ON REPRESENTATIONS OF ORDERS OVER DEDEKIND DOMAINS

D. G. HIGMAN

We study representations of $\mathfrak{o}$-*orders* $\mathfrak{O}$, that is, of $\mathfrak{o}$-regular $\mathfrak{O}$-algebras, in the case that $\mathfrak{o}$ is a Dedekind domain. Our main concern is with those $\mathfrak{O}$-modules, called $\mathfrak{O}$-*representation modules*, which are regular as $\mathfrak{o}$-modules. For any $\mathfrak{O}$-module M we denote by $D(M)$ the ideal consisting of the elements $x \in \mathfrak{o}$ such that $x \cdot \mathrm{Ext}^1(M, N) = 0$ for all $\mathfrak{O}$-modules N, where $\mathrm{Ext} = \mathrm{Ext}_{(\mathfrak{O},\mathfrak{o})}$ is the relative functor of Hochschild **(5)**. To compute $D(M)$ we need the small amount of homological algebra presented in § 1. In § 2 we show that the $\mathfrak{O}$-representation modules with rational hulls isomorphic to direct sums of right ideal components of the rational hull A of $\mathfrak{O}$, called *principal* $\mathfrak{O}$-modules, are characterized by the property that $D(M) \neq 0$. The $(\mathfrak{O}, \mathfrak{o})$-projective $\mathfrak{O}$-modules are those with $D(M) = \mathfrak{o}$. We observe that $D(M)$ divides the ideal $I(\mathfrak{O})$ of **(2)** for every M, and give another proof of the fact that $I(\mathfrak{O}) \neq 0$ if and only if A is separable. Up to this point, $\mathfrak{o}$ can be taken to be an arbitrary integral domain.

The results of the remaining sections are largely generalizations of Maranda's results for groups **(6, 7)**. In §§ 3-5 ew assume that $\mathfrak{o}$ is a local domain with prime ideal $\mathfrak{p}$, and define the *depth* of an $\mathfrak{O}$-module M to be s or ∞ according as $D(M) = \mathfrak{p}^s$ or 0. In § 3 we generalize Maranda's Theorem 2 of **(6)** by proving that an $\mathfrak{O}$-representation module M of depth s is isomorphic with an $\mathfrak{O}$-representation module N if and only if $M/\mathfrak{p}^{s+1}M$ and $N/\mathfrak{p}^{s+1}N$ are isomorphic. In § 4 it is proved, among other things, that for complete $\mathfrak{o}$, an $\mathfrak{O}$-representation module M has depth s if and only if $M/\mathfrak{p}^{s+1}M$ has depth s. This implies, for example, that M is $(\mathfrak{O}, \mathfrak{o})$-projective if and only if $M/\mathfrak{p}M$ is $(\mathfrak{O}/\mathfrak{p}\mathfrak{O}, \mathfrak{o}/\mathfrak{p})$-projective, a slight improvement of a result of Reiner **(8)**, since the "only if" part does not require a special hypothesis. In § 5, $\mathfrak{o}$ is assumed complete, and the $(\mathfrak{O}, \mathfrak{o})$-projective $\mathfrak{O}$-representation modules are characterized as being isomorphic with direct sums of indecomposable right ideal components of $\mathfrak{O}$. A generalization of Maranda's Theorem 4 of **(7)** states that if $I(\mathfrak{O}/\mathfrak{R}) = \mathfrak{o}$, two $(\mathfrak{O}, \mathfrak{o})$-projective $\mathfrak{O}$-representation modules are isomorphic if and only if their rational hulls are isomorphic. Here $\mathfrak{R}$ is the intersection of $\mathfrak{O}$ with the radical of A.

In the final § 6 we apply the local results to the case of a general Dedekind domain $\mathfrak{o}$, observing that for a principal $\mathfrak{O}$-module M, $D(M) = \Pi_{\mathfrak{p}}\mathfrak{p}^s$, where the product is over all primes $\mathfrak{p}$ of $\mathfrak{o}$, and s is the depth of M in the $\mathfrak{p}$-adic completion of $\mathfrak{o}$. We denote by S_M a complete set of non-isomorphic $\mathfrak{O}$-repre-

Received August 26, 1957; in revised form May 21, 1959.

https://doi.org/10.4153/CJM-1960-010-1 Published online by Cambridge University Press

sentation modules N with rational hulls isomorphic to that of M, and such that $D(N) = D(M)$. Although simple examples show that there may be infinitely many non-isomorphic $\mathfrak{O}$-representation modules with rational hulls isomorphic to a given indecomposable right ideal component of A, it seems possible that the cardinal $r(M)$ of S_M, which we call the *class number* of M, is finite if the class number h of $\mathfrak{o}$ is finite. As was pointed out in **(2)**, Maranda's method for the group case **(7)** can be extended to prove this if A is separable and the rational hull of M is absolutely irreducible. Two members of S_M are placed in the same *genus* if they are isomorphic in the $\mathfrak{p}$-adic completion of $\mathfrak{o}$ for all primes $\mathfrak{p}$ of $\mathfrak{o}$. We denote the number of genera in S_M by $g(M)$, and the number of classes in S_M under isomorphism in the $\mathfrak{p}$-adic completion of $\mathfrak{o}$ by $r_{\mathfrak{p}}(M)$. The final result of this paper is that $g(M) \leqslant \Pi r_{\mathfrak{p}}(M)$, the product extending over the prime divisors of $D(M) \cap I(\mathfrak{O}/\mathfrak{R})$, with the consequence that $g(M)$ is finite when $\mathfrak{o}$ has finite residue class rings.

1. The ideals $D(M)$ and $C(M)$. We need a small amount of homological algebra. For the basic notations and definitions of this subject we refer the reader to **(1** and **5)**. Throughout this paper, rings will be assumed to have identity elements, identity elements of rings and subrings will be assumed to coincide, and modules will be right unitary unless otherwise specified.

Let Q be a K-subalgebra of a K-algebra P, where K is a commutative ring with identity element. For a P-module M, we define ideals $D(M) = D_{(P,Q)}(M)$ and $C(M) = C_{(P,Q)}(M)$ by

$$D(M) = \{x \in K | x\cdot \mathrm{Ext}^1(M, N) = 0 \text{ for all } P\text{-modules } N\}$$

and

$$C(M) = \{x \in K | x\cdot \mathrm{Ext}^1(N, M) = 0 \text{ for all } P\text{-modules } N\},$$

where $\mathrm{Ext} = \mathrm{Ext}_{(P,Q)}$ is the relative functor introduced by Hochschild **(5)**. According to **(5)**, M is (P, Q)-projective (injective) if and only if $\mathrm{Ext}^1(M, N) = 0$ $(\mathrm{Ext}^1(N, M) = 0)$ for all P-modules N, hence

LEMMA 1. *$D(M) = K$, $(C(M) = K)$ if and only if M is, (P, Q)-projective (injective).*

The result we use for computing $D(M)$ in the applications to orders is the following.

LEMMA 2. *An element $x \in K$ belongs to $D(M)$ if and only if there exists a P-homomorphism $\beta : M \to M \otimes_Q P$ such that $\beta\tau = x \cdot I_M$, where $\tau : M \otimes_Q P \to M$ is the natural homomorphism, and I_M is the identity map of M.*

Before proving this we recall that $\mathrm{Ext}^1(M, N)$ can be computed as the first cohomology group of the K-complex $\mathrm{Hom}_P(X, N)$ where X is the left P-complex determined by the standard (P, Q)-projective resolution of M. This resolution is the (P, Q)-exact sequence

$$\ldots \xrightarrow{\chi_1} K_1 \otimes_Q P \xrightarrow{\chi_0} M \otimes_Q P \to 0$$

obtained by composing the natural (P, Q)-exact sequences

$$0 \to K_1 \xrightarrow{\eta_1} M \otimes_Q P \xrightarrow{\tau} M \to 0, 0 \to K_2 \xrightarrow{\eta_2} K_1 \otimes_Q P \xrightarrow{\tau_1} K_1 \to 0 \ldots .$$

In particular, $\chi_1 = \tau_2\eta_2$, so $\chi_1\tau_1 = 0$, and hence τ_1 is a 1-cocycle for $\mathrm{Hom}_P(X, K_1)$.

We shall now prove the following lemma, and then derive Lemma 2 as a Corollary.

LEMMA 3.

$$\begin{aligned} D(M) &= \{x \in K | x \cdot \mathrm{Ext}^1(M, K_1) = 0\} \\ &= \{x \in K | x \cdot \tau_1 \text{ is a coboundary}\}. \end{aligned}$$

Proof. For $\alpha \geqslant 0$, $K_\alpha \otimes_Q P$ is (P, Q)-projective, hence there corresponds to each $g \in \mathrm{Hom}_P(K_\alpha \otimes_Q P, K_1)$ an element $g' \in \mathrm{Hom}_P(K_\alpha \otimes_Q P, K_1 \otimes_Q P)$ such that $g'\tau_1 = g$. If $g'\tau_1 = 0$, then $\mathrm{Im}\, g' \subseteq \mathrm{Ker}\tau_1 = \mathrm{Im}\chi_2$. Hence for a 1-cocycle f of $\mathrm{Hom}_P(X, N)$, N a P-module, we have $g'f = 0$. Therefore mapping g onto $g'f$ defines a map $\mu_{f,\alpha} : \mathrm{Hom}_P(K_\alpha \otimes_Q P, K_1) \to \mathrm{Hom}_P(K_\alpha \otimes_Q P, N)$. These maps are readily seen to define a K-map $\mathrm{Hom}_P(X, K_1) \to \mathrm{Hom}_P(X, N)$. Since $\tau_1\tau_1' = 0$, $\mu_{f,\alpha}(\tau f) = f$. Since f can be taken as an arbitrary 1-cocycle of $\mathrm{Hom}_P(X, N)$, the lemma follows.

Proof of Lemma 2. For P-modules A and B, we shall denote by * the natural K-isomorphism $\mathrm{Hom}_Q(A, B) \approx \mathrm{Hom}_P(A \otimes_Q P, B)$.

Let

$$0 \to M \xrightarrow{\kappa} M \otimes_Q P \xrightarrow{\pi} K_1 \to 0$$

be a Q-homotopy for the sequence

$$0 \to K_1 \xrightarrow{\eta} M \otimes_Q P \xrightarrow{\tau} M \to 0,$$

κ being the natural homomorphism.

If now $x \in D(M)$, there exists by Lemma 3 an element $g \in \mathrm{Hom}_P(M, K_1)$ such that $x \cdot \tau = \chi_0 g^*$. Then, since κ^* is the identity map of $M \otimes_Q P$, and $(g\eta)^* = g^*\eta$,

$$0 = [x \cdot \tau - \chi_0 g^*]\eta = x\tau\eta - \chi_0 g^*\eta = \chi_0[x\kappa^* - g^*\eta] = \chi_0[(x \cdot \kappa) - (g\eta)]^*.$$

It follows that $\beta = x \cdot \kappa - g\eta$ is an element of $\mathrm{Hom}_P(M, M \otimes_Q P)$. Further, $\beta\tau = [(x \cdot \kappa\tau) - g\eta\tau] = x \cdot I_M$.

On the other hand, suppose such a β exists. Let $g = [x \cdot \kappa - \beta]\pi \in \mathrm{Hom}_Q(M, K_1)$. Since $[x \cdot \kappa - \beta]\tau = 0$, $g\eta = x \cdot \kappa - \beta$. Further, $\chi_0\beta^* = \chi_0\tau\beta = 0$. Hence

$$\chi_0 g^*\eta = \chi_0(g\eta)^* = \chi_0[x \cdot \kappa^* - \beta^*] = (x \cdot \tau)\eta,$$

so that $\chi_0 g^* = x \cdot \tau$ and $x \in D(M)$ by Lemma 3.

The result corresponding to Lemma 2 for $C(M)$ is

LEMMA 2'. *An element $x \in K$ belongs to $C(M)$ if and only if there exists a P-homomorphism $\gamma \in \mathrm{Hom}_P(\mathrm{Hom}_Q(P, M), M)$ such that $\xi\gamma = x \cdot I_M$, where $\xi : M \to \mathrm{Hom}_Q(P, M)$ is the natural homomorphism.*

This can be proved in the same way as Lemma 2 by first proving the result corresponding to Lemma 3, using the standard (P, Q)-injective resolution of M in place of the projective one. In the following when we state a property of $C(M)$ we shall omit the proof if it is similar to a proof of a corresponding property of $D(M)$.

To tie in the present work with **(2)** we will need the following remarks. Let $R = P \otimes_K P'$, and let S be the natural image in R of $Q \otimes_K P'$, where the $'$ denotes reciprocal ring. For an R-module W, Hochschild **(5)** defines $H^1(P, Q; W)$ to be $\mathrm{Ext}^1_{(R,S)}(P, W)$, P being considered naturally as an R-module. If $K = Q$, $H^1(P, Q; W) = H^1(P, W)$, the right-hand group being taken in the sense of cohomology of K-algebras **(1)**. If M and N are P-modules, $\mathrm{Hom}_K(M, N)$ is given the structure of an R-module, and it is proved that there exists a natural isomorphism

$$H^1(P, Q; \mathrm{Hom}_K(M, N)) \approx \mathrm{Ext}^1{}_{(P,Q)}(M, N)$$

which is readily seen to be a K-isomorphism **(5)**.

We define $D(P, Q) = \{x \in K | x \cdot H^1(P, Q; W) = 0 \text{ for all } R\text{-modules } W\}$. In other words, $D(P, Q) = D_{(R,S)}(P)$. As a consequence of the above isomorphism we have

LEMMA 4. $D(P, Q) \subseteq D(M) \cap C(M)$. *for any P-module M.*

We remark finally that it is natural to define

$$D^i(M) = \{x \in K | x \cdot \mathrm{Ext}^i(M, N) = 0 \text{ for all } P\text{-modules } N\},$$

and to define $C^i(M)$ and $D^i(P, Q)$ similarly $(i = 1, 2, \ldots,)$. Then the reduction theorem **(5)** gives $D^1(M) \subseteq D^2(M) \subseteq \ldots$, and similar inclusions for the others. Moreover, Lemmas 3 and 4 hold for arbitrary D^i, not just for $D = D^1$. The applications in this paper are restricted to the case $i = 1$.

Results equivalent to Lemmas 1-4 were established in **(3)**, but in a form not so convenient for our present purposes as the above.

2. Orders and principal modules. In this section, $\mathfrak{o}$ will denote an integral domain, and $\mathfrak{O}$ will denote an $\mathfrak{o}$-*order*, that is, an $\mathfrak{o}$-algebra which is regular as an $\mathfrak{o}$-module. Here an $\mathfrak{o}$-module is called *regular* if it is finitely generated and torsion free. It will be convenient to refer to an $\mathfrak{o}$-regular $\mathfrak{O}$-module as an $\mathfrak{O}$-*representation module.* We shall in particular determine the $\mathfrak{O}$-representation modules M such that $D(M) \neq 0$ or $C(M) \neq 0$, where

$$D(M) = D_{(\mathfrak{O},\, \mathfrak{o})}(M) \quad \text{and} \quad C(M) = C_{(\mathfrak{O},\, \mathfrak{o})}(M)$$

as defined in § 1.

First we introduce some notations useful here and in the later sections. If L is an integral domain containing $\mathfrak{o}$ as subdomain, we shall refer to the L-order

$$\mathfrak{O}_L = \mathfrak{O} \otimes_{\mathfrak{o}} L$$

as the *L-hull* of $\mathfrak{O}$. The *L-hull* of an $\mathfrak{O}$-module M is defined to be the $\mathfrak{O}_L$-module

$$M \otimes_{\mathfrak{O}} \mathfrak{O}_L = M \otimes_{\mathfrak{o}} L.$$

If k is the quotient field of $\mathfrak{o}$, k-hulls are referred to as *rational hulls*. Two $\mathfrak{O}$-modules M and N will be called *rationally equivalent* if their rational hulls are isomorphic. We shall also say that M is *rationally equivalent to an A-module* V, $A = \mathfrak{O}_k$, if the rational hull of M is isomorphic to V.

The L-hull M_L of an $\mathfrak{O}$-representation module M is an $\mathfrak{O}_L$-representation module, and, moreover, the natural homomorphisms $M \to M_L$ and

$$M \otimes_{\mathfrak{o}} \mathfrak{O} \to [M \otimes_{\mathfrak{o}} \mathfrak{O}]_L = M_L \otimes_L \mathfrak{O}_L$$

are $\mathfrak{O}$-monomorphisms. Hence the following lemma is an almost obvious consequence of Lemma 2.

LEMMA 5. *If L is a ring of quotients of $\mathfrak{o}$ and M is an $\mathfrak{O}$-representation module, then*

$$D_{(\mathfrak{O}_L, L)}(M_L) = L \cdot D_{(\mathfrak{O}, \mathfrak{o})}(M).$$

Similarly

$$C_{(\mathfrak{O}_L, L)}(M_L) = L \cdot C_{(\mathfrak{O}, \mathfrak{o})}(M).$$

An $\mathfrak{O}$-representation module M will be called a *principal $\mathfrak{O}$-module* if it is rationally equivalent to a direct sum of right ideal components of the rational hull A of $\mathfrak{O}$. On the other hand, M will be called *coprincipal* if it is rationally equivalent to a direct sum of A-module components of the A-module $\mathrm{Hom}_k(A, k)$.

THEOREM 1. *An $\mathfrak{O}$-representation module M is principal* (*coprincipal*) *if and only if $D(M) \neq 0$ ($C(M) \neq 0$).*

Proof. According to Lemma 5, $D(M) \neq 0$ is equivalent to $D_{(A,k)}(M_k) = k$, which in turn is equivalent by Lemma 1 to the (A, k)-projectiveness of M_k. But (A, k)-projectiveness coincides with A-projectiveness since k is a field, and it is well known that the A-projective modules are isomorphic with direct sums of right ideal components of A.

COROLLARY 1. $D(M) \neq 0$ ($C(M) \neq 0$) *for every $\mathfrak{O}$-representation module M if and only if the rational hull A of $\mathfrak{O}$ is semi-simple.*

Proof. Every A-representation module is the rational hull of some $\mathfrak{O}$-representation module. Hence by Theorem 1, $D(M) \neq 0$ for every $\mathfrak{O}$-representation module M if and only if every A-representation module is isomorphic with a direct sum of right ideal components of A. The latter condition is equivalent to the semi-simplicity of A.

The ideal $I(\mathfrak{O})$ defined in **(2)** coincides with the ideal $D(\mathfrak{O}, \mathfrak{o})$ as we see from the last part of § 1. The following theorem was proved more directly in **(2)**.

THEOREM 2. *A necessary and sufficient condition for $\mathfrak{O}$ to have separable rational hull is that $I(\mathfrak{O})$ be non-zero.*

Proof. By a theorem of Hochschild, $A = \mathfrak{O}_k$ is separable if and only if $H^1(A, W) = 0$ for all $A \otimes_k A'$-modules W. But $H^1(A, W) = H^1(A, k; W)$, so A is separable if and only if $1 \in D(A, k)$. Using Lemma 2 we readily obtain that $D(A, k) = k \cdot D(\mathfrak{O}, \mathfrak{o}) = k \cdot I(\mathfrak{O})$. Hence $1 \in D(A, k)$ if and only if $I(\mathfrak{O}) \neq 0$.

According to Lemma 4, $I(\mathfrak{O}) \subseteq D(M)$ for every $\mathfrak{O}$-module M. Hence it is a consequence of Theorem 2 that for separable A, $\bigcap D(M) \neq 0$, where the intersection extends over all $\mathfrak{O}$-modules M ($\mathfrak{o}$ regular or not). The result of **(4)** implies the existence of non-separable but semi-simple A such that for every A-representation module V, $\bigcap D(M) \neq 0$, where the intersection extends over all $\mathfrak{O}$-representation modules rationally equivalent to V. In fact, the Theorem of **(4)** implies the existence of such A for which every A-representation module has finite class number. It is of interest that $\bigcap D(M)$ may be zero when the intersection extends over the $\mathfrak{O}$-representation modules M rationally equivalent to a given right ideal component of A. For example, if $\mathfrak{O}$ is taken to be the Z-order of all matrices

$$X = \begin{pmatrix} x & y \\ & z \end{pmatrix}$$

with x, y, and z in the ring Z of rational integers, the $\mathfrak{O}$-representation module M_n corresponding to the matrix representation mapping X onto

$$\begin{pmatrix} x & ny \\ & z \end{pmatrix}$$

for fixed rational integer n has $D(M_n) = nZ$, as is readily seen using Lemma 2. Hence $\bigcap D(M_n) = 0$. Further, every M_n ($n = 1, 2, \ldots,$) is rationally equivalent to the same indecomposable right ideal component of the rational hull A of $\mathfrak{O}$. Of course A is not semi-simple.

The following additional remarks may be in order here. We noted at the end of § 1 that the ideal $I(\mathfrak{O}) = D(\mathfrak{O}, \mathfrak{o})$ is merely the first member of an ascending chain of ideals of $\mathfrak{o}$: $I(\mathfrak{O}) = I^1(\mathfrak{O}) \subseteq I^2(\mathfrak{O}) \subseteq \ldots,\ I^n(\mathfrak{O}) = D^n(\mathfrak{O}, \mathfrak{o})$. If $\mathfrak{o}$ satisfies the ascending chain condition, there is a first n such that $I^n(\mathfrak{O}) = I^{n+1}(\mathfrak{O}) = \ldots$, and it may be of interest to ask what is the

https://doi.org/10.4153/CJM-1960-010-1 Published online by Cambridge University Press

significance of this n for separable A. It follows from a result of **(3)** that $n = 1$ for a group ring $\mathfrak{O}$. Similar remarks apply to $D(M)$ and $C(M)$. It may also be of interest to look in the set T of $\mathfrak{O}$-representation modules rationally equivalent to a given right ideal component of A for those such that $D(M)$ is maximal for $M \in T$. When can we find $D(M) = \mathfrak{o}$, that is, M $(\mathfrak{O}, \mathfrak{o})$-projective? In this regard see Theorem 11 following.

3. The local case. In this and the next two sections we assume that $\mathfrak{o}$ is a local domain, that is, that $\mathfrak{o}$ is a principal ideal domain in which the non-units constitute the unique prime ideal $\mathfrak{p} = \pi\mathfrak{o}$. As always, $\mathfrak{O}$ denotes an $\mathfrak{o}$-order.

For an integer $s \geqslant 0$, $\mathfrak{o}^{(s)}$ will denote $\mathfrak{o}/\pi^s\cdot\mathfrak{o}$, and $\mathfrak{O}^{(s)}$ will denote the $\mathfrak{o}^{(s)}$-algebra $\mathfrak{O}/\pi^s\cdot\mathfrak{O}$. It is the main purpose of this and the next section to study relations between the representation theory of $\mathfrak{O}$, $\mathfrak{O}^{(s)}$, and the rational hull of $\mathfrak{O}$ The results are largely generalizations of results obtained by Maranda **(6**; **7)** for the group case, and extensions of some results of Reiner **(8)** also arise.

The $\mathfrak{o}$-module $M/\pi^s\cdot M$ will be denoted by $M^{(s)}$, and can be considered as an $\mathfrak{o}^{(s)}$-module. If M is an $\mathfrak{O}$-module, so is $M^{(s)}$, and $M^{(s)}$ may be considered as an $\mathfrak{O}^{(s)}$-module. For $f \in \mathrm{Hom}\dagger\,(M, N)$, $f^{(s)}$ will denote[1] the natural image in $\mathrm{Hom}\dagger(M^{(s)}, N^{(s)})$; if f is an $\mathfrak{O}$-homomorphism, so is $f^{(s)}$. We shall say that $\mathfrak{O}$-modules M and N are *isomorphic modulo* $\mathfrak{p}^s$ if $M^{(s)}$ and $N^{(s)}$ are isomorphic as $\mathfrak{O}$-modules or, what is the same thing, as $\mathfrak{O}^{(s)}$-modules.

As in § 1, we may use the standard $(\mathfrak{O}, \mathfrak{o})$-projective resolution of an $\mathfrak{O}$-module M to compute $\mathrm{Ext}^1(M, N)$. Taking into account the natural $\mathfrak{O}$-isomorphisms

$$[M \otimes_{\mathfrak{o}} \mathfrak{O}]^{(s)} \approx M^{(s)} \otimes_{\mathfrak{o}} \mathfrak{O} \approx M^{(s)} \otimes_{\mathfrak{o}^{(s)}} \mathfrak{O}^{(s)},$$

we see that if

$$\ldots \xrightarrow{\chi_1} K_1 \otimes_{\mathfrak{o}} \mathfrak{O} \xrightarrow{\chi_0} M \otimes_{\mathfrak{o}} \mathfrak{O} \to 0$$

is the standard $(\mathfrak{O}, \mathfrak{o})$-projective resolution of M, then

$$\ldots \xrightarrow{\chi_1^{(s)}} [K_1 \otimes_{\mathfrak{o}} \mathfrak{O}]^{(s)} \xrightarrow{\chi_0^{(s)}} [M \otimes_{\mathfrak{o}} \mathfrak{O}]^{(s)} \to 0$$

may be identified with the standard $(\mathfrak{O}, \mathfrak{o})$-projective resolution of $M^{(s)}$ considered as an $\mathfrak{O}$-module, or with the standard $(\mathfrak{O}^{(s)}, \mathfrak{o}^{(s)})$-projective resolution of $M^{(s)}$ considered as an $\mathfrak{O}^{(s)}$-module. We shall denote by $X^{(s)}$ the left $\mathfrak{O}$-complex determined by this resolution, and by X the left $\mathfrak{O}$-complex determined by the standard $(\mathfrak{O}, \mathfrak{o})$-projective resolution of M. Then

[1]A dagger on the homomorphism (Hom†) indicates that the homomorphism is taken with respect to the domain $\mathfrak{o}$.

https://doi.org/10.4153/CJM-1960-010-1 Published online by Cambridge University Press

$$\mathrm{Ext}^1_{(\mathfrak{O},\,\mathfrak{o})}(M^{(s)}, N^{(s)})$$

is the 1-dimensional cohomology group of Hom‡ $(X^{(s)}, N^{(s)})$, and

$$\mathrm{Ext}^1_{(\mathfrak{O}^{(s)},\,\mathfrak{o}^{(s)})}(M^{(s)}, N^{(s)})$$

is the 1-dimensional cohomology group of

$$\mathrm{Hom}_{\mathfrak{O}^{(s)}}(X^{(s)}, N^{(s)}),$$

which is merely Hom† $(X^{(s)}, N^{(s)})$ considered as an $\mathfrak{o}^{(s)}$-complex.[2] Thus

$$\mathrm{Ext}^1_{(\mathfrak{O}^{(s)},\,\mathfrak{o}^{(s)})}(M^{(s)}, N^{(s)})$$

is simply

$$\mathrm{Ext}^1_{(\mathfrak{O},\,\mathfrak{o})}(M^{(s)}, N^{(s)})$$

considered as an $\mathfrak{o}^{(s)}$-module. It follows that

$$D_{(\mathfrak{O}^{(s)},\,\mathfrak{o}^{(s)})}(M^{(s)}) = [D_{(\mathfrak{O},\,\mathfrak{o})}(M^{(s)}) + \mathfrak{p}^{(s)}]/\mathfrak{p}^{(s)},$$

a fact that is also readily seen from Lemma 2.

We now prove the following generalization of Maranda's Theorem 2 of **(6)**.

THEOREM 3. *Let M and N be $\mathfrak{O}$-representation modules, and assume that*

$$\pi^s \cdot \mathrm{Ext}^1_{(\mathfrak{O},\,\mathfrak{o})}(M, N) = 0.$$

Then M and N are isomorphic if and only if they are isomorphic modulo $\mathfrak{p}^{s+1}$.

Proof. An $\mathfrak{O}$-isomorphism $M^{(s+1)} \approx N^{(s+1)}$ is induced by an $\mathfrak{o}$-isomorphism $\beta : M \approx N$, for M and N have free $\mathfrak{o}$-module bases since $\mathfrak{o}$ is a principal ideal domain. Then for $u \in M$, $\omega \in \mathfrak{O}$, $\beta(u\omega) - \beta(u)\omega \in \pi^{s+1}\cdot N$. Let

$$*: \mathrm{Hom}_{\mathfrak{o}}(M, N) \approx \mathrm{Hom}_{\mathfrak{O}}(M \otimes_{\mathfrak{o}} \mathfrak{O}, N)$$

be the natural isomorphism, then this identity means that $\chi_0\beta^* \equiv 0 \pmod{(\pi^{s+1})}$. Hence there exists

$$f \in \mathrm{Hom}_{\mathfrak{o}}(M \otimes_{\mathfrak{o}} \mathfrak{O}, N)$$

such that

$$\chi_0\beta^* = \pi^{s+1}\cdot f.$$

Since $\chi_{\mathfrak{o}}$ and β^* are $\mathfrak{O}$-homomorphisms, so is f. And

$$0 = \chi_i\chi_0\beta^* = \pi^{s+1}\chi_1 f,$$

[2]A double dagger on the homomorphism (Hom‡) indicates that the homomorphism is taken with respect to the order $\mathfrak{O}$.

so $\chi_1 f = 0$ and f is a 1-cocycle. The assumption of the theorem therefore implies that $\pi^s f$ is a coboundary, and hence that there exists $g \in \text{Hom}\dagger\,(M, N)$ such that $\pi^s \cdot f = \chi_0 g^*$. Let $\alpha = \beta - \pi \cdot g$, then $\chi_0 \alpha^* = \chi_0 \beta^* - \pi \chi_0 g^* = \pi^{s+1} \cdot f - \pi(\pi^s f) = 0$, which implies that α is an element of $\text{Hom}\dagger\,(M, N)$. Since β induces an isomorphism $M^{(s+1)} \approx N^{(s+1)}$, $\det \beta \notin \mathfrak{p}$. But $\det \alpha \equiv \det \beta \pmod{\mathfrak{p}}$, so $\det \alpha \notin \mathfrak{p}$. Hence $\det \alpha$ is a unit in $\mathfrak{o}$ and $\alpha : M \approx N$.

Since the converse is immediate, the proof of Theorem 3 is complete.

We now define the *depth* (*codepth*) of an $\mathfrak{O}$-module M to be s if $D(M) = p^s$ $(C(U) = p^s)$ and ∞ if $D(M) = 0$ $(C(M) = 0)$. Thus by Theorem 1, the principal $\mathfrak{O}$-modules are the $\mathfrak{O}$-representation modules of finite depth (codepth).

An immediate consequence of this definition of depth and Theorem 3 is

COROLLARY 1. *A principal* (*coprincipal*) *$\mathfrak{O}$-module of depth s* (*codepth s*) *is isomorphic with an $\mathfrak{O}$-representation module N if and only if M and N are isomorphic modulo $\mathfrak{p}^{s+1}$.*

Simple examples of the sort given at the end of § 2 show that the number of non-isomorphic $\mathfrak{O}$-representation modules rationally equivalent to a given $\mathfrak{O}_k$-representation module V may be infinite, even if V is an indecomposable right ideal component of $\mathfrak{O}_k$ and $\mathfrak{o}$ has finite residue class rings. But we do have

COROLLARY 2. *If $\mathfrak{o}^{(s+1)}$ is finite, the number of non-isomorphic $\mathfrak{O}$-representation modules of depth* (*codepth*) *s and given rank is finite.*

Proof. Corollary 1 implies that the isomorphism class of M is determined by the isomorphism class of $M^{(s+1)}$. But $\mathfrak{O}^{(s+1)}$ and $M^{(s+1)}$ have only finitely many elements as finitely generated modules over the finite ring $\mathfrak{o}^{(s+1)}$.

If the rational hull of $\mathfrak{O}$ is separable, $I(\mathfrak{O}) = D(\mathfrak{O}, \mathfrak{o})$ is non-zero by Theorem 2, hence $I(\mathfrak{O}) = p^t$. We call t the *depth* of $\mathfrak{O}$. By Lemma 4, $I(\mathfrak{O}) \subseteq D(M)$ for every $\mathfrak{O}$-module M, hence $0 \leqslant \text{depth } M \leqslant t$ for every $\mathfrak{O}$-module M. Hence in this case Corollary 2 implies

COROLLARY 3. *If $\mathfrak{O}$ has separable rational hull and $\mathfrak{o}$ has finite residue class rings, then the number of non-isomorphic $\mathfrak{O}$-representation modules of given rank is finite.*

We now show that depth is preserved under the transition to the complete case. Let k_* denote the completion of the quotient field k of $\mathfrak{o}$ with respect to the valuation determined by $\mathfrak{p}$. Let $\mathfrak{o}_*$ be the valuation ring of k_*, and let $\mathfrak{p}_* = \pi \mathfrak{o}_*$ be the valuation ideal. We denote by $\mathfrak{O}_*$ and M_* the $\mathfrak{o}_*$- hulls of $\mathfrak{O}$ and M respectively.

THEOREM 4. *For an $\mathfrak{O}$-representation module M,*

$$D_{(\mathfrak{O},\, \mathfrak{o})}(M) = D_{(\mathfrak{O}_*,\, \mathfrak{o}_*)}(M_*) \cap \mathfrak{o}_*$$

and

https://doi.org/10.4153/CJM-1960-010-1 Published online by Cambridge University Press

$$C_{(\mathfrak{O},\, \mathfrak{o})}(M) = C_{(\mathfrak{O}_*,\, \mathfrak{o}_*)}(M_*) \cap \mathfrak{o}_*.$$

Proof. We use Lemma 2, according to which $x \in \mathfrak{o}$ belongs to $D(M)$ if and only if there exists an $\mathfrak{O}_k$-homomorphism

$$\beta : M_k \to [M \otimes_{\mathfrak{o}} \mathfrak{O}]_k$$

such that

(i) $\beta\tau_k = x \cdot I$, where

$$\tau_k : [M \otimes_{\mathfrak{o}} \mathfrak{O}]_k \to M_k$$

is induced by the natural homomorphism

$$\tau : M \otimes_{\mathfrak{o}} \mathfrak{O} \to M,$$

and I is the identity map of M_k, and

(ii) $\beta(M) \subseteq M \otimes_{\mathfrak{o}} \mathfrak{O}$, where M and $M \otimes_{\mathfrak{o}} \mathfrak{O}$ are identified with their natural images in M_k and $[M \otimes_{\mathfrak{o}} \mathfrak{O}]_k$ respectively.

Since $\mathfrak{o}$ is a principal ideal domain, there exist free $\mathfrak{o}$-module bases $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ of M and $M \otimes_{\mathfrak{o}} \mathfrak{O}$ respectively. If we write $\beta(u_i) = \sum a_{ij}v_j$, then for given $x \in \mathfrak{o}$, (i) may be expressed as a system of linear equations in the unknowns a_{ij}, with coefficients in $\mathfrak{o}$. The condition that x be in $D(M_*)$ means that the system has a solution in $\mathfrak{o}_*$. But a solution exists in $\mathfrak{o}_*$ if and only if one exists in $\mathfrak{o}$, that is, if and only if $x \in D(M)$, proving Theorem 4.

According to the definition of depth given above, Theorem 4 can be restated as

COROLLARY 1. *The depth of an $\mathfrak{O}$-representation module M is equal to the depth of its o^*-hull M^*.*

An important consequence for our purposes is the following extension of Corollary 1 to Theorem 1 of Maranda's paper **(7)**.

COROLLARY 2. *A principal (coprincipal) $\mathfrak{O}$-module M is isomorphic with an $\mathfrak{O}$-representation module N if and only if M_* and N_* are isomorphic.*

Proof. Suppose M has depth s, then $D(M_*) = \pi^s\mathfrak{o}_*$ by Corollary 1. Now $\mathfrak{o}^{(s+1)} \approx \mathfrak{o}_*^{(s+1)}$, and, as $\mathfrak{o}^{(s+1)}$-algebras, $\mathfrak{O}^{(s+1)} \approx \mathfrak{O}_*^{(s+1)}$. Further, as $\mathfrak{O}^{(s+1)}$-modules, $M^{(s+1)} \approx M_*^{(s+1)}$. The result now follows by Corollary 1 to Theorem 3.

We remark that a similar application of Lemma 2 proves that $I(\mathfrak{O}) = I(\mathfrak{O}_*) \cap \mathfrak{o}$, and hence that the depth of $\mathfrak{O}$ is equal to that of $\mathfrak{O}_*$.

4. The complete case. We retain the notation of § 3, and assume in addition that k is *complete*, that is, that $k = k_*$.

We need the following remark, which depends only on the fact that $\mathfrak{o}$ is a principal ideal domain.

LEMMA 6. *Given $\mathfrak{O}$-representation modules M and N, and $s \geqslant 0$, every $\mathfrak{O}$-homomorphism*

$$[M \otimes_{\mathfrak{o}} \mathfrak{O}]^{(s)} \to N^{(s)}$$

is induced by an $\mathfrak{O}$-homomorphism $M \otimes_{\mathfrak{o}} \mathfrak{O} \to N$.

Proof. Denote by * the natural isomorphism

$$\mathrm{Hom}_{\mathfrak{o}}(U, V) \approx \mathrm{Hom}_{\mathfrak{O}}(U \otimes_{\mathfrak{o}} \mathfrak{O}, V)$$

for any two $\mathfrak{O}$-modules U and V. If

$$g \in \mathrm{Hom}_{\mathfrak{O}}([M \otimes_{\mathfrak{o}} \mathfrak{O}]^{(s)}, N^{(s)}),$$

then $g = h^*$ for some $h \in \mathrm{Hom}^\dagger (M^{(s)}, N^{(s)})$. Since $\mathfrak{o}$ is a principal ideal domain, there exists $f \in \mathrm{Hom}^\dagger (M, N)$ inducing h. But then

$$f^* \in \mathrm{Hom}_{\mathfrak{O}}(M \otimes_{\mathfrak{o}} \mathfrak{O}, N)$$

induces $h^* = g$. Here we have used the natural identification of

$$M^{(s)} \otimes_{\mathfrak{o}} \mathfrak{O} \quad \text{with} \quad [M \otimes_{\mathfrak{o}} \mathfrak{O}]^{(s)}.$$

A homological extension of a result of Reiner **(8)** is the following.

THEOREM 5. *If M and N are $\mathfrak{O}$-representation modules, then*

$$\pi^s \cdot \mathrm{Ext}^1 (M^{(s+1)}, N^{(s+1)}) = 0$$

implies $\pi^s \cdot \mathrm{Ext}^1(M, N) = 0$.

Proof. Suppose that $\pi^s \cdot \mathrm{Ext}^1(M^{(s+1)}, N^{(s+1)}) = 0$, and let X be the left $\mathfrak{O}$-complex determined by the standard $(\mathfrak{O}, \mathfrak{o})$-projective resolution of M. If $f : K_1 \otimes_{\mathfrak{o}} \mathfrak{O} \to N$ is a 1-cocycle for $\mathrm{Hom}^\ddagger(X, N)$, then $f^{(s+1)}$ is a 1-cocycle for $\mathrm{Hom}^\ddagger(X^{(s+1)}, N^{(s+1)})$. (The notations X and $X^{(s+1)}$ refer to the left $\mathfrak{O}$-complexes obtained from the standard $(\mathfrak{O}, \mathfrak{o})$-projective resolutions of M and $M^{(s+1)}$ respectively, cf. the third paragraph of § 3.) Hence $\pi^s \cdot f^{(s+1)}$ is a coboundary, which means, using Lemma 6, that there exists an $\mathfrak{O}$-homomorphism $g_0 : M \otimes_{\mathfrak{o}} \mathfrak{O} \to N$ such that

$$\pi^s \cdot f \equiv \chi_0 g_0 \pmod{(\pi^{s+1})},$$

that is, there exists an $\mathfrak{o}$-homomorphism $f_1 : K_1 \otimes_{\mathfrak{o}} \mathfrak{O} \to N$ such that

$$\pi^s \cdot f = \chi_0 g_0 + \pi^{s+1} \cdot f_1.$$

Since f and $\chi_0 g_0$ are $\mathfrak{O}$-homomorphisms, so is f_1. Since f is a cocycle,

$$\pi^{s+1} \cdot \chi_1 f_1 = \pi^s \cdot \chi_1 f - \chi_1 \chi_0 g_0 = 0,$$

so f_1 is a cocycle. Hence, repetition of the above produces an $\mathfrak{O}$-homomorphism $g_1 : M \otimes_{\mathfrak{o}} \mathfrak{O} \to N$ and an $\mathfrak{o}$-homomorphism $f_2 : K_1 \otimes_{\mathfrak{o}} \mathfrak{O} \to N$ such that $\pi^s \cdot f_1 = \chi_0 g_1 + \pi^{s+1} \cdot f_2$. Then

$$\pi^s \cdot f = \chi_0(g_0 + \pi \cdot g_1) + \pi^{s+2} \cdot f,$$

and again we see that f_2 is a cocycle. Continuing in this way we obtain $\mathfrak{O}$-homomorphisms

$$g_i : K_1 \otimes_{\mathfrak{o}} \mathfrak{O} \to N (i = 0, 1, \ldots,)$$

such that

$$\pi^s \cdot f \equiv \chi_0(g_0 + \pi \cdot g_1 + \ldots + \pi^i \cdot g_i) \pmod{\mathfrak{p}^{s+i+1}}.$$

Since k is complete, we may define $g = g_0 + \pi \cdot g_1 + \ldots + \pi^i \cdot g_i + \ldots$, and conclude that $\pi^s \cdot f = \chi_0 g$ is a coboundary. Hence $\pi^s \cdot \mathrm{Ext}^1(M, N) = 0$, proving Theorem 5.

Corollary 1. *An $\mathfrak{O}$-representation module M has depth (codepth) s if and only if $M^{(s+1)}$ has depth (codepth) s (as an $\mathfrak{O}$-module).*

Proof. By Theorem 5, if $M^{(s+1)}$ has depth s, M has depth $\leqslant s$. But it follows at once from Lemma 2 and the existence of the natural isomorphism

$$[M \otimes_{\mathfrak{o}} \mathfrak{O}]^{(s)} \approx M^{(s)} \otimes_{\mathfrak{o}} \mathfrak{O}$$

that the depth of M is $\leqslant$ the depth of $M^{(t)}$ for any t.

Since by Lemma 1 the $\mathfrak{O}$-modules of depth O (codepth O) are precisely the $(\mathfrak{O}, \mathfrak{o})$-projective (injective) ones, the case $s = 0$ of Corollary 1 gives

Corollary 2. *An $\mathfrak{O}$-representation module M is $(\mathfrak{O}, \mathfrak{o})$-projective (injective) if and only if $M/\pi \cdot M$ is $(\mathfrak{O}, \mathfrak{o})$-projective (injective).*

This is a slight improvement of a result of Reiner **(8)** since the "only if" part does not require special hypotheses. Note the $(\mathfrak{O}, \mathfrak{o})$-projectiveness and $(\mathfrak{O}/\pi \cdot \mathfrak{O}, \mathfrak{o}/\pi \cdot \mathfrak{o})$-projectiveness coincide for an $\mathfrak{O}/\pi \cdot \mathfrak{O}$-module.

We observe, without including the details, that essentially the same argument used to prove Theorem 5 and Corollary 1 proves

Theorem 6. *$\mathfrak{O}$ has depth t if and only if $D(\mathfrak{O}^{(t+1)}, \mathfrak{o}) = \mathfrak{p}^t(\mathfrak{O}^{(t+1)}$ being considered as an $\mathfrak{o}$-algebra).*

The case $t = O$, combined with Hochschild's characterization of separable algebras gives the

Corollary: *$\mathfrak{O}$ has depth O if and only if $\mathfrak{O}/\pi \cdot \mathfrak{O}$ is a separable $\mathfrak{o}/\pi \cdot \mathfrak{o}$-algebra.*

For application in § 5 we need

Theorem 7 (Brauer). *If H is an $(\mathfrak{O}, \mathfrak{o})$-projective (injective) $\mathfrak{O}$-representation module, and if U is an $\mathfrak{O}$-module direct summand of $H/\pi \cdot H$, then there exists an $\mathfrak{O}$-module direct summand M of H such that $M/\pi \cdot M \approx U$.*

We will derive this theorem here as a corollary to an extension of a variant of Maranda's Theorem 3 of **(6)**.

If M is a primitive $\mathfrak{o}$-submodule of an $\mathfrak{O}$-representation module H (that is, an $\mathfrak{o}$-module direct summand), we may identify $M^{(s)} = M/\pi\cdot M$ with the $\mathfrak{o}$-submodule $(M + \pi^s\cdot H)/\pi^s\cdot H$ of $H^{(s)}$, and then we may identify $[H/M]^{(s)}$ with $H^{(s)}/M^{(s)}$. Then $M^{(s)}$ is an $\mathfrak{O}$-submodule of $H^{(s)}$ if and only if $M\mathfrak{O} \subseteq M + \pi^s\cdot H$, and every $\mathfrak{o}$-primitive $\mathfrak{O}$-submodule of $H^{(s)}$ is obtained in this way from a primitive $\mathfrak{o}$-submodule of H.

THEOREM 8. *Let M be a primitive $\mathfrak{o}$-submodule of an $\mathfrak{O}$-representation module H, such that $M\mathfrak{O} \subseteq M + \pi^{s+1}\cdot H$ and $M^{(s+1)}$ has depth s as an $\mathfrak{O}$-module. Then there exists an $\mathfrak{O}$-submodule M^* of H of depth s, and primitive as an $\mathfrak{o}$-submodule, such that $M^{*(s+1)} \approx M^{(s+1)}$.*

Proof. We construct a sequence $M_0, M_1, \ldots, M_i, \ldots$, of primitive $\mathfrak{o}$-submodules of H such that $M_i\mathfrak{O} \subseteq M_i + \pi^{s_i}\cdot H$, $s_0 = s+1$, $s_{i+1} = s_i + 1$, and $M_{i+1}{}^{(s+1)} \approx M_i{}^{(s+1)}$. The existence of an $\mathfrak{o}$-primitive $\mathfrak{O}$-submodule M^* of H with $M^{*(s+1)} \approx M^{(s+1)}$ then follows by the completeness of k. Since $M^{(s+1)}$ has depth s, Corollary 1 to Theorem 5 implies that M^* also has depth s.

The M_i are constructed inductively. Let $M_0 = M$, and assume that M_i has been constructed. Since M_i is primitive, there exists an $\mathfrak{o}$-submodule N of H such that $H = M_i \oplus N$. Then

$$M_i\mathfrak{O} \subseteq M_i + \pi^{s_i}\cdot H = M_i \oplus \pi^{s_i}\cdot N.$$

Thus, for $u \in M_i$ and $\omega \in \mathfrak{O}$, there exist unique elements $\sigma(u, \omega) \in M_i$ and $\tau(u, \omega) \in N$ such that

$$u\omega = \sigma(u, \omega) + \pi^{s_i}\cdot\tau(u, \omega).$$

We denote by T the $\mathfrak{O} \otimes_{\mathfrak{o}} \mathfrak{O}'$-module

$$\operatorname{Hom}_{\mathfrak{o}}(M_i^{(s+1)}, [H/M_i]^{(s+1)}),$$

and by τ^+ the element of $\operatorname{Hom}\dagger(\mathfrak{O}, T)$ such that $\tau^+(\omega)\{\bar{u}\}$ is the residue class modulo $\pi^{s+1}\cdot[H/M_i]$ of $M_i + \tau(u, \omega) \in H/M_i$, where $\bar{u}$ is the residue class modulo $\pi^{s+1}\cdot M_i$ of $u \in M_i$. From the associative law $u(\xi\eta) = (u\xi)\eta$ we get the identity

$$\tau(u, \xi\eta) = \tau(\sigma(u, \xi), \eta) + \tau(u, \xi)\eta,$$

which means that τ^+ is a 1-cocycle for the complex with homogeneous components $C^n(\mathfrak{O}, \mathfrak{o}; T)$ described in **(6**, § 3**)**. Since the 1-dimensional cohomology group of this complex is $\operatorname{Ext}^1(M_i{}^{(s+1)}, [H/M_i]^{(s+1)})$, and since $M_i{}^{(s+1)} \approx M^{(s+1)}$ has depth s, it follows that $\pi^s\cdot\tau^+$ is a coboundary. Hence there exists $g \in \operatorname{Hom}_{\mathfrak{o}}(H, N)$ such that for $u \in M_i$, $\omega \in \mathfrak{O}$,

$$\tau(u, \omega) = g(u\omega) - g(u)\omega + \pi^{s+1}\cdot\mu(u, \omega),$$

with $\mu(u, \omega) \in H$. Let $M_{i+1} = \{u + \pi^{s_i - s}\cdot g(u) | u \in M_i\}$, then M_{i+1} is a

https://doi.org/10.4153/CJM-1960-010-1 Published online by Cambridge University Press

primitive $\mathfrak{o}$-submodule of H since $H = M_{i+1} \oplus N$. Since $2s_i - s \geqslant s_i + 1 = s_{i+1}$, we have for $\omega \in \mathfrak{O}$ and $u \in M_i$ that

$$\begin{aligned}[u + \pi^{s_i - s} \cdot g(u)]\omega &= u\omega + \pi^{s_i - s} g(u)\omega \\ &= \sigma(u, \omega) + \pi^{s_i} \cdot \tau(u, \omega) + \pi^{s_i - s}\{g(u\omega) - \pi^s \cdot \tau(u, \omega) \\ &\qquad\qquad + \pi^{s+1} \cdot \mu(u, \omega)\} \\ &= \sigma(u, \omega) + \pi^{s_i - s} \cdot g(\sigma(u, \omega)) + \pi^{2s_i - s} \cdot g(\tau(u, \omega)) \\ &\qquad\qquad + \pi^{s_i + 1}\mu(u, \omega) \in M_{i+1} + \pi^{s_i+1} H.\end{aligned}$$

Hence $M_{i+1}\mathfrak{O} \subseteq M_{i+1} + \pi^{s_i+1} \cdot H$.

Now we define an $\mathfrak{o}$-module automorphism ϕ of H by mapping $u \in M_i$ onto $u + \pi^{s_i - s} g(u)$, and $v \in N$ onto v, so that in particular $\phi(M_i) = M_{i+1}$. For $u \in M_i$ and $\omega \in \mathfrak{O}$.

$$\begin{aligned}\phi(u\omega) = \phi(\sigma(u, \omega) + \pi^{s_i} \cdot \tau(u, \omega)) &= \sigma(u, \omega) + \pi^{s_i - s} g(\sigma(u, \omega)) + \pi^{s_i} \cdot \tau(u, \omega) \\ &\equiv \sigma(u, \omega) + \pi^{s_i - s} \cdot g(\sigma(u, \omega)) \equiv \phi(u)\omega \pmod{(\pi^{s+1})}.\end{aligned}$$

We may conclude that $M_{i+1}{}^{(s+1)} \approx M_i{}^{(s+1)}$, which means that the inductive construction of the M_i is complete.

We can deduce Theorem 7 from Theorem 8 and Corollary 2 to Theorem 5 as follows: Since H is $(\mathfrak{O}, \mathfrak{o})$-projective, so is $H/\pi \cdot H$ by Corollary 2 to Theorem 5, hence so is the $\mathfrak{O}$-module direct summand U. Theorem 8 (with $s = 0$) therefore implies the existence of an $(\mathfrak{O}, \mathfrak{o})$-projective $\mathfrak{O}$-submodule M such that $M/\pi M \approx U$.

Another application of Theorem 8 is the following

COROLLARY. *Let U be an $\mathfrak{O}^{(s+1)}$-module, that is, an $\mathfrak{O}$-module such that $\pi^{s+1} \cdot U = 0$. If U has depth (codepth) s as an $\mathfrak{O}$-module, and is finitely generated and projective as an $\mathfrak{o}^{(s+1)}$-module, there exists an $\mathfrak{O}$-representation module M^* of depth (codepth) s such that $M^{*(s+1)} \approx U$.*

Proof. Since U is a projective and finitely generated $\mathfrak{o}^{(s+1)}$-module, it is an $\mathfrak{o}^{(s+1)}$-module direct summand of a finitely generated free $\mathfrak{o}^{(s+1)}$-module V. Now there exists a regular $\mathfrak{o}$-module N such that $N^{(s+1)} \approx V$. Moreover, there exists an $\mathfrak{O}$-isomorphism $\mathrm{Hom}^\dagger(\mathfrak{O}, V) \approx [\mathrm{Hom}(\mathfrak{O}, N)]^{(s+1)}$, and an $\mathfrak{O}$-monomorphism $U \to \mathrm{Hom}^\dagger[\mathfrak{O}, V)$, namely, the composite $U \to \mathrm{Hom}^\dagger(\mathfrak{O}, U) \to \mathrm{Hom}^\dagger(\mathfrak{O}, V)$. It follows that there exists a primitive $\mathfrak{o}$-submodule M of $H = \mathrm{Hom}^\dagger(\mathfrak{O}, N)$ such that $M\mathfrak{O} \subseteq M + \pi^{s+1}H$ and $M^{(s+1)} \approx U$. The existence of M^* now follows from Theorem 8.

5. Projective modules in the complete case. We assume, as in the preceding section, that the quotient field k of $\mathfrak{o}$ is complete. Applying Theorems 3 and 7 we obtain

THEOREM 9 (Brauer). *Up to isomorphism and order of summands, every $(\mathfrak{O}, \mathfrak{o})$-projective $\mathfrak{O}$-representation module has a unique decomposition into a direct sum of indecomposable $\mathfrak{O}$-modules. If $\mathfrak{O} = \sum \oplus \mathfrak{O}_\alpha$, where the $\mathfrak{O}_\alpha$ are*

https://doi.org/10.4153/CJM-1960-010-1 Published online by Cambridge University Press

indecomposable right ideals, then $\mathfrak{O}/\pi\mathfrak{O} \approx \sum \oplus \mathfrak{O}_\alpha/\pi\mathfrak{O}_\alpha$ *is a decomposition of the* $\mathfrak{o}/\pi\mathfrak{o}$*-algebra* $\mathfrak{O}/\pi\mathfrak{O}$ *into indecomposable right ideals, and any indecomposable* $(\mathfrak{O}, \mathfrak{o})$*-projective* $\mathfrak{O}$*-representation module is isomorphic with one of the* $\mathfrak{O}_\alpha$.

Proof. Since $\mathfrak{O}/\pi\mathfrak{O}$ is a finite dimensional algebra over the field $\mathfrak{o}/\pi\mathfrak{o}$, $\mathfrak{O}/\pi\mathfrak{O} = \sum \oplus \bar{\mathfrak{O}}_\alpha$ with the $\bar{\mathfrak{O}}_\alpha$ indecomposable right ideals unique up to order and isomorphism. By Theorem 7, there exist right ideals $\mathfrak{O}_\alpha$ of $\mathfrak{O}$ such that $\mathfrak{O} = \sum \oplus \mathfrak{O}_\alpha$ and $\mathfrak{O}_\alpha/\pi\mathfrak{O}_\alpha \approx \bar{\mathfrak{O}}_\alpha$. Since the $\bar{\mathfrak{O}}_\alpha$ are indecomposable, Theorem 7 implies that the $\mathfrak{O}_\alpha$ are too.

If M is an $(\mathfrak{O}, \mathfrak{o})$-projective $\mathfrak{O}$-module, so is $M/\pi M$ by Corollary 2 to Theorem 5, and hence $M/\pi M$ is $(\mathfrak{O}/\pi\mathfrak{O}, \mathfrak{o}/\pi\mathfrak{o})$-projective. As is well known, this implies that $M/\pi M = \sum \oplus \bar{M}_\beta$, where each $\bar{M}_\beta$ is isomorphic to some $\bar{\mathfrak{O}}_\alpha$. Hence by Theorem 7, $M = \sum \oplus M_\beta$, where each M_β is isomorphic to some $\mathfrak{O}_\alpha$.

Suppose that $M = \sum \oplus N_\gamma$, with each N_γ indecomposable. Then $N_\gamma/\pi N_\gamma$ is indecomposable by Theorem 7 and $M/\pi M = \sum \oplus N_\gamma/\pi N_\gamma$. Hence by the Krull-Schmidt theorem, the $\bar{M}_\beta$ and $N_\gamma/\pi N_\gamma$ are equal in number and isomorphic in pairs. Corollary 1 to Theorem 3 (with $s = 0$) implies therefore that the M_β and N_γ are isomorphic in pairs.

There is a similar result for $(\mathfrak{O}, \mathfrak{o})$-injective $\mathfrak{O}$-representation modules, which may be proved similarly.

We now generalize Maranda's Theorem 4 of **(7)**. It will be convenient to begin with two lemmas, the second of which is a special case of our thorem.

LEMMA 7. *If* $\mathfrak{O}$ *has depth O, and the* $\mathfrak{O}$*-representation module M has irreducible rational hull, then* $M/\pi M$ *is an irreducible* $\mathfrak{O}/\pi\mathfrak{O}$*-module.*

Proof. If $\mathfrak{O}$ has depth O, every $\mathfrak{O}$-module is $(\mathfrak{O}, \mathfrak{o})$-projective, and by the Corollary to Theorem 6, $\mathfrak{O}/\pi\mathfrak{O}$ is a separable $\mathfrak{o}/\pi\mathfrak{o}$-algebra. Hence $M/\pi M$ is fully reducible, and Theorem 7 implies that it is irreducible.

LEMMA 8. *If* $\mathfrak{O}$ *has depth* O, *two* $\mathfrak{O}$*-representation modules M and N with irreducible rational hulls are isomorphic if and only if their rational hulls are isomorphic.*

Proof. If M and N have isomorphic rational hulls, we may assume that $M \subseteq N$. By Lemma 7, $N/\pi N$ is irreducible, hence $M/\pi M \approx M + \pi N/\pi N = N/\pi N$. Corollary 1 to Theorem 3 implies therefore that $M \approx N$. The converse is immediate.

We let R denote the radical of the rational hull A of $\mathfrak{O}$. Then $\mathfrak{O}/\mathfrak{R}$ is an $\mathfrak{o}$-order with rational hull isomorphic to A/R, where $\mathfrak{R} = \mathfrak{O} \cap R$.

THEOREM 10. *If* $\mathfrak{O}/\mathfrak{R}$ *has depth O, then two* $(\mathfrak{O}, \mathfrak{o})$*-projective* $\mathfrak{O}$*-representation modules are isomorphic if and only if their rational hulls are isomorphic.*

Proof. Only the "if" part requires proof, and by Theorem 9 we have only to consider indecomposable $\mathfrak{O}$-representation modules.

https://doi.org/10.4153/CJM-1960-010-1 Published online by Cambridge University Press

Let M be an indecomposable $(\mathfrak{O}, \mathfrak{o})$-projective $\mathfrak{O}$-representation module, and let $V = M_k$. Then $V = V_1 \oplus \ldots \oplus V_t$, where the V_t may be taken to be indecomposable right ideal components of A according to Theorem 1. Let

$$W_i = X_i + \sum_{j \neq i} V_j,$$

where X_i is the unique maximal A-submodule of V_i. Then $V/W_i \approx V_i/X_i$, the irreducible A-submodule determining the isomorphism class of V_i, which we shall denote by F_i. We show now that $F_i \approx F$ $(i = 1, 2, \ldots, t)$, $F = F_1$.

We note first that $\mathfrak{F}_i = M/M \cap W_i$ is an $\mathfrak{O}$-representation module with rational hull isomorphic to F_i. Since $\mathfrak{F}_i$ can be considered as an $\mathfrak{O}/\mathfrak{N}$-module, Lemma 7 implies that $\mathfrak{F}_i/\pi\mathfrak{F}_i$ is an irreducible $\mathfrak{O}/\pi\mathfrak{O}$-module. Theorem 7 implies that $M/\pi M$ is an indecomposable $\mathfrak{O}/\pi\mathfrak{O}$-module, hence, since $\mathfrak{F}_i/\pi\mathfrak{F}_i$ is isomorphic with a quotient module of $M/\pi M$, it follows that $\mathfrak{F}_i/\pi\mathfrak{F}_i$ is uniquely determined by $M/\pi M$. That is, $\mathfrak{F}_i/\pi\mathfrak{F}_i \approx \mathfrak{F}/\pi\mathfrak{F}$ $(i = 1, 2, \ldots, t)$, where $\mathfrak{F} = \mathfrak{F}_1$. Hence $\mathfrak{F}_i \approx \mathfrak{F}$ by Corollary 1 to Theorem 3, which certainly implies $F_i \approx F$.

Now let N be a second indecomposable $(\mathfrak{O}, \mathfrak{o})$-projective $\mathfrak{O}$-representation module such that $N_k \approx V$, and let $\mathfrak{G}$ correspond to N as $\mathfrak{F}$ does to M. Then we must have $\mathfrak{G}_k \approx F$, which implies $\mathfrak{G} \approx \mathfrak{F}$ by Lemma 8. Hence $\mathfrak{G}/\pi\mathfrak{G} \approx \mathfrak{F}/\pi\mathfrak{F}$, which implies $N/\pi N \approx M/\pi M$. Hence $M \approx N$ by Corollary 1 to Theorem 3.

An immediate consequence of the above proof is the following.

Corollary. *If $\mathfrak{O}/\mathfrak{N}$ has depth O, then the rational hull of an indecomposable $\mathfrak{O}$-representation module M is isomorphic with a direct sum of isomorphic indecomopsable right ideal components of A.*

It must be remarked that the completeness of $\mathfrak{o}$ is inessential for Theorem 10 and its Corollary, for, by Corollary 1 to Theorem 4, and the remark at the end of § 3, the hypotheses survive transition from the local to the complete case, and by Corollary 2 to Theorem 4, if the conclusion holds relative to the completion of a local domain $\mathfrak{o}$, it holds relative to $\mathfrak{o}$.

6. The Dedekind case. In this final section we assume that $\mathfrak{o}$ is a Dedekind domain, and denote by $\mathfrak{o}_\mathfrak{p}$ the ring of quotients of $\mathfrak{o}$ with respect to the complement of the prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, that is, the ring of $\mathfrak{p}$-integers in the quotient field k of $\mathfrak{o}$. By $\mathfrak{O}_\mathfrak{p}$ we denote the $\mathfrak{o}_\mathfrak{p}$-hull of the $\mathfrak{o}$-order $\mathfrak{O}$, and by $M_\mathfrak{p}$ the $\mathfrak{o}_\mathfrak{p}$-hull of an $\mathfrak{o}$-module M. A subscript $*$ refers to the $\mathfrak{p}$-adic completion as at the end of § 3.

According to Theorems 1 and 4, we have for any $\mathfrak{O}$-representation module M that

$$\mathfrak{o}_\mathfrak{p} . D_{(\mathfrak{O}, \mathfrak{o})}(M) = D_{(\mathfrak{O}_\mathfrak{p}, \mathfrak{o}_\mathfrak{p})}(M_\mathfrak{p}) = D_{(\mathfrak{O}_{\mathfrak{p}*}, \mathfrak{o}_{\mathfrak{p}*})}(M_{\mathfrak{p}*}) \cap \mathfrak{o}_\mathfrak{p}.$$

Therefore, if we define $d_\mathfrak{p}(M)$ to be depth $M_\mathfrak{p}$ = depth $M_{\mathfrak{p}*}$, we have

$$D(M) = \prod_\mathfrak{p} \mathfrak{p}^{d_\mathfrak{p}(M)}.$$

Similarly we have

$$C(M) = \prod_{\mathfrak{p}} \mathfrak{p}^{c_{\mathfrak{p}}(M)}$$

where $c_{\mathfrak{p}}(M)$ is defined to be codepth $M_{\mathfrak{p}}$ = codepth $M_{\mathfrak{p}*}$. Some consequences of these formulas are summarized in

THEOREM 11. *An $\mathfrak{O}$-representation module is principal (coprincipal) if and only if its $\mathfrak{o}_{\mathfrak{p}}$-hull $M_{\mathfrak{p}}$ is principal (coprincipal) for every prime $\mathfrak{p}$ of $\mathfrak{o}$. If M is principal (coprincipal), then $M_{\mathfrak{p}}$ is $(\mathfrak{O}_{\mathfrak{p}}, \mathfrak{o}_{\mathfrak{p}})$-projective (injective) for all but the finitely many primes $\mathfrak{p}$ dividing $D(M)$ $(C(M))$, and M is $(\mathfrak{O}, \mathfrak{o})$-projective (injective) if and only if $M_{\mathfrak{p}}$ is $(\mathfrak{O}_{\mathfrak{p}}, \mathfrak{o}_{\mathfrak{p}})$-projective (injective) for all $\mathfrak{p}$.*

These results, together with Corollary 2 of Theorem 5 contain results of Reiner **(8)**.

We obtain in the same way that $I(\mathfrak{O}) = \Pi_{\mathfrak{p}} p^{d_{\mathfrak{p}}(\mathfrak{O})}$, where $d_{\mathfrak{p}}(\mathfrak{O})$ is defined to be depth $\mathfrak{O}_{\mathfrak{p}}$ = depth $\mathfrak{O}_{\mathfrak{p}*}$. From this, Theorem 2, and Corollary 2 to Theorem 2 we conclude that

THEOREM 12. *If $\mathfrak{O}$ has separable rational hull, $\mathfrak{O}/\mathfrak{p}\mathfrak{O}$ is a separable $\mathfrak{o}/\mathfrak{p}$-algebra for all but the finitely many primes $\mathfrak{p}$ dividing $I(\mathfrak{O})$.*

Finally we apply our results to the question of class numbers. Let S be a complete set of non-isomorphic, rationally equivalent $\mathfrak{O}$-representation modules. Then there exists an A-representation module U, $A = \mathfrak{O}_k$, such that $M_k \approx U$ for all M in S (and we may in fact assume that M is an $\mathfrak{O}$-submodule of U for every $M \in S$). The cardinal $r = r(U)$ of the set S is called the *class number of U (with respect to $\mathfrak{O}$)*.

As was pointed out in **(2)**, Maranda's method **(7)** can be extended to prove that if A is separable, every absolutely irreducible A-representation module has finite class number, the ideal $I(\mathfrak{O}) \neq 0$ playing the role played by the group order in his arguments. On the other hand, the result of **(4)** shows the existence of non-separable semi-simple A for which every A-representation module has finite class number.

In the example of § 2, $\mathfrak{o} = Z$, the ring of rational integers, and the M_n $(n = 1, 2, \ldots,)$ are readily seen to constitute a complete set of non-isomorphic $\mathfrak{O}$-representation modules rationally equivalent to a fixed indecomposable right ideal component of A. Hence U has infinite class number. But the fact that $D(M_n) = nZ$ $(n = 1, 2, \ldots,)$ suggests the following definition for the general case: The *class number* $r(M)$ of a principal $\mathfrak{O}$-module M is defined to be the cardinal of the set S_M, where S_M is a complete set of non-isomorphic $\mathfrak{O}$-representation modules N rationally equivalent to M and such that $D(N) = D(M)$. We cannot prove here that $r(M)$ is finite when $\mathfrak{o}$ has finite ideal class number, but we reduce the problem somewhat, along the lines of the first main result of **(7)**.

https://doi.org/10.4153/CJM-1960-010-1 Published online by Cambridge University Press

Note that for an A-representation module U,

$$r(U) = \sum r(M)$$

where the sum is over a set of $M \in S$ such that $D(M)$ takes on exactly once each possible value. In the case of separable A, $I(\mathfrak{O}) \neq O$, and hence, since $D(M)$ divides $I(\mathfrak{O})$ for every M, the number of summands is finite. That the sum need not be finite in the general case is shown by the example mentioned in the preceding paragraph.

We shall say that two $\mathfrak{O}$-representation modules M and N are *equivalent at a prime* $\mathfrak{p}$ of $\mathfrak{o}$ if their $\mathfrak{o}_\mathfrak{p}$-hulls are isomorphic, or what is the same thing by Corollary 2 to Theorem 4, if their $\mathfrak{o}_{\mathfrak{p}*}$-hulls are isomorphic. We shall say that M and N belong to the same *genus* if they are equivalent at all primes $\mathfrak{p}$ of $\mathfrak{o}$. If M is a principal $\mathfrak{O}$-module, we let

$$r_\mathfrak{p}(M) = \text{the number of classesin } S_M \text{ under equivalence at } \mathfrak{p},$$

and

$$g(M) = \text{the number of genera in } S_M.$$

From the definitions, we have $g(m) \leqslant \Pi_\mathfrak{p} r_\mathfrak{p}(M)$. By Theorems 10 and 11 we have that $r_\mathfrak{p}(M) = 1$ when $\mathfrak{p} \times \mathrm{D}(M) \cap I(\mathfrak{O}/\mathfrak{R})$, where $\mathfrak{R} = \mathfrak{O} \cap R$, R being the radical of A. Corollary 2 to Theorem 3 implies that $r_\mathfrak{p}(M)$ is finite when $\mathfrak{o}$ has finite residue class rings. Hence

THEOREM 13. *Assume that A/R is separable. Then for a principal $\mathfrak{O}$-module M, $g(M) \leqslant \Pi r_\mathfrak{p}(M)$, the product extending over the primes $\mathfrak{p}$ dividing $D(M) \cap I(\mathfrak{O}/\mathfrak{R})$, and $g(M)$ is finite if $\mathfrak{o}$ has finite residue class rings.*

For an A-module U, let $g(U)$ denote the number of genera of $\mathfrak{O}$-representation modules with rational hull isomorphic to U, and for a prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, let $r_\mathfrak{p}(U)$ denote the number of classes under equivalence at $\mathfrak{p}$ of such $\mathfrak{O}$-representation modules. As was pointed out in **(2)**, if A is separable and U is absolutely irreducible, Maranda's method's **(7)** can be extended to prove that

$$g(U) = \prod_\mathfrak{p} r_\mathfrak{p}(U) \tag{1}$$

where the product extends over all prime ideals $\mathfrak{p}$ of $\mathfrak{o}$, and

$$r(U) = h \cdot g(U) \tag{2}$$

where h is the ideal class number of $\mathfrak{o}$. We leave open the general questions of when equality holds in the formula of Theorem 13, and when $r(M) = h \cdot g(M)$ for an $\mathfrak{O}$-representation module M. (See also **(9)** in this regard.)

Taking M to be a coprincipal $\mathfrak{O}$-module instead of principal, and replacing $D(M)$ by $C(M)$ in the above definitions, we obtain numbers $s(M)$, $s_\mathfrak{p}(M)$, and $h(M)$, between which relations hold analogous to those for $r(M)$, $r_\mathfrak{p}(M)$, and $g(M)$.

https://doi.org/10.4153/CJM-1960-010-1 Published online by Cambridge University Press

There are also some relations between left and right which should be mentioned. If M is an $\mathfrak{O}$-representation module, $M^+ = \text{Hom}\dagger\ (M, \mathfrak{o})$ is a left $\mathfrak{O}$-representation module, and since $\mathfrak{o}$ is a Dedekind domain, $\text{Hom}\dagger\ (M^+, \mathfrak{o}) \approx M$ and $\text{Hom}\dagger\ (M, N) \approx \text{Hom}\dagger\ (N^+, M^+)$ for $\mathfrak{O}$-representation modules M and N. Now it can be verified that $\text{Ext}^1(M, N) \approx \text{Ext}^1(N^+, M^+)$, and hence that $D(M) = C(M^+)$ and $C(M) = D(M^+)$. Moreover, relations such as $r(M) = s(M^+)$ and $g(M) = h(M^+)$ hold.

References

1. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton, 1956).
2. D. G. Higman, *On orders in separable algebras*, Can. J. Math., *7* (1955), 509–515.
3. ——— *Relative cohomology*, Can. J. Math., *9* (1957), 19–34.
4. D. G. Higman and J. E. MacLaughlin, *Finiteness of class numbers of representations of algebras over function fields*, to appear in the Michigan Journal of Mathematics.
5. G. Hochschild, *Relative homological algebra*, Trans. Amer. Math. Soc., *82* (1956), 246–269.
6. J-M. Maranda, *On p-adic integral representations of finite groups*, Can. J. Math., *5* (1953), 344–355.
7. ——— *On the èquivalence of representations of finite groups by groups of automorphisms of modules over Dedekind rings*, Can. J. Math., *7* (1955), 516–526.
8. I. Reiner, *Maschke modules over Dekedind rings*, Can. J. Math., *8* (1956), 329–334.
9. ——— *On class numbers of representations of an order*, A. M. S. Notices, *5* (1958), Abstract no. 548–142, 584.

University of Michigan

https://doi.org/10.4153/CJM-1960-010-1 Published online by Cambridge University Press