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ISOMETRICALLY EQUIVALENT COMPOSITION OPERATORS ON SPACES OF ANALYTIC VECTOR-VALUED FUNCTIONS

WILLIAM E. HORNOR

Department of Mathematics, University of Southern Mississippi, Hattiesburg, Mississippi 39406, USA

JAMES E. JAMISON

Department of Mathematical Sciences, University of Memphis, Memphis, Tennessee 38152, USA

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1. Introduction. Let X be a Banach space and let $B(X)$ denote the space of bounded operators on X. Two elements $S, T \in B(X)$ are *isometrically equivalent* if there exists an invertible isometry V such that $TV = VS$. If X is a Hilbert space, then V is a unitary operator and S and T are said to be *unitarily equivalent*.

The unitary equivalence of operators on Hilbert spaces has a rich history. However, necessary and sufficient conditions for the unitary equivalence of two operators are known only for a relatively small class of operators. Only recently has this problem been considered for operators in Banach spaces which are not Hilbert spaces. The equivalence of certain integral operators on L^p ([0,1]) was investigated by G. Kalisch [**9**] while Campbell and Jamison [**4**] considered the isometric equivalence of weighted composition operators acting on $L^p(\mu)$ spaces. Randall Campbell-Wright [**5**] was the first to investigate the isometric equivalence of composition operators on the Hardy spaces H^p on the unit disc.

In [**8**] the authors extended R. Campbell-Wright's results to various classical Banach spaces of analytic functions in the unit disk or ball. In this paper we consider the isometric equivalence of composition operators on two spaces of analytic vector-valued functions. Let $\mathcal{K}$ denote a separable complex Hilbert space and let $H^p_{\mathcal{K}}, p \neq 2$, be the associated Hardy space of analytic functions with values in $\mathcal{K}$. The isometric equivalence problem is solved for composition operators in this setting using the results of P. K. Lin [**11**] on surjective isometries of these spaces. We also consider the spaces $S^p_{\mathcal{K}}$ of functions analytic on the disk with derivatives in the Hardy space $H^p_{\mathcal{K}}$. We characterize the isometries of this space, extending results of the scalar case due to Novinger-Oberlin [**13**] and solve the isometric equivalence problem for composition operators on this space.

2. Composition operators on $H^p_{\mathcal{B}}$. Suppose that $\mathcal{B}$ is a complex Banach space, and denote by $H^0_{\mathcal{B}}$ the space of all holomorphic $\mathcal{B}$-valued functions defined on $\mathbb{D}$, the unit disk. A function $F \in H^0_{\mathcal{B}}$ is said to *belong to* $H^p_{\mathcal{B}}$ if

$$\|F\|_{H^p_{\mathcal{B}}} := \left[\sup_{0<r<1} \int_0^{2\pi} \left\|F\left(re^{it}\right)\right\|^p_{\mathcal{B}} \frac{dt}{2\pi}\right]^{\frac{1}{p}} < \infty.$$

For $F \in H^p_{\mathcal{B}}$, membership in $H^p_{\mathcal{B}}$ is equivalent to the existence of a harmonic majorant for $\|F(z)\|^p_{\mathcal{B}}$. (See [**14**] for a proof.) With the norm $\|\cdot\|_{p,\mathcal{B}}$ defined above, $H^p_{\mathcal{B}}$ is a Banach space. The characterization of $H^p_{\mathcal{B}}$ in terms of the existence of harmonic majorants is particularly useful in establishing the following analogue of the Littlewood Subordination Theorem for $H^p_{\mathcal{B}}$.

Suppose $F \in H^p_{\mathcal{B}}$ and that φ is an analytic self-mapping of $\mathbb{D}$. Whenever $G(z)$ is a harmonic majorant of $\|F(z)\|^p_{\mathcal{B}}$, it is clear that $G \circ \varphi(z)$ is a harmonic majorant of $\|F \circ \varphi(z)\|^p_{\mathcal{B}}$. Thus every analytic self-mapping φ of the disk induces a *composition operator* $C_\varphi : H^p_{\mathcal{B}} \to H^p_{\mathcal{B}}$ by $(C_\varphi F)(z) = F(\varphi(z))$. Note that for each Λ in the unit ball of $\mathcal{B}^*$ we have

$$\|\Lambda F\|^p_p = \sup_{0<r<1} \int_0^{2\pi} \left|\Lambda F(re^{it})\right|^p \frac{dt}{2\pi} \leq \sup_{0<r<1} \int_0^{2\pi} \left\|F(re^{it})\right\|^p_{\mathcal{B}} \frac{dt}{2\pi} = \|F\|^p_{H^p_{\mathcal{B}}},$$

and for $re^{i\theta} \in \mathbb{D}$ we have

$$\left|\Lambda F\left(re^{i\theta}\right)\right| = \left|\int_0^{2\pi} (\Lambda F)^*(e^{it}) P_r(\theta - t) \frac{dt}{2\pi}\right| \leq \frac{1+r}{1-r} \left\|(\Lambda F)\right\|_p,$$

where $(\Lambda F)^*$ is the representation of ΛF as a function on $\Gamma = \partial\mathbb{D}$, and $P_r(\theta)$ is the Poisson kernel. Thus we obtain the inequality

$$\max_{0<\theta<2\pi} \left\|F(re^{i\theta})\right\|_{\mathcal{B}} = \sup\left\{\left|\Lambda\left(F(re^{i\theta})\right)\right| : 0 \leq \theta < 2\pi,\ \|\Lambda\|_{\mathcal{B}^*} = 1\right\} \leq \frac{1+r}{1-r} \|F\|_{H^p_{\mathcal{B}}}.$$

We observe that convergence of a sequence $\{F_n\}$ in $H^p_{\mathcal{B}}$ implies convergence in the compact-open topology of $H^0_{\mathcal{B}}$.

Suppose that φ is as above and $\{F_n\}$ is a sequence in $H^p_{\mathcal{B}}$ converging to 0 for which the sequence $\left\{C_\varphi F_n\right\} = \left\{F_n \circ \varphi\right\}$ converges to some $G \in H^p_{\mathcal{B}}$. Now for each $r \in (0, 1)$, we have

$$\int_0^{2\pi} \left\|G\left(re^{it}\right)\right\|^p_{\mathcal{B}} \frac{dt}{2\pi} = \lim_{n\to\infty} \int_0^{2\pi} \left\|F_n\left(\varphi\left(re^{it}\right)\right)\right\|^p_{\mathcal{B}} \frac{dt}{2\pi} = 0$$

since $\{F_n\}$ converges uniformly to 0 on the compact set $\partial(r\mathbb{D})$. It follows that G is the zero function, and continuity of C_φ follows from the Closed Graph Theorem. We formalize the above as a theorem.

THEOREM 1. *For each analytic self-map φ of the unit disk, the composition operator $C_\varphi : H^p_{\mathcal{B}} \to H^p_{\mathcal{B}}$ is bounded.*

3. Composition operators on $S^p_{\mathcal{B}}$. For $p \geq 1$, we denote by S^p the space of anayltic functions on $\mathbb{D}$ with first derivative in the classical Hardy space H_p. We define a norm $\|\cdot\|_{S^p}$ on S^p by

$$\|f\|_{S^p} = |f(0)| + \|f'\|_p,$$

where $\|\cdot\|_p$ denotes the usual norm on H^p. Equipped with this norm, S^p is a Banach space. Functions of class S^p extend continuously to the closed disk, and the boundary-value function $f(e^{i\theta})$ is absolutely continuous for each $f \in S^p$. (See, for example,

[7].) For a Banach space $\mathcal{B}$, we define the Banach space $S^p_{\mathcal{B}}$ as the linear space of all $F \in H^0_{\mathcal{B}}$ for which $F' \in H^p_{\mathcal{B}}$ with norm

$$\|F\|_{S^p_{\mathcal{B}}} = \|F(0)\|_{\mathcal{B}} + \|F'\|_{H^p_{\mathcal{B}}}.$$

In this section we characterize those analytic self-maps of $\mathbb{D}$ that give rise to bounded composition operators on $S^p_{\mathcal{B}}$ for a broad class of Banach spaces $\mathcal{B}$. We reduce the problem to that of characterizing those φ for which C_φ is bounded on S^p. This problem has been solved by MacCluer [**12**] who gives a Carleson measure criterion for the boundedness of C_φ in the scalar-valued setting. Our method requires that we restrict our attention to Banach spaces $\mathcal{B}$ having the analytic Radon-Nikodym property.

Definition 2. A Banach space $\mathcal{B}$ is said to have the *analytic Radon-Nikodym property* (*aRNP*) if for every $F \in H^\infty_{\mathcal{B}}$ the strong radial limits $\lim_{r \to 1^-} F(re^{i\theta})$ exist for almost all $\theta \in [0, 2\pi)$.

If a Banach space $\mathcal{B}$ has the aRNP, then it possesses the (formally) stronger property that for any $f \in H^p_{\mathcal{B}}$, $1 \le p \le \infty$, strong nontangential limits $F(e^{i\theta})$ exist for almost every $\theta \in [0, 2\pi)$, and $F \in H^p_{\mathcal{B}}$ can be recovered from its boundary values via the Poisson representation

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} F(e^{it}) P_r(\theta - t)\,dt.$$

Setting $F_r(e^{i\theta}) = F(re^{i\theta})$, we have $F_r \to F$ in $L^p_{\mathcal{B}}(\Gamma)$ as $r \to 1^-$, where $L^p_{\mathcal{B}}(\Gamma)$ denotes the Lebesgue space of Bochner p-integrable $\mathcal{B}$-valued functions on the unit circle Γ. As in the scalar setting, the identification of each element F of $H^p_{\mathcal{B}}$ with its boundary value function is an isometric isomorphism between $H^p_{\mathcal{B}}$ and the closed subspace of $L^p_{\mathcal{B}}(\Gamma)$ consisting of those functions whose negative Fourier coefficients vanish. It now follows that the polynomials with $\mathcal{B}$-valued coefficients are dense in $H^p_{\mathcal{B}}$ for $1 \le p < \infty$. Jensen's inequality is valid in this setting and, in particular, $\log\|F(e^{i\theta})\|_{\mathcal{B}} \in L^1_{\mathcal{B}}(\Gamma)$. A good reference for the above is [**2**]. See also [**3**] and [**15**].

When $\mathcal{B}$ has the aRNP, members of $S^p_{\mathcal{B}}$ extend smoothly to $\bar{\mathbb{D}}$ just as in the scalar case. Our proof makes use of the following result of Barbee [**2**].

Theorem 3. *Suppose that $\mathcal{B}$ is a Banach space and F is an analytic $\mathcal{B}$-valued function with $F' \in H^1_{\mathcal{B}}$. Then F has strong radial limits at every point of the unit circle.*

The proof of Barbee's result relies on the classical Fejér-Riesz inequality and basic properties of the Bochner integral.

Theorem 4. *Suppose that $\mathcal{B}$ is a Banach space with the aRNP and F is a $\mathcal{B}$-valued analytic function in the unit disc with $F' \in H^1_{\mathcal{B}}$. Then F extends continuously to the closed unit disk and the boundary function is strongly absolutely continuous.*

Proof. Let $\varepsilon > 0$ be given, and choose r so that $\|F' - F'_R\|_{L^1_{\mathcal{B}}(\Gamma)} < \frac{\varepsilon}{2}$ whenever $r \le R < 1$. Now there exists a $\delta > 0$ such that

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$$\sum_{k=1}^{n} \| F(re^{i\beta_k}) - F(re^{i\alpha_k}) \|_{\mathcal{B}} < \frac{\varepsilon}{2}$$

for any collection of disjoint line segments $(\alpha_1, \beta_2), \ldots, (\alpha_n, \beta_n)$ satisfying

$$\sum_{k=1}^{n} (\beta_k - \alpha_k) < \delta.$$

With r and δ as above, by employing Cauchy's Integral Theorem we obtain for each k

$$\begin{aligned}
&\| F(e^{i\beta_k}) - F(e^{i\alpha_k}) \|_{\mathcal{B}} \leq \\
&\left\| \int_r^1 F'(se^{i\beta_k}) e^{i\beta_k} ds - \int_r^1 F'(se^{i\alpha_k}) e^{i\alpha_k} ds \right\|_{\mathcal{B}} + \left\| F(re^{i\beta_k}) - F(re^{i\alpha_k}) \right\|_{\mathcal{B}} = \\
&\lim_{R \to 1^-} \left\| \int_r^R F'(se^{i\beta_k}) e^{i\beta_k} ds - \int_r^R F'(se^{i\alpha_k}) e^{i\alpha_k} ds \right\|_{\mathcal{B}} + \left\| F(re^{i\beta_k}) - F(re^{i\alpha_k}) \right\|_{\mathcal{B}} = \\
&\lim_{R \to 1^-} \left\| \int_{\alpha_k}^{\beta_k} F'(Re^{i\theta}) Re^{i\theta} d\theta - \int_{\alpha_k}^{\beta_k} F'(re^{i\theta}) re^{i\theta} d\theta \right\|_{\mathcal{B}} + \left\| F(re^{i\beta_k}) - F(re^{i\alpha_k}) \right\|_{\mathcal{B}} \leq \\
&\lim_{R \to 1^-} \int_{\alpha_k}^{\beta_k} \| (RF'_R - F')(e^{i\theta}) \|_{\mathcal{B}} d\theta + \int_{\alpha_k}^{\beta_k} \| (F'_r - F')(e^{i\theta}) \|_{\mathcal{B}} d\theta + \| F(re^{i\beta_k}) - F(re^{i\alpha_k}) \|_{\mathcal{B}} = \\
&\int_{\alpha_k}^{\beta_k} \| (F'_r - F')(e^{i\theta}) \|_{\mathcal{B}} d\theta + \| F(re^{i\beta_k}) - F(re^{i\beta_k}) \|_{\mathcal{B}}
\end{aligned}$$

and the strong absolute continuity of the boundary function follows.

The next result, due to MacCluer [**12**], classifies the self-maps of $\mathbb{D}$ that induce bounded composition operators on S^p, where $p \geq 1$. For $\xi \in \partial\mathbb{D}$ and $\delta > 0$, set

$$\mathcal{S}(\xi, \delta) = \left\{ z \in \bar{\mathbb{D}} : |z - \xi| < \delta \right\}.$$

If $\varphi : \Gamma \to \bar{\mathbb{D}}$ is absolutely continuous with $\varphi' \in L^p$, we define a Borel measure $\mu = \mu_{\varphi,p}$ on Γ by

$$\mu(E) = \int_E |\varphi'|^p d\sigma,$$

where σ denotes normalized Lebesgue measure on Γ. We consider as well the image measure

$$\mu \circ \varphi^{-1}(B) = \mu(\varphi^{-1} B),$$

a Borel measure on $\bar{\mathbb{D}}$.

THEOREM 5. *Suppose that φ is an analytic self-map of $\mathbb{D}$. Then the composition operator C_φ is bounded on S^p if and only if $\varphi \in S^p$ and there exists a constant C such that*

$$\mu \circ \varphi^{-1}(\mathcal{S}(\xi, \delta)) \leq C\delta,$$

for all $\xi \in \partial\mathbb{D}$ and $\delta > 0$.

We note that the polynomials with coefficients in $\mathcal{B}$ are dense in $S^p_{\mathcal{B}}$ for $1 \le p < \infty$ when $\mathcal{B}$ has the aRNP and, by Theorem 4, we can identify each member F of $S^p_{\mathcal{B}}$ with its boundary function, a strongly absolutely continuous function which we also denote by F. We denote by $S^p_{\mathcal{B}}(\Gamma)$ the Banach space of boundary value functions of elements of $S^p_{\mathcal{B}}$ where $\|F\|_{S^p_{\mathcal{B}}(\Gamma)} = \|F\|_{S^p_{\mathcal{B}}}$. We combine this observation with MacCluer's theorem to obtain the following result.

THEOREM 6. *Suppose that $\mathcal{B}$ has the aRNP. Then the composition operator induced by a self-map φ of the unit disk is continuous on $S^p_{\mathcal{B}}$ if and only if the composition operator induced by φ is continuous on S^p. Consequently C_φ is continuous on S^p_β if and only if the conditions in Theorem 5 hold.*

Proof. By considering functions of the form $F(z) = f(z)e$, where $f \in S^p$ and e is a unit vector in $\mathcal{B}$, we see that the conditions from Theorem 5 are necessary for the boundedness of C_φ on S^p_β.

Next, suppose that the conditions from Theorem 5 hold. By the Carleson measure theorem, there exists a constant C such that

$$\int_{\bar{\mathbb{D}}} |f|^p d\mu \circ \varphi^{-1} \le C \int_{\partial\mathbb{D}} |f(e^{i\theta})|^p d\sigma,$$

for every $f \in H^p$. We suppose that F is a polynomial $S^p_{\mathcal{B}}$ and use F, F', φ, and φ' to denote the respective continuous extensions of these functions to $\bar{\mathbb{D}}$. Now $\log\|F'(e^{i\theta})\|_{\mathcal{B}} \in L^1(\Gamma)$ and $F'(e^{i\theta}) \in L^p_{\mathcal{B}}(\Gamma)$; also the outer function

$$g(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\|F'(e^{i\theta})\|_{\mathcal{B}} d\theta\right\}$$

satisfies $|g(e^{i\theta})| = \|F'(e^{i\theta})\|_{\mathcal{B}}$, for almost every $e^{i\theta} \in \Gamma$. We also have $|g(z)| \ge \|F'(z)\|_{\mathcal{B}}$, for all $z \in \mathbb{D}$. (See [**1**].) Since F' is a polynomial, $|g(e^{i\theta})|$ is continuous on Γ. Now setting $\mu = \mu_{\varphi,p}$, we have

$$\begin{aligned}
\|F\circ\varphi\|^p_{S^p_{\mathcal{B}}} - 2^{p-1}\|F\circ\varphi(0)\|^p_{\mathcal{B}} &\le 2^{p-1}\|(F\circ\varphi)'\|^p_{\mathcal{B}} \\
&= 2^{p-1}\int_{\partial\mathbb{D}} \|F'\circ\varphi(e^{i\theta})\|^p_{\mathcal{B}} d\mu \\
&= 2^{p-1}\int_{\bar{\mathbb{D}}} \|F'\|^p_{\mathcal{B}} d\mu\circ\varphi^{-1} \\
&\le 2^{p-1}\int_{\bar{\mathbb{D}}} |g|^p d\mu\circ\varphi^{-1} \\
&\le 2^{p-1}C\int_{\partial\mathbb{D}} |g(e^{i\theta})|^p d\sigma \\
&= 2^{p-1}C\int_{\partial\mathbb{D}} \|F'(e^{i\theta})\|^p_{\mathcal{B}} d\sigma \\
&\le 2^{p-1}C\|F\|_{S^p_{\mathcal{B}}},
\end{aligned}$$

and it follows that

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$$\|F\circ\varphi\|^p_{S^p_{\mathcal{B}}}\leq 2^{p-1}\Big(C\|F\|_{S^p_{\mathcal{B}}}+\|F\circ\varphi(0)\|^p_{\mathcal{B}}\Big).$$

Since the mapping $F\mapsto F(\varphi(0))$ is continuous on $S^p_{\mathcal{B}}$, it follows that C_φ is bounded on $S^p_{\mathcal{B}}$.

4. Isometric equivalence of composition operators. For the remainder of the paper we assume that $\mathcal{B}=\mathcal{K}$, a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle_\kappa$. As they are reflexive, complex Hilbert spaces have the Radon-Nikodym Property, hence the (weaker) aRNP, and the boundedness of composition operators on $H^p_{\mathcal{K}}$ and $S^p_{\mathcal{K}}$ is described by Theorems 1 and 6. We assume in this section that $1\leq p<\infty$. Our first result, a characterization of isometric equivalence of composition operators on $H^p_{\mathcal{K}}$, is a generalization of a result of R. Campbell-Wright for the classical Hardy space. The following result of P. Lin [**11**] allows us to apply Campbell-Wright's technique, in our more general setting.

THEOREM 7. *Any surjective isometry of $H^p_{\mathcal{K}}$, where $1\leq p<\infty$ and $p\neq 2$, has the form*

$$(TF)(z)=\big(\varphi_a'(z)\big)^{\frac{1}{p}}U\cdot F(\lambda\varphi_a(z)),$$

where U is a unitary operator on $\mathcal{K}$, λ is a unimodular constant, $a\in\mathbb{D}$, and φ_a is the disk automorphism

$$\varphi_a(z)=\Big(\frac{a-z}{1-\bar{a}z}\Big).$$

THEOREM 8. *Suppose that C_φ and C_ψ are isometrically equivalent composition operators on $H^p_{\mathcal{K}}, 1\leq p<\infty, p\neq 2$. Then $\varphi(z)=e^{-i\theta}\psi(e^{i\theta}z)$, for some real θ. Conversely, if φ and ψ are so related, the operators C_φ and C_ψ are isometrically equivalent.*

Proof. Suppose that T is a surjective isometry, as in Theorem 7, that satisfies $TC_\varphi=C_\psi T$. Choose a nonzero $k\in\mathcal{K}$. Now, for any $F\in H^p_{\mathcal{K}}$ and $z\in\mathbb{D}$, we have

$$\langle TC_\varphi F(z),k\rangle_{\mathcal{K}}=\big(\varphi_a'(z)\big)^{\frac{1}{p}}\langle F(\varphi(\lambda\varphi_a(z))),U^*k\rangle_{\mathcal{K}}$$

and

$$\langle C_\psi TF(z),k\rangle_{\mathcal{K}}=\big(\varphi_a'(\psi(z))\big)^{\frac{1}{p}}\langle F(\lambda\varphi_a(\psi(z))),U^*k\rangle_{\mathcal{K}}.$$

Letting $F(z)\equiv U^*k$, we conclude that

$$\big(\varphi_a'(z)\big)^{\frac{1}{p}}=\big(\varphi_a'(\psi(z))\big)^{\frac{1}{p}},$$

from which it follows that either $a = 0$ or $\psi(z) \equiv z$. Since $H^p_{\mathcal{K}}$ separates points, we conclude that either $\varphi(z) \equiv \psi(z) \equiv z$, or $a = 0$ and $\varphi(z) \equiv -\lambda\psi(-\bar{\lambda}z)$.

The proof of the converse is trivial.

We consider next the isometric equivalence problem for bounded composition operators on $S^p_{\mathcal{K}}$. We must first characterize the surjective isometries of these spaces. In doing so we employ a technique of Novinger and Oberlin from [**13**]. To adapt their method to our setting, we make use of the following result of Deeb and Khalil [**6**].

Definition 9. If $\mathcal{B}$ is a Banach space, then an $x \in \mathcal{B}$ with $\|x\| = 1$ is called a *smooth* point of the unit ball of $\mathcal{B}$ if there exists a unique $\Lambda \in \mathcal{B}^*$ such that $\|\Lambda\| = 1$ and $\Lambda(x) = 1$. A Banach space is said to be *smooth* if every boundary point of its unit ball is a point of smoothness.

Theorem 10. *Let $\mathcal{B}$ be a Banach space for which $\mathcal{B}^*$ is separable and suppose that $F \in L^p_{\mathcal{B}}(\Gamma)$ with $\|F\|_{L^p_{\mathcal{B}}(\Gamma)} = 1$. Then the following conditions are equivalent:*

(i) *F is a smooth point of the unit ball of $L^p_{\mathcal{B}}(\Gamma)$.*

(ii) *$F(e^{i\theta})/\|F(e^{i\theta})\|_{\mathcal{B}}$ is a smooth point of the unit ball of $\mathcal{B}$, for almost all $e^{i\theta} \in \Gamma$ for which $F(e^{i\theta}) \neq 0$.*

The following characterization of smoothness is useful in our characterization of the surjective isometries of $S^p_{\mathcal{K}}$. A good reference is Köthe [**10**, p. 350].

Theorem 11. *A Banach space $\mathcal{B}$ is smooth if and only if its norm is weakly differentiable at every point except the origin. That is, $\mathcal{B}$ is smooth exactly when*

$$\lim_{t \to 0} \frac{\|x + ty\|_{\mathcal{B}} - \|x\|_{\mathcal{B}}}{t}$$

exists for each $y \in \mathcal{B}$ and all nonzero $x \in \mathcal{B}$.

Corollary 12. *Every Hilbert space is smooth.*

Corollary 13. *For every separable Hilbert space $\mathcal{K}$ and each $p \geq 1$, $S^p_{\mathcal{K}}$ is smooth.*

Theorem 14. *$T : S^p_{\mathcal{K}} \to S^p_{\mathcal{K}}$, where $1 \leq p < \infty$ and $p \neq 2$, is a surjective linear isometry if and only if there exist unitary operators U, V on $\mathcal{K}$ and $\eta \in \mathrm{Aut}(\mathbb{D})$ such that*

$$TF(z) = VF(0) + U\int_0^z (\eta'(\xi))^{1/p} F'(\eta(\xi))d\xi.$$

Proof. Sufficiency follows almost immediately, for $TF(0) = VF(0)$ and

$$(TF)'(\xi) = (\eta'(\xi))^{1/p} UF'(\eta(\xi)).$$

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Therefore

$$\begin{aligned}\|TF\|_{S^p_{\mathcal{K}}} &= \|VF(0)\|_{\mathcal{K}} + \|(\eta')^{1/p} UF' \circ \eta\|_{H^p_{\mathcal{K}}} \\ &= \|F(0)\|_{\mathcal{K}} + \|(\eta')^{1/p} F' \circ \eta\|_{H^p_{\mathcal{K}}} \\ &= \|F(0)\|_{\mathcal{K}} + \|F'\|_{H^p_{\mathcal{K}}} \\ &= \|F\|_{S^p_{\mathcal{K}}}.\end{aligned}$$

Next, suppose that $T : S^p_{\mathcal{K}} \to S^p_{\mathcal{K}}$ is a surjective isometry. Let n be a positive integer and $x \in \mathcal{K}$ with $\|x\|_{\mathcal{K}} = 1$. Let χ_n denote the mapping $z \mapsto z^n$ and, for $t \in \mathbb{R}$, set $F_t = (1 + t\chi_n)x$. Clearly $F_t \in S^p_{\mathcal{K}}$ with $\|F_t(0)\|_{\mathcal{K}} = 1$. Now

$$\begin{aligned}\|F'_t\|^p_{H^p_{\mathcal{K}}} &= \lim_{r \to 1^-} \frac{1}{2\pi} \sup_{0<r<1} \int_0^{2\pi} |t|^p \|nr^{n-1} e^{ins} x\|^p_{\mathcal{K}} ds \\ &= \lim_{r \to 1^-} n^p |t|^p r^{np} \\ &= n^p |t|^p,\end{aligned}$$

so that $\|F_t\|_{S^p_{\mathcal{K}}} = 1 + n|t|$. Also,

$$TF_t(z) = Tx + T(\chi_n x),$$

where for each $x \in \mathcal{K}$ the constant function on $\mathbb{D}$ whose only value is x is also denoted by x. Note that

$$\begin{aligned}1 + n|t| &= \|F_t\|_{S^p_{\mathcal{K}}} \\ &= \|TF_t\|_{S^p_{\mathcal{K}}} \\ &= \|Tx(0) + tT(\chi_n x)(0)\|_{\mathcal{K}} + \|(Tx)' + t(T(\chi_n x))'\|_{H^p_{\mathcal{K}}} \\ &\leq \|Tx(0)\|_{\mathcal{K}} + \|(Tx)'\|_{H^p_{\mathcal{K}}} + |t| \|T(\chi_n x)(0)\|_{\mathcal{K}} + \|T(\chi_n x)'\|_{H^p_{\mathcal{K}}} \\ &= \|Tx\|_{S^p_{\mathcal{K}}} + |t| \|T(\chi_n x)\|_{S^p_{\mathcal{K}}} = 1 + n|t|.\end{aligned}$$

It follows that the (weak) inequality is an equality, and consequently that

$$\|Tx(0) + tT(\chi_n x)(0)\|_{\mathcal{K}} = \|Tx(0)\|_{\mathcal{K}} + |t| \|T(\chi_n x)(0)\|_{\mathcal{K}},$$

$$\|(Tx)' + tT(\chi_n x)'\|_{H^p_{\mathcal{K}}} = \|(Tx)'\|_{H^p_{\mathcal{K}}} + |t| \|(T(\chi_n x))'\|_{H^p_{\mathcal{K}}}.$$

Since $\mathcal{K}$ has the aRNP, the polynomials with coefficients from $\mathcal{K}$ are dense in $S^p_{\mathcal{K}}$, and so we can choose a positive integer n such that $T(\chi_n x)' \neq 0$. Now the function $t \mapsto \|(Tx)' + tT(\chi_n x)'\|_{H^p_{\mathcal{K}}}$ must be differentiable for all t provided only that $(Tx)' \neq 0$, while the right-hand side of the above equality is clearly nondifferentiable

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at $t = 0$. It follows that $(Tx)' = 0$, and consequently that Tx is a constant function with

$$\|Tx\|_{S^p_{\mathcal{K}}} = \|Tx(0)\|_{\mathcal{K}} = 1.$$

Since $\|Tx(0)\|_{\mathcal{K}} = 1$ and

$$\|Tx(0) + tT(\chi_n x)(0)\|_{\mathcal{K}} = 1 + |t|\,\|T(\chi_n x)(0)\|_{\mathcal{K}}$$

for all real t, a differentiability of norm argument shows that

$$T(\chi_n x)(0) = 0,$$

for all $n \geq 1$.

Next, let $F \in S^p_{\mathcal{K}}$, and for each $t \in \mathbb{R}$ define $G_t \in S^p_{\mathcal{K}}$ by $G_t = x + t(F - F(0))$. Note that

$$\|G_t\|_{S^p_{\mathcal{K}}} = \|x\|_{\mathcal{K}} + |t|\,\|F'\|_{H^p_{\mathcal{K}}} = 1 + |t|\,\|F'\|_{H^p_{\mathcal{K}}},$$

$$\|TG_t\|_{S^p_{\mathcal{K}}} = \left\|(Tx)(0) + t[TF(0) - (TF(0))(0)]\right\|_{\mathcal{K}} + |t|\,\|(TF)'\|_{H^p_{\mathcal{K}}}.$$

Now $\|TG_t\|_{S^p_{\mathcal{K}}} = \|G_t\|_{S^p_{\mathcal{K}}}$ is equivalent to

$$1 + |t|\left(\|F'\|_{H^p_{\mathcal{K}}} - \|(TF)'\|_{H^p_{\mathcal{K}}}\right) = \|(Tx)(0) + t[TF(0) - (TF(0))(0)]\|_{\mathcal{K}}$$

and, since $(Tx)(0) \neq 0$, the expression on the right-hand side is a differentiable function of t at 0. We conclude that

$$\|(TF)'\|_{H^p_{\mathcal{K}}} = \|F'\|_{H^p_{\mathcal{K}}}$$

and

$$TF(0) = (TF(0))(0) = TF(0).$$

Denote by $zS^p_{\mathcal{K}}$ the subspace of $S^p_{\mathcal{K}}$ consisting of those functions in $S^p_{\mathcal{K}}$ that vanish at 0. Letting D denote the differentiation operator and I the integration operator $IF(z) = \int_0^z F(\xi)d\xi$, it is easy to show that $D : zS^p_{\mathcal{K}} \to H^p_{\mathcal{K}}$ is a surjective isometry whose inverse is I. But then the operator $DTI : H^p_{\mathcal{K}} \to H^p_{\mathcal{K}}$ is a surjective isometry. It follows from Theorem 7 that there exist a unitary operator U on $\mathcal{K}$ and a $\theta \in \mathbb{R}$ such that

$$DTI(F)(z) = (\varphi_a'(z))^{\frac{1}{p}} U \cdot F(e^{i\theta}\varphi_a(z)),$$

for all $F \in H^p_{\mathcal{K}}$. But $I(F') = F - F(0)$, $TI(F') = TF - TF(0)$ and $DTI(F') = (TF)'$, since $TF(0)$ is a constant function. Hence

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$$TF(z) = \int_0^z \left(\varphi_a'(\xi)\right)^{\frac{1}{p}} U \cdot F\left(e^{i\theta}\varphi_a(\xi)\right) d\xi + TF(0).$$

Choosing a unitary operator V on $\mathcal{K}$ such that $VF(0) = TF(0)$, our result follows.

Having characterized the surjective isometries of $S^p_{\mathcal{K}}$, determining when two (bounded) composition operators on $S^p_{\mathcal{K}}$ are isometrically equivalent is relatively straightforward. Our last result extends a theorem from [**8**] dealing with isometric equivalence of composition operators on S^p.

THEOREM 15. *Suppose that φ and ψ are self-maps of the unit disc inducing bounded composition operators on $S^p_{\mathcal{K}}$, where $\mathcal{K}$ is a separable Hilbert space and $p \geq 1$. Then C_φ and C_ψ are isometrically equivalent if and only if there exists a $\theta \in \mathbb{R}$ such that $\psi(z) = e^{-i\theta}\varphi(e^{i\theta}z)$, for all $z \in \mathbb{D}$.*

Proof. Suppose that $TC_\varphi = C_\psi T$, where $T : S^p_{\mathcal{K}} \to S^p_{\mathcal{K}}$ is the surjective isometry given by

$$TF(z) = VF(0) + U\int_0^z (\eta'(\xi))^{1/p} F'(\eta(\xi)) d\xi.$$

Note that the intertwining is equivalent to the equality of

$$VF(\varphi(0)) + U\int_0^z (\eta'(\xi))^{1/p} (F \circ \varphi)'(\eta(\xi)) d\xi$$

and

$$VF(0) + U\int_0^{\psi(z)} (\eta'(\xi))^{1/p} F'(\eta(\xi)) d\xi,$$

for all $F \in S^p_{\mathcal{K}}$ and $z \in \mathbb{D}$. Differentiating each expression yields the equality

$$(\eta')^{1/p} \cdot (F \circ \varphi)' \circ \eta = (\eta' \circ \psi)^{1/p} \cdot F' \circ (\eta \circ \psi) \cdot \psi'.$$

Choosing a nonzero vector $k \in \mathcal{K}$ and setting $F(z) = zk$, this equality becomes

$$(\eta')^{1/p} \cdot \varphi' \circ \eta = (\eta' \circ \psi)^{1/p} \cdot \psi'.$$

Taking $F(z) = z^2 k$, the equality becomes

$$(\eta')^{1/p} \cdot \varphi \circ \eta \cdot \varphi' \circ \eta = (\eta' \circ \psi)^{1/p} \cdot (\eta \circ \psi) \cdot \psi'.$$

The remainder of the proof of the necessity of the condition now follows as in [**8**]. The proof of sufficiency is almost immediate.

The basic theme of our paper [**8**], inspired by the work of Campbell-Wright on the Hardy spaces, was that on a broad class of Banach spaces of (scalar-valued) analytic functions on the unit disc, isometric equivalence of composition operators is "rare" and occurs only in a trivial sense. We have demonstrated above that this situation persists in a broader context of spaces of vector-valued analytic functions.

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