\epsilon)$ such that

$$\|\bar{K}(I + \epsilon H)^{-p}a\| \leqq c_p(\epsilon)\|a\|, \quad a \in \mathfrak{h}.$$

Now introduce the family of bounded self-adjoint operators

$$K_{\epsilon} = (I + \epsilon H)^{-p}\bar{K}(I + \epsilon H)^{-p}. \tag{2.10}$$

It follows that $K_{\epsilon}\mathfrak{h} \subseteq \mathfrak{h}_p$.

Second remark that the norm

$$\|a\|'_{\alpha} = \|(I + H)^{\alpha}a\|, \quad a \in \mathfrak{h}_{\alpha},$$

satisfies the bounds

$$\|a\| \leqq \|a\|'_{\alpha} \leqq 2^{\alpha/2}\|a\|_{\alpha}. \tag{2.11}$$

Therefore (2.6) implies (2.7) and hence (2.8) is valid for $\alpha \in [0, 1]$.

Third, consider the unitary groups V^{ϵ} generated by the K_{ϵ}. One has $V_t^{\epsilon}\mathfrak{h}_{\alpha} \subseteq \mathfrak{h}_{\alpha}$ for all $t \in \mathbf{R}$ and $\alpha \in [0, 1]$ by power series expansion. Then since

$$\mathfrak{h}_{\alpha} = V_t^{\epsilon}(V_{-t}^{\epsilon}\mathfrak{h}_{\alpha})$$

one has $V_t^{\epsilon}\mathfrak{h}_{\alpha} = \mathfrak{h}_{\alpha}$. But

$$\begin{aligned}
\frac{d}{dt}(\|V_t^{\epsilon}a\|'_{\alpha})^2 &= \frac{d}{dt}((I + H)^{\alpha}V_t^{\epsilon}a, (I + H)^{\alpha}V_t^{\epsilon}a) \\
&= i((I + \epsilon H)^{-p}V_t^{\epsilon}a, \\
&\qquad\qquad (\mathrm{ad}(I + H)^{2\alpha})(K)(I + \epsilon H)^{-p}V_t^{\epsilon}a) \\
&\leqq \alpha k(\|(I + \epsilon H)^{-p}V_t^{\epsilon}a\|'_{\alpha})^2 \\
&\leqq \alpha k(\|V_t^{\epsilon}a\|'_{\alpha})^2
\end{aligned}$$

where the first inequality uses (2.6), (2.11), and Lemma 2.6, and is valid for $0 \leqq \alpha \leqq 1/2$. Now integrating this differential inequality gives

$$(2.12) \quad \|V_t^\epsilon a\|'_\alpha \leqq e^{|t|\alpha k/2}\|a\|'_\alpha, \quad a \in \mathfrak{h}_\infty$$

and it is important that this bound is uniform in ϵ.

Next if $a \in \mathfrak{h}_\infty$ then

$$\begin{aligned} \|(K_\epsilon - K)a\| &\leqq \|K((I + \epsilon H)^{-p} - I)a\| \\ &\quad + \|((I + \epsilon H)^{-p} - I)Ka\| \\ &\leqq c_p\|((I + \epsilon H)^{-p} - I)a\|_p \\ &\quad + \|((I + \epsilon H)^{-p} - I)Ka\| \\ &\underset{\epsilon \to 0}{\to} 0 \end{aligned}$$

where we have used $\|(I + \epsilon H)^{-1}\| \leqq 1$ and (2.9). Since $\mathfrak{h}_\infty$ is a core of $\bar{K}$ it follows immediately that V_t^ϵ converges strongly to V_t, for each $t \in \mathbf{R}$, and the convergence is uniform for t in compact subsets of $\mathbf{R}$ (see, for example, [**2**] Section 3.1.3).

Now if E denotes the spectral measure of $(I + H)$ then for $a \in \mathfrak{h}_\infty$ one deduces from (2.12) that

$$\begin{aligned} (e^{|t|k/4}\|a\|'_{1/2})^2 &\geqq \int_N^\infty d(V_t^\epsilon a, E(\lambda)V_t^\epsilon a)\lambda \\ &\geqq N^{1-2\alpha}\int_N^\infty d(V_t^\epsilon a, E(\lambda)V_t^\epsilon a)\lambda^{2\alpha} \\ &= N^{1-2\alpha}(\|(I - E_N)V_t^\epsilon a\|'_\alpha)^2 \end{aligned}$$

where E_N denotes the projection onto the subspace for which $(I + H) \leqq NI$. Thus if $0 \leqq \alpha < 1/2$

$$\begin{aligned} (2.13) \quad \|(V_t^\epsilon - V_t^\delta)a\|'_\alpha &\leqq (1 + N)^\alpha\|E_N(V_t^\epsilon - V_t^\delta)a\|' \\ &\quad + 2e^{|t|k/4}\|a\|'_{1/2}/N^{1/2-\alpha} \end{aligned}$$

and

$$\begin{aligned} (2.14) \quad \|(V_t^\epsilon - 1)a\|'_\alpha &\leqq (1 + N)^\alpha\|E_N(V_t^\epsilon - I)a\| \\ &\quad + (e^{|t|k/4} + 1)\|a\|'_{1/2}/N^{1/2-\alpha}. \end{aligned}$$

It immediately follows from (2.13) and (2.14) that V^ϵ is $\|\cdot\|'_\alpha$-convergent to V as $\epsilon \to 0$. Thus $V_t\mathfrak{h}_\alpha \subseteq \mathfrak{h}_\alpha$, and $V_t\mathfrak{h}_\alpha = \mathfrak{h}_\alpha$ by the group property. But then it follows from (2.12), (2.13) and (2.14), that the restriction of V to $\mathfrak{h}_\alpha$ is $\|\cdot\|'_\alpha$-continuous and satisfies the bounds

$$\|V_t a\|'_\alpha \leqq e^{|t|\alpha k/2}\|a\|'_\alpha, \quad a \in \mathfrak{h}_\alpha,$$

for $0 \leqq \alpha < 1/2$. Next it follows from these bounds and monotone convergence that $V_t\mathfrak{h}_{1/2} = \mathfrak{h}_{1/2}$ and

$$\|V_t a\|'_{1/2} \leqq e^{|t|k/4}\|a\|'_{1/2}, \quad a \in \mathfrak{h}_{1/2}.$$

Finally, since $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_{1/2}$ and

$$(V_t - I)a = i\int_0^t ds V_s Ka, \quad a \in \mathfrak{h}_\infty,$$

one has

$$\|(V_t - I)a\|'_{1/2} \leqq |t|e^{|t|k/4}\|Ka\|'_{1/2}, \quad a \in \mathfrak{h}_\infty.$$

But $\mathfrak{h}_\infty$ is $\|\cdot\|'_{1/2}$-dense in $\mathfrak{h}_{1/2}$ and hence this establishes that V_t is $\|\cdot\|'_{1/2}$-continuous.

Remark 2.7. The original Glimm-Jaffe commutator theorem [**7**] proved essential self-adjointness from a commutator condition of index one under the assumption $H \geqq 0$. Other versions based on similar assumptions were subsequently given in [**17**] [**5**] [**15**] [**16**]. An invariance property of $\mathfrak{h}_{1/2}$ was first proved by Faris and Lavine [**5**] by a differential inequality method similar to the above. But the modified method using the K_ϵ approximation technique was introduced by Glimm and Jaffe [**8**] [**9**] for the proof of invariance and commutation properties. Similar applications were subsequently given by Driessler and Fröhlich [**3**] [**6**]. The invariance of the fractional spaces $\mathfrak{h}_\alpha$ and the $\|\cdot\|_\alpha$-continuity of V on $\mathfrak{h}_\alpha$ appear to be new.

Our second result on symmetric operators is based on the stronger form of the commutator condition with index one and does not require $H \geqq 0$. The essential self-adjointness statement is essentially Theorem 1 of [**4**].

THEOREM 2.8. *Let K be a symmetric operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ and suppose that*

$$(2.15) \quad |(a, (\mathrm{ad}\, H)(K)b)| \leqq k\|a\| \cdot \|b\|_1, \quad a, b \in \mathfrak{h}_\infty$$

for some $k \geqq 0$.

It follows that K is essentially self-adjoint.

Moreover, $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_1$ and the unitary group $V_t = \exp\{it\bar{K}\}$ has the following properties for each $\alpha \in [0, 1]$:

1. $V_t\mathfrak{h}_\alpha = \mathfrak{h}_\alpha, \quad t \in \mathbf{R}$,
2. $\|V_t a\|_\alpha \leqq e^{|t|\alpha k}\|a\|_\alpha, \quad t \in \mathbf{R}, a \in \mathfrak{h}_\alpha$,
3. $V|_{\mathfrak{h}_\alpha}$ *is* $\|\cdot\|_\alpha$*-continuous.*

Proof. The essential self-adjointness follows from Theorem 2.1, with $\alpha = 0$, applied to $\pm iK$. Next note that

$$(2.16)\quad |(a, (\text{ad } H^2)(K)b)| \leqq |(Ha, (\text{ad } H)(K)b)| + |((\text{ad } H)(K)a, Hb)| \leqq k(\|Ha\| \cdot \|b\|_1 + \|a\|_1 \cdot \|Hb\|) \leqq 2k\|a\|_1 \cdot \|b\|_1.$$

Since the C^∞-elements of H and H^2 are the same it follows that the hypotheses of Theorem 2.5 are fulfilled with respect to the positive self-adjoint operator H^2. Hence the statements about V follow essentially from Theorem 2.5 when one notes that the fractional space $\mathfrak{h}_\alpha$ of H corresponds to the space $\mathfrak{h}_{\alpha/2}$ of H^2. The estimates on $V_t a$ can be improved, however, because it is no longer necessary to introduce the equivalent norm $\|\cdot\|'_\alpha$. For example one has the following:

LEMMA 2.9. *Let K be a symmetric operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ and suppose that*

$$|(a, (\text{ad } H^2)(K)b| \leqq 2k\|a\|_1 \cdot \|b\|_1, \quad a, b \in \mathfrak{h}_\infty$$

then for each $\alpha \in [0, 1]$

$$|(a, (\text{ad}(I + H^2)^\alpha)(K)b)| \leqq 2\alpha k\|a\|_\alpha \cdot \|b\|_\alpha, \quad a, b \in \mathfrak{h}_\infty.$$

The proof is identical to that of Lemma 2.6.

Next repeating the derivation of inequality (2.12), but using (2.16) and Lemma 2.9 one deduces that

$$\|V_t^\epsilon a\|_\alpha \leqq e^{|t|\alpha k}\|a\|_\alpha, \quad a \in \mathfrak{h}_\alpha.$$

The proof then continues as in Theorem 2.5.

Finally we note that the index one in the commutator bounds (2.6) and (2.15) is necessary for the essential self-adjointness of K. Example 1 in Section X.5 of **[22]** gives counterexamples to any relaxation of this condition, even with $H \geqq 0$. Nevertheless self-adjointness follows even from a mixed estimate of index one, i.e., an estimate of the type

$$|(a, (\text{ad } H)(K)b)| \leqq \sum_{i=1}^{n} k_i\|a\|_{(1+\beta_i)/2} \cdot \|b\|_{(1-\beta_i)/2}, \quad a, b \in \mathfrak{h}_\infty,$$

with $\beta_i \in [0, 1]$. The invariance properties are, however, more delicate. If one has an intermediate bound

$$|(a, (\text{ad } H)(K)b)| \leqq k\|a\|_{(1+\beta)/2} \cdot \|b\|_{(1-\beta)/2}, \quad a, b \in \mathfrak{h}_\infty,$$

for $\beta \in [0, 1]$ then this estimate together with symmetry gives

$$|(a, (\text{ad } H^2)(K)b)| \geqq 2k\|a\|_{1-\gamma} \cdot \|b\|_{1+\gamma}, \quad a, b \in \mathfrak{h}_\infty,$$

where $\gamma = (1 - |\beta|)/2$. Therefore by a slight variation of the proof of Lemma 2.6 one finds

$$|(a, (\text{ad}(I + H^2)^\alpha)(K)b)| \leqq 2k_{\alpha,\gamma}\|a\|_\alpha \cdot \|b\|_\alpha, \quad a, b \in \mathfrak{h}_\infty,$$

valid for $\alpha \in [0, 1 - |\gamma| \rangle$. Hence one has the invariance of $\mathfrak{h}_\alpha$ under V_t for each $\alpha \in [0, (1 + |\beta|)/2\rangle$.

c. *Positive operators.* The results of the previous subsection can be extended if the symmetric operator K is positive. First it is only necessary to have a bound on the double commutator $(\text{ad } H)^2(K)$ and second it suffices that this bound has index two. It is the latter point which is of greatest interest in applications.

Again there are two results based on a weak bound of index two and a strong bound, respectively.

THEOREM 2.10. *Let K be a positive operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ and suppose that*

(2.17) $$|(a, (\text{ad } H)^2(K)b)| \leqq k\|a\|_1 \cdot \|b\|_1$$

for some $k \geqq 0$.

It follows that K is essentially self-adjoint.

If, in addition, $H \geqq 0$ then $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_1$ and for each $\alpha \in [0, 1]$ the contraction semigroup $T_t = \exp\{-t\bar{K}\}$ has the following properties.

1. $T_t\mathfrak{h}_\alpha \subseteq \mathfrak{h}_\alpha, \quad t \geqq 0,$
2. $\|T_t a\|'_\alpha \leqq e^{t\alpha^2 k/2}\|a\|'_\alpha, \quad t \geqq 0, a \in \mathfrak{h}_\alpha,$
3. $T|_{\mathfrak{h}_\alpha}$ *is* $\|\cdot\|_\alpha$*-continuous.*

Proof. Since $K \geqq 0$ it suffices, for essential self-adjointness, to prove that $R(I + \epsilon K)$ is dense in $\mathfrak{h}$ for sufficiently small $\epsilon > 0$. But if there is an $a \in \mathfrak{h}$ orthogonal to the range of $I + \epsilon K$ then by the calculation used to derive (2.2) one has

(2.18) $$\|S_t a\|^2 = -\epsilon \operatorname{Re}(S_t a, KS_t a) + \epsilon \operatorname{Re}(a, (\text{ad } S_t)(K)S_t a)$$
$$\leqq (\epsilon/2)\{(a, (\text{ad } S_t)(K)S_t a) + ((\text{ad } S_t)(K)S_t a, a)\}$$

where $S_t = \exp\{-tH^2\}$. Now consider the operators

$$C_{2,t} = (\text{ad } S_t)^2(K)$$

defined on $\mathfrak{h}_\infty$. We will prove that the $C_{2,t}$ have bounded closures $\bar{C}_{2,t}$ and

$$\omega = \sup\{\|\bar{C}_{2,t}\|; 0 < t < 1/2\} < +\infty.$$

Then one has

$$(a, (\text{ad } S_t)(K)S_t a) + ((\text{ad } S_t)(K)S_t a, a) = -(a, \bar{C}_{2,t}a)$$

by continuity, and hence

$$\|a\|^2 = \lim_{t \to 0} \|S_t a\|^2 \leqq (\epsilon/2)\omega\|a\|^2$$

by (2.18). But if $\epsilon\omega < 2$ this implies $a = 0$. Hence K is essentially self-adjoint.

Next we establish the boundedness properties of the $C_{2,t}$.

If $c, d \in \mathfrak{h}_\infty$, then

$$(c, (\text{ad } S_t)^2(K)d)$$
$$= t^2 \int_0^1 dr \int_0^1 ds (S_{t(r+s)}c, (\text{ad } H^2)^2(K)S_{t(2-r-s)}d)$$

and hence by linear algebra, and (2.17), one finds

$$(2.19) \quad |(c, (\text{ad } S_t)^2(K)d)|$$
$$\leqq kt^2 \int_0^1 dr \int_0^1 ds \sum_{m=0}^{2} \binom{2}{m} \|H^{2-m}S_{t(r+s)}\|_1 \cdot \|H^m S_{t(2-r-s)}d\|_1$$
$$\leqq kt^2 \int_0^1 dr \int_0^1 ds \sum_{m=0}^{2} \binom{2}{m} \|S_{t(r+s)}\|_{3-m} \cdot \|S_{t(2-r-s)}\|_{1+m}.$$

But

$$\|S_s c\|_\beta \leqq c_\beta s^{-\beta/2}\|c\|$$

for $0 < s \leqq 1$ and $\beta \geqq 0$, where

$$c_\beta = \sup\{ (1 + \lambda)^{\beta/2}e^{-\lambda}, \lambda \geqq 0\}$$

by spectral theory. Substituting these bounds into (2.19) then gives

$$|(c, (\text{ad } S_t)^2(K)d)| \leqq k'\|c\| \cdot \|d\|$$

for some $k' \geqq 0$ and all $0 < t \leqq 1/2$. The boundedness properties of $C_{2,t}$ are then apparent.

Next we establish that $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_1$ when $H \geqq 0$. This requires an estimate on the commutator $(\text{ad } H)(K)$, which does not rely on the assumption $H \geqq 0$.

LEMMA 2.11. *Let K be an operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ and suppose that*

$$|(a, Kb)| \leqq k_0\|a\|_{\alpha_0} \cdot \|b\|_{\beta_0},$$
$$|(a, (\text{ad } H)^2(K)b)| \leqq k_2\|a\|_{\alpha_2} \cdot \|b\|_{\beta_2}, \quad a, b \in \mathfrak{h}_\infty,$$

for some $k_i \geqq 0$ and $\alpha_i, \beta_i \geqq 0$. Then

$$|(a, (\text{ad } H)(K)b)| \leqq k_1\|a\|_{\alpha_1} \cdot \|b\|_{\beta_1}, \quad a, b \in \mathfrak{h}_\infty,$$

for some $k_1 \geqq 0$ where $\alpha_1 = \alpha_0 \vee \alpha_2$ and $\beta_1 = \beta_0 \vee \beta_2$.

Proof. First for each $t > 0$ define $T_t(K)$ as an operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ by setting

$$T_t(K)a = U_tKU_{-t}a, \quad a\ b \in \mathfrak{h}_\infty$$

where $U_t = \exp\{itH\}$. Then

$$|(a, T_t(K)b)| \leqq k_0\|U_{-t}a\|_{\alpha_0} \cdot \|U_{-t}b\|_{\beta_0}$$
$$= k_0\|a\|_{\alpha_0} \cdot \|b\|_{\beta_0}, \quad a, b \in \mathfrak{h}_\infty.$$

Next for $s, t > 0$ define sesquilinear forms

$$F_{s,t}(a, b; K) = \sum_{p \geqq 0} e^{-s/t}\frac{(s/t)^p}{p!}(a, T_{pt}(K)b)$$

over $\mathfrak{h}_\infty \times \mathfrak{h}_\infty$. Then one has the bounds

$$|F_{s,t}(a, b; K)| \leqq k_0\|a\|_{\alpha_0} \cdot \|b\|_{\beta_0}.$$

But one calculates by rearrangement of convergent power series that

$$F_{s,t}(a, b; K) = \sum_{p \geqq 0} \frac{(-s)^p}{p!}(a, D_t^p(K)b)$$

where we have set $D_t = (I - T_t)/t$. Hence

$$\frac{d^p}{ds^p}F_{s,t}(a, b; K) = (-1)^p F_{s,t}(a, b; D_t^p(K)).$$

Now

$$(a, D_t^2(K)b) = -\frac{1}{t^2}\int_0^t dr \int_0^t ds(U_{-r-s}a, (\text{ad } H)^2(K)U_{-r-s}b).$$

Hence

$$|(a, D_t^2(K)b)| \leqq k_2\|a\|_{\alpha_2} \cdot \|b\|_{\beta_2}, \quad a, b \in \mathfrak{h}_\infty.$$

Consequently

$$|F_{s,t}(a, b; D_t^2K))| \leqq k_2\|a\|_{\alpha_2} \cdot \|b\|_{\beta_2}, \quad a, b \in \mathfrak{h}_\infty.$$

Therefore one deduces from the Taylor series expansion

$$F_{s,t}(a, b; K) = (a, Kb) - s(a, D_t(K)b)$$
$$+ \int_0^s dr(r - s)F_{r,t}(a, b; D_t^2(K))$$

that

$$s|(a, D_t(K)b)| \leqq 2k_0\|a\|_{\alpha_0} \cdot \|b\|_{\beta_0} + (k_2s^2/2)\|a\|_{\alpha_2} \cdot \|b\|_{\beta_2}$$

for all $a, b \in \mathfrak{h}_\infty$ and $s > 0$. Finally

$$|(a, (\text{ad } H)(K)b)| = \lim_{t \to 0+} |(a, D_t(K)b)|$$

and hence

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$$|(a, (\text{ad } H)(K)b)| \leqq k_1\|a\|_{\alpha_1} \cdot \|b\|_{\beta_1}$$

with $\alpha_1 = \alpha_0 \vee \alpha_2$, $\beta_1 = \beta_0 \vee \beta_2$ and

$$k_1 = (2k_0/s) + (sk_2/2)$$

where $s > 0$.

Now we return to the proof that $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_1$ when $H \geqq 0$.

First K must satisfy the bound (2.9). Thus

$$|(a, Kb)| \leqq c_p\|a\| \cdot \|b\|'_p, \quad a, b \in \mathfrak{h}_\infty.$$

Second by the assumption of the theorem

$$|(a, (\text{ad } H)^2(K)b)| \leqq k\|a\|'_1 \cdot \|b\|'_1, \quad a, b \in \mathfrak{h}_\infty.$$

Hence by Lemma 2.11 there is a k_1 such that

$$|(a, (\text{ad } H)(K)b)| \leqq k_1\|a\|'_1 \cdot \|b\|'_p, \quad a, b \in \mathfrak{h}_\infty.$$

Then one has

$$\begin{aligned} |((I + H)a, Kb)| &\leqq |((I + H)^{-1}a, (\text{ad}(I + H)^2)(K)b)| \\ &\quad + |((I + H)^{-1}a, K(I + H)^2b)| \\ &\leqq |((I + H)^{-1}a, (\text{ad } H)^2(K)b)| \\ &\quad + 2|((I + H)^{-1}a, (\text{ad } H)(K)(I + H)b)| \\ &\quad + |((I + H)^{-1}a, K(I + H)^2b)| \\ &\leqq k\|a\| \cdot \|b\|'_1 + 2k_1\|a\| \cdot \|b\|'_{p+1} \\ &\quad + c_p\|a\| \cdot \|b\|'_{p+2} \end{aligned}$$

for all $a, b \in \mathfrak{h}_\infty$. But since $\mathfrak{h}_\infty$ is a core for H it follows that $Kb \in D(I + H)$, i.e., $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_1$.

Next we discuss the properties of the semigroup T. For this we need the double commutator analogue of Lemma 2.6.

LEMMA 2.12. *Let K be a closable operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$. Suppose $H \geqq 0$ and*

$$|(a, (\text{ad } H)^2(K)b)| \leqq k\|a\|'_1 \cdot \|b\|'_1, \quad a, b \in \mathfrak{h}_\infty.$$

Then, for each $\alpha \in [0, 1]$

$$|(a, (\text{ad } (I + H)^\alpha)^2(K)b)| \leqq \alpha^2k\|a\|'_\alpha \cdot \|b\|'_\alpha, \quad a, b \in \mathfrak{h}_\infty.$$

Proof. The proof is an easy extension of the proof of Lemma 2.6.

Next we reintroduce the approximants

$$K_\epsilon = (I + \epsilon H)^{-p}K(I + \epsilon H)^{-p},$$

$\epsilon > 0$, used in the proofs of Theorems 2.5 and 2.8. We assume $p \geqq 1$ and hence $K_\epsilon \mathfrak{h}_\infty \subseteq \mathfrak{h}_1$. Note that since $K \geqq 0$ one has $K_\epsilon \geqq 0$. Now we prove that $K_\epsilon + \omega I$ is $\|\cdot\|'_\alpha$-dissipative for $\alpha \in [0, 1]$ for ω sufficiently large.

LEMMA 2.13. *Let K be a positive operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$. Suppose $H \geqq 0$ and*

$$|(a, (\mathrm{ad}\,(I + H)^\alpha)^2(K)b)| \leqq \omega\|a\|'_\alpha \cdot \|b\|'_\alpha, \quad a, b \in \mathfrak{h}_\infty,$$

for some $\alpha \in [0, 1]$. If $K_\epsilon = (I + \epsilon H)^{-p}K(I + \epsilon H)^{-p}$ with $\epsilon > 0$ and $p \geqq 1$ then $K_\epsilon + (\omega/2)I$ is $\|\cdot\|'_\alpha$-dissipative.

Proof. Let $b \in \mathfrak{h}_\infty$ and take $a \in \mathfrak{h}^*_\alpha = \mathfrak{h}_{-\alpha}$ such that

$$a(b) = \|a\|'_{-\alpha} \cdot \|b\|'_\alpha.$$

Now $(I + H)^{-\alpha}$ defines a unitary map from $\mathfrak{h}^*_\alpha$ into $\mathfrak{h}^*$ and $(I + H)^\alpha$ is a unitary map from $\mathfrak{h}_\alpha$ into $\mathfrak{h}$. Thus

$$c = (I + H)^{-\alpha}a \in \mathfrak{h} \quad \text{and} \quad d = (I + H)^\alpha b \in \mathfrak{h}.$$

But since

$$(c, d) = a(b) = \|a\|'_{-\alpha} \cdot \|b\|'_\alpha = \|c\| \cdot \|d\|$$

it follows that

$$c = d = (I + H)^\alpha b \in \mathfrak{h}_\infty.$$

Now with $c_\epsilon = (I + \epsilon H)^{-p}c$ one has

$$\begin{aligned}
\mathrm{Re}\, a(K_\epsilon b) &= \mathrm{Re}(\,(I + H)^\alpha c, K_\epsilon(I + H)^{-\alpha}c) \\
&= \mathrm{Re}(c, K_\epsilon c) + \mathrm{Re}(c, (\mathrm{ad}(I + H)^\alpha)(K_\epsilon)(I + H)^{-\alpha}c) \\
&\geqq (1/2)(c_\epsilon, (I + H)^{-\alpha}(\mathrm{ad}(I + H)^\alpha)^2(K)(I + H)^{-\alpha}c_\epsilon) \\
&\geqq -(\omega/2)\,\|c_\epsilon\|^2 \geqq -(\omega/2)\,\|c\|^2 \\
&= -(\omega/2)\,\|a\|'_{-\alpha} \cdot \|b\|'_\alpha,
\end{aligned}$$

i.e., $K_\epsilon + (\omega/2)I$ is $\|\cdot\|'_\alpha$-dissipative.

Now we are in a position to prove the second half of Theorem 2.10. We proceed as in the proof of Theorem 2.5 and introduce

$$K_\epsilon = (I + \epsilon H)^{-p}K(I + \epsilon H)^{-p}$$

with $p \geqq 1$ large enough that K_ϵ is bounded. Then $K_\epsilon \geqq 0$ and it is easily verified by series expansion that the associated contraction semigroups

$$T^\epsilon_t = \exp\{-tK_\epsilon\}$$

map $\mathfrak{h}_\alpha$ into $\mathfrak{h}_\alpha$ for each $\alpha \in [0, 1]$. But it follows from (2.17), Lemma 2.12, and Lemma 2.13, that $K_\epsilon + (\alpha^2 k/2)I$ is $\|\cdot\|'_\alpha$-dissipative for $\alpha \in [0, 1]$. Hence

$$\|T_t a\|'_\alpha \leqq e^{t\alpha^2 k/2}\|a\|'_\alpha, \quad t \geqq 0, a \in \mathfrak{h}_\infty$$

for all $\alpha \in [0, 1]$. The remainder of the proof is identical to the proof of Theorem 2.5.

Remark 2.14. The semigroup T on $\mathfrak{h}$ is a bounded holomorphic semigroup of angle $\pi/2$ and it follows by interpolation theory that the restriction of T to each $\mathfrak{h}_\alpha$, $\alpha \in [0, 1]$, is a bounded holomorphic semigroup.

Remark 2.15. It is unclear whether the hypotheses of Theorem 2.10 and the condition $H \geqq 0$ imply that $\mathfrak{h}_\alpha$, $\alpha \in [0, 1]$, is invariant under the unitary group $V_t = \exp\{it\bar{K}\}$.

Next we have a version of Theorem 2.10 based on the stronger commutation condition of index 2 which does not require $H \geqq 0$.

THEOREM 2.16. *Let K be a positive operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ and suppose that*

(2.20) $$|(a, (\text{ad } H)^2(K)b)| \leqq k\|a\| \cdot \|b\|_2, \quad a, b \in \mathfrak{h}_\infty$$

for some $k \geqq 0$.

It follows that K is essentially self-adjoint and $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_2$. Moreover, for each $\alpha \in [0, 2]$ the contraction semigroup $T_t = \exp\{-t\bar{K}\}$ has the following properties:

1. $T_t\mathfrak{h}_\alpha \subseteq \mathfrak{h}_\alpha$, $t \geqq 0$,
2. $\|T_t a\|_\alpha \leqq e^{t\alpha^2 k/2}\|a\|_\alpha$, $t \geqq 0$, $a \in \mathfrak{h}_\alpha$
3. $T|_{\mathfrak{h}_\alpha}$ *is* $\|\cdot\|_\alpha$-*continuous.*

Proof. It follows from (2.20) by Observation 2.4 that

$$|(a, (\text{ad } H)^2(K)b)| \leqq k\|a\|_\beta \cdot \|b\|_{2-\beta}, \quad a, b \in \mathfrak{h}_\infty,$$

for $\beta \in [0, 2]$, In particular K is essentially self-adjoint, by Theorem 2.10. But applying this bound with $\beta = 0, 1, 2$ one also has

(2.21) $$|(a, (\text{ad } H^2)^2(K)b)| \leqq 4k\|a\|_2 \cdot \|b\|_2, \quad a, b \in \mathfrak{h}_\infty.$$

Hence one can now apply Theorem 2.10 with respect to the positive operator H^2 to obtain the properties of T. Alternatively, one can repeat the arguments used to prove the theorem with H replaced by H^2. For example it follows from (2.21) by an extension of the proof of Lemma 2.6 that

$$|(a, (\text{ad } (I + H^2)^{\alpha/2})^2(K)b)| \leqq \alpha^2 k\|a\|_\alpha \cdot \|b\|_\alpha, \quad a, b \in \mathfrak{h}_\alpha,$$

for all $\alpha \in [0, 2]$. Hence by the proof of Lemma 2.13 one concludes that $K_\epsilon + (\alpha^2 k/2)I$ is $\|\cdot\|_\alpha$-dissipative for $\alpha \in [0, 2]$. This gives the bound on $\|T_t^\epsilon a\|_\alpha$ for the approximating semigroups

$$T_t^\epsilon = \exp\{-tK_\epsilon\}$$

and the properties of $T|_{\mathfrak{h}_\alpha}$ are deduced by the same arguments used in the proof of Theorem 2.5.

Finally since K satisfies (2.9) it follows from (2.20) and Lemma 2.11 that there is a $k_1 \geqq 0$ such that

$$|(a, (\text{ad } H)(K)b)| \leqq k_1\|a\| \cdot \|b\|_{p\vee 2}, \quad a, b \in \mathfrak{h}_\infty.$$

Then one deduces that $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_2$ by the estimate

$$|(H^2a, Kb)| \leqq |(a, (\text{ad } H)^2(K)b)| + 2|(a, (\text{ad } H)(K)Hb)| + |(a, KH^2b)|$$

which establishes continuity of $a \in \mathfrak{h}_\infty \mapsto (H^2a, Kb)$, for each $b \in \mathfrak{h}_\infty$.

d. *Multiple commutators.* The preceding results on generators and invariance properties were based on estimates for first- and second-order commutators. Bounds on higher order commutators lead to improved results of invariance **[1]** **[6]**. The simplest result is for dissipative operators.

THEOREM 2.17. *Let K be a dissipative operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ and suppose that*

$$|(a, (\text{ad } H)^p(K)b)| \leqq k_p\|a\| \cdot \|b\|_p, \quad a, b \in \mathfrak{h}_\infty, p = 1, 2, \dots, m.$$

Then $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_m$ and the contraction semigroup T generated by $\bar{K}$ satisfies the following properties, for $p = 1, 2, \dots, m$;

1. $T_t\mathfrak{h}_p \subseteq \mathfrak{h}_p, \quad t \geqq 0,$

2. $\|T_ta\| \leqq \exp\left\{t \sum_{q=1}^{p} \binom{p}{q} k_q\right\}\|a\|_p, \quad t \geqq 0, a \in \mathfrak{h}_p,$

3. $T|_{\mathfrak{h}_p}$ *is $\|\cdot\|_p$-continuous.*

Proof. The property $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_p$ follows by continuity from the identity

$$(H^pa, Kb) = (a, KH^pb) + \sum_{q=1}^{p} \binom{p}{q}(a, (\text{ad } H)^q(K)H^{p-q}b), \quad a, b \in \mathfrak{h}_\infty.$$

The rest of the proof is an extension of the argument used to prove Theorem 2.3. First one proves $\|\cdot\|_p$-dissipativity of $K + \omega I$ for large $\omega \geqq 0$ as follows. One has

$$\text{(2.22)} \quad \|(I + iH)^p(I + \epsilon(K + \omega I))a\| \geqq \|(I + \epsilon(K + \omega I))(I + iH)^pa\| - \epsilon\|(\text{ad } (I + iH)^p)(K)a\|.$$

But then

$$(\text{ad } (I + iH)^p)(K)a = \sum_{q=1}^{p} \binom{p}{q} (\text{ad } iH)^q(K)(I + iH)^{p-q}a$$

and hence

$$\| (\text{ad } (I + iH)^p)(K)a\| \leqq \sum_{q=1}^{p} \binom{p}{q} k_q \|a\|_p. \tag{2.23}$$

Now (2.22) and (2.23) together with dissipativity of K establish that $K + \omega I$ is $\|\cdot\|_p$-dissipative whenever

$$\omega \geqq \sum_{q=1}^{p} \binom{p}{q} k_q.$$

Next one argues that $R(I + \epsilon(K + \omega I))$ is $\|\cdot\|_p$-dense in $\mathfrak{h}_p$ for small $\epsilon > 0$. For this suppose there is an $f \in \mathfrak{h}_p^*$ such that

$$f((I + \epsilon(K + \omega I))a) = 0, \quad a \in \mathfrak{h}_\infty.$$

Now setting $R_p = (I + iH)^{-p}$ one has $R_p^* f \in \mathfrak{h}$ and

$$(R_p^* f, (I + \epsilon K)a) = -\epsilon\omega(R_p^* f, a) + \epsilon(R_p^* f)((\text{ad } (I + iH)^p)(K)R_p a).$$

Then using (2.23) and dissipativity of K one finds

$$\begin{aligned} &|(R_p^* f, (I + \epsilon K)a)| \\ &\leqq \epsilon\omega\|R_p^* f\| \cdot \|a\| + \epsilon\left(\sum_{q=1}^{p} \binom{p}{q} k_q\right)\|R_p^* f\| \cdot \|a\| \\ &\leqq \epsilon\left(\omega + \sum_{q=1}^{p} \binom{p}{q} k_q\right)\|R_p^* f\| \cdot \|(I + \epsilon K)a\|. \end{aligned}$$

Since $R(I + \epsilon K)$ is dense in $\mathfrak{h}$ it follows that $R_p^* f = 0$ for sufficiently small $\epsilon > 0$.

Finally one concludes that the $\|\cdot\|_p$-closure of K generates a semigroup with the three properties stated in the theorem. An easy argument then shows that this semigroup is the restriction of T to $\mathfrak{h}_p$.

If K is symmetric similar conclusions can be reached for the unitary group generated by $i\bar{K}$ but the combinatorics are more complicated.

THEOREM 2.18. *Let K be a symmetric operator from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ and suppose that*

$$|(a, (\text{ad } H)^p(K)b)| \leqq k_p\|a\| \cdot \|b\|_p, \quad a, b \in \mathfrak{h}_\infty, p = 1, 2, \ldots, 2m. \tag{2.24}$$

Then $K\mathfrak{h}_\infty \subseteq \mathfrak{h}_{2m}$ *and for each* $\alpha \in [0, 2m]$ *the unitary group V generated by* $i\bar{K}$ *has the following properties*:

1. $V_t\mathfrak{h}_\alpha = \mathfrak{h}_\alpha, \quad t \in \mathbf{R}$,
2. $\|V_t a\|_\alpha \leqq e^{|t|\alpha k}\|a\|_\alpha, \quad t \in \mathbf{R}, a \in \mathfrak{h}_\alpha$,

for some $k \geqq 0$,

3. $V|_{\mathfrak{h}_\alpha}$ *is* $\|\cdot\|_\alpha$*-continuous.*

Proof. The proof is based upon the observation that

$$|(a, (\text{ad } H)^p(K)b)| \leqq k_p\|a\|_q \cdot \|b\|_{p-q}, \quad q \in [0, p],$$

by Observation 2.4. Then it follows by rearrangement that

$$|(a, (\text{ad } (I + H^2)^m)(K)b)| \leqq k\|a\|_m \cdot \|b\|_m, \quad a, b \in \mathfrak{h}_\infty$$

for a suitable $k \geqq 0$, and consequently, by Lemma 2.6,

$$|a, (\text{ad } (I + H^2)^\alpha)(K)b)| \leqq k\|a\|_\alpha \cdot \|b\|_\alpha, \quad a, b, \in \mathfrak{h}_\infty$$

for $\alpha \in [0, m]$. The rest of the proof then follows the proofs of Theorems 2.5 and 2.10.

Finally one has similar results for positive symmetric operators.

THEOREM 2.19. *Let K be a positive symmetric operator from* $\mathfrak{h}_\infty$ *into* $\mathfrak{h}$ *and suppose that*

$$|(a, (\text{ad } H)^p(K)b)| \leqq k_p\|a\| \cdot \|b\|_p, \quad a, b \in \mathfrak{h}_\infty, p = 2, 3, \ldots, 2m.$$

Then the semigroup T generated by $\bar{K}$ *has the following properties for each* $\alpha \in [0, 2m]$:

1. $T_t\mathfrak{h}_\alpha \subseteq \mathfrak{h}_\alpha, \quad t \geqq 0$,
2. $\|T_t a\|_\alpha \leqq e^{t\alpha^2 k}\|a\|_\alpha, \quad t \geqq 0, a \in \mathfrak{h}_\alpha$,
3. $T|_{\mathfrak{h}_\alpha}$ *is* $\|\cdot\|_\alpha$*-continuous.*

Proof. Now one uses the multiple commutator conditions, and Observation 2.4, to deduce that

$$|(a, (\text{ad}(I + H^2)^m)^2(K)b)| \leqq k\|a\|_m \cdot \|b\|_m, \quad a, b \in \mathfrak{h}_\infty,$$

for a suitable $k \geqq 0$. Then the proof follows the line of reasoning used to prove Theorems 2.10 and 2.16.

3. Lie groups. In this section we extend the previous results to operators approximately invariant under the unitary action of a Lie group.

Let U be a strongly continuous unitary representation of the Lie group G on the Hilbert space $\mathscr{H}$ and for each positive integer n let $\mathscr{H}_n$ denote the C^n-elements of U, i.e, those $a \in \mathscr{H}$ for which the function

$g \in G \mapsto U(g)a \in \mathscr{H}$ is n-times continuously differentiable on G. Furthermore let $\mathscr{H}_\infty$ denote the C^∞-elements of U. Then

$$\mathscr{H}_\infty = \bigcap_{n \geqq 1} \mathscr{H}_n.$$

Next if $\mathfrak{g}$ is the Lie algebra of G there is a representation dU of $\mathfrak{g}$ on $\mathscr{H}_\infty$ such that the representative $dU(X)$ of $X \in \mathfrak{g}$ is the infinitesimal generator of the one-parameter group $t \mapsto U(e^{tX})$. The representation dU extends uniquely to a representation of the universal enveloping algebra of the complexification of $\mathfrak{g}$ and it can be used to give several operator-theoretic characterizations of the subspaces $\mathscr{H}_n$.

First if $X_1, \ldots, X_d$ is an arbitrary basis of $\mathfrak{g}$ then

$$\mathscr{H}_n = \bigcap_{|\alpha| = n} D(dU(X_1)^{\alpha_1} \ldots dU(X_d)^{\alpha_d})$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \geqq 0$, and $|\alpha| = a_1 + \ldots + \alpha_d$ (see, for example, [**11**] Proposition 1.1). Moreover if $\rho_0(a) = \|a\|$ and

$$\rho_n(a) = \max_{1 \leqq j_i \leqq d} \|dU(X_{j_1}) \ldots dU(X_{j_n})a\|, \quad n \geqq 1,$$

then $\mathscr{H}_n$ is a Banach space with respect to the norm

$$\|a\|_n' = \sum_{m=0}^{n} \rho_m(a)$$

(see [**11**] Corollary 1.1). In addition $\mathscr{H}_\infty$ is a Fréchet space with respect to the family of norms $\{ \|\cdot\|_n'; n \geqq 0\}$.

An alternative description of the $\mathscr{H}_n$ can be given in terms of the Laplacian Δ determined by the basis $X_1, \ldots, X_d$. The operator Δ is defined as the closure of the sum

$$-\textstyle\sum_{i=1}^{d} (dU(X_i))^2.$$

It is both self-adjoint and positive. Then it follows from [**18**] that

$$\mathscr{H}_n = D((I + \Delta)^{n/2})$$

and the norm $\|\cdot\|_n'$ on $\mathscr{H}_n$ is equivalent to the norm

$$a \mapsto \|a\|_n = \|(I + \Delta)^{n/2}a\|$$

(see [**11**] Proposition 1.3). Hence for $\alpha \geqq 0$ we introduce the fractional spaces

$$\mathscr{H}_\alpha = D((I + \Delta)^{\alpha/2})$$

with the norm

$$\|a\|_\alpha = \|(I + \Delta)^{\alpha/2}a\|$$

and if $\alpha < 0$ we define $\mathscr{H}_\alpha$ to be the completion of $\mathscr{H}$ with respect to the norm

$$\|a\|_\alpha = \|(I + \Delta)^{\alpha/2}a\|.$$

Again each $\mathscr{H}_\alpha$ is a Hilbert space with respect to the norm $\|\cdot\|_\alpha$ and the operators $(I + \Delta)^{\beta/2}$ define unitary maps from $\mathscr{H}_\alpha$ into $\mathscr{H}_{\alpha-\beta}$. Moreover the dual $\mathscr{H}_\alpha^*$ of $\mathscr{H}_\alpha$ is identifiable as $\mathscr{H}_{-\alpha}$.

Now it is straightforward to use this characterization of the differentiable structure of $(\mathscr{H}, G, U)$ in terms of the Laplacian to extend the results of Section 2. Again we consider the three classes of dissipative, symmetric, and positive, operators.

a. *Dissipative operators.* First one has a direct analogue of Theorem 2.1.

THEOREM 3.1. *Let K be a dissipative operator from $\mathscr{H}_\infty$ into $\mathscr{H}$ and suppose that*

$$(3.1) \qquad |(a, (\operatorname{ad} dU(X_i))(K)b)| \leqq k_\alpha \|a\|_\alpha \cdot \|b\|_{1-\alpha}, \quad a, b \in \mathscr{H}_\infty,$$

for some $\alpha \in [0, 1\rangle$ and a $k_\alpha \geqq 0$.

It follows that the closure $\bar{K}$ of K generates a strongly continuous contraction semigroup.

Proof. It follows immediately from (3.1) that

$$|(a, (\operatorname{ad} \Delta)(K)b)| \leqq k_\alpha \|a\|_{1+\alpha} \cdot \|b\|_{1-\alpha} + k_\alpha \|a\|_\alpha \cdot \|b\|_{2-\alpha}$$

for all $a, b \in \mathscr{H}_\infty$. Now the proof is the same as the proof of Theorem 2.1 but with $S_t = \exp\{-t\Delta\}$.

Next we consider the limiting case $\alpha = 0$ in Theorem 3.1.

If $\alpha = 0$ in (3.1) then it follows by continuity, as in the proof of Theorem 2.3, that $K\mathscr{H}_\infty \subseteq \mathscr{H}_1$. It further follows by the proof of this latter theorem that $K + \omega I$ is $\|\cdot\|_1'$-dissipative for $\omega \geqq k_0$. But it is not evident that the semigroup $T_t = \exp\{-t\bar{K}\}$ leaves $\mathscr{H}_1$ invariant. This follows, however, if one has appropriate higher-order commutator estimates.

THEOREM 3.2. *Let K be a dissipative operator from $\mathscr{H}_\infty$ into $\mathscr{H}$ and suppose that*

$$(3.2) \qquad |(a, (\operatorname{ad} dU(e \cdot X))^p(K)b)| \leqq c_p \|a\| \cdot \|b\|_p, \quad a, b \in \mathscr{H}_\infty,$$

for all $e \in \mathbf{R}^d$ with $|e| = 1$, and all $p = 1, 2, \ldots, m$ where $m \geqq 2$ and $c_p \geqq 0$. Then the contraction semigroup T generated by $\bar{K}$ satisfies the following properties for $p = 1, 2, \ldots, m$;

1. $T_t\mathscr{H}_p \subseteq \mathscr{H}_p$

2. *there exists* $\omega_p \geqq 0$ *such that*

$$\|T_t a\|'_p \leqq e^{\omega_p t}\|a\|'_p, \quad t \geqq 0,\ a \in \mathscr{H}_p,$$

3. $T|_{\mathscr{H}_p}$ *is* $\|\cdot\|'_p$*-continuous.*

Proof. The proof follows the reasoning used in Theorems 2.3 and 2.17. We will just outline the argument.

First it follows easily from (3.2) that $K\mathscr{H}_\infty \subseteq \mathscr{H}_m$. Then setting $H_i = dU(X_i)$ one has for $a \in \mathscr{H}_\infty$

$$\|H_{i_1} \dots H_{i_p}(I + \epsilon(K + \omega I))a\| \geqq (1 + \epsilon\omega)\|H_{i_1} \dots H_{i_p}a\| \\ -\epsilon\|(\text{ad } H_{i_1} \dots H_{i_p})(K)a\|$$

by dissipativity of K. Then by (3.2), and linear algebra there is an $\omega_p \geqq 0$ such that

$$\|(\text{ad } H_{i_1} \dots H_{i_p})(K)a\| \leqq \omega_p\|a\|'_p.$$

Thus if $\omega \geqq \omega_p$ it follows that

$$\|(I + \epsilon(K + \omega I))a\|'_p \geqq \|a\|'_p,$$

i.e., $K + \omega I$ is $\|\cdot\|'_p$-dissipative. Next one proves that $R(I + \epsilon(K + \omega I))$ is $\|\cdot\|'_p$-dense in $\mathscr{H}_p$.

Suppose there is an $f \in \mathscr{H}_p$ such that

$$f((I + \epsilon(K + \omega_p I))a) = 0, \quad a \in \mathscr{H}_\infty.$$

Then if p is even set

$$R_p = (I + \Delta)^{-p/2},$$

and if p is odd set

$$R_p = (I + \Delta)^{-(p+1)/2}.$$

Thus $R_p\mathscr{H} = \mathscr{H}_p$, or $\mathscr{H}_{p+1}$. In both cases to prove $f = 0$ it suffices to prove $R_p^* f = 0$ because the range of R_p is $\|\cdot\|_p$-dense in $\mathscr{H}_p$. But the proof that $R_p^* f = 0$ is similar to the proof of the comparable property in Theorem 2.17.

One concludes from these observations that the $\|\cdot\|'_p$-closure of K generates a semigroup with the properties stated in the theorem, and this semigroup must be the restriction of T to $\mathscr{H}_p$.

b. *Symmetric operators.* Again there are two results based on the weak and strong form of the commutator bounds of index one.

THEOREM 3.3. *Let* K *be a symmetric operator from* $\mathscr{H}_\infty$ *into* $\mathscr{H}$ *and suppose that*

(3.3) $\quad |(a, (\text{ad } dU(X_i))(K)b)| \leqq k\|a\|_{1/2} \cdot \|b\|_{1/2}, \quad a, b \in \mathscr{H}_\infty,$

for some $k \geqq 0$, *and all* $i = 1, 2, \ldots, d$.
It follows that K *is essentially self-adjoint.*

This follows by applying Theorem 3.1, with $\alpha = 1/2$, to $\pm iK$.

THEOREM 3.4. *Let* K *be a symmetric operator from* $\mathscr{H}_\infty$ *into* $\mathscr{H}$ *and suppose that*

$$(3.4) \qquad |(a, (\mathrm{ad}\, dU(X_i))(K)b)| \leqq k\|a\| \cdot \|b\|_1, \quad a, b \in \mathscr{H}_\infty,$$

for some $k \geqq 0$ *and all* $i = 1, 2, \ldots, d$. *Then the unitary group* $V_t = \exp\{it\bar{K}\}$ *has the following properties for each* $\alpha \in [0, 1]$;

1. $V_t\mathscr{H}_\alpha = \mathscr{H}_\alpha, \quad t \in \mathbf{R}$,
2. $\|V_ta\|_\alpha \leqq e^{|t|\alpha kd}\|a\|_\alpha, \quad t \in \mathbf{R}, a \in \mathscr{H}_\alpha$
3. $V|_{\mathscr{H}_\alpha}$ *is* $\|\cdot\|_\alpha$*-continuous.*

Proof. Set $H_i = dU(X_i)$. The proof is based on the observation that

$$\begin{aligned}
&|(a, (\mathrm{ad}\, \Delta)(K)b)| \\
&\leqq \sum_{i=1}^d \{ |(H_ia, (\mathrm{ad}\, H_i)(K)b)| + |((\mathrm{ad}\, H_i)(K)a, H_ib)| \} \\
&\leqq \sum_{i=1}^d k\|H_ia\| \cdot \|b\|_1 + k\|a\|_1 \cdot \|H_ib\| \\
&\leqq 2kd\|a\|_1 \cdot \|b\|_1.
\end{aligned}$$

Now the result follows from Theorem 2.5 with H replaced by Δ (see in addition the proof of Theorem 2.10).

c. *Positive operators.* The extension of the results of Section 2c is a little less straightforward because for non-abelian G one also needs bounds on single commutators.

THEOREM 3.5. *Let* K *be a positive operator from* $\mathscr{H}_\infty$ *into* $\mathscr{H}$ *and suppose that*

$$(3.5) \qquad |(a, (\mathrm{ad}\, dU(X_i)(K)b)| \leqq k_1\|a\|_1 \cdot \|b\|_1,$$

$$(3.6) \qquad |(a, (\mathrm{ad}\, dU(X_i))(\mathrm{ad}\, dU(X_j))(K)b)| \leqq k_2\|a\|_1 \cdot \|b\|_1, \quad a, b \in \mathscr{H}_\infty,$$

for some $k_1, k_2 \geqq 0$ *for all* $i, j = 1, 2, \ldots, d$.
It follows that K *is essentially self-adjoint.*

Proof. This follows from the proof of the first statement in Theorem 2.12 once one replaces S by the semigroup $S_t = \exp\{-t\Delta\}$. But for this one needs an estimate on $(\mathrm{ad}\, \Delta)^2(K)$. Now setting $H_i = dU(X)$

$$(\text{ad } \Delta)^2(K) = \sum_{i=1}^{d} \sum_{j=1}^{d} (\text{ad } H_i^2)(\text{ad } H_j^2)(K)$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \{H_i(\text{ad } H_i)(H_j(\text{ad } H_j)(K) + (\text{ad } H_j)(K)H_j)$$

$$+ (\text{ad } H_i)(H_j(\text{ad } H_j)(K) + (\text{ad } H_j)(K)H_j)H_i\}.$$

Now using the structure relations for $\mathfrak{g}$ one can re-express the summand as a linear combination of terms $H^m(\text{ad } H)^n(K)H^{2-m}$ where $n = 2$ and $m = 0, 1, 2$, or $n = 1$ and $m = 1$. Here H represents an H_i, H^2 a product H_iH_j, $(\text{ad } H)(K)$ represents $(\text{ad } H_i)(K)$ etc. Hence from (3.5) and (3.6) one obtains a bound

$$|(a, (\text{ad } \Delta)^2(K)b)| \leqq C_0\|a\|_3 \cdot \|b\|_1 + C_1\|a\|_2 \cdot \|b\|_2 + C_2\|a\|_1 \cdot \|b\|_3.$$

This suffices for the argument used to prove Theorem 2.12.

Remark 3.6. If G is abelian then the only terms occurring in the calculation of $(\text{ad } \Delta)^2(K)$ are of the form $H^m(\text{ad } H)^2(K)H^{2-m}$. No terms $H(\text{ad } H)(K)H$ occur. Hence the bound (3.5) is unnecessary for abelian G.

Alternatively if one has a bound

$$|(a, Kb)| \leqq k_0\|a\|_1 \cdot \|b\|_1, \quad a, b \in \mathscr{H}_\infty,$$

then (3.5) follows from (3.6) by Lemma 2.11.

COROLLARY 3.7. *Let K be a positive operator from $\mathscr{H}_\infty$ into $\mathscr{H}$ and suppose that*

$$|(a, Kb)| \leqq k_0\|a\|_1 \cdot \|b\|_1,$$

$$|(a, (\text{ad } dU(X_i))(\text{ad } dU(X_j))(K)b)| \leqq k_2\|a\|_1 \cdot \|b\|_1, \quad a, b \in \mathscr{H}_\infty,$$

for some $k_0, k_2 \geqq 0$ and all $i, j = 1, 2, \ldots, d$.

It follows that K is essentially self-adjoint.

Finally one has generalization of Theorem 2.16.

THEOREM 3.8. *Let K be a positive operator from $\mathscr{H}_\infty$ into $\mathscr{H}$ and suppose that*

(3.7) $$|(a, (\text{ad } dU(X_i))(K)b)| \leqq k_1\|a\|_1 \cdot \|b\|_1,$$

(3.8) $$|(a, (\text{ad } dU(X_i))(\text{ad } dU(X_j))(K)b)| \leqq k_2\|a\| \cdot \|b\|_2, \quad a, b \in \mathscr{H}_\infty,$$

for some $k_1, k_2 \geqq 0$ and all $i, j = 1, 2, \ldots, d$. Then for each $\alpha \in [0, 2]$ the contraction semigroup $T_t = \exp\{-t\bar{K}\}$ has the following properties;

1. $T_t\mathscr{H}_\alpha \subseteq \mathscr{H}_\alpha \quad t \geqq 0,$

2. *there is an* ω (*independent of* α) *such that*

$$\|T_t a\|_\alpha \leqq e^{t\alpha^2\omega}\|a\|_\alpha \quad t \geqq 0, a \in \mathscr{H}_\alpha$$

3. $T|_{\mathscr{H}_\alpha}$ *is* $\|\cdot\|_\alpha$-*continuous.*

Proof. The proof is based upon the reasoning used to establish Theorem 2.16 but with H^2 replaced by Δ. Again it is necessary to have appropriate estimates on $(\mathrm{ad}\ \Delta)^2(K)$. But by Observation 2.4 and (3.8) one has

(3.9) $$|(a, (\mathrm{ad}\ dU(X_i))(\mathrm{ad}\ dU)(X_j))(K)b)| \leqq k_2\|a\|_\beta \cdot \|b\|_{2-\beta}, a, b \in \mathscr{H}_\infty,$$

for all $\beta \in [0, 2]$. Now estimating $|(a, (\mathrm{ad}\ \Delta)^2(K)b)|$ as in the proof of Theorem 3.5 but using (3.7) and (3.9), with $\beta = 0, 1, 2$, one finds a bound

$$|(a, (\mathrm{ad}\ \Delta)^2(K)b)| \leqq k\|a\|_2 \cdot \|b\|_2 \quad a, b \in \mathscr{H}_\infty,$$

for some $k \geqq 0$. Now one can apply the arguments used in the proof of Theorem 2.10 with H replaced by Δ.

Remark 3.9. If G is abelian (3.7) is unnecessary (see Remark 3.6).

COROLLARY 3.10. *Let K be a positive operator from $\mathscr{H}_\infty$ into $\mathscr{H}$ and suppose that*

(3.9) $$|(a, Kb)| \leqq k_0\|a\|_1 \cdot \|b\|_1,$$
$$|(a, (\mathrm{ad}\ dU(X_i))(\mathrm{ad}\ dU(X_j))(K)b)| \leqq k_2\|a\| \cdot \|b\|_2, \quad a, b \in \mathscr{H}_\infty,$$

for some $k_0, k_2 \geqq 0$ and all $i, j = 1, 2, \ldots, d$. Then the conclusions of Theorem 3.8 *are valid.*

The corollary follows because (3.7) follows from (3.9), (3.10), and Observation 2.4.

4. Commutation theorems. In this section we return to the context of Section 2 and examine pairs of symmetric operators K_1, K_2 satisfying the criteria of essential self-adjointness given in Theorems 2.5, 2.8, and 2.16. Our aim is to show that if K_1 and K_2 formally commute, i.e., if $(\mathrm{ad}\ K_1)(K_2) = 0$ as a form on $\mathfrak{h}_\infty \times \mathfrak{h}_\infty$, then $\bar{K}_1$ and $\bar{K}_2$ commute, i.e., all bounded functions of $\bar{K}_1$ and $\bar{K}_2$ commute. But to establish this result we need to impose form bounds

$$|(a, K_i b)| \leqq k\|a\|_p \cdot \|b\|_q, \quad a, b \in \mathfrak{h}_\infty$$

on the K_i. Therefore it is natural to slightly reformulate our basic hypotheses in terms of forms. Then, following **[8]**, we can use the form bounds combined with commutator bounds to prove the existence of operators. This is particularly convenient in applications. Thus we now assume that k, k_1, k_2 etc. are sesquilinear forms over $\mathcal{D} \times \mathcal{D}$ where $\mathcal{D} \subseteq \mathfrak{h}_\infty$ is a core of H and $H\mathcal{D} \subseteq \mathcal{D}$. Then to emphasize that we are dealing with forms we now explicitly define the commutators as forms, e.g.,

$$(\text{ad } H)(k)(a, b) = k(Ha, b) - k(a, Hb), \quad a, b \in \mathcal{D}.$$

In general a sesquilinear form does not determine an operator, unless it is closable, but Glimm and Jaffe [**8**] established that it does if one has a weak commutator bound of index one. We will demonstrate a similar result using a double commutator bound of index two. But first we give a slight generalization of the Glimm-Jaffe theorem.

PROPOSITION 4.1. *Let* $H \geqq 0$ *and consider a sesquilinear form* k *over* $\mathcal{D} \times \mathcal{D}$ *satisfying*

$$(4.1) \qquad |(\text{ad } H)(k)(a, b)| \leqq k_1 \|a\|_{1/2} \cdot \|b\|_{1/2}, \quad a, b \in \mathcal{D},$$

for some $k_1 \geqq 0$. *Then for each* $\alpha \geqq 0$ *the following conditions are equivalent*:

1. $|k(a, b)| \leqq k_0' \|a\|_\alpha \cdot \|b\|_\alpha, \quad a, b \in \mathcal{D},$
2. $|k(a, b)| \leqq k_0'' \|a\| \cdot \|b\|_{2\alpha}, \quad a, b \in \mathcal{D}.$

Moreover, if these conditions are satisfied then k *determines a unique linear operator* K *such that* $D(K) = \mathfrak{h}_{2\alpha}$ *and*

$$k(a, b) = (a, Kb), \quad a, b \in \mathcal{D}.$$

Proof. Both Conditions 1 and 2 can be extended to $\mathfrak{h}_\infty \times \mathfrak{h}_\infty$ by continuity. Then setting $R = (I + H)^{-1}$ Condition 1 can be reformulated as

$$|k(R^\alpha a, R^\alpha b)| \leqq k_0' \|a\| \cdot \|b\|, \quad a, b \in \mathfrak{h}_\infty$$

and Condition 2 as

$$|k(a, R^{2\alpha} b)| \leqq k_0'' \|a\| \cdot \|b\|, \quad a, b \in \mathfrak{h}_\infty.$$

But if $\alpha = n + \beta$ with n integer and $0 \leqq \beta < 1$ then

$$k(R^\alpha a, R^\alpha b) - k(a, R^{2\alpha} b) = \sum_{m=1}^{n} (\text{ad } H)(k)(R^{\alpha - m + 1} a, R^{\alpha + m} b) + (\text{ad}(I + H)^\beta)(k)(R^\beta a, R^{2\alpha} b)$$

where the sum is absent if $n = 0$, i.e., if $\alpha < 1$. Hence it follows from (4.1) and Lemma 2.6 that

$$|k(R^\alpha a, R^\alpha b) - k(a, R^{2\alpha} b)| \leqq k_1 \sum_{m=1}^{n} \|R^{\alpha-m+1}a\|_{1/2} \cdot \|R^{m+\alpha}a\|_{1/2} + k_1'\|R^\beta a\|_{\beta/2} \cdot \|R^{2\alpha}b\|_{\beta/2} \leqq (nk_1 + k_1')\|a\| \cdot \|b\|$$

for some $k_1' \geqq 0$. Therefore $1 \Leftrightarrow 2$.

Finally if Condition 2 is satisfied then there exists a bounded operator K_α such that

$$(a, K_\alpha b) = k(a, R^{2\alpha} b) \quad \text{for all } a, b \in \mathfrak{h}_\infty.$$

Then one can define K on $\mathfrak{h}_{2\alpha}$ by $K = K_\alpha(I + H)^{2\alpha}$.

Next we establish an analogue of this result based on a weak double commutator bound of index two.

PROPOSITION 4.2. *Let $H \geqq 0$ and consider a symmetric sesquilinear form k over $\mathcal{D} \times \mathcal{D}$ satisfying*

$$(4.2) \quad |(\text{ad } H)^2(k)(a, b)| \leqq k_2\|a\|_1 \cdot \|b\|_1, \quad a, b \in \mathcal{D},$$

for some $k_2 \geqq 0$. Then for each $\alpha \in [0, 1]$ the following conditions are equivalent

1. $|k(a, b)| \leqq k_0'\|a\|_\alpha \cdot \|b\|_\alpha \quad a, b \in \mathcal{D}$
2. $|k(a, b)| \leqq k_0''\|a\| \cdot \|b\|_{2\alpha} \quad a, b, \in \mathcal{D}$.

Moreover, if these conditions are satisfied then k determines a unique linear operator K such that $D(K) = \mathfrak{h}_{2\alpha}$ and $k(a, b) = (a, Kb)$, $a, b \in \mathcal{D}$, and

$$(4.3) \quad |(a, (\text{ad } (I + H)^\alpha)(K)b)| \leqq k_1\|a\| \cdot \|b\|_{2\alpha}, \quad a, b \in \mathcal{D},$$

for some $k_1 \geqq 0$.

Proof. $2 \Rightarrow 1$. Since k is symmetric this follows from Observation 2.4 without use of (4.2)

$1 \Rightarrow 2$. First we prove a weaker form of (4.3)

$$(4.4) \quad |(\text{ad } (I + H)^\alpha)^2(k)(a, b)| \leqq k_1'\|a\|_\alpha \cdot \|b\|_\alpha \quad a, b \in \mathcal{D},$$

by exploiting Condition 1 and (4.2).

Since $\alpha \in [0, 1]$ it follows from Lemma 2.12 and (4.2) that

$$(4.5) \quad |(\text{ad } (I + H)^\alpha)^2(k)(a, b)| \leqq \alpha^2 k_2\|a\|_\alpha' \cdot \|b\|_\alpha' \quad a, b \in \mathcal{D}.$$

Next we repeat the proof of Lemma 2.11 with U_t replaced by

$$U_t^\alpha = \exp\{it(I + H)^\alpha\}$$

and using the form k in place of the operator K. Thus for $s, t > 0$ we define sesquilinear forms

$$F_{s,t}(a, b; k) = \sum_{p \geqq 0} e^{-s/t}\frac{(s/t)^p}{p!} k(U^{\alpha}_{-pt}a, U^{\alpha}_{-pt}b)$$

where it is understood that k has been extended to $\mathfrak{h}_\alpha \times \mathfrak{h}_\alpha$. Then from Condition 1 one has

$$|F_{s,t}(a, b; k)| \leqq k'_0\|a\|_\alpha \cdot \|b\|_\alpha.$$

But using (4.5) one estimates as in the proof of Lemma 2.11 that

$$\left|\frac{d^2}{ds^2}F_{s,t}(a, b; k)\right| \leqq \alpha^2 k_2\|a\|'_\alpha \cdot \|b\|'_\alpha \quad a, b \in \mathfrak{h}_\alpha.$$

Therefore one deduces from the Taylor series expansion of $F_{s,t}$ that

$$|k(U^{\alpha}_{-t}a, U^{\alpha}_{-t}b) - k(a, b)|/t \leqq k'_1\|a\|_\alpha \cdot \|b\|_\alpha, \quad a, b \in \mathfrak{h}_\alpha$$

for a suitable $k'_1 \geqq 0$ and all $t \geqq 0$. Then (4.4) follows by taking the limit $t \to 0$.

Next with $R = (I + H)^{-1}$ one has

$$(4.6) \quad |k(a, R^{2\alpha}b) - k(R^\alpha a, R^\alpha b)| \leqq |(\text{ad } (I + H)^\alpha)(k)(R^\alpha a, R^{2\alpha}b)| \leqq k'_1\|a\| \cdot \|b\|$$

by (4.4). This establishes that $1 \Leftrightarrow 2$.

But replacing k by $(\text{ad } (I + H)^\alpha)(k)$ in (4.6) and using (4.5) one concludes that (4.3) and (4.4) are equivalent.

Finally if Condition 2 is satisfied then there is a bounded operator K such that

$$(a, K_\alpha b) = k(a, R^{2\alpha}b) \quad \text{for all } a, b \in \mathfrak{h}_\infty$$

and one can define K on $\mathfrak{h}_{2\alpha}$ by $K = K_\alpha(I + H)^{2\alpha}$.

One can immediately deduce versions of Propositions 4.1 and 4.2 without the assumption $H \geqq 0$.

COROLLARY 4.3. *Let k by a symmetric sesquilinear form over $\mathscr{D} \times \mathscr{D}$ and suppose that*

$$(4.7) \quad |(\text{ad } H)(k)(a, b)| \leqq k_1\|a\| \cdot \|b\|_1, \quad a, b \in \mathscr{D},$$

for some $k_1 \geqq 0$. Then for each $\alpha \geqq 0$ the following conditions are equivalent;

1. $|k(a, b)| \leqq k'_0\|a\|_\alpha \cdot \|b\|_\alpha \quad a, b \in \mathscr{D},$
2. $|k(a, b)| \leqq k''_0\|a\| \cdot \|b\|_{2\alpha} \quad a, b \in \mathscr{D}.$

Moreover, if these conditions are satisfied then k determines a unique operator K.

Proof. It follows from (4.7) that

$$|(\text{ad } H^2)(k)(a, b)| \leqq 2k_1\|a\|_1 \cdot \|b\|_1.$$

Therefore the hypotheses of Proposition 4.1 are fulfilled with respect to the positive operator H^2. (The norm $\|\cdot\|_\alpha$ with respect to H corresponds to the norm $\|\cdot\|_{\alpha/2}$ with respect to H^2). The result follows immediately.

COROLLARY 4.4. *Let k be a symmetric sesquilinear form over* $\mathscr{D} \times \mathscr{D}$ *and suppose that*

$$(4.8) \quad |(\text{ad } H)^2(k)(a, b)| \leqq k_2\|a\| \cdot \|b\|_2, \quad a, b \in \mathscr{D},$$

for some $k \geqq 0$. *Then for each* $\alpha \in [0, 2]$ *the following conditions are equivalent*;

1. $|k(a, b)| \leqq k_0'\|a\|_\alpha \cdot \|b\|_\alpha \quad a, b \in \mathscr{D}$,
2. $|k(a, b)| \leqq k_0''\|a\| \cdot \|b\|_{2\alpha} \quad a, b \in \mathscr{D}$.

Moreover, if these conditions are satisfied then k determines an operator K and

$$(4.9) \quad |(a, (\text{ad } (I + H^2)^{\alpha/2})(K)b)| \leqq k_1\|a\| \cdot \|b\|_{2\alpha}, \quad a, b \in \mathscr{D},$$

for some $k_1 \geqq 0$.

Proof. It follows from (4.8) and Observation 2.4 that

$$|(\text{ad } H)^2(k)(a, b)| \leqq k_2\|a\|_p \cdot \|b\|_{2-p}, \quad a, b \in \mathscr{D},$$

for $p = 0, 1, 2$. Therefore

$$|(\text{ad } H^2)^2(k)(a, b)| \leqq 4k_2\|a\|_2 \cdot \|b\|_2, \quad a, b \in \mathscr{D},$$

and the corollary follows from Proposition 4.2, with H replaced by H^2.

After these preliminaries on form bounds we now state and prove the commutation theorems for operators satisfying the criteria of the theorems in Section 2. Again Glimm and Jaffe **[8]** **[9]** were the first to prove results of this nature but subsequent generalizations of their results were given by Driessler and Fröhlich **[3]** **[6]**. These papers all deal with the commutation of two operators satisfying the conditions of Theorem 2.8, but a partial result was given by Driessler and Summers **[4]** for one operator satisfying the conditions of Theorem 2.8 and the second satisfying the conditions of Theorem 2.16. The next theorem gives a complete result for all such operators.

THEOREM 4.5. *Let* K_1, K_2 *be symmetric operators from* $\mathfrak{h}_\infty$ *into* $\mathfrak{h}$ *and suppose they each satisfy one of the following two conditions*;

1. $$\begin{cases} |(a, Kb)| \leqq k_0\|a\|_{1/2} \cdot \|b\|_{1/2} \\ |(a, (\text{ad } H)(K)b)| \leqq k_1\|a\| \cdot \|b\|_1 \quad a, b \in \mathfrak{h}_\infty, \end{cases}$$

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2. $$\begin{cases} K \geqq 0 \\ |(a, Kb)| \leqq k_0\|a\|_1 \cdot \|b\|_1 \\ |(a, (\text{ad } H)^2(K)b)| \leqq k_2\|a\| \cdot \|b\|_2 \quad a, b \in \mathfrak{h}_\infty. \end{cases}$$

Further assume that

(4.9) $(K_1a, K_2b) = (K_2a, K_1b) \quad a, b \in \mathfrak{h}_\infty.$

It follows that the self-adjoint closures $\bar{K}_1$, *and* $\bar{K}_2$, *commute.*

Proof. The proof divides into three cases. Case 1; K_1 and K_2 both satisfy Condition 1. Case 2; K_1 and K_2 both satisfy Condition 2. Case 3; K_1 satisfies Condition 1 and K_2 satisfies Condition 2.

Case 1. First it follows from Corollary 4.3 that

$$|(a, K_ib)| \leqq k_0''\|a\| \cdot \|b\|_1, \quad a, b \in \mathfrak{h}_\infty,$$

for some $k_0'' \geqq 0$. Thus $\mathfrak{h}_1 \subseteq D(\bar{K}_i)$ and

$$\|\bar{K}_ib\| \leqq k_0''\|b\|_1, \quad b \in \mathfrak{h}_1,$$

by continuity. Therefore one deduces from (4.9) that

$$(\bar{K}_1a, \bar{K}_2b) = (\bar{K}_2a, \bar{K}_1b), \quad a, b \in \mathfrak{h}_1.$$

Next let

$$V_t^j = \exp\{itK_j\}, \quad j = 1, 2,$$

then $V_t^j\mathfrak{h}_1 = \mathfrak{h}_1$ by Theorem 2.8. Hence if $a, b \in \mathfrak{h}_1$ one has

$$\begin{aligned}
&(a, (V_s^1V_t^2 - V_t^2V_s^1)b) \\
&= i\int_0^s du(a, V_u^1(\bar{K}_1V_t^2 - V_t^2\bar{K}_1)V_{s-u}^1b) \\
&= \int_0^s du \int_0^t dv\{ (\bar{K}_2V_{-v}^2V_{-u}^1a, \bar{K}_1V_{t-v}^2V_{s-u}^1b) \\
&\qquad\qquad - (\bar{K}_1V_{-v}^2V_{-u}^1a, \bar{K}_2V_{t-v}^2V_{s-u}^1b) \} \\
&= 0.
\end{aligned}$$

Therefore the unitary groups V^1 and V^2 commute and hence all bounded functions of $\bar{K}_1$ and $\bar{K}_2$ commute (see, for example, [**22**] Chapter VIII).

Case 2. Now it follows from Corollary 4.4 that

$$|(a, K_ib)| \leqq k_0''\|a\| \cdot \|b\|_2, \quad a, b \in \mathfrak{h}_\infty,$$

for some $k_0'' \geqq 0$. Thus $\mathfrak{h}_2 \subseteq D(\bar{K}_i)$ and

$$\|\bar{K}_ib\| \leqq k_0''\|b\|_2, \quad b \in \mathfrak{h}_2,$$

by continuity. Therefore one deduces from (4.9) that

$$(\bar{K}_1 a, \bar{K}_2 b) = (\bar{K}_2 a, \bar{K}_1 b), \quad a, b \in \mathfrak{h}_2.$$

Next let

$$T_t^i = \exp\{-t\bar{K}_i\}, \quad i = 1, 2,$$

then $T_t^i \mathfrak{h}_2 \subseteq \mathfrak{h}_2$ by Theorem 2.16. Hence one can compute as in Case 1 and conclude that the semigroups T^1 and T^2 commute. But then all bounded functions of $\bar{K}_1$ and $\bar{K}_2$ commute.

Case 3. It follows as before that $\mathfrak{h}_2 \subseteq \mathfrak{h}_1 \subseteq D(\bar{K}_1)$, $\mathfrak{h}_2 \subseteq D(\bar{K}_2)$, and

$$\|\bar{K}_1 b\| \leqq k_0' \|b\|_1 \leqq k_0' \|b\|_2,$$
$$\|\bar{K}_2 b\| \leqq k_0'' \|b\|_2,$$

for some $k_0', k_0'' \geqq 0$ and all $b \in \mathfrak{h}_2$. Thus one again deduces from (4.9) that

$$(\bar{K}_1 a, \bar{K}_2 b) = (\bar{K}_2 a, \bar{K}_1 b), \quad a, b \in \mathfrak{h}_2.$$

Next for $\alpha > 0$ define K_α by

$$K_\alpha = \frac{1}{\alpha} \int_0^\alpha dt U_t K U_{-t}$$

where $U_t = \exp\{itH\}$ and $K = K_1$. Then a simple computation establishes that K_α satisfies Condition 1.

In particular $\bar{K}_\alpha$ is self-adjoint, by Theorem 2.8, and $\mathfrak{h}_2 \subseteq \mathfrak{h}_1 \subseteq D(\bar{K}_\alpha)$ by Corollary 4.3 and the previous reasoning. But one also has

$$(\text{ad } H)^2(K_\alpha) = (i\alpha)^{-1}\{U_\alpha(\text{ad } H)(K)U_{-\alpha} - (\text{ad } H)(K)\}.$$

Therefore

$$|(a, (\text{ad } H)^2(K_\alpha)b)| \leqq (2/\alpha)k_1\|a\| \cdot \|b\|_1, \quad a, b \in \mathfrak{h}_\infty.$$

Thus if $V_t^\alpha = \exp\{it\bar{K}_\alpha\}$ then $V_t^\alpha \mathfrak{h}_2 = \mathfrak{h}_2$ by Theorem 2.18. But it also follows from Theorem 2.8 that $V_t^\alpha \mathfrak{h}_1 = \mathfrak{h}_1$ and

$$\|V_t^\alpha a\|_1 \leqq e^{|t|k_1}\|a\|_1, \quad a \in \mathfrak{h}_1. \tag{4.10}$$

Next setting $T_t = \exp\{-t\bar{K}_2\}$ one calculates as in Case 1 above that

$$(a, (V_s^\alpha T_t - T_t V_s^\alpha)b)$$
$$= i \int_0^s du \int_0^t dv \{ (\bar{K}_2 T_v V_{-u}^\alpha a, \bar{K}_\alpha T_{t-v} V_{s-u}^\alpha b)$$
$$- (\bar{K}_\alpha T_v V_{-u}^\alpha a, \bar{K}_2 T_{t-v} V_{s-u}^\alpha b) \}$$

for all $a, b \in \mathfrak{h}_2$. But if $\bar{K}_\alpha$ is replaced by $\bar{K}$ the integrand is zero, by commutation of K and K_2. Therefore

(4.11) $|(a, (V_s^\alpha T_t - T_t V_s^\alpha)b)|$

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$$\leqq \int_0^s du \int_0^t dv \{ \|\bar{K}_2 T_v V^\alpha_{-u} a\| \cdot \|(\bar{K}_\alpha - \bar{K})T_{t-v} V^\alpha_{s-u} b\| + \|(\bar{K}_\alpha - \bar{K})T_v V^\alpha_{-u} a\| \cdot \|\bar{K}_2 T_{t-v} V^\alpha_{s-u} b\| \}.$$

Now

$$\|\bar{K}_2 T_v V^\alpha_{-u} a\|^2 \leqq (V^\alpha_{-u} a, T_{v/2}\bar{K}_2 T_{v/2} V^\alpha_{-u} a)\, \|\bar{K}_2 T_v\| \leqq k_0 \|T_{v/2} V^\alpha_{-u} a\|_1^2 c_1 v^{-1},$$

by Condition 2 and spectral theory, where

$$c_1 = \sup_{\lambda \geqq 0} \lambda e^{-\lambda}.$$

Moreover

$$\|T_{v/2} V^\alpha_{-u} a\|_1 \leqq e^{vk_2/4} \|V^\alpha_{-u} a\|_1 \leqq e^{vk_2/4} e^{|u|k_1} \|a\|_1$$

by Theorem 2.16 and (4.10). Thus combination of these estimates establishes there is a $c \geqq 0$ such that

$$(4.12) \quad \|\bar{K}_2 T_v V^\alpha_{-u} a\| \leqq c v^{-1/2} \|a\|_1, \quad a \in \mathfrak{h}_2,$$

for all $|u| \leqq 1$, and $0 \leqq v \leqq 1$.

Next note that

$$\|(U_t K U_{-t} - K)a\| \leqq \int_0^{|t|} ds \|(\text{ad } H)(K) U_{-s} a\| \leqq |t| k_1 \|a\|_1, \quad a \in \mathfrak{h}_\infty$$

by Condition 1. Therefore

$$\|(K_\alpha - K)a\| \leqq \alpha^{-1} \int_0^\alpha dt \|(U_t K U_{-t} - K)a\| \leqq (\alpha/2) k_1 \|a\|_1, \quad a \in \mathfrak{h}_\infty.$$

Consequently

$$\|(\bar{K}_\alpha - \bar{K}) T_v V^\alpha_{-u} a\| \leqq (\alpha/2) k_1 \|T_v V^\alpha_{-u} a\|_1$$

and arguing as above one concludes there is a $c' \geqq 0$ such that

$$(4.13) \quad \|(\bar{K}_\alpha - \bar{K}) T_v V^\alpha_{-u} a\| \leqq \alpha c' \|a\|_1, \quad a \in \mathfrak{h}_2,$$

for all $|u| \leqq 1$ and $0 \leqq v \leqq 1$.

Combination of (4.11), (4.12), and (4.13), allows us to conclude that

$$(4.14) \quad |(a, (V^\alpha_s T_t - T_t V^\alpha_s) b)| \leqq 4\alpha c c' \|a\|_1 \cdot \|b\|_1$$

for all $a, b \in \mathfrak{h}_2$ and $|s| \leqq 1$, $0 \leqq t \leqq 1$.

Finally for $a \in \mathfrak{h}_\infty$

$$\|(K_\alpha - K)a\| \leqq \alpha^{-1} \int_0^\alpha dt \{ \|(U_t - I)Ka\| + \|K(U_{-t} - I)a\| \}$$

$$\leqq \alpha^{-1} \int_0^\alpha dt\{ \| (U_t - I)Ka\| + k_0' \| (U_{-t} - I)a\|_1\}$$

where we have used Condition 1 and Corollary 4.3. Thus

$$\lim_{\alpha \to 0} \| (K_\alpha - K)a\| = 0, \quad a \in \mathfrak{h}_\infty$$

and hence

$$V_t^\alpha \to V_t = \exp\{it\bar{K}\} \quad \text{as } \alpha \to 0$$

(see, for example, [**2**] Section 3.1.3). Therefore

$$(a, (V_s T_t - T_t V_s)b) = 0, \quad a, b \in \mathfrak{h}_2$$

for $|s| \leqq 1$ and $0 \leqq t \leqq 1$, by (4.14). But it then follows from boundedness, density, and the group property, that V and T commute. Thus $\bar{K}_1$ and $\bar{K}_2$ commute.

If $H \geqq 0$ and one considers operators K_1, K_2 satisfying weak commutator bounds of the type occurring in Theorem 2.5 and 2.10 the commutation properties are less evident. But one has the following partial analogue of Theorem 4.5.

THEOREM 4.6. *Let $H \geqq 0$ and consider two symmetric operators K_1, K_2 from $\mathfrak{h}_\infty$ into $\mathfrak{h}$ satisfying*

$$(4.15) \quad \begin{cases} |(a, K_i b)| \leqq k_0 \|a\|_{1/2} \cdot \|b\|_{1/2} \\ |(a, (\operatorname{ad} H)(K_i)b)| \leqq k_1 \|a\|_{1/2} \cdot \|b\|_{1/2}, \quad a, b \in \mathfrak{h}_\infty. \end{cases}$$

Further assume that

$$(4.16) \quad (K_1 a, K_2 b) = (K_2 a, K_1 b), \quad a, b \in \mathfrak{h}_\infty.$$

It follows that the self-adjoint closures $\bar{K}_1$, and $\bar{K}_2$, commute.

Proof. First it follows from Proposition 4.1 that

$$|(a, K_i b)| \leqq k_0'' \|a\| \cdot \|b\|_1, \quad a, b \in \mathfrak{h}_\infty,$$

for some $k_0'' \geqq 0$. Thus $\mathfrak{h}_1 \subseteq D(\bar{K}_i)$ and

$$(4.17) \quad \|\bar{K}_i b\| \leqq k_0'' \|b\|_1, \quad b \in \mathfrak{h}_1,$$

by continuity. Thus one deduces from (4.16) that

$$(4.18) \quad (\bar{K}_1 a, \bar{K}_2 b) = (\bar{K}_2 a, \bar{K}_1 b), \quad a, b \in \mathfrak{h}_1.$$

Next let

$$V_t^j = \exp\{itK_j\}, j = 1, 2.$$

Then $V_t^j \mathfrak{h}_{1/2} = \mathfrak{h}_{1/2}$, by Theorem 2.5 and

$$\|V_t^j a\|_{1/2} \leqq \exp\{\omega |t|\} \|a\|_{1/2}, \quad a \in \mathfrak{h}_{1/2}$$

where $\omega = k_1/4$. The rest of the proof is divided into two parts.

OBSERVATION 4.7. *For $a, b \in \mathfrak{h}_1$ and $t \in \mathbf{R}$*

$$(a, V_t^1 \bar{K}_2 b) = (\bar{K}_2 a, V_t^1 b).$$

Proof. For each $\epsilon > 0$ define

$$\bar{K}_\epsilon = (I + \epsilon H)^{-1} \bar{K}_2 (I + \epsilon H)^{-1}.$$

Then it follows straightforwardly with the aid of (4.17) that

$$\lim_{\epsilon \to 0} \|K_\epsilon a - \bar{K}_2 a\| = 0, \quad a \in \mathfrak{h}_1.$$

Next if $a, b \in \mathfrak{h}_1$ then

$$(a, V_t^1 K_\epsilon b) - (K_\epsilon a, V_t^1 b) = i \int_0^t ds \{ (\bar{K}_1 V_{-s}^1 a, K_\epsilon V_{t-s}^1 b) - (K_\epsilon V_{-s}^1 a, \bar{K}_1 V_{t-s}^1 b) \}.$$

But one also has

$$(\bar{K}_1 (I + \epsilon H)^{-1} V_{-s}^1 a, \bar{K}_2 (I + \epsilon H)^{-1} V_{t-s}^1 b)$$
$$= (\bar{K}_2 (I + \epsilon H)^{-1} V_{-s}^1 a, \bar{K}_1 (I + \epsilon H)^{-1} V_{t-s}^1 b)$$

by (4.18). Therefore

$$(4.19) \quad (a, V_t^1 K_\epsilon b) - (K_\epsilon a, V_t^1 b)$$
$$= i \int_0^t ds \{ ((\mathrm{ad}\,(I + \epsilon H)^{-1})(\bar{K}_1) V_{-s}^1 a, \bar{K}_2 (I + \epsilon H)^{-1} V_{t-s}^1 b) - (\bar{K}_2 (I + \epsilon H)^{-1} V_{-s}^1 a, (\mathrm{ad}\,(I + \epsilon H)^{-1})(\bar{K}_1) V_{t-s}^1 b) \}.$$

Now consider the first term in the integrand. One has

$$|((\mathrm{ad}(I + \epsilon H)^{-1})(\bar{K}_1) V_{-s}^1 a, \bar{K}_2 (I + \epsilon H)^{-1} V_{t-s}^1 b)|$$
$$\leqq \epsilon |((I + \epsilon H)^{-1} (\mathrm{ad}\, H)(\bar{K}_1)(I + \epsilon H)^{-1} V_{-s}^1 a, \bar{K}_2 (I + \epsilon H)^{-1} V_{t-s}^1 b)|$$
$$\leqq \epsilon k_1 \|(I + \epsilon H)^{-1} V_{-s}^1 a\|_{1/2} \cdot \|(I + \epsilon H)^{-1} \bar{K}_2 (I + \epsilon H)^{-1} V_{t-s}^1 b\|_{1/2}$$

where we have used (4.15). But

$$\|(I + \epsilon H)^{-1} V_{-s}^1 a\|_{1/2} \leqq \|V_{-s}^1 a\|_{1/2} \leqq e^{\omega |s|} \|a\|'_{1/2}$$

by Theorem 2.5. On the other hand

$$\epsilon \|(I + \epsilon H)^{-1} \bar{K}_2 (I + \epsilon H)^{-1} V_{t-s}^1 b\|_{1/2}$$
$$\leqq \epsilon^{1/2} \|\bar{K}_2 (I + \epsilon H)^{-1} V_{t-s}^1 b\|$$

$$\leqq \epsilon^{1/2} k_0 \|(I + \epsilon H)^{-1} V^1_{t-s} b\|_1$$

where the last estimate uses (4.15) and Proposition 4.1. But if E denotes the spectral measure of H then

$$\epsilon \|(I + \epsilon H)^{-1} V^1_{t-s} b\|_1^2$$
$$\leqq \int_0^\infty d(V^1_{t-s} b, E(x) V^1_{t-s} b)(1 + x)\epsilon(1 + x)(1 + \epsilon x)^{-1}.$$

Now $x \to \epsilon(1 + x)(1 + \epsilon x)^{-1}$ is uniformly bounded on $[0, \infty\rangle$ and tends pointwise to zero as $\epsilon \to 0$. Moreover

$$\int_0^\infty d(V^1_{t-s} b, E(x) V^1_{t-s} b)(1 + x) = (\|V^1_{t-s} b\|'_{1/2})^2 < +\infty$$

by Theorem 2.6. Therefore

$$\lim_{\epsilon \to 0} \epsilon^{1/2} \|(I + \epsilon H)^{-1} V^1_{t-s} b\|_1 = 0$$

by the Lebesgue dominated convergence theorem. This establishes that the first term in the integrand of (4.19) is uniformly bounded on $[0, t]$ and tends pointwise to zero as $\epsilon \to 0$. The second term has similar properties, by symmetry and interchange of s and $t - s$. Therefore

$$\lim_{\epsilon \to 0} \{ (a, V^1_t K_\epsilon b) - (K_\epsilon a, V^1_t b) \} = 0$$

by another application of the Lebesgue dominated convergence theorem. But since

$$\|K_\epsilon a - \bar{K}_2 a\| \to 0 \quad \text{and} \quad \|K_\epsilon b - \bar{K}_2 b\| \to 0$$

this establishes the observation.

OBSERVATION 4.8. *For all* $s, t \in \mathbf{R}$

$$(\text{ad } V^1_s)(V^2_t) = 0.$$

Proof. For each $\delta > 0$ define K_δ by

$$K_\delta = \delta^{-1} \int_0^\delta dt U_t \bar{K}_2 U_t$$

where $U_t = \exp\{itH\}$. Then

$$(\text{ad } H)(K_\delta) = (i\delta)^{-1}(U_\delta \bar{K}_2 U_{-\delta} - \bar{K}_2)$$

and hence

$$\| (\text{ad } H)(K_\delta) a\| \leqq 2(k_0/\delta) \|a\|_1, \quad a \in \mathfrak{h}_\infty$$

by (4.15) and Proposition 4.1. Thus if

$$V^\delta_t = \exp\{itK_\delta\}$$

then $V^\delta_t \mathfrak{h}_1 = \mathfrak{h}_1$, $t \in \mathbf{R}$, by Theorem 2.8. But one also has

$$|(a, (\text{ad } H)(K_\delta)b)| \leqq k_1\|a\|_{1/2} \cdot \|b\|_{1/2}, \quad a, b \in \mathfrak{h}_\infty$$

by (4.15). Hence $V_t^\delta \mathfrak{h}_{1/2} = \mathfrak{h}_{1/2}$, $t \in \mathbf{R}$, and

(4.20) $\|V_t^\delta a\|'_{1/2} \leqq e^{|t|\omega}\|a\|'_{1/2}, \quad t \in \mathbf{R}, \alpha \in \mathfrak{h}_{1/2},$

by Theorem 2.5. Moreover

$$(a, (K_\delta - \bar{K}_2)b) = \delta^{-1} \int_0^\delta dt \int_0^t ds\, i(a, U_s(\text{ad } H)(\bar{K}_2)U_{-s}b)$$

and consequently

(4.21) $|((a, (K_\delta - \bar{K}_2)b)| \leqq (k_1\delta/2)\|a\|_{1/2} \cdot \|b\|_{1/2}.$

In particular

$$|(a, (V_t^2 - V_t^\delta)b)| = \left|\int_0^t ds(a, V_s^2(K_\delta - \bar{K}_2)V_{t-s}^\delta b)\right|$$

$$\leqq (k_1\delta/2)\left|\int_0^t ds\|V_{-s}^2 a\|_{1/2} \cdot \|V_{t-s}^\delta b\|_{1/2}\right|$$

$$\leqq (k_1\delta/2)e^{\omega|t|}\|a\|'_{1/2} \cdot \|b\|'_{1/2}$$

by another application of Theorem 2.5. Hence

$$\lim_{\delta \to 0} (a, (V_t^2 - V_t^\delta)b) = 0$$

for all $a, b \in \mathfrak{h}_{1/2}$ and then, by continuity, for all $a, b \in \mathfrak{h}$. Thus

$$(a, (\text{ad } V_s^1)(V_t^2)b) = \lim_{\delta \to 0} (a, (\text{ad } V_s^1)(V_t^\delta)b).$$

But if $a, b \in \mathfrak{h}_1$ then

$$(a, (\text{ad } V_s^1)(V_t^\delta)b) = i \int_0^t du\{(K_\delta V_{-u}^\delta a, V_s^1 V_{t-u}^\delta b) - (V_{-u}^\delta a, V_s^1 K_\delta V_{t-u}^\delta b)\}$$

$$= i \int_0^t du\{(K_\delta - \bar{K}_2)V_{-u}^\delta a, V_s^1 V_{t-u}^\delta b) - (V_{-u}^\delta a, V_s^1(K_\delta - \bar{K}_2)V_{t-u}^\delta b)\}$$

by Observation 4.8, where it is important that $V^\delta \mathfrak{h}_1 = \mathfrak{h}_1$. Finally one estimates that

$$|(a, (\text{ad} V_s^1)(V_t^\delta)b)| \leqq (k_1\delta/2) \int_0^{|t|} du\{\|V_{-u}^\delta a\|_{1/2} \cdot \|V_s^1 V_{t-u}^\delta b\|_{1/2} + \|V_{-s}^1 V_{-u}^\delta a\|_{1/2} \cdot \|V_{t-u}^\delta b\|_{1/2}\}$$

$$\leqq k_1\delta e^{\omega(|s|+|t|)}\|a\|'_{1/2} \cdot \|b\|'_{1/2}, \quad a, b \in \mathfrak{h}_{1/2}$$

by (4.20), (4.21), and Theorem 2.5. Therefore

$$(a, (\text{ad } V_s^1)(V_t^2)b) = \lim_{\delta\to 0} (a, (\text{ad } V_s^1)(V_t^\delta)b) = 0$$

for all $a, b \in \mathfrak{h}_{1/2}$ and then, by continuity, for all $a, b \in \mathfrak{h}$.

Since the unitary groups V^1, V^2 commute the operators $\bar{K}_1$, $\bar{K}_2$ commute and the proof is complete.

Remark 4.9. Although no proof of Theorem 4.6 seems to appear in the literature the result is referred to in notes added in proof to **[3]** and **[6]** and is attributed to Nelson.

5. Lie groups and commutation. Combination of the results of Sections 3 and 4 immediately give a commutation theorem for self-adjoint operators associated with a unitary representation of a Lie group.

THEOREM 5.1. *Let K_1, K_2 be symmetric operators from $\mathscr{H}_\infty$ into $\mathscr{H}$ and suppose they each satisfy one of the following two conditions*:

1. $$\begin{cases} |(a, Kb)| \leqq k_0\|a\|_{1/2} \cdot \|b\|_{1/2}, \\ |(a, (\text{ad } dU(X_i))(K)b)| \leqq k_2\|a\| \cdot \|b\|_1, \\ i = 1, \dots, d, \ a, b \in \mathscr{H}_\infty \end{cases}$$

$$K \geqq 0$$

2. $$\begin{cases} |(a, Kb)| \leqq k_0\|a\|_1 \cdot \|b\|_1, \\ |(a, (\text{ad } dU(X_i))(\text{ad } dU(X_j))(K)b)| \leqq k_2\|a\| \cdot \|b\|_2, \\ i, j = 1, \dots, d, \ a, b \in \mathscr{H}_\infty. \end{cases}$$

Further assume that

$$(K_1a, K_2b) = (K_2a, K_1b), \quad a, b \in \mathscr{H}_\infty.$$

It follows that the self-adjoint closures $\bar{K}_1$, and $\bar{K}_2$, commute.

Proof. It follows from Theorem 3.4 and Corollary 3.10 that $\bar{K}_1$ and $\bar{K}_2$ are self-adjoint. But if K satisfies Condition 1 then it also satisfies Condition 1 of Theorem 4.5 with $H = \Delta$, the Laplacian associated with the representation. This is verified in the proof of Theorem 3.4. Alternatively if K satisfies Condition 2 then it satisfies Condition 2 of Theorem 4.5 with $H = \Delta$. This follows because K must satisfy the hypotheses of Theorem 3.8, as a consequence of Observation 2.4 and Lemma 2.11. Then the bound on $(\text{ad } \Delta)^2(K)$ is verified in the proof of Theorem 3.8.

Now Theorem 5.1 is a direct corollary of Theorem 4.5.

6. Partial differential operators. As an illustration of the foregoing theory we discuss various applications to general partial differential operators.

Let U be a strongly continuous unitary representation of the Lie group G on the Hilbert space $\mathscr{H}$ and adopt the notation of Section 3. Moreover set $H_i = dU(X_i)$. Next let $\mathscr{B}$ denote the C^*-algebra of a bounded operator on $\mathscr{H}$. Then G acts as a σ-weakly continuous group of $*$-automorphisms of $\mathscr{B}$ through the action

$$\tau_g(A) = U(g)AU(g)^{-1}, \quad A \in \mathscr{B}, g \in G.$$

Consequently for each X_i there exists a σ-weakly continuous one-parameter group

$$t \mapsto U(e^{tX_i})AU(e^{-tX_i})$$

of $*$-automorphisms of $\mathscr{B}$. Let δ_{X_i} denote the generator of this subgroup. Then δ_{X_i} is a σ-weak densely defined, σ-weakly – σ-weakly closed, derivation with

$$\delta_{X_i}(A)^* = -\delta_{X_i}(A^*).$$

Moreover $A \in D(\delta_{X_i})$ if, and only if, $(\text{ad } H_i)(A)$ is a bounded sesquilinear form over $D(H_i) \times D(H_i)$ or, equivalently, $AD(H_i) \subseteq D(H_i)$ and $(\text{ad } H_i)(A)$ has a bounded closure (see, for example, [**2**] Proposition 3.2.55). If $A \in D(\delta_{X_i})$ then $\delta_{X_i}(A)$ is equal to the closure of $(\text{ad } H_i)(A)$.

One can associate with the δ_{X_i} a C^n-structure on $\mathscr{B}$ similar to the structure on $\mathscr{H}$ defined in Section 3 with the H_i. For example, we set

$$\mathscr{B}_n = \bigcap_{|\alpha|=n} D(\delta_{X_i}^{\alpha_1} \dots \delta_{X_d}^{\alpha_d})$$

and

$$\|A\|'_n = \sum_{m=0}^{n} \rho_m(A)$$

where $\rho_0(A) = \|A\|$ and

$$\rho_n(A) = \max_{1 \leqq j_k \leqq d} \|\delta_{X_{j_1}} \dots \delta_{X_{j_n}} A\|.$$

Next we examine partial differential operators associated with the system $(\mathscr{H}, G, U)$. By this we mean polynomials in the generators H_i with coefficients in $\mathscr{B}$. Nelson and Stinespring [**19**] established essential self-adjointness results for elliptic operators with constant coefficients and similar results for second-order elliptic operators on Banach space. The Banach space results were then extended to all orders, and significantly generalized by Langlands [**13**] [**14**], but again for operators with constant coefficients. The subsequent discussion is more restrictive in that it only applies to first- and second-order operators, but more general insofar as it allows coefficients in $\mathscr{B}$. We begin with first-order operators.

If $A_i = A_i^* \in \mathscr{B}$, $i = 1, \ldots, d$ then one can define a symmetric sesquilinear form over $\mathscr{H}_1 \times \mathscr{H}_1$ by

$$K = i \sum_{j=1}^{d} (H_j A_j + A_j H_j).$$

But if $A_i \in \mathscr{B}_1$ then K defines a symmetric operator on $\mathscr{H}_1$ because

$$K = i \sum_{j=1}^{d} (2A_j H_j + \delta_{X_j}(A_j))$$

and one has an estimate

(6.1) $\quad ||Ka|| \leqq k||a||_1, \quad a \in \mathscr{H}_1.$

Next if $A_i \in \mathscr{B}_2$ then a straightforward calculation, using the structure relations of $\mathfrak{g}$, establishes that

$$|(a, (\text{ad } H_i)(K)b)| \leqq k_1||a|| \cdot ||b||_1, \quad a, b \in \mathscr{H}_\infty$$

for some $k_1 \geqq 0$ and all $i = 1, \ldots, d$. Hence K is essentially self-adjoint on $\mathscr{H}_\infty$ by Theorem 3.3. But it then follows from (6.1) that K is essentially self-adjoint on any $||\cdot||_1$-dense subspace of $\mathscr{H}_1$. Note that it also follows from Theorem 3.3 that the unitary group $V_t = \exp\{it\bar{K}\}$ leaves the subspaces $\mathscr{H}_\alpha$, $\alpha \in [0, 1]$, invariant and $V|_{\mathscr{H}_\alpha}$ is $||\cdot||_\alpha$-continuous. Finally it follows by the methods of Section 2d that if $A_i \in \mathscr{B}_{n+1}$ then V leaves $\mathscr{H}_\alpha$, $\alpha \in [0, n]$, invariant. Smoothness properties of the coefficients are reflected by smoothness properties of V.

Next we examine second-order partial differential operators. Let $A_{ij} = A_{ji}^*$, $A_i = A_i^*$, $A = A^*$ be elements of $\mathscr{B}$ then

$$K = \sum_{i,j=1}^{d} H_i A_{ij} H_j + i \sum_{j=1}^{d} (A_j H_j + H_j A_j) + A$$

is a symmetric sesquilinear form over $\mathscr{H}_1 \times \mathscr{H}_1$. Moreover if $A_{ij}, A_i \in \mathscr{B}_1$ then K defines a symmetric operator on $\mathscr{H}_2$ satisfying estimates

(6.2) $\quad ||Ka|| \leqq k_0||a||_2, \quad a \in \mathscr{H}_2,$

$$|(a, (\text{ad } H_i)(K)b)| \leqq k_1||a||_1 \cdot ||b||_1, \quad a, b \in \mathscr{H}_1,$$

But if $A_{ij}, A_i \in \mathscr{B}_2$ and $A \in \mathscr{B}_1$ one has a further estimate

$$|(a, (\text{ad } H_i)(\text{ad } H_j)(K)b)| \leqq k_2||a||_1 \cdot ||b||_1, \quad a, b \in \mathscr{H}_1.$$

Thus the commutator estimates of Theorem 3.5 are verified. Consequently if K is lower semi-bounded then it is essentially self-adjoint on $\mathscr{H}_\infty$ by Theorem 3.5. Strong ellipticity is a simple sufficient condition for semi-boundedness.

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The operator K is defined to be *strongly elliptic* if there is an $\epsilon > 0$ such that

$$(A_{ij} - \epsilon\|A_i\|^2\delta_{ij}I)$$

is positive definite in the sense

$$\sum_{i,j=1}^{d} (a_i, (A_{ij} - \epsilon\|A_i\|^2\delta_{ij}I)a_j) \geqq 0, \quad a_i \in \mathscr{H}.$$

Note that if $A_i = 0$ then this condition reduces to positive-definiteness of (A_{ij}) which clearly ensures that

$$K \geqq A \geqq -\|A\|I.$$

The general case is a small perturbation of this special case.

Set $\lambda_i = \|A_i\|^2$ and adopt the convention that

$$A_i/\sqrt{\lambda_i} = 0 \quad \text{if } A_i = 0.$$

Then for each $\delta > 0$ one has

$$\begin{aligned}\delta \sum_{i,j=1}^{d} (H_ia, A_{ij}H_ja) = \delta &\sum_{i,j=1}^{d} (H_ia, (A_{ij} - \epsilon\lambda_i\delta_{ij}I)H_ja) \\ &+ \sum_{j=1}^{d} \|(i\sqrt{\epsilon\delta\lambda_j}H_j \pm (1/\sqrt{\epsilon\delta\lambda_j})A_j)a\|^2 \\ &\mp i \sum_{j=1}^{d} (a, (H_jA_j + A_jH_j)a) \\ &- \sum_{j=1}^{d} \|A_ja\|^2/(\epsilon\delta\lambda_j).\end{aligned}$$

Therefore one has the perturbation estimate

$$\left|\sum_{i=1}^{d} (a, (H_iA_i + A_iH_i)a)\right|$$

$$\leqq \delta \sum_{i,j=1}^{d} (H_ia, A_{ij}H_ja) + (\epsilon\delta)^{-1} \sum_{j=1}^{d} \|A_ja\|^2/\lambda_j$$

and in particular

$$(a, Ka) \geqq -(d/\epsilon)\|a\|^2 + (a, Aa) \geqq -((d/\epsilon) + \|A\|)\|a\|^2.$$

Thus K is lower semi-bounded. Hence K is essentially self-adjoint on $\mathscr{H}_\infty$

or, by (6.2), essentially self-adjoint on any $\|\cdot\|_2$-dense subspace of $\mathscr{H}_2$.

Finally if $A_{ij} \in \mathscr{B}_3$, and A_i, $A \in \mathscr{B}_2$ one has a further estimate

$$|(a, (\text{ad } H_i)(\text{ad } H_j)(K)b)| \leqq k_2' \|a\| \cdot \|b\|_2, \quad a, b \in \mathscr{H}_2$$

and hence the semigroup $T_t = \exp\{-t\bar{K}\}$ maps $\mathscr{H}_\alpha$ into $\mathscr{H}_\alpha$, for $\alpha \in [0, 2]$, and $T|_{\mathscr{H}_\alpha}$ is $\|\cdot\|_\alpha$-continuous by Theorem 3.8.

To conclude we discuss the more specific setting of partial differential operators on $\mathbf{R}^d$. Thus we choose $\mathscr{H} = L^2(\mathbf{R}^d)$, set $p_j = i\partial/\partial x_j$ the self-adjoint operator of differentiation, defined through multiplication on the Fourier transform, and set q_j the operator of multiplication by x_j. Now if $G = \mathbf{R}^d$ and U denotes the action of G on $\mathscr{H}$ by translations we can choose X_j such that $H_j = p_j$. Next if $A(q) \in \mathscr{B}$ denotes the operator of multiplication by a bounded function

$$x \in \mathbf{R}^d \mapsto A(x)$$

then $A(q) \in \mathscr{B}_1$ if, and only if, $x \mapsto A(x)$ is Lipschitz, i.e.,

$$|A(x) - A(y)| \leqq c|x - y|, \quad |x - y| \leqq 1,$$

for some $c \geqq 0$. Moreover $A(q) \in \mathscr{B}_{n+1}$ if $x \mapsto A(x)$ is n-times continuously differentiable and the n-th order derivatives are Lipschitz. Thus the above considerations give self-adjointness and smoothness results for first and second order partial differential operators with differentiable coefficients, e.g.

$$(6.3) \qquad K = \sum_{i,j=1}^{d} p_i A_{ij}(q) p_j + A(q)$$

is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$ if $A_{ij}(q) \in \mathscr{B}_2$, $(A_{ij}(x))$ is positive-definite for each $x \in \mathbf{R}^d$, and $A(q) \in \mathscr{B}$. (No differentiability of A is necessary because it is bounded.) Moreover if $A_{ij} \in \mathscr{B}_3$ and $A \in \mathscr{B}_2$ then the semigroup $T_t = \exp\{-t\bar{K}\}$ maps the subspaces $\mathscr{H}_\alpha$ into $\mathscr{H}_\alpha$, for $\alpha \in [0, 2]$, and $T|_{\mathscr{H}_\alpha}$ is $\|\cdot\|_\alpha$-continuous. Note that it also follows from Theorems 3.5 and 3.8 that these conclusions are still valid if one adds to K a positive operator

$$K_n = \sum_{|\alpha| \leqq n} c_\alpha p^\alpha$$

with constant coefficients $c_\alpha \in \mathbf{C}^{|\alpha|}$.

One drawback of such results is that they require boundedness of the coefficients but such restrictions can be removed by the choice of other groups. For example, there is a unitary action of the Heisenberg group on $L^2(\mathbf{R}^d)$ given by the Schrödinger representation. Then we can choose a basis X_i, $i = 0, 1, \ldots, 2d$ of g such that $H_0 = I$, $H_j = p_j$, $H_{d+j} = q_j$, $j = 1, \ldots, d$. But for this action $\mathscr{H}_\infty$ is equal to the Schwartz space

$\mathscr{S}(\mathbf{R}^d)$ (see, for example, [**21**] Example 5.1) and hence (6.3) is defined as a form over $\mathscr{H}_\infty \times \mathscr{H}_\infty$ whenever $A_{ij}(q)$ and $A(q)$ are polynomially bounded in q. Moreover the hypotheses of Theorem 3.5 are satisfied whenever A_{ij} satisfies the previous hypotheses but $A \geqq 0$, $x \mapsto A(x)$ is once-differentiable, and

$$|\nabla A(x)| \leqq c_1|x| + c_2 \quad \text{for some } c_1, c_2 \geqq 0.$$

Hence one obtains results for A which are $O(x^2)$ at infinity.

Finally we give an improvement of an example discussed by Driessler and Summers [**4**]. Let K be a positive fourth-order polynomial in the p_j, q_j. Then it follows that K is essentially self-adjoint on $\mathscr{S}(\mathbf{R}^d)$, and the contraction semigroup $T_t = \exp\{-t\bar{K}\}$ maps each $\mathscr{H}_\alpha$ into itself, for $\alpha \geqq 0$, and $T|_{\mathscr{H}_\alpha}$ is $\|\cdot\|_\alpha$-continuous. The self-adjointness statement follows easily from Theorem 2.6 by choosing

$$H = \sum_{i=1}^{d} (p_i^2 + q_i^2) + I$$

to be the Laplacian of the Schrödinger representation, i.e., the harmonic oscillator Hamiltonian. Then it follows from the Heisenberg commutation relations that $(\mathrm{ad}\, H)^p(K)$ is also a fourth-order polynomial in the p_i, q_i for each $p = 1, 2, \ldots$. Therefore

$$\|(\mathrm{ad}\, H)^p(K)a\| \leqq k_p\|a\|_2, \quad a \in \mathscr{S}(\mathbf{R}^d) = \mathscr{H}_\infty,$$

by Nelson's estimates [**18**] used in Section 3 for a general Lie group (see [**11**] Proposition 1.3). Now the statements concerning K follow from Theorems 2.16, and 2.19.

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