Canad. Math. Bull. Vol. **25** (4), 1982

HAUSDORFF OBSTRUCTIONS TO PACKING $(N-1)$-BALLS IN N-SPACE

BY

J. B. WILKER

ABSTRACT. An arbitrary collection of $(N-1)$-flats in $\mathbb{R}^N$ is called a frame and an arbitrary assignment of $(N-1)$-balls to these $(N-1)$-flats is called a loading. Any loading in which the designated $(N-1)$-balls are mutually disjoint is called a packing. For the frame consisting of the $(N-1)$-flats perpendicular to a given line, every loading is automatically a packing. Although this is obviously not the most general frame to admit a packing, we show two senses in which all frames which admit packings are "at most one-dimensional." Our principal tool is the Hausdorff measure-theoretic dimension.

1. **Introduction.** Let $\mathcal{P}=\{\pi\}$ be the set of all $(N-1)$-flats in Euclidean N-space $(N\geq 2)$ and let $\mathcal{B}=\{\beta\}$ be the set of all $(N-1)$-balls. An arbitrary collection of $(N-1)$-flats $\mathcal{F}\subset\mathcal{P}$ is called a *frame* and an arbitrary assignment of $(N-1)$-balls to these $(N-1)$-flats is called a *loading* of the frame. More precisely, a loading L of the frame $\mathcal{F}$ is a mapping $L:\mathcal{F}\to\mathcal{B}$ such that for every $\pi\in\mathcal{F}$ the $(N-1)$-ball $L(\pi)$ lies in the $(N-1)$-flat π. The loading L is called a *packing* of $\mathcal{F}$ if $L(\pi)\cap L(\pi')=\phi$ wherever π and π' are distinct $(N-1)$-flats in $\mathcal{F}$.

A problem of Erdös [2], [9] indicates that every loading of the full frame $\mathcal{P}$ must fail, in a spectacular way, to be a packing. On the other hand it is possible to construct quite large frames which do admit packings. One such construction [9] uses the first coordinate from a Peano curve to produce a frame with c different $(N-1)$-flats in each of c different directions.

We would like to characterize those frames which do admit packings. As yet we do not have general sufficient conditions for a frame to admit a packing but we do have an infinite collection of necessary conditions. We formulate our result as follows.

For each direction $\mathbf{d}$, let $l(\mathbf{d})$ be the line through the origin with direction $\mathbf{d}$ and let $S(\mathbf{d})\subset l(\mathbf{d})$ be the set of points where the $(N-1)$-flats of $\mathcal{F}$ with normal direction $\mathbf{d}$ meet $l(\mathbf{d})$:

$$S(\mathbf{d})=\{\pi\cap l(\mathbf{d}):\pi\in\mathcal{F},\ \pi\perp\mathbf{d}\}.$$

For each real number p with $0\leq p\leq 1$ let D_p be the set of unit vectors $\mathbf{d}$ for

AMS Subject Classification Primary 52A45 Scondary 28A75.

Received by the editors, July 21, 1980 and, in revised form, December 15, 1980.

which the Hausdorff dimension of the set $S(\mathbf{d})$ is greater than or equal to p:

$$D_p = \{\mathbf{d} : \|\mathbf{d}\| = 1, \dim S(\mathbf{d}) \geq p\}.$$

Since D_p is a subset of the $(N-1)$-sphere either $D_p = \phi$ in which case $\dim D_p = -\infty$ or $D_p \neq \phi$ in which case $0 \leq \dim D_p \leq N-1$. Let $q(p) = \dim D_p$ be the Hausdorff dimension of the set D_p. It is clear that the function $q:[0,1] \to [0, N-1] \cup \{-\infty\}$ depends only on the frame $\mathcal{F}$ and not on the choice of coordinates.

THEOREM 1. *Let $\mathcal{F}$ be a frame in Euclidean N-space $(N \geq 2)$. Suppose that $\mathcal{F}$ admits a packing by $(N-1)$-balls. Then for every real number p with $0 \leq p \leq 1$ we have $p + q(p) \leq 1$.*

If we have a frame with the property that $p + q(p) > 1$ for some p with $0 \leq p \leq 1$ then we say that there is a Hausdorff obstruction to packing in dimension p.

The full frame $\mathcal{P}$ fails to admit a packing because $q(1) = N-1$. The frame consisting of all the $(N-1)$-flats tangent to an $(N-1)$-sphere admits a packing in dimension $N=2$ but not in higher dimensions because $q(0) = N-1$. Two frames which admit a packing in all dimensions but are borderline cases in the sense of Theorem 1 are the parallel pencil consisting of all the $(N-1)$-flats perpendicular to a fixed line $(q(1) = 0)$ and the intersecting pencil consisting of all the $(N-1)$-flats perpendicular to a fixed circle $(q(0) = 1)$. There is an application of Theorem 1 in [10].

It would be interesting to relate the excess, $p + q(p) - 1$, to the extent of forced overlap in loadings of frames which do not admit packings. It would also be interesting to consider analogous questions of loading and packing in codimension greater than one. These questions remain open but in the last section we prove a second result, Theorem 2, which suggests a possible line of attack. This result was obtained first by the referee of an earlier version of this paper who based his argument on the methods of [1], [5] and [7]. See also [3]. We obtain this result differently, by using Lemma 4 which is required for Theorem 1, and thereby illustrate a second method for treating this type of question.

2. **Hausdorff preliminaries.** In this section we introduce some notation and provide a summary of results from Hurewicz and Wallman [6] chapter VII, Rogers [8] and Federer [4] §2.10.

Let (X, d) be a metric space and $h:[0,\infty) \to [0,\infty)$ an arbitrary function. These ingredients allow us to construct the Hausdorff measure $\mu^h : \mathcal{P}(X) \to [0, \infty]$ satisfying

(i) $\mu^h(\phi) = 0$;

(ii) if $A \subset B$, $\mu^h(A) \leq \mu^h(B)$;

(iii) if A_i $(i=1,2,3,\ldots)$ is a sequence of sets $\mu^h(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu^h(A_i)$;
(iv) if $A_1 \subset A_2 \subset A_3 \ldots$ is a nest of sets $\mu^h(\bigcup_{i=1}^{\infty} A_i) = \sup_i \mu^h(A_i)$.

Properties (i)–(iii) are the defining properties of a measure (outer measure) and are immediate from the definition given below. Property (iv) depends on the fact that μ^h is a regular measure.

The details of the construction of μ^h are as follows. For each non-void subset $S \subset X$, diameter $S = d(S) = \sup_{x,y \in S} d(x, y)$; $d(\phi) = 0$. The function h gives a pre-measure on the bounded subsets of X by $h(S) = h(d(S))$ if $S \neq \phi$ and $h(\phi) = 0$. If $\delta > 0$ and $A \subset X$, a δ-cover of A is a sequence of sets S_i $(i=1,2,3,\ldots)$ such that $d(S_i) \le \delta$ and $\bigcup_{i=1}^{\infty} S_i \supset A$. The size-$\delta$ approximating measure μ_δ^h is defined on $\mathscr{P}(X)$ by $\mu_\delta^h(A) = \inf \sum_{i=1}^{\infty} h(S_i)$ where the infimum is taken over all δ-covers of A. (This entails $\mu_\delta^h(A) = \infty$ if A does not admit a δ-cover.) Finally μ^h is defined on $\mathscr{P}(X)$ by $\mu^h(A) = \lim_{\delta \to 0^+} \mu_\delta^h(A)$.

If $h(t) = t^p$ $(0 \le p < \infty)$, $\mu^h = \mu^p$ is called the *Hausdorff p-dimensional measure* or *p-measure* for short. If $p = 0$, the p-measure is just the counting measure. For a fixed set A there is a single number p_0 such that $\mu^p(A) = \infty$ if $p < p_0$ and $\mu^p(A) = 0$ if $p > p_0$. This number $p_0 = \dim A$ is called the *Hausdorff dimension* of A. It is easy to check that for $p < \dim A$, $\mu^p(A)$ is not even σ-finite. We adopt the convention that $\dim \phi = -\infty$.

If (X_1, d_1) and (X_2, d_2) are two metric spaces then a function $f: X_1 \to X_2$ is called Lipschitz if there is a constant $k > 0$ such that for all $x, y \in X_1$, $d_2(f(x), f(y)) \le k d_1(x, y)$. A first step towards Theorem 1 is

LEMMA 1. *Let* $f:[a, b] \to \mathbb{R}^N$ *be a Lipschitz function. Let* $Y = \{y \in \mathbb{R}^N : \dim f^{-1}(y) \ge p\}$ *and suppose* $\dim Y = q$. *Then* $p + q \le 1$.

Proof. We have $\dim[a, b] = 1$ and $\dim f([a, b]) \le 1$ so $p \le 1$ and $q \le 1$. If $p + q > 1$, then $p > 0$, $q > 0$ and we can choose p' and q' with $0 < p' < p$, $0 < q' < q$ and $p' + q' > 1$. This implies that $\mu^{q'}(Y) = \infty$ and for any $y \in Y$, $\mu^{p'}(f^{-1}(y)) = \infty$.

Let $Y_n = \{y \in Y : \mu_{1/n}^{p'}(f^{-1}(y)) > 1\}$. Since $Y_1 \subset Y_2 \subset Y_3 \ldots$ and $\bigcup Y_n = Y$, there is an integer n_1 such that $\mu^{q'}(Y_{n_1}) > (k\sqrt{N} + 1)^N$ and hence an integer n_2 such that $\mu_{1/n_2}^{q'}(Y_{n_1}) > (k\sqrt{N} + 1)^N$ where k is a Lipschitz constant for f.

Let $n \ge \max(n_1, n_2)$ and partition a big cube containing Y into cubes of diameter $1/n$ lying in $[k\sqrt{N} + 1]^N$ families such that any two cubes of the same family are separated by a distance of at least k/n. If ν of these cubes meet Y_n then $\nu(1/n)^{q'} > (k\sqrt{N} + 1)^N$ and there is one family containing ν' cubes such that $\nu'(1/n)^{q'} > 1$. Let $y_1, y_2, \ldots, y_{\nu'}$ be points where these ν' cubes meet Y_n. Since $d(y_i, y_j) > k/n$ no two of these points lie in the image of any sub-interval of $[a, b]$ of length $1/n$.

Partition $[a, b]$ into $\approx n(b-a)$ subintervals of length $1/n$. By summing over

https://doi.org/10.4153/CMB-1982-059-4 Published online by Cambridge University Press

the intervals of this partition we obtain

$$
\begin{aligned}
n(b-a)\frac{1}{n^{p'+q'}} &\approx \sum_I \frac{1}{n^{p'+q'}} \\
&\geq \sum_{i=1}^{\nu'} \sum_{I\cap f^{-1}(y_i)\neq\varnothing} \frac{1}{n^{p'+q'}} \\
&= \sum_{i=1}^{\nu'} \frac{1}{n^{q'}} \sum_{I\cap f^{-1}(y_i)\neq\varnothing} \frac{1}{n^{p'}} \\
&> \sum_{i=1}^{\nu'} \frac{1}{n^{q'}} = \nu'\frac{1}{n^{q'}} > 1.
\end{aligned}
$$

The first inequality holds because we may be dropping certain terms from the sum; the second, because $\mu^{p'}_{1/n}(f^{-1}(y_i))>1$ and the third, because of the argument in the preceding paragraph. On the other hand, since $p'+q'>1$, $n(b-a)\ 1/n^{p'+q'}\to 0$ as $n\to\infty$. This contradiction shows that $p+q\leq 1$ as required.

3. **More lemmas for Theorem 1**

LEMMA 2. (*The first condensation*). *Let* $\mathbf{d}$ *be a unit vector in Euclidean* N*-space*; l, *a line through the origin with direction* $\mathbf{d}$ *and* S *a subset of* l *with Hausdorff dimension* p, $0<p<1$. *Suppose that for every point* $s\in S$ *the* $(N-1)$*-flat* π_s *through* s *perpendicular to* l *carries an* $(N-1)$*-ball* β_s *with centre* $\mathbf{x}(s)$ *and radius* $r(s)>0$. *Then for any number* p' *with* $0<p'<p$ *there is a cylinder* $C(\mathbf{d})$ *with centre* $\mathbf{x}(\mathbf{d})$, *central axis parallel to* $\mathbf{d}$ *and height = radius =* $r(\mathbf{d})>0$ *such that* $\dim S'\geq p'$ *where* $S'=\{s\in S:\pi_s\cap C(\mathbf{d})=\beta_s\cap C(\mathbf{d})\}$.

Proof. Let π^* be the $(N-1)$-dimensional subspace with unit normal $\mathbf{d}$ and let π denote the orthogonal projection of the N-space onto π^*. Let $\mathbf{x}_j$ $(j=1,2,3,\ldots)$ be a sequence which is dense in π^*. For each triple (i,j,k) where i and j are positive integers and k is an integer define

$$
\begin{aligned}
S_{ijk}=\{s\in S:\ &\text{(i)}\ \ r(s)>i^{-1},\\
&\text{(ii)}\ \ \|\pi(\mathbf{x}(s))-\mathbf{x}_j\|<(2i)^{-1}\ \ \text{and}\\
&\text{(iii)}\ \ k(2i)^{-1}\leq\mathbf{x}(s)\cdot\mathbf{d}\leq(k+1)(2i)^{-1}\}.
\end{aligned}
$$

Then S is the union of the countable collection of sets S_{ijk}.

Since $p'<p=\dim S$, $\mu^{p'}(S)$ is not σ-finite and there is a triple (i_0,j_0,k_0) such that $\mu^{p'}(S_{i_0j_0k_0})=\infty$. Let the cylinder $C(\mathbf{d})$ have centre $\mathbf{x}(\mathbf{d})=\mathbf{x}_{j_0}+(k_0+\frac{1}{2})(2i_0)^{-1}\mathbf{d}$ and height = radius = $r(\mathbf{d})=(2i_0)^{-1}$. Then $S'\supset S_{i_0j_0k_0}$ and $\dim S'\geq p'$.

LEMMA 3. (*The second condensation*). *Let* D *be a subset of the unit sphere in Euclidean* N*-space and suppose that* D *has Hausdorff dimension* q, $0<q\leq N-1$. *Suppose that for every point* $\mathbf{d}\in D$ *there is a cylinder* $C(\mathbf{d})$ *in the* N*-space*

with centre $\mathbf{x}(\mathbf{d})$, *central axis parallel to* $\mathbf{d}$ *and height* $\le$ *radius* $= r(\mathbf{d})$, $0 < r(\mathbf{d}) \le \frac{1}{2}$. *Then for any number* q' *with* $0 < q' < q$ *there is a cylinder* C_0 *such that* $\dim D' \ge q'$ *where* $D' = \{\mathbf{d} \in D : C_0$ *pierces both flat faces of* $C(\mathbf{d})\}$.

Proof. The unit sphere in Euclidean N-spaces can be covered by a finite number of caps with angular radius $\theta_0 = \tan^{-1} \frac{1}{2}$ and therefore we may choose a unit vector $\mathbf{d}_0$ such that $\dim D_0 = q$ where $D_0 = \{\mathbf{d} \in D : \mathbf{d} \cdot \mathbf{d}_0 \ge \cos \theta_0\}$. Let π^* be the $(N-1)$-dimensional subspace with unit normal $\mathbf{d}_0$ and let π denote the orthogonal projection of the N-space onto π^*. Then if $\mathbf{x} \in \pi^*$, $\mathbf{d} \in D_0$ and $\|\pi(\mathbf{x}(\mathbf{d})) - \mathbf{x}\| \le \frac{1}{2} r(\mathbf{d})$ it follows that the line through $\mathbf{x}$ parallel to $\mathbf{d}_0$ pierces both flat faces of $C(\mathbf{d})$.

Let $\mathbf{x}_j$ $(j = 1, 2, 3, \ldots)$ be a sequence which is dense in π^*. For each triple of positive integers (i, j, k) define

$$\begin{aligned} D_{ijk} = \{\mathbf{d} \in D_0 \colon\ &\text{(i)}\ \ r(d) > i^{-1} \\ &\text{(ii)}\ \ \|\pi(\mathbf{x}(\mathbf{d})) - \mathbf{x}_j\| \le (4i)^{-1} \quad \text{and} \\ &\text{(iii)}\ \ \|\pi(\mathbf{x}(\mathbf{d})) - \mathbf{x}(\mathbf{d})\| \le k\}. \end{aligned}$$

Then D_0 is the union of the countable collection of sets D_{ijk}.

Since $q' < q = \dim D_0$, $\mu^{q'}(D_0)$ is not σ-finite and there is a triple (i_0, j_0, k_0) such that $\mu^{q'}(D_{i_0 j_0 k_0}) = \infty$. Let the cylinder C_0 have centre $\mathbf{x}_{j_0}$, axis of length $2(k_0 + 1)$ parallel to $\mathbf{d}_0$ and radius $(4i_0)^{-1}$. Then $D' \supset D_{i_0 j_0 k_0}$ and $\dim D' \ge q'$.

LEMMA 4. *Let* C_0 *be a cylinder in Euclidean* N*-space with radius* $\rho > 0$ *and central axis* l_0 *parallel to* $\mathbf{d}_0$ *where* $\|\mathbf{d}_0\| = 1$. *Let* β_1 *and* β_2 *be non-intersecting* $(N-1)$*-balls with normal directions* $\mathbf{d}_1$ *and* $\mathbf{d}_2$ *which cut across* C_0 *and meet* l_0 *at* $\mathbf{x}_1$ *and* $\mathbf{x}_2$. *Then if* $\mathbf{d}_1$ *and* $\mathbf{d}_2$ *are normalized so that* $\mathbf{d}_1 \cdot \mathbf{d}_0 = \mathbf{d}_2 \cdot \mathbf{d}_0 = 1$ *we have* $\|\mathbf{d}_1 - \mathbf{d}_2\| \le \rho^{-1} \|\mathbf{x}_1 - \mathbf{x}_2\|$.

Proof. Without loss of generality we assume that $\mathbf{d}_1 \ne \mathbf{d}_2$ and we work in a 3-space containing $\mathbf{d}_0$, $\mathbf{d}_1$ and $\mathbf{d}_2$. If these vectors are linearly dependent and in particular if $N = 2$, this involves the choice of an arbitrary third dimension.

The planes of β_1 and β_2 are $(\mathbf{x} - \mathbf{x}_1) \cdot \mathbf{d}_1 = 0$ and $(\mathbf{x} - \mathbf{x}_2) \cdot \mathbf{d}_2 = 0$ respectively and these meet in a line m parallel to $\mathbf{d}_1 \times \mathbf{d}_2$. Since β_1 and β_2 are non-intersecting m does not enter C_0 and the distance from m to l_0 must be greater than ρ. Using the standard formula for the distance between skew lines we obtain

$$\left| \frac{(\mathbf{x} - \mathbf{x}_1) \cdot [(\mathbf{d}_1 \times \mathbf{d}_2) \times \mathbf{d}_0]}{\|(\mathbf{d}_1 \times \mathbf{d}_2) \times \mathbf{d}_0\|} \right| \ge \rho$$

where $(\mathbf{x} - \mathbf{x}_1) \cdot \mathbf{d}_1 = (\mathbf{x} - \mathbf{x}_2) \cdot \mathbf{d}_2 = 0$. Because of the normalization of $\mathbf{d}_1$ and $\mathbf{d}_2$ we obtain

$$(\mathbf{d}_1 \times \mathbf{d}_2) \times \mathbf{d}_0 = (\mathbf{d}_1 \cdot \mathbf{d}_0)\mathbf{d}_2 - (\mathbf{d}_2 \cdot \mathbf{d}_0)\mathbf{d}_1 = \mathbf{d}_2 - \mathbf{d}_1$$

https://doi.org/10.4153/CMB-1982-059-4 Published online by Cambridge University Press

and

$$\begin{aligned}(\mathbf{x}-\mathbf{x}_1)\cdot[(\mathbf{d}_1\times\mathbf{d}_2)\times\mathbf{d}_0] &= (\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{d}_2-\mathbf{d}_1)=\mathbf{x}\cdot\mathbf{d}_2-\mathbf{x}_1\cdot\mathbf{d}_2\\ &= \mathbf{x}_2\cdot\mathbf{d}_2-\mathbf{x}_1\cdot\mathbf{d}_2=(\mathbf{x}_2-\mathbf{x}_1)\cdot\mathbf{d}_2=\pm\|\mathbf{x}_2-\mathbf{x}_1\|\,\mathbf{d}_0\cdot\mathbf{d}_2\\ &= \pm\|\mathbf{x}_2-\mathbf{x}_1\|.\end{aligned}$$

With these simplifications, our inequality for the distance from m to l_0 becomes

$$\left|\frac{\pm\|\mathbf{x}_2-\mathbf{x}_1\|}{\|\mathbf{d}_2-\mathbf{d}_1\|}\right|\geq\rho$$

and this yields the desired result.

4. **Proof of Theorem 1.** Let $\mathcal{F}$ be a frame in Euclidean N-space ($N\geq 2$) which admits a packing $L:\mathcal{F}\to\mathcal{B}$ by $(N-1)$-balls. Furure reference to $(N-1)$-balls will be to those in the set $L(\mathcal{F})$. Suppose, contrary to Theorem 1, that there is a real number p with $0\leq p\leq 1$ such that $p+q(p)>1$.

Step 1. If $p=0$, set $p'=p$. For every line $l(\mathbf{d})$ with $\mathbf{d}\in D_p$ set $C(\mathbf{d})$ equal to one of the $(N-1)$-balls perpendicular to $l(\mathbf{d})$. Let $S'(\mathbf{d})$ be the point of intersection of the $(N-1)$-flat of this $(N-1)$-ball with $l(\mathbf{d})$.

If $p>0$, choose p' with $0<p'<p$ and $p'+q(p)>1$. For every line $l(\mathbf{d})$ with $\mathbf{d}\in D_p$ carry out the condensation of Lemma 2 with $S=S(\mathbf{d})$ to obtain $C(\mathbf{d})$ and $S'(\mathbf{d})$.

Step 2. Since $p+q(p)>1$ and $p\leq 1$ we have $0<q(p)\leq N-1$. Choose q' with $0<q'<q$ such that $p'+q'>1$. Carry out the condensation of Lemma 3 using $D=D_p$ and the cylinders $C(\mathbf{d})$, $\mathbf{d}\in D$, which were constructed at Step 1. The result of this construction is a cylinder C_0 with radius $\rho>0$ and finite central axis l_0 parallel to the unit vector $\mathbf{d}_0$.

Step 3. The axis l_0 is isometric to a finite interval $[a,b]$ of the real line. The $(N-1)$-flat tangent to the unit sphere of Euclidean N-space at $\mathbf{d}_0$ is isometric to $\mathbb{R}^{N-1}$. We construct a Lipschitz function $f:[a,b]\to\mathbb{R}^{N-1}$.

If $s\in[a,b]$ corresponds to a point where l_0 is cut by an $(N-1)$-ball with unit normal $\mathbf{d}$ satisfying $\mathbf{d}\cdot\mathbf{d}_0\geq\cos\theta_0$, we define $f(s)=(\mathbf{d}\cdot\mathbf{d}_0)^{-1}\mathbf{d}$. By Lemma 4 this function is Lipschitz with constant ρ^{-1}. We extend it by continuity to the closure of its domain and then by linear interpolation to the rest of $[a,b]$.

Step 4. The sets $S'(\mathbf{d})$ are projected by the $(N-1)$-flats perpendicular to $\mathbf{d}$ to sets of the same dimension on l_0. Also the gnomic projection used in defining f carries subsets of the spherical cap around $\mathbf{d}_0$ to sets of the same dimension in $\mathbb{R}^{N-1}$.

Let p' and q' be as defined in Step 1 and Step 2 respectively. Let $Y=\{y\in\mathbb{R}^{N-1}:\dim f^{-1}(y)\geq p'\}$. Then $\dim Y\geq q'$. The fact that $p'+q'>1$ contradicts Lemma 1 and thereby proves Theorem 1.

5. **A further property of frames which admit packings.** If 0 is an origin for our N-space and π is an $(N-1)$-flat which does not pass through 0, then π is

determined by the point $F(\pi)\in\pi$ which lies at the foot of the perpendicular from 0 to π. If $\mathcal{F}$ is a frame with the property that no $(N-1)$-flat of $\mathcal{F}$ passes through 0 then $\mathcal{F}$ is determined by $F(\mathcal{F})=\{F(\pi):\pi\in\mathcal{F}\}$. By using $N+1$ centres $0_0, 0_1, 0_2, \ldots, 0_N$ which do not lie in a single $(N-1)$-flat we can write an arbitrary frame $\mathcal{F}$ as the union of $N+1$ frames $\mathcal{F}_i=\{\pi\in\mathcal{F}:0_i\notin\pi\}$ of the type described above.

THEOREM 2. *Let $\mathcal{F}$ be a frame determined by the point set $F(\mathcal{F})$. If $\mathcal{F}$ admits a packing then $F(\mathcal{F})$ lies in the union of countably many Lipschitz curves, each of finite length.*

Proof. Let $C_n (n=1,2,3,\ldots)$ be an enumeration of the right-cylinders of length 1 with rational radius $\rho_n<1$, rational centre and rational direction. For each $(N-1)$-flat π of the frame $\mathcal{F}$, the $(N-1)$-ball $L(\pi)$ of the given packing $L(\mathcal{F})$ is pierced at least once by a cylinder of the form C_n. We will use this fact to include $F(\pi)$ in a Lipschitz curve $f_n:[0,1]\to\mathbb{R}^N$.

Let the axis of C_n be written $\mathbf{a}(s)=\mathbf{c}+s\mathbf{d}_0$ $(0\le s\le 1)$. If the axis cuts π at $\mathbf{a}(s)$ and π has normal $\mathbf{d}(s)$ satisfying $\mathbf{d}(s)\cdot\mathbf{d}_0=1$ then Lemma 4 guarantees that the mapping $s\to\mathbf{d}(s)$ is Lipschitz with constant ρ_n^{-1} at the points where it is defined. The mapping $\mathbf{d}$ can be extended to all of $[0,1]$ by continuity and linear interpolation. Then

$$f_n(s)=\frac{\mathbf{a}(s)\cdot\mathbf{d}(s)}{\mathbf{d}(s)\cdot\mathbf{d}(s)}\mathbf{d}(s)$$

is a Lipschitz curve passing through $F(\pi)$.

REFERENCES

1. A. S. Besicovitch, *On the set of directions of linear segments on a convex surface*, Proc. Amer. Math. Soc. Symp. Pure Math., Convexity **7** (1963) 24–25.
2. P. Erdös, P. 236, Can. Math. Bull. **17** (1974) 620 and **19** (1976) 124–125.
3. G. Ewald, D. G. Larman and C. A. Rogers, *The directions of the line segments and of the r-dimensional balls on the boundary of a convex body in Euclidean space*, Mathematika **17** (1970) 1–20.
4. H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.
5. R. J. Gardner, *The measure of boundary sets of families of compact convex sets*, J. London Math. Soc. (2) **9** (1975) 317–325.
6. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, 1964.
7. T. J. McMinn, *On the line segments of a convex surface in E_3*, Pacific J. Math., **10** (1960), 943–946.
8. C. A. Rogers, *Hausdorff Measures*, Cambridge University Press, 1970.
9. J. B. Wilker, *Russian dolls*, Canad. Math. Bull. **21** (1978) 237–240.
10. ——, *Space curves that point almost everywhere*, Trans. Amer. Math. Soc. 250 (1979) 263–274.

UNIVERSITY OF TORONTO
TORONTO, ONTARIO, CANADA