

DESCENDING CHAINS, THE LILYPOND MODEL, AND MUTUAL-NEAREST-NEIGHBOUR MATCHING

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Abstract

We consider a hard-sphere model in \mathbb{R}^d generated by a stationary point process N and the *lilypond growth protocol*: at time 0, every point of N starts growing with unit speed in all directions to form a system of balls in which any particular ball ceases its growth at the instant that it collides with another ball. Some quite general conditions are given, under which it is shown that the model is well defined and exhibits no percolation. The absence of percolation is attributable to the fact that, under our assumptions, there can be no *descending chains* in N . The proof of this fact forms a significant part of the paper. It is also shown that, in the absence of descending chains, mutual-nearest-neighbour matching can be used to construct a bijective point map as defined by Thorisson.

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1. Introduction

We consider the lilypond model introduced and studied in [5] and [3]. It is based on a point process N on \mathbb{R}^d and defined according to the *lilypond growth protocol* as follows. The points of N , also called *germs*, start growing at the same time with unit speed in all directions, so that in t time units, in the absence of any interaction with growth around other points, a given germ grows into a ball (also called a *grain*) of radius t . The other critical feature of the lilypond growth protocol concerns the cessation of growth: any grain stops growing at the instant that it touches another grain (which may itself be either growing or have ceased growing at an earlier time). Any such instant is called a *collision time*. The union of all grains obtained in this way constitutes the *lilypond model generated by N* . It can be described by the *marked* point process $\{(\mathbf{x}, R(\mathbf{x})) : \mathbf{x} \in N\}$, where $R(\mathbf{x})$ is the radius of the grain with centre the germ \mathbf{x} .

When N is a (stationary) lattice, the lilypond model *percolates*, i.e. the union of all grains contains an unbounded (connected) component (think of grains as being ‘connected’ if and only if they either touch or are ‘linked’ by a sequence of touching grains). However, it was proved in [5] and Section 8.3 of [12] that percolation can almost surely (a.s.) not occur when N is a homogeneous Poisson process. In this paper, we prove this result for a much broader class of point processes N . This class includes Poisson cluster processes, Cox processes, and Gibbs processes satisfying suitable (exponential) moment conditions. While the result is not

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unexpected, its proof seems to be far from straightforward. Without the strong independence properties of the Poisson process we have to resort here to more general point process arguments.

Under suitable nonlattice-type conditions that hold when N is a Poisson process as in [5] and [3], the absence of percolation in the lilypond model is a consequence of a more basic feature, namely that N a.s. has no *descending chains* (see [8]). By a descending chain we mean an *infinite* sequence $\mathbf{x}_1, \mathbf{x}_2, \dots$ of different points of N for which $|\mathbf{x}_{i-1} - \mathbf{x}_i| \geq |\mathbf{x}_i - \mathbf{x}_{i+1}|$ for all $i \geq 2$. In fact, we devote a significant part of this paper to proving that our assumptions imply that N is *descending chain free* in the sense that descending chains a.s. do not exist.

A *bijective point shift* (of N) in the sense of [16] shifts the ‘typical point’ of N to a (possibly) different point of N without changing the Palm distribution of N . The existence of nontrivial bijective point shifts is an intriguing invariance property of the Palm distribution of N , but when N is in two or more dimensions it is at first not clear whether a stationary point process will actually have this property. For Poisson processes, the problem was resolved in [4] and [8], while the general case was discussed in [6]. *Mutual-nearest-neighbour matching*, as described in [4] and [8], is one rather straightforward way of constructing bijective point maps. The absence of descending chains guarantees the success of this procedure.

Olle Häggström (personal communication) has indicated to us an example of a stationary nonlattice point process possessing descending chains (see [8] for a related example). Following a proposal of Venkat Anantharam and Francois Baccelli (personal communication), in Subsection 3.4 we give another example based on Poisson line processes. These examples show that there must exist some nontrivial criteria for excluding descending chains. Finding the most general form of such criteria remains an open problem.

The paper is organized as follows. Section 2 contains a detailed discussion of the lilypond model as based on a (deterministic) locally finite point pattern. In particular, we will prove that the model exists and is unique whenever the underlying pattern has no descending chain. In Section 3, we first recall some basic definitions from point process theory before discussing specific nonlattice-type conditions that are needed to prove the absence of percolation in a (stochastic) lilypond model. This section also contains a simple argument proving that a Poisson process a.s. has no descending chain. This argument is considerably generalized in Section 4, where we state and prove the main result of this paper, on the absence of descending chains for a large class of stationary point processes. Section 5 contains the applications to the lilypond model while, in Section 6, we apply our main result to mutual-nearest-neighbour matching. Section 7 gives examples of Cox processes, Poisson cluster processes, and Gibbs processes that are descending chain free.

We work in d -dimensional space \mathbb{R}^d with norm $|\cdot|$. By $d(\mathbf{x}, A) := \inf_{\mathbf{y} \in A} \{|\mathbf{y} - \mathbf{x}|\}$ we define the distance between a point $\mathbf{x} \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$, where $\inf \emptyset := \infty$. $|A|_d$ denotes the volume of a (measurable) set A and $B(\mathbf{x}, r) \equiv B_r(\mathbf{x}) := \{\mathbf{y} : |\mathbf{y} - \mathbf{x}| \leq r\}$ denotes the ball with centre \mathbf{x} and radius $r \geq 0$. The *unit ball* $B_1(\mathbf{0})$ has volume $\kappa_d = |B_1(\mathbf{0})|_d$, while its boundary S^{d-1} (the unit sphere) has surface content $d\kappa_d$.

2. Hard-sphere models and the lilypond protocol

2.1. Some basic definitions

This section builds on the informal description of a lilypond model given in the introduction, by describing phenomena that are excluded from the description of the lilypond model based on point sets φ that arise from a Poisson process, as in [5] and [3]. While some facets may be regarded as mathematical, pathological cases, they point to better formal descriptions of the

concepts that underlie germ–grain models in general. Before starting even this informal work, it is helpful to define some basic properties of hard-sphere models.

Our discussion centres on germ–grain models in which to each point or germ x there is associated a grain that is in fact a ball $B_{R(x)}(x)$ with centre x and radius $R(x)$; one view of such a process is as a marked point process in which each point x in the process N has a $[0, \infty)$ -valued label $R(x)$. To avoid trivialities, every φ we consider has cardinality greater than or equal to 2.

Definition 2.1. Let φ be a locally finite subset of \mathbb{R}^d with associated mapping $x \mapsto R(\varphi, x) \equiv R(x) \in [0, \infty)$ defined on φ .

- (a) $\{(x, R(x)) : x \in \varphi\}$ is a *hard-sphere model* on φ if, for any point pair $\{x, y\} \subseteq \varphi$ with $x \neq y$, we have $R(x) + R(y) \leq |x - y|$.
- (b) Distinct germs $x, y \in \varphi$ in a hard-sphere model are *grain-neighbours* if $R(x) + R(y) = |x - y|$, i.e. if $B_{R(x)}(x) \cap B_{R(y)}(y) \neq \emptyset$ or, more precisely, their intersection is the one-point set

$$B_{R(x)}(x) \cap B_{R(y)}(y) = \left\{ \frac{R(y)x + R(x)y}{R(x) + R(y)} \right\}.$$

- (c) For grain-neighbours x and y , y is a *smaller grain-neighbour* of x when $R(y) \leq R(x)$.
- (d) A hard-sphere model is a *lilypond model* if every $x \in \varphi$ has a smaller grain-neighbour.

In this definition, the inequality $R(x) + R(y) \leq |x - y|$ in part (a) states that, in a hard-sphere model, the interiors of different balls do not intersect. Part (b) states that, for grain-neighbours x and y , equality holds in this weak inequality. Part (c) uses ‘smaller’ in the weak (nonstrict inequality) sense; this is particularly relevant in our discussion of the possible absence of percolation in Section 4. Part (d) asserts that every germ in a lilypond model has at least one grain-neighbour and, amongst its grain-neighbours, one that is (weakly) smaller. These two facets of part (d) reflect the ‘growth termination’ conditions in the description given in Section 1.

Definition 2.2. Consider a hard-sphere model on φ and let $x \in \varphi$. A nontrivial *cluster* $\mathcal{C}(\varphi, x) \equiv \mathcal{C}(x) \subseteq \varphi$ is a subset of φ with the following properties:

- (i) $x \in \mathcal{C}(x)$;
- (ii) $\mathcal{C}(x)$ contains all the grain-neighbours of y for every $y \in \mathcal{C}(x)$;
- (iii) there exists, for every $z \in \mathcal{C}(x)$, a sequence y_0, y_1, \dots, y_n of distinct elements of $\mathcal{C}(x)$, with $y_0 = x$ and $y_n = z$, such that y_{r-1} and y_r are grain-neighbours for every $r = 1, \dots, n$.

It is now almost trivial to observe that the clusters of a hard-sphere model with germ set φ furnish a partition of φ into equivalence classes of germs. It is not so simple (see below) to decide whether a cluster is necessarily finite (in the sense that $\text{card } \mathcal{C}(x) < \infty$).

Lemma 2.1. *The germ sets of the grain clusters of a hard-sphere model on φ constitute a partition of all points in φ .*

In fact, more is true of the clusters in a finite lilypond model, because each cluster is a finite set (and of cardinality greater than or equal to 2), meaning that $\min_{y \in \mathcal{C}(x)} R(y) = R(z_x)$ for some $z_x \in \mathcal{C}(x)$. Then, by Definition 2.1(d), z_x has a smaller grain-neighbour that, by Lemma 2.1,

must belong to $\mathcal{C}(\mathbf{x})$, and must have the same radius as \mathbf{z}_x , i.e. the two grains constitute a *doublet* (we call grain-neighbours \mathbf{x} and \mathbf{y} a doublet when $R(\mathbf{x}) = R(\mathbf{y}) = \frac{1}{2}|\mathbf{x} - \mathbf{y}|$).

Lemma 2.2. *Every finite cluster in a lilypond model on φ contains at least one doublet.*

Finally we record a relationship that need not be reflexive, although when it is some stronger results ensue (see Section 2.4).

Definition 2.3. Given a locally finite point set $\varphi \subset \mathbb{R}^d$, every $\mathbf{x} \in \varphi$ has a not necessarily uniquely determined nearest neighbour $\eta(\varphi, \mathbf{x})$ in φ , for which

$$|\mathbf{x} - \eta(\varphi, \mathbf{x})| = \inf_{z \in \varphi \setminus \{\mathbf{x}\}} |\mathbf{x} - z|.$$

Distinct elements \mathbf{x} and \mathbf{y} of φ are *mutual nearest neighbours* when

$$|\mathbf{x} - \mathbf{y}| = \inf_{z \in \varphi \setminus \{\mathbf{x}\}} |\mathbf{x} - z| = \inf_{z \in \varphi \setminus \{\mathbf{y}\}} |\mathbf{y} - z|.$$

Thus, in a lilypond model, a pair of mutual nearest neighbours constitutes a doublet, but not every doublet is a pair of mutual nearest neighbours.

2.2. A naive algorithm for lilypond models on finite sets

When φ is a finite set, there certainly exists a simple algorithm (it need not be efficient(!)) that constructs a well-defined lilypond model based on φ : we call it the naive algorithm. Starting from the germ set φ at time 0, around each and every germ \mathbf{x} a grain starts growing at unit rate. Take ‘snapshots’ of the growing (and grown) grains every time a collision occurs between a growing grain and another grain (so that the growing grain ceases growth at the time of the snapshot). Then the set of all snapshot times coincides with the set of all radii of the grains. Since φ is a finite set, this procedure generates a finite set of times, and since φ is a bounded set, there is a maximum time within which this finite set of snapshot times is located. The procedure is thus determinate.

In [3], an algorithm was given that determines the radii $R(\mathbf{x})$ for $\mathbf{x} \in \varphi$ when φ is finite and, in practice, for cases of local finiteness as well; for such cases the existence of the lilypond model is discussed shortly. The naive algorithm can be shown to be equivalent to both the algorithm in Section 2 of [3] and our Definition 2.1(d) for finite φ .

2.3. Descending chains: a deterministic example

As described, both the naive algorithm and the definition of a grain cluster $\mathcal{C}(\mathbf{x})$ depend on the finiteness of φ to ensure their uniqueness. To see what may go awry without finiteness, we now describe a countably infinite set, using \mathbb{N} to denote the positive integers $\{1, 2, \dots\}$ and \mathbb{N}_0 the nonnegative integers $\{0\} \cup \mathbb{N}$.

Definition 2.4. A countable but locally finite set $\varphi = \{\mathbf{x}_0, \dots, \mathbf{x}_i, \mathbf{x}_{i+1}, \dots\}$ is a *descending chain* (of germs) when

$$|\mathbf{x}_{i-1} - \mathbf{x}_i| \geq |\mathbf{x}_i - \mathbf{x}_{i+1}| \quad (i \in \mathbb{N}).$$

If, in addition,

$$d(\mathbf{x}_i, \varphi \setminus \{\mathbf{x}_0, \dots, \mathbf{x}_i\}) = |\mathbf{x}_{i+1} - \mathbf{x}_i| \quad (i \in \mathbb{N}_0), \tag{2.1}$$

we call φ a *strong descending chain*.

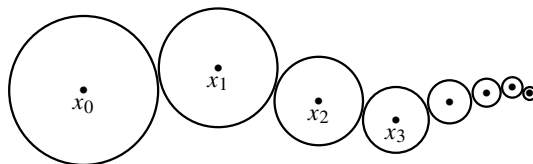


FIGURE 1: Part of an infinite set of points generating a topologically connected lilypond model.

Given a strong descending chain $\varphi = \{x_i, i \in \mathbb{N}_0\}$ (Figure 1 illustrates part of such a set φ), $\lim_{i \rightarrow \infty} |x_{i+1} - x_i|$ exists and equals $2r \geq 0$, say. Then $|x_i - x_{i+1}| - 2r \downarrow 0$ and, therefore, the quantities

$$r_m := r + \sum_{i=1}^{\infty} (-1)^{i+1} (|x_{m+i-1} - x_{m+i}| - 2r) \quad (m \in \mathbb{N}_0) \tag{2.2}$$

are well defined and positive, and satisfy

$$\begin{aligned} r_m &= r + \sum_{i=1}^{\infty} (-1)^{i+1} (|x_{m+i-1} - x_{m+i}| - 2r) \\ &= |x_m - x_{m+1}| - r - \sum_{i=1}^{\infty} (-1)^{i+1} (|x_{m+i} - x_{m+1+i}| - 2r) = |x_m - x_{m+1}| - r_{m+1}, \end{aligned}$$

i.e.

$$r_i + r_{i+1} = |x_i - x_{i+1}| \quad (i \in \mathbb{N}_0). \tag{2.3}$$

Having made this observation, it is now easy to state conditions necessary and sufficient for $\{(x_i, r_i)\}$ to constitute a lilypond model.

Lemma 2.3. *Let $\{x_i\}$ be a strong descending chain of germs and define the sequence $\{r_i\}$ according to (2.2), where $2r = \lim_{i \rightarrow \infty} |x_i - x_{i+1}|$. Then $\{(x_i, r_i)\}$ constitutes a lilypond model if and only if $\{r_i\}$ is monotone decreasing.*

Proof. We first note that

$$r_{i-2} \geq r_i \quad (i \geq 2). \tag{2.4}$$

Assume that $\{r_i\}$ is monotone decreasing. By (2.3), we then find that, for any $i, j \in \mathbb{N}_0$ with $i < j$,

$$|x_i - x_j| \geq |x_i - x_{i+1}| = r_i + r_{i+1} \geq r_i + r_j.$$

Hence, $\{(x_i, r_i)\}$ is a hard-sphere model. Since $\{r_i\}$ is decreasing, (2.3) implies that it is a lilypond model.

Conversely, we assume now that $\{(x_i, r_i)\}$ is a lilypond model. Let us further assume that $\{r_i\}$ is not decreasing. Then there is some $j \in \mathbb{N}$ such that

$$r_0 \geq \dots \geq r_{j-1} < r_j. \tag{2.5}$$

By the lilypond property, there must be some $i \in \mathbb{N}_0 \setminus \{j - 1, j\}$ such that

$$|\mathbf{x}_{j-1} - \mathbf{x}_i| = r_{j-1} + r_i \quad \text{and} \quad r_{j-1} \geq r_i. \tag{2.6}$$

Assume first that $j \geq 2$ and that $i \leq j - 2$. By (2.5), we then have $r_{j-2} = r_{j-1} < r_j$, contradicting (2.4). Hence, we can assume that $i \geq j + 1$. We now distinguish two cases. If $i - j$ is even then (2.4) implies that $r_i \leq r_j$. Since the equality $r_i = r_j$ would contradict (2.5) and (2.6), we deduce that $r_i < r_j$. Hence,

$$|\mathbf{x}_{j-1} - \mathbf{x}_i| = r_{j-1} + r_i < r_{j-1} + r_j = |\mathbf{x}_{j-1} - \mathbf{x}_j|,$$

contradicting (2.1). If $i - j$ is odd then (2.4) implies that $r_i \leq r_{j-1} < r_j$. This yields a contradiction as above.

By the preceding lemma, $\{\mathbf{x}_i\}$ is a *descending chain of germs* when $\{r_i\}$ is monotone. Hence, the union set $\bigcup_i B_{r_i}(\mathbf{x}_i)$ is topologically connected and unbounded, meaning that the lilypond model *percolates*. More generally, if a locally finite set does not contain a descending chain of germs, then the lilypond model cannot contain a descending chain of grains. Consequently, percolation could only occur along an *ascending* chain of grains. However, the latter behaviour is not possible for a stationary point process satisfying some suitable condition of nonarithmeticity (see Section 5 and [5]).

Trivially, by using (2.3), the inequalities $r_m \leq \frac{1}{2}|\mathbf{x}_{m-1} - \mathbf{x}_m|$ (for all $m \in \mathbb{N}_0$) hold if and only if $r_m \leq r_{m-1}$, i.e. if $\{\mathbf{x}_i\}$ defines a descending chain of grains, and the model therefore percolates. A sufficient condition, in terms of the distances $\Delta_i := |\mathbf{x}_i - \mathbf{x}_{i+1}|$, is that $\{\Delta_i\}$ be a convex decreasing sequence; this follows from writing

$$r_m - r_{m+1} = \sum_{i=1}^{\infty} (-1)^{i+1} (\Delta_{m+i-1} - \Delta_{m+i})$$

and observing that this infinite sum is nonnegative when the terms $(\Delta_{m+i-1} - \Delta_{m+i})$ decrease in i .

When $|\mathbf{x}_{i-1} - \mathbf{x}_i| > |\mathbf{x}_i - \mathbf{x}_{i+1}|$ for all i , the radii (2.2) are all strictly decreasing, and there is no smallest collision time. Therefore, for any $\varepsilon > 0$, the (open) time interval $(r, r + \varepsilon)$ contains infinitely many collision times. The naive algorithm for determining $R(\mathbf{x}_i)$ thus fails on two accounts: first, the set of snapshot times is countably infinite and, second, there is no smallest snapshot time, only an infimum.

Observe also that, when the r_m decrease strictly, each r_m depends on *all* points $\{\mathbf{x}_n, n \geq m\}$ rather than just a finite subset of φ . Hence, the algorithm in [3] that, for practical purposes, determines $R(\mathbf{x}_i)$ would in principle fail.

2.4. Existence of the lilypond model

We now prove that, when φ is descending chain free, the lilypond growth protocol leads to a uniquely determined hard-sphere model, i.e. the lilypond model is then well defined (meaning that, given φ , all the radii $R(\mathbf{x})$ are determined uniquely and satisfy Definition 2.1).

Proposition 2.1. *Let φ be a locally finite subset of \mathbb{R}^d with $\text{card } \varphi \geq 2$. When φ has no descending chain, there is a uniquely determined lilypond model $\{(\mathbf{x}, R(\mathbf{x})) : \mathbf{x} \in \varphi\}$ based on φ .*

Proof. We use a version of mutual-nearest-neighbour matching (see [8]) to construct a sequence $\{(\mathbf{x}, R(\mathbf{x})) : \mathbf{x} \in \varphi_n\}$, $n \in \mathbb{N}_0$, such that $\varphi_n \subseteq \varphi_{n+1} \subseteq \varphi$ for any $n \in \mathbb{N}_0$, and ultimately determine $R(\mathbf{x})$ for all $\mathbf{x} \in \varphi$. We then check that these, and only these, $R(\mathbf{x})$ values satisfy the conditions of Definition 2.1.

We proceed recursively; start by setting $\varphi_0 := \emptyset$. At step $n \geq 0$, define the set $\nu_n(\mathbf{x})$, for all $\mathbf{x} \in \varphi \setminus \varphi_n$, by

$$\nu_n(\mathbf{x}) = \{\mathbf{x}\} \cup \{\mathbf{y} \in \varphi \setminus (\varphi_n \cup \{\mathbf{x}\}) : \mathbf{y} \text{ is a mutual nearest neighbour of } \mathbf{x} \text{ in } \varphi \setminus \varphi_n\}.$$

Then $\nu_n(\mathbf{x})$ is nonempty and contains at most a finite number of points (the maximum such number depends on the dimension d). Let

$$\psi_n := \{\mathbf{x} \in \varphi \setminus \varphi_n : \nu_n(\mathbf{x}) \neq \{\mathbf{x}\}\},$$

i.e. ψ_n contains all mutual-nearest-neighbour pairs to be found in the complement in φ of φ_n (in particular, ψ_0 is the union of all mutual-nearest-neighbour sets). It is possible that ψ_n is empty. In this case, we define $\varphi_{n+1} = \varphi_n$ (and in fact $\varphi_m = \varphi_n$ for all $m \geq n + 1$). Otherwise, we proceed, as below, to construct φ_{n+1} from φ_n and determine $R(\mathbf{x})$ for $\mathbf{x} \in \varphi_{n+1} \setminus \varphi_n$.

For $\mathbf{x} \in \psi_n$, define

$$r_n(\mathbf{x}) := \inf_{z \in \varphi_n} \{|\mathbf{x} - z| - R(z)\},$$

where $\inf \emptyset := \infty$, let

$$s'_n(\mathbf{x}) = \frac{1}{2}|\mathbf{x} - \mathbf{y}| \quad (\mathbf{y} \in \nu_n(\mathbf{x}) \setminus \{\mathbf{x}\}) \quad \text{and} \quad s''_n(\mathbf{x}) = \min_{y \in \nu_n(\mathbf{x})} r_n(\mathbf{y}),$$

and finally let

$$R_n(\mathbf{x}) := \min\{s'_n(\mathbf{x}), s''_n(\mathbf{x})\}.$$

We now add certain points in ψ_n to φ_n to construct φ_{n+1} , distinguishing, for each $\mathbf{x} \in \psi_n$, the two cases $R_n(\mathbf{x}) = s'_n(\mathbf{x})$ and $R_n(\mathbf{x}) < s'_n(\mathbf{x})$. In the former case, we augment φ_n by adding all points of $\nu_n(\mathbf{x})$ to φ_{n+1} , and set $R(\mathbf{y}) := R_n(\mathbf{x})$ for any $\mathbf{y} \in \nu_n(\mathbf{x})$. In the other case, we augment φ_n by including those points $\mathbf{y} \in \nu_n(\mathbf{x})$ for which $R_n(\mathbf{x}) = r_n(\mathbf{y})$, and define $R(\mathbf{y}) := R_n(\mathbf{x})$. Applying this argument for all $\mathbf{x} \in \psi_n$ yields the $(n + 1)$ th hard-sphere model $\{(\mathbf{x}, R(\mathbf{x})) : \mathbf{x} \in \varphi_{n+1}\}$.

Define

$$\varphi_\infty := \bigcup_{n=1}^\infty \varphi_n, \quad \psi := \varphi \setminus \varphi_\infty = \bigcap_{n=1}^\infty (\varphi \setminus \varphi_n).$$

Since $\psi \subseteq \varphi$ is a countable set, ψ is either empty, finite and nonempty, or countably infinite. In the first case we are done, while if ψ contains just one point, i.e. $\psi = \{\mathbf{x}\}$ for some $\mathbf{x} \in \varphi$, it is enough to define $R(\mathbf{x}) := \inf_{y \in \varphi_\infty} (|\mathbf{x} - y| - R(y))$.

Otherwise, ψ contains several points. Let us show that it contains no pair of mutual nearest neighbours. To prove this, we take some $\mathbf{x} \in \psi$ and consider the set $\nu(\mathbf{x})$ of all mutual nearest neighbours of \mathbf{x} in ψ . Certainly, from nearest-neighbour properties, the compact set

$$A(\mathbf{x}) := \bigcup_{y \in \nu(\mathbf{x})} B_{|\mathbf{x}-y|}(y)$$

contains no point of $\psi \setminus (\nu(\mathbf{x}) \cup \{\mathbf{x}\})$. Because ψ is the intersection of the decreasing sets $\varphi \setminus \varphi_n$, $n \in \mathbb{N}$, there must be an $n \in \mathbb{N}$ such that $A(\mathbf{x})$, which is compact, contains no points of

$\varphi \setminus \varphi_n$ other than those in $\nu(\mathbf{x}) \cup \{\mathbf{x}\}$. If $\nu(\mathbf{x})$ were nonempty then some points that are both mutual nearest neighbours and in $\nu(\mathbf{x})$ would have been removed in step $n + 1$. Hence, $\nu(\mathbf{x})$ is empty and, thus, ψ contains no pairs of mutual nearest neighbours.

Since a finite set with at least two elements contains at least one pair of mutual nearest neighbours, it follows that ψ cannot be finite and contain at least two points: the only possibility is that it contains infinitely many points. We show that, for this to occur, and for ψ to have no pairs of mutual nearest neighbours, it must have a descending chain $\{\mathbf{x}_i\}$. Let \mathbf{x}_0 be any point of ψ and let \mathbf{x}_1 be one of its nearest neighbours in ψ . To proceed by induction, assume that $\mathbf{x}_0, \dots, \mathbf{x}_n$ ($n \geq 2$) are different points in ψ satisfying $|\mathbf{x}_{i+1} - \mathbf{x}_i| \leq |\mathbf{x}_i - \mathbf{x}_{i-1}|$ for all $i \in \{1, \dots, n - 1\}$. Let \mathbf{x}_{n+1} be a nearest neighbour of \mathbf{x}_n . Then $|\mathbf{x}_{n+1} - \mathbf{x}_n| \leq |\mathbf{x}_n - \mathbf{x}_{n-1}|$ (otherwise \mathbf{x}_n and \mathbf{x}_{n-1} would be mutual nearest neighbours). To show that $\mathbf{x}_{n+1} \notin \{\mathbf{x}_0, \dots, \mathbf{x}_{n-1}\}$, assume, on the contrary, that $\mathbf{x}_{n+1} = \mathbf{x}_j$ for some $j \in \{0, \dots, n - 1\}$. If \mathbf{x}_n were not a nearest neighbour of \mathbf{x}_j then there would exist $\mathbf{x} \in \psi$ such that

$$|\mathbf{x}_j - \mathbf{x}| < |\mathbf{x}_j - \mathbf{x}_n| = |\mathbf{x}_{n+1} - \mathbf{x}_n| \leq |\mathbf{x}_{j+1} - \mathbf{x}_j|,$$

contradicting the fact that \mathbf{x}_{j+1} is a nearest neighbour of \mathbf{x}_j . Hence, \mathbf{x}_n and \mathbf{x}_j are mutual nearest neighbours, contrary to what we already know of ψ . Thus, ψ and therefore also φ each have a descending chain, contradicting the assumption on φ . This completes the construction of $R(\mathbf{x})$ for all $\mathbf{x} \in \varphi$.

Let $\{(\mathbf{x}, R'(\mathbf{x})) : \mathbf{x} \in \varphi\}$ be a lilypond model based on φ , as in Definition 2.1. We prove, by induction, that $R'(\mathbf{x}) = R(\mathbf{x})$ for any $\mathbf{x} \in \varphi_n$ and any $n \in \mathbb{N}$. The set ψ_0 consists of all pairs of mutual nearest neighbours in φ . Take $\mathbf{x} \in \psi_0$ and $\mathbf{y} \in \nu_0(\mathbf{x}) \setminus \{\mathbf{x}\}$. If $R'(\mathbf{x}) \neq \frac{1}{2}|\mathbf{x} - \mathbf{y}|$ then the hard-sphere property implies that at least one of the radii $R'(\mathbf{x})$ and $R'(\mathbf{y})$, $R'(\mathbf{x})$ say, must be strictly smaller than $\frac{1}{2}|\mathbf{x} - \mathbf{y}|$. The lilypond property then implies that there is some point $\mathbf{z} \in \varphi \setminus \{\mathbf{x}\}$ such that $R'(\mathbf{z}) \leq R'(\mathbf{x})$ and $R'(\mathbf{z}) + R'(\mathbf{x}) = |\mathbf{x} - \mathbf{z}|$. Since this contradicts the nearest-neighbour property of \mathbf{y} , we obtain

$$R'(\mathbf{x}) = R'(\mathbf{y}) = \frac{1}{2}|\mathbf{x} - \mathbf{y}| = R(\mathbf{x}) = R(\mathbf{y}).$$

Therefore, the statement of the induction hypothesis holds for $n = 1$.

Now fix $n \geq 1$ and consider a point $\mathbf{x} \in \psi_n$. Assume first that $R_n(\mathbf{x}) = s'_n(\mathbf{x})$, i.e.

$$r_n(\mathbf{y}) \geq s'_n(\mathbf{x}), \quad \mathbf{y} \in \nu_n(\mathbf{x}). \tag{2.7}$$

If $R'(\mathbf{x}) < \frac{1}{2}|\mathbf{x} - \mathbf{y}|$ for some $\mathbf{y} \in \nu_n(\mathbf{x}) \setminus \{\mathbf{x}\}$ then, as we have just seen, either \mathbf{x} or \mathbf{y} cannot have a (weakly) smaller grain-neighbour in $\varphi \setminus \varphi_n$. Suppose, for definiteness, that this is true of \mathbf{y} . Then there must be a grain-neighbour $\mathbf{w} \in \varphi_n$ of \mathbf{y} such that $R'(\mathbf{w}) \leq R'(\mathbf{y})$. Using the induction hypothesis $R'(\mathbf{w}) = R(\mathbf{w})$, we obtain

$$|\mathbf{w} - \mathbf{y}| - R_n(\mathbf{w}) = R'(\mathbf{y}) < \frac{1}{2}|\mathbf{x} - \mathbf{y}|,$$

contradicting (2.7). Were $R'(\mathbf{x}) > \frac{1}{2}|\mathbf{x} - \mathbf{y}|$ for some $\mathbf{y} \in \nu_n(\mathbf{x}) \setminus \{\mathbf{x}\}$, then $R'(\mathbf{y}) < \frac{1}{2}|\mathbf{x} - \mathbf{y}|$, which would again contradict (2.7). Hence, $R'(\mathbf{x}) = \frac{1}{2}|\mathbf{x} - \mathbf{y}| = R(\mathbf{x})$.

Now consider the case $s''_n(\mathbf{x}) < s'_n(\mathbf{x})$ and let $\mathbf{y} \in \nu_n(\mathbf{x})$ such that $r_n(\mathbf{y}) < s'_n(\mathbf{x})$. The hard-sphere property and the induction hypothesis imply that $R'(\mathbf{y}) \leq r_n(\mathbf{y})$. To show that this inequality is in fact an equality we assume, on the contrary, that $R'(\mathbf{y}) < r_n(\mathbf{y})$. Then none of the points in φ_n is a grain-neighbour of \mathbf{y} . Since $r_n(\mathbf{y}) < s'_n(\mathbf{x})$ (whence $R'(\mathbf{y}) < \frac{1}{2}|\mathbf{x} - \mathbf{y}|$) and $\{\mathbf{x}, \mathbf{y}\}$ is a pair of mutual nearest neighbours in $\varphi \setminus \varphi_n$, the point \mathbf{y} cannot have a smaller

grain-neighbour. Hence, $R'(y) = r_n(y) = R(y)$, and the induction hypothesis holds for $n + 1$. Consequently, when a lilypond model based on φ exists, it is unique.

To check that the hard-sphere model $\{(\mathbf{x}, R(\mathbf{x})) : \mathbf{x} \in \varphi\}$ is indeed a lilypond model, take $n \in \mathbb{N}_0$ and $\mathbf{x} \in \varphi_{n+1} \setminus \varphi_n$. If $R_n(\mathbf{x}) = s'_n(\mathbf{x})$ then \mathbf{x} has at least one grain-neighbour of equal radius. Assume that $n \geq 1$ and $R_n(\mathbf{x}) < s'_n(\mathbf{x})$. Then there exists a $\mathbf{z} \in \varphi_n$ satisfying $R(\mathbf{x}) = |\mathbf{x} - \mathbf{z}| - R(\mathbf{z})$. To prove that \mathbf{z} is a smaller grain-neighbour of \mathbf{x} , suppose, on the contrary, that $R(\mathbf{x}) < R(\mathbf{z})$. Then,

$$|\mathbf{x} - \mathbf{z}| = R(\mathbf{x}) + R(\mathbf{z}) < 2R(\mathbf{z}).$$

Let $i \in \{1, \dots, n\}$ be such that $\mathbf{z} \in \varphi_i \setminus \varphi_{i-1}$ and take some $\mathbf{w} \in v_{i-1}(\mathbf{z}) \setminus \{\mathbf{z}\}$. By construction, we then have $R(\mathbf{z}) \leq \frac{1}{2}|\mathbf{z} - \mathbf{w}|$, meaning that

$$|\mathbf{x} - \mathbf{z}| < 2R(\mathbf{z}) \leq |\mathbf{z} - \mathbf{w}|.$$

Since $\mathbf{x} \notin \varphi_{i-1}$, the points \mathbf{z} and \mathbf{w} cannot be mutual nearest neighbours in $\varphi \setminus \varphi_i$. This contradiction shows that \mathbf{z} is a smaller grain-neighbour of \mathbf{x} . Hence, all points in φ_∞ do have a smaller grain-neighbour. If $\psi = \varphi \setminus \varphi_\infty$ contains just a single point \mathbf{x} then, by definition, there are some $i \in \mathbb{N}$ and $\mathbf{z} \in \varphi_i \setminus \varphi_{i-1}$ such that $|\mathbf{x} - \mathbf{z}| = R(\mathbf{x}) + R(\mathbf{z})$. As above, it follows that \mathbf{z} is a smaller grain-neighbour of \mathbf{x} .

The following property of the above construction will be useful in Section 5.

Corollary 2.1. *Let φ be as in Proposition 2.1, let $n \geq 1$, and let $\mathbf{x} \in \varphi_{n+1} \setminus \varphi_n$. Let $\mathbf{y} \in \varphi_{n+1}$ be a grain-neighbour of \mathbf{x} . Then \mathbf{y} is a smaller grain-neighbour of \mathbf{x} , i.e. $R(\mathbf{y}) \leq R(\mathbf{x})$.*

Proof. If $\mathbf{y} \in \varphi_n$ then the last step of the preceding proof shows that \mathbf{y} is a smaller grain-neighbour of \mathbf{x} . Assume now that $\mathbf{y} \in \varphi_{n+1}$ and choose some $\mathbf{z} \in v_n(\mathbf{y})$. By construction, $R(\mathbf{y}) \leq \frac{1}{2}|\mathbf{y} - \mathbf{z}|$. Assuming, in addition, that $R(\mathbf{y}) > R(\mathbf{x})$, we obtain

$$|\mathbf{x} - \mathbf{y}| = R(\mathbf{x}) + R(\mathbf{y}) < 2R(\mathbf{y}) \leq |\mathbf{y} - \mathbf{z}|.$$

However, \mathbf{y} and \mathbf{z} cannot then be mutual nearest neighbours in $\varphi \setminus \varphi_n$. This contradiction shows that $R(\mathbf{y}) \leq R(\mathbf{x})$.

Remark 2.1. In the first (submitted) version of this paper, the authors conjectured that the lilypond model exists and is unique for any locally finite φ having at least two points. Since then, this has been proved in [7] (using a very different approach) for quite general spaces.

3. Stationary point processes

3.1. Palm probabilities

Let N be a (simple) point process on \mathbb{R}^d , defined on some probability space (Ω, \mathcal{F}, P) . Such a process is defined as a random variable taking values in the space \mathbf{N} of all locally finite subsets of \mathbb{R}^d equipped with the smallest σ -field \mathcal{N} containing the sets $\{\varphi \in \mathbf{N} : \varphi(B) = k\}$, for any bounded Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ and any $k \in \mathbb{N}_0$, where $\varphi(B) := \text{card}(\varphi \cap B)$. Hence, φ can be interpreted as a counting measure: it can be written as a finite or countably infinite sum $\sum_i \delta_{\mathbf{x}_i}$ of Dirac measures located at some points $\mathbf{x}_i \in \varphi$. Accordingly, we can and will consider N to be a random (counting) measure on $\mathcal{B}(\mathbb{R}^d)$.

Throughout the paper we consider a *stationary* point process N , i.e. a point process whose distribution is invariant under all translations. We assume that $N \neq \emptyset$ a.s. and that the *intensity*

$$\lambda_N := E[N([0, 1]^d)]$$

is finite. Quite often it is useful to work under the *Palm probability measure* \mathbb{P}_0^N of N on (Ω, \mathcal{F}) , under which $\mathbb{P}_0^N(\mathbf{0} \in N) = 1$, where $\mathbf{0}$ denotes the origin of \mathbb{R}^d . The Palm theory needed in this paper is contained in the *refined Campbell theorem* (see [14, Theorem 6.12] or [2, Proposition 12.1.IV]), stating that, for any nonnegative random field $\{Y(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$ that is jointly stationary with N ,

$$\mathbb{E} \left[\int h(\mathbf{x}) Y(\mathbf{x}) N(d\mathbf{x}) \right] = \mathbb{E} \left[\sum_{\mathbf{x} \in N} Y(\mathbf{x}) h(\mathbf{x}) \right] = \lambda_N \mathbb{E}_0[Y(0)] \int h(\mathbf{x}) d\mathbf{x}, \tag{3.1}$$

where $h : \mathbb{R}^d \rightarrow [0, \infty)$ is measurable and \mathbb{E}_0 denotes expectation with respect to \mathbb{P}_0^N . If N is Poisson, then $N \setminus \{\mathbf{0}\}$ is Poisson under \mathbb{P}_0^N . We can also use (3.1) with N replaced by some other point process M that has finite intensity λ_M and that is jointly stationary with N . If we denote the Palm probability measure associated with M by \mathbb{P}_0^M , then, for all measurable and nonnegative functions $h : \mathbb{N} \times \mathbb{R}^d \rightarrow [0, \infty)$, we have

$$\mathbb{E} \left[\int h(N - \mathbf{x}, \mathbf{x}) M(d\mathbf{x}) \right] = \lambda_M \mathbb{E}_0^M \left[\int h(N, \mathbf{x}) d\mathbf{x} \right], \tag{3.2}$$

where $N - \mathbf{x} := \{y - \mathbf{x} : y \in N\}$ and \mathbb{E}_0^M denotes expectation with respect to \mathbb{P}_0^M .

More details on stationary point processes and their associated Palm probabilities can be found in [2] and [14].

3.2. Factorial moment measures

For any $\varphi \in \mathbb{N}$ and $n \in \mathbb{N}$ we let $\varphi^{(n)}$ denote the set of all n -tuples $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \varphi^n$ with pairwise different entries. Identify $\varphi^{(n)}$ with the measure

$$\varphi^{(n)}(B) := \sum_{(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \varphi^{(n)}} 1((\mathbf{x}_1, \dots, \mathbf{x}_n) \in B) \quad (B \in \mathcal{B}((\mathbb{R}^d)^{(n)})),$$

where $1(\cdot)$ denotes the indicator function, and define the n th *factorial moment measure* of a point process N , on such a set B , by

$$\begin{aligned} \alpha^{(n)}(B) &:= \mathbb{E} \left[\sum_{(\mathbf{x}_1, \dots, \mathbf{x}_n) \in N^{(n)}} 1((\mathbf{x}_1, \dots, \mathbf{x}_n) \in B) \right] \\ &= \mathbb{E}(N(B_1)[N(B_1) - 1] \cdots [N(B_1) - n + 1]) \quad \text{if } B = (B_1)^{(n)} \text{ with } B_1 \in \mathcal{B}(\mathbb{R}^d). \end{aligned}$$

Now assume that N is a stationary point process satisfying

$$\alpha^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \leq c^n d\mathbf{x}_1 \cdots d\mathbf{x}_n \quad (n \in \mathbb{N}) \tag{3.3}$$

for some $c > 0$. A Poisson process, for instance, has $\alpha^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) = \lambda_N^n d\mathbf{x}_1 \cdots d\mathbf{x}_n$ for all $n \in \mathbb{N}$. For $b > 0$, let $C(b)$ be the set of all $\varphi \in \mathbb{N}$ containing a descending chain $\mathbf{x}_1, \mathbf{x}_2, \dots$ such that $b \geq |\mathbf{x}_1| \geq |\mathbf{x}_2 - \mathbf{x}_1| \geq |\mathbf{x}_3 - \mathbf{x}_2| \geq \dots$, and let $C_n(b)$ denote the set of all $\varphi \in \mathbb{N}$ containing points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ such that $b \geq |\mathbf{x}_1| \geq |\mathbf{x}_2 - \mathbf{x}_1| \geq \dots \geq |\mathbf{x}_n - \mathbf{x}_{n-1}|$. These sets $C_n(b)$ are decreasing and, by inspection, $\mathbb{P}(C(b)) \leq \lim_{n \rightarrow \infty} \mathbb{P}(C_n(b))$. Since clearly

$$\mathbb{P}(N \in C_n(b)) \leq \mathbb{E} \left[\int \cdots \int 1(b \geq |\mathbf{x}_1| \geq |\mathbf{x}_2 - \mathbf{x}_1| \geq \dots \geq |\mathbf{x}_n - \mathbf{x}_{n-1}|) N(d\mathbf{x}_1 \cdots d\mathbf{x}_n) \right],$$

we find, from (3.3) and a change of variables, that

$$\begin{aligned} P(N \in C_n(b)) &\leq \alpha^n \int \cdots \int 1(b \geq |\mathbf{x}_1| \geq |\mathbf{x}_2 - \mathbf{x}_1| \geq \cdots \geq |\mathbf{x}_n - \mathbf{x}_{n-1}|) \, d\mathbf{x}_1 \cdots d\mathbf{x}_n \\ &= \frac{c^n}{n!} (b^d \kappa_d)^n \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

(recall that κ_d is the volume of the unit ball in \mathbb{R}^d). Hence, $P(N \in C(b)) = 0$, which in turn implies (see also the proof of Theorem 4.1) that N a.s. contains no descending chain. This argument does not require any independence properties of N ; it will be generalized in Section 4.

3.3. Nonlattice-type processes

A finite or countably infinite set $\varphi \subseteq \mathbb{R}^d$ is called *nonlattice* if, for any $m \geq 2$ and any mutually different points $\mathbf{x}_1, \dots, \mathbf{x}_m \in \varphi$, the equality

$$\sum_{1 \leq i < j \leq m} c_{ij} |\mathbf{x}_i - \mathbf{x}_j| = 0,$$

for some integers c_{ij} ($1 \leq i < j \leq m$), implies that $c_{ij} = 0$ for all $i < j$.

The nonlattice property of a set is not affected by shifts. Hence, the point process N is P-a.s. nonlattice if and only if it is P_0^N -a.s. nonlattice. In this case, we simply call N *nonlattice*.

Lemma 3.1. *Assume that $d \geq 2$ and that N is a stationary point process satisfying*

$$\alpha^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \ll d\mathbf{x}_1 \cdots d\mathbf{x}_n \quad (n \in \mathbb{N}) \tag{3.4}$$

i.e. assume that, for each $n \in \mathbb{N}$, $\alpha^{(n)}$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^{nd} . Then N is nonlattice.

Proof. Let $m \geq 2$ and $c_{ij} \in \mathbb{R}$, $1 \leq i < j \leq m$, and assume that $c_{ij} \neq 0$ for at least one pair (i, j) . Consider the event A that there is an m -tuple $(\mathbf{x}_1, \dots, \mathbf{x}_m) \in N^{(m)}$ such that

$$\sum_{1 \leq i < j \leq m} c_{ij} |\mathbf{x}_i - \mathbf{x}_j| = 0.$$

It is enough to prove that $P(A) = 0$. Obviously, the indicator of A can be bounded from above by

$$\sum_{(\mathbf{x}_1, \dots, \mathbf{x}_m) \in N^{(m)}} 1(f(\mathbf{x}_1, \dots, \mathbf{x}_m) = 0),$$

where

$$f(\mathbf{x}_1, \dots, \mathbf{x}_m) := \sum_{1 \leq i < j \leq m} c_{ij} |\mathbf{x}_i - \mathbf{x}_j|. \tag{3.5}$$

Taking expectations and using (3.4), we obtain

$$P(A) \leq \int \cdots \int 1(f(\mathbf{x}_1, \dots, \mathbf{x}_m) = 0) g_m(\mathbf{x}_1, \dots, \mathbf{x}_m) \, d\mathbf{x}_1 \cdots d\mathbf{x}_m$$

for some nonnegative measurable function g_m . Lemma 3.2 now implies the assertion.

Lemma 3.2. Assume that $d \geq 2$. Let $m \in \mathbb{N}$, with $m \geq 2$, and let $c_{ij} \in \mathbb{R}$, $1 \leq i < j \leq m$, with $c_{ij} \neq 0$ for at least one pair (i, j) . Then the set of all $(\mathbf{x}_1, \dots, \mathbf{x}_m) \in (\mathbb{R}^d)^m$ satisfying $\sum_{1 \leq i < j \leq m} c_{ij} |\mathbf{x}_i - \mathbf{x}_j| = 0$ has (md) -dimensional Lebesgue measure 0.

Proof. Define a function $f: (\mathbb{R}^d)^m \rightarrow \mathbb{R}$ by (3.5). This function is smooth on the set A of all $(\mathbf{x}_1, \dots, \mathbf{x}_m) \in (\mathbb{R}^d)^m$ satisfying $\mathbf{x}_i \neq \mathbf{x}_j$ for any $i \neq j$. A simple calculation shows that the gradient $f'(\mathbf{x})$ of f is given by

$$f'(\mathbf{x}_1, \dots, \mathbf{x}_m) = \left(\sum_{i \neq 1} a_{1i} \frac{\mathbf{x}_1 - \mathbf{x}_i}{|\mathbf{x}_1 - \mathbf{x}_i|}, \dots, \sum_{i \neq m} a_{mi} \frac{\mathbf{x}_m - \mathbf{x}_i}{|\mathbf{x}_m - \mathbf{x}_i|} \right) \quad (\mathbf{x}_1, \dots, \mathbf{x}_m \in A),$$

where

$$a_{ij} := \begin{cases} c_{ij} & \text{if } i < j, \\ c_{ji} & \text{if } i > j. \end{cases}$$

Let $\{A_n\}$, $n \in \mathbb{N}$, be a sequence of compact sets whose union is A . Then

$$\begin{aligned} & \{(\mathbf{x}_1, \dots, \mathbf{x}_m) \in A : f'(\mathbf{x}_1, \dots, \mathbf{x}_m) \neq 0\} \\ &= \bigcup_{n=1}^{\infty} \left\{ (\mathbf{x}_1, \dots, \mathbf{x}_m) \in A_n : |f'(\mathbf{x}_1, \dots, \mathbf{x}_m)| \geq \frac{1}{n} \right\} \end{aligned}$$

is a countable union of compact sets. By the implicit function theorem, each of these sets is a smooth $(md - 1)$ -dimensional manifold and, hence, has Lebesgue measure 0. Since the complement of A clearly has Lebesgue measure 0, it now suffices to show that

$$\{(\mathbf{x}_1, \dots, \mathbf{x}_m) \in A : f'(\mathbf{x}_1, \dots, \mathbf{x}_m) = 0\}$$

has Lebesgue measure 0. To this end, we assume without loss of generality that $a_{21} = c_{12} \neq 0$. Then

$$\begin{aligned} & \int \cdots \int 1((\mathbf{x}_1, \dots, \mathbf{x}_m) \in A) 1(f'(\mathbf{x}_1, \dots, \mathbf{x}_m) = 0) \, d\mathbf{x}_1 \cdots d\mathbf{x}_m \\ & \leq \int \cdots \int 1((\mathbf{x}_1, \dots, \mathbf{x}_m) \in A) 1\left(a_{21} \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|} = - \sum_{i=3}^m a_{2i} \frac{\mathbf{x}_2 - \mathbf{x}_i}{|\mathbf{x}_2 - \mathbf{x}_i|}\right) d\mathbf{x}_1 \cdots d\mathbf{x}_m. \end{aligned}$$

Using polar coordinates $t \geq 0$ and $\mathbf{u} \in S^{d-1}$, such that $\mathbf{x}_1 - \mathbf{x}_2 = t\mathbf{u}$, for the inner integral we find that the latter volume equals

$$\begin{aligned} & d\kappa_d \int \cdots \int \int \int 1((\mathbf{x}_2 + t\mathbf{u}, \mathbf{x}_2, \dots, \mathbf{x}_m) \in A) 1\left(a_{21}\mathbf{u} = \sum_{i=3}^m a_{2i} \frac{\mathbf{x}_2 - \mathbf{x}_i}{|\mathbf{x}_2 - \mathbf{x}_i|}\right) \\ & \quad \times t^{d-1} \nu(d\mathbf{u}) \, dt \, d\mathbf{x}_2 \cdots d\mathbf{x}_m, \end{aligned}$$

where ν denotes the spherical Lebesgue measure on the unit sphere S^{d-1} . Since $d \geq 2$, the measure ν is diffuse, which in turn implies that the inner integral above vanishes.

3.4. Descending chains: a stochastic example

Following a proposal by Venkat Anantharam and Francois Baccelli (personal communication, Oberwolfach, December 2003) we construct a stationary, ergodic nonlattice point process N , with finite intensity, that has descending chains. The essential idea is to piece together, via a zigzag path, segments of point processes located on the elements of two independent unit-rate Poisson line processes X_1 and X_2 , respectively consisting of vertical and horizontal lines, so that the intersections of any line of X_1 or X_2 with the lines of the other constitute a Poisson process with unit rate.

Independently label each line $L \in (X_1 \cup X_2)$ to be of type $k \in \mathbb{N}$ with probability p_k . On each such line L of type k , we place points Φ_L according to a (generic) one-dimensional point process Φ_k . The latter is assumed to have ‘regenerative cycles’ of points, for which the length of the cycles is D_k and the intensity of points (per unit length) is $\lambda_k \approx 3^k$ (meaning that there are about $\lambda_k D_k$ points in each cycle). Within a cycle, the points constitute a finite strictly descending chain in the positive direction of the axis to which the line is parallel; the points are almost equally spaced, separated by a distance arbitrarily slightly different from $1/\lambda_k$.

The zigzag paths are constructed by switching from a line of type $k - 1$ to a line of type k at the points of intersection of $L_{k-1} \in X_1$, say, with $L_k \in X_2$, subject to the condition that the points on L_k closest to the intersection are located within the first half of a cycle. Then, because the line processes are Poisson with unit rate, the length τ_k of such a zigzag segment on L_k is exponentially distributed with $P(\tau_k > x) = \exp(-\frac{1}{2} p_{k+1} x)$.

Each zigzag path is used to construct a set of points consisting, roughly speaking, of the members of Φ_{L_k} on the segment; the provisos are that, of the points on an L_k segment, not the first but the second point after the intersection with L_{k-1} is included, and the point closest to L_{k+1} is the last point included. These provisos ensure that at least one point from L_k is included, that the distance from the last point of L_k to the first point of L_{k+1} in the neighbourhood of the intersection lies between $1/\lambda_{k+1}$ and $(4/\lambda_{k+1}^2 + (\frac{1}{2})^2/\lambda_k^2)^{1/2}$, and that this upper bound is less than $1/\lambda_k$ when $\lambda_k < (\sqrt{3}/4)\lambda_{k+1}$.

Thus, when this condition holds, points on successive segments of each zigzag path are part of a longer (finite) descending chain as long as each segment of length τ_k , say, is contained within the same regenerative cycle of Φ_{L_k} , where the intersection between L_{k-1} and L_k occurs. To ensure this, it is enough that $\tau_k < \frac{1}{2} D_k$. Consequently, an infinite descending chain of points exists and follows on from the points on some line L_K if, for all segments $\{\tau_k, k \geq K\}$ it holds that $\tau_k < \frac{1}{2} D_k$. These events are mutually independent and, by the Borel–Cantelli lemma, we have

$$P(\tau_k < \frac{1}{2} D_k \text{ for all } k > K, \text{ for some finite } K) = 1$$

if and only if

$$\sum_{k \geq 1} \exp(-\frac{1}{2} p_{k+1} \frac{1}{2} D_k) < \infty.$$

To ensure this, it is enough that $p_{k+1} D_k = O(k)$.

Finally, the points N of Φ_L over all possible lines L constitute a planar point process of finite intensity if and only if $\sum_k p_k \lambda_k < \infty$.

Setting $\lambda_k = 3^k$ is enough to satisfy the intersection property, and setting $p_k = c/k^2 \lambda_k$, where $1/c = \sum_k 1/k^2 \lambda_k$, ensures that the planar point process has finite intensity. Choosing $D_k = k/p_k = k^3 3^k / c$ ensures that the critical sum resulting from the Borel–Cantelli lemma is finite.

It is routine that a stationary Poisson line process exists, and it is likewise routine to construct stationary deterministic regenerative cycles of points Φ_k . These can be given arbitrarily small perturbations to ensure that they are still composed of finite descending chains of points as described, and that the resulting union point process N satisfies the absolute continuity condition (3.4). The example is now constructed as claimed.

4. Absence of descending chains

In this section, we introduce assumptions on the stationary point process N that ensure that N a.s. has no descending chain. To make our assumptions more flexible, we use a random field $\xi = \{\xi_x, x \in \mathbb{R}^d\}$ that takes values in some measurable space (E, \mathcal{E}) and is *jointly* stationary with N . In more detail, we assume that ξ is a random element in a subset \mathbf{W} of the path space $E^{\mathbb{R}^d}$, which is invariant under shifts and equipped with a σ -field \mathcal{W} rendering all mappings $w \mapsto w(x), x \in \mathbb{R}^d$, measurable. As in [16], assume that $(x, w) \mapsto T_x w$ is measurable with respect to both $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{W}$ and \mathcal{W} , where the shift operator $T_x : \mathbf{W} \rightarrow \mathbf{W}$ is defined by $T_x w(y) := w(x + y), y \in \mathbb{R}^d$. The shift operators on \mathbf{N} are denoted by the same symbols and are defined by $T_x \varphi := \varphi - x$. Joint stationarity of N and ξ then requires that the distribution of the pair $(T_x N, T_x \xi)$ does not depend on $x \in \mathbb{R}^d$.

Our basic assumption on N is that

$$\lambda_N P_{\mathbf{0}}^N((N \setminus \{\mathbf{0}\}, \xi) \in \cdot) = E \left[\mu(N, \xi) \int 1((N \cup \varphi, \xi) \in \cdot) Q_{\mathbf{0}}(d\varphi) \right], \tag{4.1}$$

where μ is a nonnegative measurable function on $\mathbf{N} \times \mathbf{W}$ and $Q_{\mathbf{0}}$ is a probability measure on $(\mathbf{N}, \mathcal{N})$. We call μ the Gibbs–Cox component and $Q_{\mathbf{0}}$ the cluster component of N . This framework encompasses some examples that we will discuss in more detail in Section 7:

- (i) When $Q_{\mathbf{0}}(\{\emptyset\}) = 1$ and $\mu(N, \xi)$ does not depend on N , N is a Cox process.
- (ii) When $Q_{\mathbf{0}}(\{\emptyset\}) = 1$ and $\mu(N, \xi)$ does not depend on ξ , N is a Gibbs process.
- (iii) When $\mu \equiv \lambda_N$, N is a Poisson cluster process.
- (iv) When $\mu \equiv \lambda_N$ and $Q_{\mathbf{0}}(\{\emptyset\}) = 1$, N is a stationary Poisson process.

We next formulate additional assumptions on μ and $Q_{\mathbf{0}}$. Let

$$e(x, \varphi, w) := \mu(T_x \varphi, T_x w) \quad ((x, \varphi, w) \in \mathbb{R}^d \times \mathbf{N} \times \mathbf{W})$$

and, for any $x_1, \dots, x_n \in \mathbb{R}^d$, define

$$e_n(x_1, \dots, x_n) := e(x_1, N, \xi) e(x_2, N \cup \{x_1\}, \xi) \cdots e(x_n, N \cup \{x_1, \dots, x_{n-1}\}, \xi).$$

Assume that

$$E[e_n(x_1, \dots, x_n)] \leq c_n \quad (n \in \mathbb{N}, x_1, \dots, x_n \in \mathbb{R}^d) \tag{4.2}$$

for some sequence (c_n) . Our assumption on $Q_{\mathbf{0}}$ is that

$$\int \varphi^{(n)}(dx_1, \dots, dx_n) Q_{\mathbf{0}}(d\varphi) \leq b_n dx_1 \cdots dx_n \quad (n \in \mathbb{N}) \tag{4.3}$$

for some sequence (b_n) . Our main result requires the following assumptions on the sequences (c_n) and (b_n) :

$$\limsup_{n \rightarrow \infty} \frac{\sqrt[n]{c_n}}{n} < \infty, \tag{4.4}$$

$$\limsup_{n \rightarrow \infty} \frac{\sqrt[n]{b_n}}{n} < \infty. \tag{4.5}$$

Theorem 4.1. *Let assumptions (4.1), (4.2), and (4.3) be satisfied and assume, moreover, that (4.4) and (4.5) hold. Then N is descending chain free.*

Our proof of this theorem uses the following two lemmas, which are of a combinatorial nature. The first lemma is easily proved by induction.

Let $d_n \equiv d_n(i_1, \dots, i_n)$ denote the number of distinct partitions of n distinguishable objects into i_r subsets of size r ($r = 1, \dots, n$). Then, elementary combinatorics (see, e.g. Theorem 13.2 of [1]) give

$$d_n(i_1, \dots, i_n) = \frac{n!}{(\prod_{r=1}^n i_r!) (\prod_{r=1}^n (r!)^{i_r})}. \tag{4.6}$$

The following lemma is readily checked.

Lemma 4.1. *The numbers $d_n(i_1, \dots, i_n) \in \mathbb{N}$, where $i_1 + 2i_2 + \dots + ni_n = n$ and $n \in \mathbb{N}_0$, are uniquely determined by $d_0 = 1$, $d_1(1) = 1$, $d_2(2, 0) = d_2(0, 1) = 1$, and the recurrence relation*

$$\begin{aligned} d_{n+1}(i_1, \dots, i_{n+1}) &= 1(i_1 \geq 1)d_n(i_1 - 1, i_2, \dots, i_n) \\ &\quad + 1(i_2 \geq 1)(i_1 + 1)d_n(i_1 + 1, i_2 - 1, i_3, \dots, i_n) \\ &\quad + 1(i_3 \geq 1)(i_2 + 1)d_n(i_1, i_2 + 1, i_3 - 1, i_4, \dots, i_n) \\ &\quad + \dots + 1(i_{n+1} \geq 1)(i_n + 1)d_n(i_1, \dots, i_{n-1}, i_n + 1). \end{aligned} \tag{4.7}$$

Recursion (4.7) generates partitions of the set $\{1, \dots, n\}$ and can be taken as another argument for the validity of (4.6).

Consider a sequence $x_j, j \in \mathbb{N}$, of nonnegative numbers satisfying $\sum_{j=1}^{\infty} x_j/j! < \infty$, and let

$$G_n := \frac{1}{n!} \sum_{i_1+2i_2+\dots+ni_n=n} d_n(i_1, \dots, i_n) x_1^{i_1} \dots x_n^{i_n} \quad (n \in \mathbb{N}_0).$$

By Theorem 13.1 of [1],

$$\exp\left(\sum_{j=1}^{\infty} \frac{x_j z^j}{j!}\right) = \sum_{j=0}^{\infty} G_j z^j \quad (|z| \leq 1).$$

We need a related result to compute the power series with general coefficient

$$F_n := \sum_{i_1+2i_2+\dots+ni_n=n} \binom{i_1 + \dots + i_n}{i_1, \dots, i_n} \left(\frac{x_1}{1!}\right)^{i_1} \dots \left(\frac{x_n}{n!}\right)^{i_n} \quad (n \in \mathbb{N}_0).$$

Note that $G_n \leq F_n$.

Lemma 4.2. Let (a_n) be sequence of nonnegative numbers with $\sum_{n=1}^\infty a_n < 1$. Define F_n , for $n \in \mathbb{N}_0$, by $F_0 = 1$ and

$$F_n := \sum_{i_1+2i_2+\dots+ni_n=n} \binom{i_1 + \dots + i_n}{i_1, \dots, i_n} a_1^{i_1} \dots a_n^{i_n} \quad (n \in \mathbb{N}).$$

Then,

$$\sum_{n=0}^\infty F_n = \left(1 - \sum_{n=1}^\infty a_n\right)^{-1}. \tag{4.8}$$

Proof. For any $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, define

$$F_n^{(k)} := \sum_{i_1+2i_2+\dots+ki_k=n} \binom{i_1 + \dots + i_k}{i_1, \dots, i_k} a_1^{i_1} \dots a_k^{i_k},$$

with $F_0^{(k)} := 1$. Note that

$$F_n^{(k)} = F_n \quad (n \leq k). \tag{4.9}$$

We claim that

$$a_k F_{n-k+1}^{(k)} + \dots + a_1 F_n^{(k)} = F_{n+1}^{(k)} \quad (n \in \mathbb{N}_0), \tag{4.10}$$

with $F_i^{(k)} := 0$ for $i < 0$. For $k = 1$, we have $F_n^{(1)} = a_1^n$, implying (4.10). The general case can be proved by induction on k , in a straightforward calculation. By letting

$$F^{(k,m)} := \sum_{n=0}^m F_n^{(k)} \quad (m \in \mathbb{N}_0) \quad \text{and} \quad F^{(k)} := \sum_{n=0}^\infty F_n^{(k)}$$

and summing (4.10) over $n \in \{0, \dots, m\}$, we obtain

$$(a_k + \dots + a_1) F^{(k,m)} \geq F^{(k,m)} - 1,$$

meaning that $F^{(k,m)} \leq (1 - a_1 - \dots - a_k)^{-1}$. Letting $m \rightarrow \infty$ gives

$$F^{(k)} \leq (1 - a_1 - \dots - a_k)^{-1} < \infty.$$

Therefore, we can sum (4.10) over all $n \in \mathbb{N}_0$ to obtain

$$F^{(k)} = (1 - a_1 - \dots - a_k)^{-1} < \infty.$$

By (4.9),

$$\sum_{n=0}^k F_n = \sum_{n=0}^k F_n^{(k)} \leq (1 - a_1 - \dots - a_k)^{-1},$$

meaning that

$$\sum_{n=0}^\infty F_n \leq \left(1 - \sum_{n=1}^\infty a_n\right)^{-1}. \tag{4.11}$$

On the other hand, $F_n \geq F_n^{(k)}$ for any $n \geq k$, implying that

$$\sum_{n=0}^{\infty} F_n \geq \sum_{n=0}^{\infty} F_n^{(k)} = F^{(k)} = (1 - a_1 - \dots - a_k)^{-1}.$$

Together with (4.11), this implies the assertion.

If $\sum_{n=1}^{\infty} a_n < \infty$ then (4.8) applies with a_n replaced by $a_n z^n$ and F_n by $F_n z^n$ for all $n \geq 0$ and all sufficiently small $z > 0$. Now, setting $a_1 = \dots = a_p = 1$ and $a_n = 0$ for $n > p$, and appealing to (4.10), yields the *p*th-order Fibonacci sequence (up to a shift of the index). Then Lemma 4.2 is a classical result in combinatorics (see, e.g. [9, p. 270]). If $\sum_{n=1}^{\infty} a_n \leq 1$ then the generating function version of (4.8) is standard in renewal theory.

Proof of Theorem 4.1. For any a and b such that $0 \leq a < b$, let $C(a, b)$ denote the set of all $\varphi \in \mathbf{N}$ containing a descending chain (x_n) such that $\lim_{n \rightarrow \infty} |x_{n+1} - x_n| \geq a$ and $b \geq |x_1| \geq |x_2 - x_1|$. We will show that there exists an $\varepsilon_0 > 0$ such that $P(N \in C(a, b)) = 0$ whenever $b - a \leq \varepsilon_0$. For such a and b , we then also have $P(N \in C_1(a, b)) = 0$, where $C_1(a, b)$ is the set of all $\varphi \in \mathbf{N}$ containing a descending chain (x_n) such that $b \geq |x_1| \geq a + \frac{1}{2}(b - a) \geq |x_2 - x_1|$ and $\lim_{n \rightarrow \infty} |x_{n+1} - x_n| \geq a$. Stationarity then implies that even $P(N \in C_2(a, b)) = 0$, where $C_2(a, b)$ is the set of all $\varphi \in \mathbf{N}$ containing a descending chain (x_n) such that $a + \frac{1}{2}(b - a) \geq |x_2 - x_1|$ and $\lim_{n \rightarrow \infty} |x_{n+1} - x_n| \geq a$. This is clearly enough to conclude that N a.s. has no descending chain.

To prove that $P(N \in C(a, b)) = 0$ for sufficiently small $b - a$, we start by noting that

$$1(\varphi \in C(a, b)) \leq \int 1(a \leq |x| \leq b) 1(T_x \varphi \setminus \{\mathbf{0}\} \in C(a, |x|)) \varphi(d\mathbf{x}) \quad (\varphi \in \mathbf{N}). \tag{4.12}$$

Using this, assumption (4.1), and the refined Campbell theorem (3.1), gives

$$\begin{aligned} P(N \in C(a, b)) &\leq E \left[\int 1(a \leq |x_1| \leq b) 1(T_{x_1} N \setminus \{\mathbf{0}\} \in C(a, |x_1|)) N(d\mathbf{x}_1) \right] \\ &= E \left[\iint 1(a \leq |x_1| \leq b) 1(N \cup \varphi_1 \in C(a, |x_1|)) \mu(N, \xi) Q_0(d\varphi_1) d\mathbf{x}_1 \right]. \end{aligned} \tag{4.13}$$

Using (4.12), (3.1), and (4.1) again gives

$$\begin{aligned} P(N \in C(a, b)) &\leq E \left[\iiint 1(a \leq |x_2| \leq |x_1| \leq b) 1(N \cup \varphi_2 \cup T_{x_2} \varphi_1 \in C(a, |x_2|)) \right. \\ &\quad \left. \times e_2(\mathbf{0}, -x_2) Q_0(d\varphi_2) Q_0(d\varphi_1) dx_2 dx_1 \right] \\ &\quad + E \left[\iiint 1(a \leq |x_2| \leq |x_1| \leq b) 1((N \cup T_{x_2} \varphi_1) \setminus \{\mathbf{0}\} \in C(a, |x_2|)) \right. \\ &\quad \left. \times e_1(-x_2) \varphi_1(dx_2) Q_0(d\varphi_1) dx_1 \right]. \end{aligned} \tag{4.14}$$

Iterating further yields

$$\begin{aligned}
 & P(N \in C(a, b)) \\
 & \leq E \left[\int \cdots \int 1(a \leq |\mathbf{x}_3| \leq |\mathbf{x}_2| \leq |\mathbf{x}_1| \leq b) 1(N \cup \varphi_3 \cup T_{\mathbf{x}_3}(\varphi_2 \cup T_{\mathbf{x}_2}\varphi_1) \in C(a, |\mathbf{x}_3|)) \right. \\
 & \quad \left. \times e_3(\mathbf{0}, -\mathbf{x}_3, -\mathbf{x}_2 - \mathbf{x}_3) Q_0(d\varphi_3) Q_0(d\varphi_2) Q_0(d\varphi_1) d\mathbf{x}_3 d\mathbf{x}_2 d\mathbf{x}_1 \right] \\
 & + E \left[\int \cdots \int 1(a \leq |\mathbf{x}_3| \leq |\mathbf{x}_2| \leq |\mathbf{x}_1| \leq b) 1(N \cup T_{\mathbf{x}_3}(\varphi_2 \cup T_{\mathbf{x}_2}\varphi_1) \setminus \{\mathbf{0}\} \in C(a, |\mathbf{x}_3|)) \right. \\
 & \quad \left. \times e_2(-\mathbf{x}_3, -\mathbf{x}_2 - \mathbf{x}_3)(\varphi_2 \cup T_{\mathbf{x}_1}\varphi_1)(d\mathbf{x}_3) Q_0(d\varphi_2) Q_0(d\varphi_1) d\mathbf{x}_2 d\mathbf{x}_1 \right] \\
 & + E \left[\int \cdots \int 1(a \leq |\mathbf{x}_3| \leq |\mathbf{x}_2| \leq |\mathbf{x}_1| \leq b) 1(N \cup \varphi_2 \cup T_{\mathbf{x}_3}(T_{\mathbf{x}_2}\varphi_1 \setminus \{\mathbf{0}\}) \in C(a, |\mathbf{x}_3|)) \right. \\
 & \quad \left. \times e_2(\mathbf{0}, -\mathbf{x}_2 - \mathbf{x}_3) d\mathbf{x}_3 \varphi_1(d\mathbf{x}_2) Q_0(d\varphi_2) Q_0(d\varphi_1) d\mathbf{x}_1 \right] \\
 & + E \left[\int \cdots \int 1(a \leq |\mathbf{x}_3| \leq |\mathbf{x}_2| \leq |\mathbf{x}_1| \leq b) \right. \\
 & \quad \times 1((N \cup T_{\mathbf{x}_3}(T_{\mathbf{x}_2}\varphi_1 \setminus \{\mathbf{0}\})) \setminus \{\mathbf{0}\} \in C(a, |\mathbf{x}_3|)) \\
 & \quad \left. \times e_1(-\mathbf{x}_2 - \mathbf{x}_3)(T_{\mathbf{x}_2}\varphi_1 \setminus \{\mathbf{0}\})(d\mathbf{x}_3) \varphi_1(d\mathbf{x}_2) Q_0(d\varphi_1) d\mathbf{x}_1 \right], \tag{4.15}
 \end{aligned}$$

where we have also used the joint stationarity of N and ξ . Replacing the second indicator function on the right-hand side of (4.13) by 1 gives

$$P(N \in C(a, b)) \leq \kappa_d(b - a)c_1,$$

where we have used (4.2) for $n = 1$. Similarly, from (4.14) we find that

$$P(N \in C(a, b)) \leq \frac{\kappa_d^2(b - a)^2}{2}(c_2 + c_1b_1),$$

where we have used both (4.3) and (4.2). From (4.15), we obtain (via a change of variables)

$$P(N \in C(a, b)) \leq \frac{\kappa_d^3(b - a)^3}{3!}(c_3 + 3c_2b_1 + c_1b_2).$$

In general, we have

$$P(N \in C(a, b)) \leq \frac{\kappa_d^n(b - a)^n}{n!} \sum_{i_1+2i_2+\cdots+ni_n=n} d_n(i_1, \dots, i_n)c_{i_1+\cdots+i_n}b_1^{i_2} \cdots b_{n-1}^{i_n}, \tag{4.16}$$

for some coefficients $d_n(i_1, \dots, i_n) \in \mathbb{N}_0$. A careful check of the above recursion shows that the latter coefficients satisfy equation (4.7) and, hence, are given by (4.6). By assumption (4.4)

and Stirling’s formula, there exists a $c > 0$ such that $c_n \leq c^n n!$ for all $n \in \mathbb{N}$. Using this in (4.16) yields

$$\begin{aligned}
 P(N \in C(a, b)) &\leq \kappa_d^n (b - a)^n \sum_{i_1+2i_2+\dots+ni_n=n} \binom{i_1 + \dots + i_n}{i_1, \dots, i_n} c^{i_1+\dots+i_n} f_1^{i_2} \dots f_{n-1}^{i_n} \\
 &= \sum_{i_1+2i_2+\dots+ni_n=n} \binom{i_1 + \dots + i_n}{i_1, \dots, i_n} \\
 &\quad \times (c\kappa_d(a - b))^{i_1} (c\kappa_d^2(a - b)^2 f_1)^{i_2} \dots (c\kappa_d^n(a - b)^n f_{n-1})^{i_n}, \quad (4.17)
 \end{aligned}$$

where $f_i := b_i / (i + 1)!$, $i = 1, \dots, n - 1$. By assumption (4.5) and Stirling’s formula, the series $\sum_{n=1}^\infty f_n z^n$ converges for all sufficiently small $z > 0$. Hence, we can apply Lemma 4.2 to conclude that the right-hand side of (4.17) is the n th summand of a convergent series for sufficiently small $b - a$.

We have shown that $P(N \in C(a, b)) = 0$ whenever $b - a \leq \varepsilon_0$, where $\varepsilon_0 > 0$ depends on c and the sequence (b_n) but not on a and b . As we have seen above, this implies that N is descending chain free.

Remark 4.1. Our proof shows that a stationary point process N satisfying assumption (4.1) has the property (3.4). Lemma 3.1 then implies that N is nonlattice.

5. Absence of percolation in the lilypond model

In this section, we fix a stationary point process $N \neq \emptyset$ with finite intensity and assume that N is descending chain free. We consider the a.s. uniquely defined lilypond model $\{(\mathbf{x}, R(N, \mathbf{x})) : \mathbf{x} \in N\}$ based on N . Recall that if the union set

$$\bigcup_{\mathbf{x} \in N} B_{R(N, \mathbf{x})}(\mathbf{x})$$

contains an unbounded connected component, then the lilypond model *percolates*.

Theorem 5.1. *If the point process N is a.s. descending chain free and nonlattice, then there is a.s. no percolation in the lilypond model defined by N .*

From now on we assume that the hypotheses of Theorem 5.1 are satisfied.

Lemma 5.1. *Each point of N a.s. has exactly one smaller grain-neighbour.*

Proof. Assume that $\varphi \in \mathbf{N}$ has no descending chain. We can and do assume that the lilypond model $\{(\mathbf{x}, R(\mathbf{x})) : \mathbf{x} \in \varphi\}$, based on φ , has been generated by the matching procedure in the proof of Proposition 2.1. If $\mathbf{x} \in \varphi$ has more than one smaller grain-neighbour then, for two different grain-neighbours \mathbf{y} and \mathbf{z} of \mathbf{x} , one of the following statements must hold:

- (i) $R(\mathbf{x}) = R(\mathbf{y}) = R(\mathbf{z})$ and, thus, $2R(\mathbf{x}) = |\mathbf{x} - \mathbf{y}| = |\mathbf{x} - \mathbf{z}|$;
- (ii) $R(\mathbf{x}) = R(\mathbf{y})$ and $R(\mathbf{z}) < R(\mathbf{x})$, so $\frac{1}{2}|\mathbf{x} - \mathbf{y}| = R(\mathbf{x}) = |\mathbf{x} - \mathbf{z}| - R(\mathbf{z})$; or
- (iii) $R(\mathbf{y}) < R(\mathbf{x})$ and $R(\mathbf{z}) < R(\mathbf{x})$, so $R(\mathbf{x}) = |\mathbf{x} - \mathbf{y}| - R(\mathbf{y}) = |\mathbf{x} - \mathbf{z}| - R(\mathbf{z})$.

In the first case, we have $|\mathbf{x} - \mathbf{y}| = |\mathbf{x} - \mathbf{z}|$. In the other two cases, we are using Corollary 2.1. In case (ii), there must also exist an $n \in \mathbb{N}$ such that $\mathbf{z} \in \varphi_n$ and $\mathbf{x} \notin \varphi_n$. By construction, $R(\mathbf{z})$ is a finite linear combination of the distances $|\mathbf{w} - \mathbf{w}'|$, for $\mathbf{w}, \mathbf{w}' \in \varphi_n$, with rational

coefficients. In case (iii), there must exist an $n \in \mathbb{N}$ such that $\mathbf{y}, \mathbf{z} \in \varphi_n$ and $\mathbf{x} \notin \varphi_n$ with $R(\mathbf{y})$ and $R(\mathbf{z})$ linear functions (with rational coefficients) of the distances $|\mathbf{w} - \mathbf{w}'|$, for $\mathbf{w}, \mathbf{w}' \in \varphi_n$. If φ is nonlattice, none of these three cases is possible.

Exactly as in [5], we now conclude the following result.

Lemma 5.2. *Each cluster of N a.s. contains at most one doublet.*

Proof of Theorem 5.1. To show that there are no infinite clusters containing a doublet, consider the set of all doublets pertaining to infinite clusters. From each such doublet, choose one point according to some deterministic, shift-invariant rule. This choice yields a stationary point process M with a finite intensity λ_M . Let $i \in \mathbb{N}$. To each $\mathbf{x} \in M$ attach a finite point process $\Psi_i(\mathbf{x})$ containing exactly i points of the cluster $\mathcal{C}(N, \mathbf{x})$, again using some deterministic and shift-invariant rule. Then, for all $i \in \mathbb{N}$, $\{(\mathbf{x}, \Psi_i(\mathbf{x})) : \mathbf{x} \in M\}$ is a marked point process that is *jointly* stationary with N . Furthermore, by Lemma 5.2, $\Psi_i(\mathbf{x}) \cap \Psi_i(\mathbf{y}) = \emptyset$ for all different points $\mathbf{x}, \mathbf{y} \in M$. In an abuse of notation, we write $\Psi_i(\mathbf{x}) \equiv \Psi_i(N, \mathbf{x})$ and note that $\Psi_i(N, \mathbf{x}) = \Psi_i(N - \mathbf{x}, \mathbf{0}) + \mathbf{x}$. Let \mathbb{P}_0^M denote the Palm probability measure associated with M . Using the refined Campbell theorem (3.2), we conclude that, for all measurable $B \subseteq \mathbb{R}^d$,

$$\begin{aligned} \mathbb{E}[N(B)] &\geq \mathbb{E}\left[\sum_{\mathbf{x} \in M} \Psi_i(B, \mathbf{x})\right] \\ &= \mathbb{E}\left[\sum_{\mathbf{x} \in M} \Psi_i(N - \mathbf{x}, \mathbf{0}, B - \mathbf{x})\right] \\ &= \lambda_M \int \mathbb{E}_0^M[\Psi_i(B - \mathbf{x}, \mathbf{0})]d\mathbf{x} \\ &= \lambda_M \mathbb{E}_0^M\left[\iint 1(\mathbf{y} + \mathbf{x} \in B) \Psi_i(d\mathbf{y}, \mathbf{0}) d\mathbf{x}\right] \\ &= |B|_d \lambda_M \mathbb{E}_0^M[\Psi_i(\mathbb{R}^d, \mathbf{0})] = |B|_d i \lambda_M, \end{aligned}$$

where, recall, \mathbb{E}_0^M denotes expectation with respect to \mathbb{P}_0^M . Hence,

$$\lambda_N \geq i \lambda_M \quad (i \in \mathbb{N}). \tag{5.1}$$

This is possible only if $\lambda_M = 0$, i.e. $\mathbb{P}(M = \emptyset) = 1$.

It remains to prove that there are no infinite clusters without a doublet. However, by Lemma 5.1, such an infinite cluster would contain a descending chain, contradicting our assumptions on N .

By Lemma 2.2, each finite cluster contains a doublet. This proves the following corollary.

Corollary 5.1. *Each cluster of N a.s. is finite and contains exactly one doublet.*

6. Mutual-nearest-neighbour matching

Let $\mathbf{N}_0 := \{\varphi \in \mathbf{N} : \mathbf{0} \in \varphi\}$ and $\mathcal{N}_0 := \mathcal{N} \cap \mathbf{N}_0$. Following [16] we call an \mathcal{N}_0 -measurable mapping $\pi : \mathbf{N}_0 \rightarrow \mathbb{R}^d$ a *point map* if $\pi(\varphi) \in \varphi$ for each $\varphi \in \mathbf{N}_0$. Such a point map is bijective if, for every $\varphi \in \mathbf{N}_0$, $\mathbf{x} \mapsto \pi(\varphi - \mathbf{x}) + \mathbf{x}$ is a bijection of φ . In this case,

$$\mathbb{P}_0^N(N - \pi(N) \in \cdot) = \mathbb{P}_0^N(N \in \cdot),$$

as proved in [11] and Theorem 9.4.1 of [16]. A bijective point map can therefore be used to shift the typical point of N to another point of N without biasing the Palm distribution. Thorisson ([4], [16]) has asked whether there exists a point map π satisfying

$$P_0^N(\pi(N) \neq \mathbf{0}) = 1. \tag{6.1}$$

For a Poisson process N , the above question has been resolved in [4] and [8]. The general case was treated in [6]. Below we use *mutual-nearest-neighbour matching* (see [4] and [8]) to give a straightforward construction of a bijective point map satisfying (6.1); in the absence of descending chains this matching is successful.

Let \mathbf{N}' denote the set of all $\varphi \in \mathbf{N}$ such that $|\mathbf{x} - \mathbf{y}| \neq |\mathbf{y} - \mathbf{z}|$ whenever \mathbf{x}, \mathbf{y} , and \mathbf{z} are three different points of φ ; all nonlattice sets φ belong to \mathbf{N}' . Pick $\varphi \in \mathbf{N}'$. Let $S_\varphi \subseteq \varphi$ denote the set of all those points of φ that are not members of a pair of mutually nearest neighbours, and inductively define

$$S_{n+1}\varphi := S(S_n\varphi) \quad (n \in \mathbb{N}_0),$$

with $S_0\varphi := \varphi$. Set

$$S_\infty\varphi := \bigcap_{n=1}^\infty S_n\varphi \quad \text{and} \quad \mathbf{N}_\infty := \{\varphi \in \mathbf{N}' : S_\infty\varphi = \emptyset\},$$

and define a point map π as follows. If $\varphi \in \mathbf{N}_0 \setminus \mathbf{N}_\infty$ then set $\pi(\varphi) := \mathbf{0}$, while if $\varphi \in \mathbf{N}_0 \cap \mathbf{N}_\infty$ then $\pi(\varphi) \in \varphi \setminus \{\mathbf{0}\}$ is the uniquely determined point satisfying $\{\mathbf{0}, \pi(\varphi)\} \in S_n\varphi \setminus S_{n+1}\varphi$ for some (uniquely determined) $n \in \mathbb{N}_0$. (Any point of $S_n\varphi$ has a uniquely determined nearest neighbour.) It is convenient to call π the *mutual-nearest-neighbour map*.

Theorem 6.1. *Let N be a stationary, descending-chain-free point process on \mathbb{R}^d with finite positive intensity, and assume that $P(N \in \mathbf{N}') = 1$. Then the mutual-nearest-neighbour map π is a bijective point map satisfying (6.1).*

Proof. The set \mathbf{N}' is shift invariant. Furthermore, the mappings S_n ($n \in \mathbb{N}_0 \cup \{\infty\}$) are shift covariant, in the sense that

$$S_n(\varphi - \mathbf{x}) = S_n\varphi - \mathbf{x} \quad (\mathbf{x} \in \mathbb{R}^d, \varphi \in \mathbf{N}').$$

In particular, \mathbf{N}_∞ is a shift-invariant set. Let $\varphi \in \mathbf{N}_0$ and define the mapping $h_\varphi : \varphi \rightarrow \varphi$ by $h_\varphi(\mathbf{x}) := \pi(\varphi \setminus \{\mathbf{x}\}) + \mathbf{x}$. By definition and the above shift invariance, $h_\varphi \circ h_\varphi$ is the identity on φ . Hence, h_φ is a bijection.

It remains to show that

$$P_0^N(N \in \mathbf{N}_\infty) = P(N \in \mathbf{N}_\infty) = 1. \tag{6.2}$$

Let $\varphi \in \mathbf{N}'$ and set $\psi := S_\infty\varphi$. As in the proof of Proposition 2.1, it follows that ψ cannot contain any pairs of mutually nearest neighbours (in ψ). Assume now that, in addition, ψ contains infinitely many points. Then, as in the proof of Proposition 2.1, it follows that ψ (and, hence, φ) must have a descending chain. Define $S_\infty\varphi := \varphi$ whenever $\varphi \notin \mathbf{N}'$. Then $S_\infty N$ is a stationary point process. The event $\{S_\infty N \neq \emptyset\}$ is therefore a.s. contained in the event that $S_\infty N$ contains infinitely many points. Since $P(N \notin \mathbf{N}') = 0$, it follows that the event $\{S_\infty N \neq \emptyset\}$ is a.s. contained in the event that N contains a descending chain. This shows that (6.2) holds.

7. Examples

7.1. Cox processes

A *Cox process* N is a Poisson process with a random intensity measure ξ (see, e.g. [2]). Formally, N is defined by requiring that the conditional distribution of N given ξ is that of an inhomogeneous Poisson process with intensity measure ξ . Then N and ξ have the same intensity measure, i.e. $E[N(\cdot)] = E[\xi(\cdot)]$. Now assume, in addition, that

$$\xi(B) = \int_B \xi_x \, d\mathbf{x} \quad (B \in \mathcal{B}(\mathbb{R}^d)) \tag{7.1}$$

for some stationary, nonnegative random field $\{\xi_x : \mathbf{x} \in \mathbb{R}^d\}$. Then we may assume that N and $\{\xi_x\}$ are jointly stationary. The intensity of N is given by $\lambda_N = E[\xi_0]$, which must be assumed finite and positive.

Theorem 7.1. *Let N be a Cox process directed by the stationary random measure ξ , as in (7.1), and assume that*

$$\limsup_{n \rightarrow \infty} \frac{\sqrt[n]{E[\xi_0^n]}}{n} < \infty.$$

Then N is descending chain free.

Proof. Using the fundamental properties of a Poisson process, it is easy to prove that (4.1) holds with $\mu(N, \xi) = \xi_0$ and $Q_0(\{\emptyset\}) = 1$. By recursive use of the Hölder inequality and stationarity of $\{\xi_x\}$, we have

$$E[\xi_{x_1} \cdots \xi_{x_n}] \leq E[\xi_0^n] \quad (n \in \mathbb{N}).$$

Hence, (4.2) holds with $c_n := E[\xi_0^n]$. The assertion is now implied by Theorem 4.1.

7.2. Poisson cluster processes

Here, we assume that

$$N = \bigcup_{\mathbf{x} \in N_c} (N^{\mathbf{x}} + \mathbf{x}).$$

is a *Poisson cluster process* (see, e.g. [2]) based on a Poisson process N_c of *parents* or *cluster centres* with finite intensity λ_c and a family $\{N^{\mathbf{x}} : \mathbf{x} \in N_c\}$ of point processes on \mathbb{R}^d that are conditionally independent, given N_c . The conditional distribution of $N^{\mathbf{x}}$ given N_c is assumed to be the same for all $\mathbf{x} \in N_c$ and is denoted by Q . Then N is a stationary point process. We further assume that the mean number λ'_c of cluster points,

$$\lambda'_c := \int (\text{card } \psi) Q(d\psi),$$

is positive and finite, implying that the clusters are a.s. finite sets. Then N is a stationary point process with intensity

$$\lambda_N = \lambda_c \lambda'_c.$$

We can write Q in the form

$$Q = p_0 \delta_{\emptyset} + \sum_{n=1}^{\infty} p_n \Pi_n,$$

where $\{p_n, n \in \mathbb{N}_0\}$ is the cluster size distribution and the Π_n are probability measures concentrated on $\{\varphi \in \mathbf{N} : \varphi(\mathbb{R}^d) = n\}$. For any $m \in \mathbb{N}$, we write $m^{[n]} := m(m-1) \cdots (m-n+1)$ for $n \leq m$.

Theorem 7.2. *Let N be a Poisson cluster process as described above, and assume that*

$$\begin{aligned} & \iint 1((\mathbf{x}_2 - \mathbf{x}_1, \dots, \mathbf{x}_{k+1} - \mathbf{x}_1) \in \cdot) \varphi^{(k+1)}(d(\mathbf{x}_1, \dots, \mathbf{x}_{k+1})) \Pi_n(d\varphi) \\ & \leq M^k n^{[k+1]} \int \cdots \int 1((\mathbf{x}_1, \dots, \mathbf{x}_k) \in \cdot) d\mathbf{x}_1 \cdots d\mathbf{x}_k \quad (n \geq k + 1) \end{aligned} \tag{7.2}$$

for some $M > 0$. Assume, moreover, that the cluster size distribution $\{p_n, n \in \mathbb{N}_0\}$ has a finite exponential moment, i.e.

$$\sum_{n=1}^{\infty} p_n z^n < \infty \quad \text{for some } z > 1. \tag{7.3}$$

Then N is descending chain free.

Proof. It is well known (see, e.g. [2]) that (4.1) is satisfied by

$$Q_{\mathbf{0}}(\cdot) := \frac{1}{\lambda'_c} \int \left[\sum_{\mathbf{x} \in \psi} 1(T_{\mathbf{x}}\psi \setminus \{\mathbf{0}\} \in \cdot) \right] Q(d\psi)$$

and $\mu \equiv \lambda_N$. A straightforward computation shows that (4.3) holds with

$$b_n = \frac{M^n}{\lambda'_c} \sum_{m=n+1}^{\infty} p_m m^{[n+1]}.$$

To check (4.5), we take $q \in (0, 1)$ and compute

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{b_n q^n}{n!} & \leq \frac{1}{\lambda'_c} \sum_{m=2}^{\infty} m p_m \sum_{n=0}^{m-1} \binom{m-1}{n} q^n M^n \\ & = \frac{1}{\lambda'_c} \sum_{m=2}^{\infty} m p_m (1 + qM)^{m-1}. \end{aligned}$$

Taking q sufficiently small that $1 + qM < z$, where z is as in assumption (7.3), the above series converges and (4.5) follows. The assertion is now implied by Theorem 4.1.

We can always assume (see [2]) that

$$\Pi_n(\cdot) = P(\{\mathbf{X}_{n,1}, \dots, \mathbf{X}_{n,n}\} \in \cdot) \quad (n \in \mathbb{N})$$

for some symmetrically distributed random vectors $\mathbf{X}_{n,1}, \dots, \mathbf{X}_{n,n}$. Since, for $n \geq k$,

$$\iint 1((\mathbf{x}_1, \dots, \mathbf{x}_k) \in \cdot) \varphi^{(k)}(d(\mathbf{x}_1, \dots, \mathbf{x}_k)) \Pi_n(d\varphi) = n^{[k]} P((\mathbf{X}_{n,1}, \dots, \mathbf{X}_{n,k}) \in \cdot),$$

(7.2) is a reasonable weak assumption. A Neyman–Scott process (see [2]), for instance, has this property, if the individual cluster points have a bounded density.

7.3. Gibbs processes

The last class of point processes we discuss consists of *Gibbsian* point processes. A mathematical (rather than a physicist's) definition requires the existence of a measurable function $e: \mathbb{R}^d \times \mathbf{N} \rightarrow [0, \infty)$ satisfying

$$\mathbb{E} \left[\sum_{\mathbf{x} \in N} f(\mathbf{x}, N \setminus \{\mathbf{x}\}) \right] = \mathbb{E} \left[\int e(\mathbf{x}, N) f(\mathbf{x}, N) \, d\mathbf{x} \right], \tag{7.4}$$

for all measurable functions $f: \mathbb{R}^d \times \mathbf{N} \rightarrow [0, \infty)$ (see, e.g. [15]). Theorem 4.1 specializes as follows.

Theorem 7.3. *Let N be a Gibbs process as described above, and assume that $e(\mathbf{x}, \varphi) = \mu(T_{\mathbf{x}}\varphi)$ for some measurable function $\mu: \mathbf{N} \rightarrow [0, \infty)$. Assume, moreover, that there exists a $c > 0$ such that*

$$\mathbb{E}[e(\mathbf{x}_1, N)e(\mathbf{x}_2, N \cup \{\mathbf{x}_1\}) \dots e(\mathbf{x}_n, N \cup \{\mathbf{x}_1, \dots, \mathbf{x}_{n-1}\})] \leq c^n n!$$

for any $n \in \mathbb{N}$ and any $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^d$. Then N is descending chain free.

An interesting class of Gibbsian point processes with bounded μ can be defined in terms of a pair potential with a *hard-core* radius $r_0 > 0$. In this case,

$$\mu(\varphi) = \prod_{\mathbf{y} \in \varphi} \theta(|\mathbf{y}|) \quad (\varphi \in \mathbf{N}), \tag{7.5}$$

where $\theta: \mathbb{R} \rightarrow [0, \infty)$ is measurable and satisfies $\theta(r) = 0$ for all $r \leq r_0$. The pair potential $U(r) := -\ln \theta(r)$ satisfies $U(r) = -\infty$ for $r \leq r_0$ and is further assumed to satisfy $|U(r)| \leq U_0(|r|)$ for all $r \geq r_0$, where $U_0: [r_0, \infty) \rightarrow [0, \infty)$ is nonincreasing and satisfies $\int_{r_0}^{\infty} r^{d-1} U_0(r) \, dr < \infty$ and $\lim_{r \rightarrow \infty} r^d U_0(r) = 0$. By an existence result of [13], we can now actually assume that N is a stationary Gibbs process with μ given by (7.5). It follows immediately, from a fundamental local absolute continuity property of Gibbs processes (see, e.g. [13] and [15]), that $\mathbb{P}(N \in \mathbf{N}_h) = 1$, where \mathbf{N}_h denotes the set of all those $\varphi \in \mathbf{N}$ whose points are at least a distance r_0 apart from each other. It has been shown in [10] (in the case $d = 2$) that the absolute value of $\sum_{\mathbf{y} \in \varphi} U(|\mathbf{y}|)$ is bounded in $\varphi \in \mathbf{N}_h$ by a constant that only depends on the function U_0 . By (7.4), the intensity of N can be written as

$$\lambda_N = \mathbb{E}[\mu(N)].$$

Hence, we indeed have $\lambda_N < \infty$. Since

$$\mu(\varphi) = \exp\left(-\sum_{\mathbf{y} \in \varphi} U(|\mathbf{y}|)\right) > 0 \quad (\varphi \in \mathbf{N}_h),$$

we have $\lambda_N > 0$. Therefore, we may assume (after conditioning) that $\mathbb{P}(N \neq \emptyset) = 1$.

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