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# A NOTE ON SUPERSOLUBLE GROUPS

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Let $H$ be a subgroup of a group $G$ (all groups considered throughout this article are finite); then $H$ will be called primitive if the subgroup

$$\hat{H} = \bigcap_{H<K\leqq G} K$$

is distinct from $H$. Such subgroups, which are also called meet-irreducible, arise naturally in connection with minimal permutation representations of groups and in other contexts; for example, every subgroup of a group $G$ can be written as an intersection of primitive subgroups of $G$, and the set of all primitive subgroups of $G$ is characterized by its minimality with respect to this property. While maximal subgroups are always primitive, most groups contain non-maximal subgroups which are primitive (see remark at end of article). Note that a subgroup $H$ of an abelian group $G$ is primitive if, and only if, $G/H$ is cyclic of prime-power order. Thus, every abelian group lies in the class

$$\mathscr{X} = \{\text{groups } G\text{: every primitive } H \leqq G \text{ has prime power index}\},$$

as indeed do all nilpotent groups. The main purpose of this note is to prove the following theorem.

THEOREM. *Every $G \in \mathscr{X}$ is supersoluble.*

*Proof.* 1. The first of our three steps is to establish a basis for induction; this consists of three assertions, namely

(a) let $H$ be a subgroup of an arbitrary group $G$ and $K$ a primitive subgroup of $H$; then there is a primitive subgroup $X \leqq G$ with $K = H \cap X$;

(b) if $G \in \mathscr{X}$ and $N \lhd G$, then $G/N \in \mathscr{X}$;

(c) if $G \in \mathscr{X}$ and $N \lhd G$, then $N \in \mathscr{X}$.

For the proof of (a), choose primitive subgroups $X_1, \ldots, X_n$ of $G$ such that $K = \bigcap_{i=1}^{n} X_i$; then we have

$$K = K \cap H = \bigcap_{i=1}^{n} (X_i \cap H).$$

Thus, since $K$ is primitive in $H$, $K = X_i \cap H$ for some $i$. As to (b), it is clear that if $H/N$ is primitive in $G/N$, then $H$ is a primitive subgroup of $G$ of the same index. To prove (c), let $H$ be primitive in $N$ and choose $X$ primitive in $G$ with $H = N \cap X$ (possible by (a)); then $|G:XN||XN:X| = |G:X|$, a prime power, since $G \in \mathscr{X}$. Thus

$$|N:H| = |N:N \cap X| = |XN:X|$$

is also a prime-power, as required.

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2. We now claim that every $G \in \mathscr{X}$ is soluble. Assume that this is false and let $G$ be a minimal counterexample. Then $G$ is a non-abelian simple group, by virtue of 1(b) and 1(c). Let $S$ be a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime dividing $|G|$, with $|S| = p^n$, say. Further, let $M$ be a maximal subgroup of $S$, and let $N$ be the normalizer of $M$ in $G$ (the case $n = 1$ does not need separate treatment). Then $p$ divides $|N/M|$ to the first power only, so we can assert (by Burnside's normalizer-centralizer Theorem [**1**, IV, 2.6]; see also [**1**, IV, 2.7 and 2.8]) the existence of a normal complement $C/M$ for $S/M$ in $N/M$. Now $|N:C| = p$, so $C$ is maximal, and therefore primitive, in $N$. In accordance with 1(a), let $X$ be a primitive subgroup of $G$ with $C = X \cap N$. By hypothesis, there is a prime $q$ and a natural number $r$ such that $|G:X| = q^r$. Now $M \leqq X$, so $p^{n-1}$ divides $|X|$, whence, if $p = q$, we must have $r = 1$, implying that $X \lhd G$, which is impossible. Hence $p \neq q$, and we can find a Sylow $p$-subgroup $T$ of $G$ lying in $X$ and containing $M$. Since $T$ normalizes $M$, $T \leqq N \cap X = C$. Thus, $|T| = p^n$ is a divisor of $|C|$, while $|G:C|$ is divisible by $|N:C| = p$. This leads to the contradiction that $p^{n+1}$ divides $|G|$.

3. Finally, we prove the assertion that every $G \in \mathscr{X}$ is supersoluble. Again, suppose this to be false and let $G$ be a minimal counterexample. Thus $G$ is soluble by 2 above, but not supersoluble. Thus, in accordance with a theorem of B. Huppert (see [**1**, VI, 9.5]), we can choose a maximal subgroup $M$ of $G$ with $|G:M| = p^s$, with $p$ a prime and $s$ a natural number $\geqq 2$. Now by 1(b) and induction, the core of $M$ is trivial, so if $N$ is a minimal normal subgroup of $G$, the usual argument for soluble groups allows us to conclude that $MN = G$ and $M \cap N = \{e\}$, whence $|N| = p^s$. Now let $H$ be any maximal subgroup of $N$ and let $X$ be a primitive subgroup of $G$ such that $X \cap N = H$, which is possible by 1(a). By hypothesis, $|G:X|$ is a prime-power which, moreover, is divisible by

$$|XN:X| = |N:X \cap N| = |N:H| = p,$$

so $|G:X|$ is in fact a $p$-power. But since $N \lhd G, H = N \cap X \lhd X$, so the normalizer of $H$ in $G$ has non-trivial $p$-power index in $G$ (since $\{e\} < H < N$). Thus, since the set $\mathscr{M}$ of maximal subgroups of $N$ is a union of $G$-conjugacy classes each of cardinality divisible by $p$, $||\mathscr{M}||$ must be a multiple of $p$. But this number is equal to

$$\frac{(p^s - 1)(p^s - p)\ldots(p^s - p^{s-2})}{(p^{s-1} - 1)(p^{s-1} - p)\ldots(p^{s-1} - p^{s-1})} = p^{s-1} + \ldots + p + 1.$$

This is a contradiction and the theorem is proved.

*Example.* Since $S_3 \in \mathscr{X}$, $\mathscr{X}$ properly contains the class of nilpotent groups. Further, since the supersoluble group $S_3 \times Z_3$ of order 18 contains a primitive subgroup of order 3, the converse of the theorem is false. This example also shows that $\mathscr{X}$ is not closed under direct products, and so is not a formation.

The present article was inspired by some discussions with J. F. Humphreys on Lagrangian groups (groups having subgroups of all possible orders), the results of which we hope will appear in a future article. By way of an example, we state without proof one application of the above theorem to such groups and their analogues. Define

$$\mathscr{Y} = \{\text{groups } G : \forall H \leqq G \text{ and } \forall n \,|\, |G:H|,\ \exists K \leqq G, \text{ such that } K \geqq H \text{ and } |G:K| = n\}.$$

Then $\mathscr{Y}$ consists of those groups $G$ every subgroup of which is the intersection of a set of subgroups of $G$ of pairwise coprime, prime-power indices. The groups in $\mathscr{X}$ are given by the same condition, but with the pairwise coprime requirement relaxed; hence $\mathscr{Y} \subseteq \mathscr{X}$. Denoting by $\bar{\ }$ the operation of subgroup closure, we can prove that $\overline{\mathscr{X}} \subseteq \mathscr{Y}$, and hence that $\overline{\mathscr{X}} = \overline{\mathscr{Y}}$. Whether $\mathscr{X} = \mathscr{Y}$ we have not as yet been able to decide; however, if $G$ is a group of minimal order in $\mathscr{X} \backslash \mathscr{Y}$, then $|G|$ is divisible by $p^4$, where $p$ is the largest prime dividing $|G|$.

Another problem which may be worthy of attention is to characterize the elements of the class $\mathscr{Z}$ of groups all of whose primitive subgroups are maximal. Beyond making the obvious remarks that $S_3 \in \mathscr{Z}$ and the nilpotent groups in $\mathscr{Z}$ are abelian of square-free exponent, while the soluble groups in $\mathscr{Z}$ are all supersoluble, we have not pursued this question.

## Reference

1. B. Huppert, *Endliche Gruppen*, Vol. I (Springer-Verlag, Berlin, 1967).

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