On Graphical Integration.

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1. The theory of graphical integration is founded on the graphical integration of parabolic arcs. The details and the different technical applications of these problems are developed in the fundamental work of J. Massau[1] ; we find there very simple pure constructive methods for the case of parabolas till the 3rd order.

Massau's investigations were recently described and completed by M. d'Ocagne.[2] The principal idea is the introduction of the " centre d'ordonnée moyenne " (U in the figure), defined by the equation :

$$hY = \int_a^b f(x)dx,$$

where $y = f(x)$ is the equation of the given curve, whose arc is $\widehat{AB}$, h the length of the segment $\overline{ab}$, and Y the ordinate of U (see Fig. 1).

In the case of Π_2 and Π_3 (parabolas of the 2nd and 3rd order), the point U can be constructed in a very easy manner. If C, M, W are the mid-points of $\widehat{AB}$, $\overline{AB}$, and $\overline{TS}$ respectively (T, S being the meeting points of the tangents at A, B with the middle ordinate), we have

$$\overline{MU} = \tfrac{2}{3}\overline{MC} = \tfrac{1}{3}\overline{MW},$$

so that U is the centre of gravity of the triangle ABW.[3]

In this manner the graphical integration of the arc is transformed into the treatment of rectilinear segments. We have indeed

$$\text{area } ab\widehat{BA} = \text{area } abB'A' = \overline{ab} \cdot Y.$$

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[1] *Mémoire sur l'intégration graphique et ses applications (Annales de l'Assoc. des ingénieurs sortis des écoles spéciales de Gand,* 1878–1890), chap. II., § 2, 3, 4.

[2] *Calcul graphique et Nomographie (Encyclop. scientif. ;* Paris, O. Doin, 1908), Part I., chap. II.

[3] See d'Ocagne, p. 95–101.

With the aid of the "polar distance" one gets to any initial point A_1 (indeterminate by reason of the constant of integration) the point B_1 and the directions of the tangents at A_1 and B_1—in other words, the elements of the inscribed and of the circum-
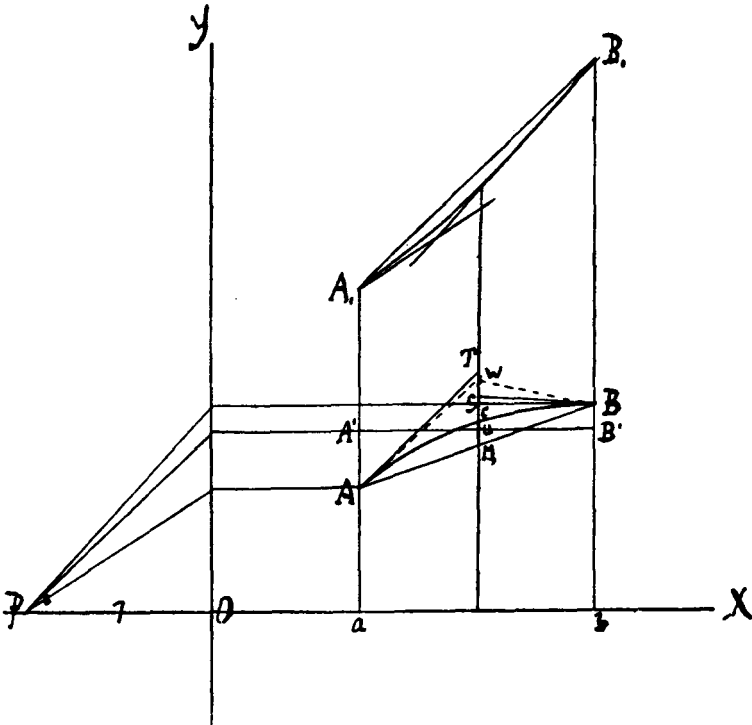

Fig. 1.

scribed polygons. Our Fig. 1 shows the steps of the construction. Similar methods were given by Professor C. Runge in his lectures at the University of Göttingen, and applied to a practical problem by A. Willers in his doctor-dissertation.[4]

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[4] *Die Torsion eines Rotationskörpers um seine Axe* (Göttingen, 1907, p. 14–17; and *Zeitschrift für Math. und Phys.*, 1907).

By interchanging the roles of the axes, one can define in the same way the "centre d'abscisse moyenne," and prove that this centre lies on the same ordinate-line with the point of meeting of the two tangents at A_1 and B_1 respectively.[3]

Massau has shown that the corresponding formulas in the case of Π_4 are more complicated, and do not lead to so simple a method for the construction of these centres. In practice, indeed, arcs of 2nd and 3rd degree are sufficient.

2. In this paper we give a method which is founded on the construction of the ordinate-line of the "centre d'abscisse moyenne." We remarked that this is also the ordinate-line of the meeting point of the two tangents at the end points of the integral curve. Its distance from the ordinate-axis is indeed a characteristical constant for every parabolic arc of a given order. The graphical work connected with this method is not so simple in regard to the execution; it is a natural application of the operations given in the elementary chapters of Graphical Arithmetics. We show that one can proceed in this manner till the case of Π_4 and Π_5; for higher arcs the method looses its simplicity. We could extend the results to higher cases, but only with the application of very complicated constructions of graphical interpolation (*i.e.* the graphical determination of the coefficients of empirically given parabolas).[5] These steps are so long and so difficult that they have no value for practical work.

We fix the principal point of our method in the circumstance that it gives the elements of the inscribed and of the circumscribed polygon at the same time.

3. Let us take

    (*a*) the equation of the given parabola $y = f(x)$;

    (*b*) the equation of the integral curve

$$Y = F(x) = \int f(x)dx + c \ ;$$

    (*c*) -1 and $+1$, the abscissae of the end points of our arc ;

    (*d*) X, the abscissa of the meeting point of the two tangents at the end points of the arc of the integral curve.

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[3] See d'Ocagne, p. 95–101.   [5] See d'Ocagne, p. 75–77.

A simple analytical calculation gives then the following solution

$$X = \frac{Y_{(-1)} - Y_{(+1)} + y_{(-1)} + y_{(+1)}}{y_{(+1)} - y_{(-1)}}$$

By the application of this characteristical formula we find in the different cases :

I. Case of Π_1 : [6]

$y = a_0 x + a_1,$
$X = 0.$

II. Case of Π_2 :

$y = a_0 x^2 + a_1 x + a_2,$

$$X = \frac{2}{3} \frac{a_0}{a_1} = \frac{2}{3} \frac{y_{(+1)} + y_{(-1)} - 2y_{(0)}}{y_{(+1)} - y_{(-1)}}.$$

III. Case of Π_3 :

$y = a_0 x^3 + a_1 x^2 + a_2 x + a_3,$

$$X = \frac{2}{3} \frac{a_1}{a_0 + a_2} = \frac{2}{3} \frac{y_{(+1)} + y_{(-1)} - 2y_{(0)}}{y_{(+1)} - y_{(-1)}}.$$

IV. Case of Π_4 :

$y = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4$

$$X = \frac{2}{15} \frac{6a_0 + 5a_2}{a_1 + a_3} = \frac{1}{15} \left[\frac{8a_0 + 6a_2}{a_1 + a_3} + 4\frac{a_0 + a_2}{a_1 + a_3} \right]$$

$$= \frac{1}{15} \frac{y'_{(+1)} - y'_{(-1)} + 8[y_{(+1)} + y_{(-1)} - 2y_{(0)}]}{y_{(+1)} - y_{(-1)}}.$$

V. Case of Π_5 :

$y = a_0 x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5,$

$$X = \frac{2}{15} \frac{6a_1 + 5a_3}{a_0 + a_2 + a_4} = \frac{1}{15} \frac{y'_{(+1)} - y'_{(-1)} + 8[y_{(+1)} + y_{(-1)} - 2y_{(0)}]}{y_{(+1)} - y_{(-1)}}.$$

Remarks :—

(1) The end formulas are of the same form in the case of Π_2 and Π_3, also Π_4 and Π_5.

(2) In the case of Π_2 and Π_3 the elements of the graphical work are the ordinates of the end points and of the middle point. In the case of Π_4 and Π_5 one uses five data, so that one must firstly draw the tangents at the end points, and determine then with the

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[6] See R. F. Muirhead. " A method of successive graphical integrations " (*Proc. Ed. Math. Soc.*, Vol. XXIX., 1910-11).

aid of the "polar distance" the values of $y'(+1)$ and $y'(-1)$. One finds a simple constructive method for this detail in the work : O. Lesser : "Graphische Darstellungen im Mathematikunterricht der höheren Schulen" (Leipzig, Freytag, 1908), p. 70.

(3) The construction of the value of X, the first step of the graphical work, should be done with the well-known elements of Graphical Arithmetics. This work is not so simple as Massau's constructions are in the corresponding cases, but it reposes on quite natural premises and on the use of the "polar distance."
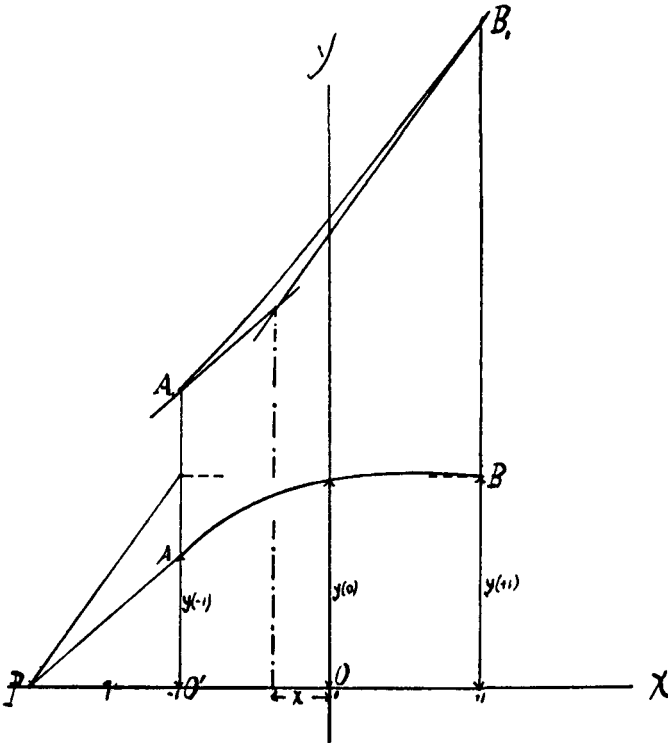

Fig. 2.

(4) Figure 2 shows the steps of the graphical integration after the abscissa X has been separately calculated.