Can. J. Math., Vol. XL, No. 3, 1988, pp. 666-694

THE QUANTIFICATIONAL TANGENT CONES

DOUG WARD

1. Introduction. Nonsmooth analysis has provided important new mathematical tools for the study of problems in optimization and other areas of analysis [**1**, **2**, **6-12**, **28**]. The basic building blocks of this subject are local approximations to sets called tangent cones.

Definition 1.1. Let E be a real, locally convex, Hausdorff topological vector space (abbreviated l.c.s.). A *tangent cone* (on E) is a mapping $A:2^E \times E \to 2^E$ such that $A(C, x)$ is a (possibly empty) cone for all nonempty C in 2^E and x in E.

In the sequel, we will say that a tangent cone has a certain property (e.g. "A is closed" or "A is convex") if $A(C, x)$ has that property for all nonempty sets C and all x in C. (If $A(C, x)$ is empty, it will be counted as having the property trivially.)

Quite a number of tangent cone definitions have been proposed (see for example [**8**, **13**, **14**, **17-20**, **23**, **34-39**]), two of the most useful of which are reviewed below:

Definition 1.2. Let E be a l.c.s., C a nonempty subset of E, and $x \in C$. Denote by $N(y)$ the class of neighborhoods of $y \in E$.

(a) The *contingent cone* to C at x is the set

$$K(C, x) := \{y \in E : \forall\, Y \in N(y),\ \forall\, \lambda > 0,\ \exists\, t \in (0, \lambda),\ \exists\, y' \in Y,\ x + ty' \in C\}.$$

(b) The *Clarke tangent cone* to C at x is the set

$$T(C, x) := \{y \in E : \forall\, Y \in N(y),\ \exists\, X \in N(x),\ \exists\, \lambda > 0,\ \forall\, x' \in X \cap C,\ \forall\, t \in (0, \lambda),\ \exists\, y' \in Y,\ x' + ty' \in C\}.$$

Both K and T are closed tangent cones. These two cones coincide if, in particular, C is a convex set or smooth manifold. In general, however, they can be quite different. The notable properties of the contingent cone include the following:

(i) K is *isotone* with respect to inclusion; that is, if $C_1 \subset C_2$ and $x \in C_1$, then $K(C_1, x) \subset K(C_2, x)$.

(ii) K *preserves unions*, that is,

Received July 3, 1987.

https://doi.org/10.4153/CJM-1988-029-6 Published online by Cambridge University Press

$$K(C_1 \cup C_2, x) = K(C_1, x) \cup K(C_2, x)$$

for all nonempty C_1 and C_2 in 2^E and x in $C_1 \cup C_2$.

The Clarke tangent cone, on the other hand, has neither of these properties, but it does possess several properties not shared by the contingent cone:

(iii) T is convex.

(iv) T is *product-preserving*; that is, if C_1 and C_2 are nonempty and $x_1 \in C_1$, $x_2 \in C_2$, then

$$T(C_1, x_1) \times T(C_2, x_2) = T(C_1 \times C_2, (x_1, x_2)).$$

(v) T satisfies the inclusion

$$T(C_1, x) \cap T(C_2, x) \subset T(C_1 \cap C_2, x)$$

if, for example, E is finite-dimensional, C_1 and C_2 are closed, and

$$T(C_1, x) - T(C_2, x) = E$$

[**2**, **6**, **46**]. (For conditions guaranteeing this inclusion in infinite dimensions, see [**11**, **26**].)

The convexity of the Clarke tangent cone makes it a very useful approximant. Properties (iv) and (v) are important in the proofs of subdifferential calculus formulae and optimality conditions in nonsmooth optimization [**42**, **45-47**]. The applications of property (i) in optimization are highlighted in [**38**, **39**, **43**]. For more information about the properties of these cones, see [**1-2**, **12**, **28**].

One might ask whether there exists some tangent cone possessing all of the properties listed in (i) through (v). This question has a trivial affirmative answer. One could, for example, define a tangent cone which maps every (C, x) in $2^E \times E$ to the set $\{0\}$. Let us then revise the question and ask if there exists a tangent cone that agrees with K and T on convex sets and smooth manifolds and has the properties mentioned in (i) through (v). In the attempt to build the definition of such a "supercone", we confront a further question: what parts of the definition of a tangent cone will give the cone these various properties?

The investigation of such questions can lead to a better understanding of the potential and the limitations of nonsmooth analysis. In order to gain insight into their answers, we will define in Section 2 of this paper a general class of tangent cones containing T, K and other familiar tangent cones as specific cases. The relationships between the quantifications in the definitions of these cones and the properties of the cones are discussed in Sections 2 and 3. The information gathered in Sections 2 and 3 is used in Section 4 as an aid in the formulation of tangent cone "impossibility theorems"; results listing incompatible combinations of properties. Section 5 contains a brief discussion of the directional derivatives and subgradient sets that can be defined in association with tangent cones.

Finally, the definitions and properties of a number of tangent cones, along with a summary of the results of Sections 2 and 3, are listed in Tables 6.1 and 6.2.

The notation used in this paper will follow that of [**24**]. For example, "cl" will be used to denote closure of sets and functions, "int" will denote "interior", "ri" will denote "relative interior", and "co" will denote "convex hull". For $f:E \to [-\infty, \infty]$, epi f and dom f will stand for the epigraph and effective domain of f, respectively. In addition, for a tangent cone A, we define

$$A(f, x) := A(\text{epi}\, f, (x, f(x))).$$

If f is convex and finite at $x \in E$, $\partial f(x)$ will denote the usual subgradient of f at x.

2. Quantificational tangent cones.

Definition 2.1. A *quantificational tangent cone* (or "q-cone") is a mapping $A:2^E \times E \to 2^E$ that associates with each nonempty $C \subset E$ and $x \in C$ a set of the form

$$\begin{aligned} A(C, x) = \{y \in E \mid * \; Y \in N(y), \# \; X \in N(x), \$ \; \lambda > 0, \\ \exists \; W \in J(X), \exists \; Z \in M(Y), \#' \; x' \in W \cap C, \$' \; t \in (0, \lambda), \\ *' \; y' \in Z, x' + ty' \in C\}, \end{aligned}$$

where

(i) $J(X)$ and $M(Y)$ are given classes of subsets of X and Y, respectively;

(ii) $*, *', \#, \#', \$, \$' \in \{\exists, \forall\}$;

(iii) $* \neq *', \# \neq \#', \$ \neq \$'$.

We will be primarily concerned here with the tangent cones for which $J(X)$ consists only of the set X itself or the set $\{x\}$ and $M(Y)$ consists only of the set Y or the set $\{y\}$. We will refer to these special cases as "$J(X) = X$", "$J(X) = x$", "$M(Y) = Y$", and "$M(Y) = y$", respectively. One other special case of M considered in the literature is that in which $M(Y)$ is the class of all nonempty compact subsets of Y [**8**].

Notice that T can be obtained by setting $* = \forall$, $\# = \exists$, $\$ = \exists$, $J(X) = X$, and $M(Y) = Y$. Similarly, K is obtained from Definition 2.1 by setting $* = \forall$, $\$ = \forall$, $J(X) = x$, and $M(Y) = Y$, with either choice of $\#$. Many, but by no means all, of the most commonly studied tangential approximants are q-cones. The weak tangent cones of [**3**, **5**, **9-11**], the prototangent cone and quasi-strict tangent cone of [**23**], the cone defined by Treiman in [**34**], and the tangential approximants compared in [**20**] are among those not obtainable from Definition 2.1.

There are eighteen q-cones whose definitions contain $J(X) = X$ or

$J(X) = x$ combined with $M(Y) = Y$ or $M(Y) = y$. Following [**17**], we will refer to these as the *principal series* of tangent cones. Six of these eighteen have $* = \forall$ and $M(Y) = Y$, including T, K, and the following tangent cones:

(a) $k(C, x) := \{y : \forall\, Y \in N(y), \exists\, \lambda > 0, \forall\, t \in (0, \lambda),$
$$\exists\, y' \in Y, x + ty' \in C\}.$$

(b) $T^*(C, x) := \{y : \forall\, Y \in N(y), \exists\, X \in N(x), \forall\, \lambda > 0,$
$$\forall\, x' \in X \cap C, \exists\, t \in (0, \lambda), \exists\, y' \in Y, x' + ty' \in C\}.$$

(c) $D(C, x) := \{y : \forall\, Y \in N(y), \forall\, X \in N(x), \exists\, \lambda > 0,$
$$\exists\, x' \in X \cap C, \forall\, t \in (0, \lambda), \exists\, y' \in Y, x' + ty' \in C\}.$$

(d) $D^*(C, x) := \{y : \forall\, Y \in N(y), \forall\, X \in N(x), \forall\, \lambda > 0,$
$$\exists\, x' \in X \cap C, \exists\, t \in (0, \lambda), \exists\, y' \in Y, x' + ty' \in C\}.$$

The cone k is sometimes called the *Ursescu tangent cone* and is studied in [**35**, **13**, **21**, **23**, **40-46**]. Treiman [**33**] has shown that $T(C, x) = T^*(C, x)$ if C is a locally closed subset of a Banach space. Notice that $A \subset D^*$ for all q-cones A. In fact, we will see that D and D^* are "too large" for most applications.

For any q-cone A whose definition contains $* = \forall$ and $M(Y) = Y$, we define RA to be the q-cone obtained by replacing "$M(Y) = Y$" with "$M(Y) = y$" in the definition of A, and we call IA the q-cone obtained by replacing "$* = \forall$" with "$* = \exists$" in the definition of A. Observe that in general $IA \subset RA \subset A$, and that the cones in the principal series are related by the inclusions

$$\begin{array}{ccccccccccccc}
T & \subset & T^* & \subset & K & \subset & D^* & & & k & \subset & K \\
\cup & & \cup & & \cup & & \cup & & & \cup & & \cup \\
RT & \subset & RT^* & \subset & RK & \subset & RD^* & & \text{and} & Rk & \subset & RK, \\
\cup & & \cup & & \cup & & \cup & & & \cup & & \cup \\
IT & \subset & IT^* & \subset & IK & \subset & ID & & & Ik & \subset & IK
\end{array}$$

as well as

$$\begin{array}{ccccccc}
T & \subset & k & \subset & D & \subset & D^* \\
\cup & & \cup & & \cup & & \cup \\
RT & \subset & Rk & \subset & RD & \subset & RD^* \\
\cup & & \cup & & \cup & & \cup \\
IT & \subset & Ik & \subset & ID & \subset & ID^*
\end{array}$$

Many of these q-cones are discussed in [**8-11**, **13**, **14**, **16-18**, **21**, **23**, **28**, **35-47**]. A quite complete listing of their names is given in [**17**]. It is shown in [**40**], by an argument paralleling that of [**33**], that

$$IT(C, x) = IT^*(C, x) \quad \text{and} \quad RT(C, x) = RT^*(C, x)$$

whenever C is a closed subset of a l.c.s. (see also [**17**, Proposition 5]). Just as D^* is the "largest" q-cone, IT is the "smallest"; the inclusion $IT \subset A$ is true for any q-cone A.

In the remainder of this section, we establish some basic properties of q-cones and make precise the relationships between A, RA, and IA. In our proofs, we will often use the symbols *, #, and $ to represent either $\forall$ or $\exists$, just as in Definition 2.1. This will help us to avoid tedious case-by-case analyses and will also serve to highlight which quantifications are important in a given proof and which are not.

We begin by showing that the members of a large class of q-cones are indeed tangent cones. In the statement of Proposition 2.2 below, the inclusion "$\alpha M(Y) \subset M(\alpha Y)$" means that if $\alpha > 0$ and Z is in $M(Y)$, then αZ is in $M(\alpha Y)$.

PROPOSITION 2.2. *Suppose A is a q-cone such that $\alpha M(Y) \subset M(\alpha Y)$ for all $\alpha > 0$. Then A is a tangent cone. In particular, each member of the principal series is a tangent cone.*

Proof. Let C in E, x in C, and $\alpha > 0$ be given and suppose $y \in A(C, x)$. Then $* \ \alpha^{-1}Y \in N(y)$, $\# \ X \in N(x)$, $\$ \ \alpha\lambda > 0$, there exist W in $J(X)$ and Z in $M(\alpha^{-1}y)$, $\#' \ x' \in W \cap C$, $\$' \ t \in (0, \alpha\lambda)$, $*' \ y' \in Z$, $x' + ty' \in C$. Now $\alpha y' \in \alpha Z$, and αZ is in $M(Y)$ by hypothesis. Therefore $* \ Y \in N(\alpha y)$, $\# \ X \in N(x)$, $\$ \ \lambda > 0$, there exist W in $J(X)$ and αZ in $M(Y)$, $\#' \ x' \in W \cap C$, $\$' \ t/\alpha \in (0, \lambda)$, $*' \ \alpha y' \in \alpha Z$,

$$x' + ty' = x' + (t/\alpha)\alpha y' \in C.$$

Thus $\alpha y \in A(C, x)$ and $A(C, x)$ is a cone.

It is generally true that if $* = \forall$ and $M(Y) = Y$ in the definition of a q-cone A, then A will be closed and the corresponding IA will be open. We demonstrate these facts in Theorems 2.3 and 2.5.

THEOREM 2.3. *Suppose A is a q-cone in whose definition $* = \forall$ and M is such that if $Y_1 \subset Y_2$ and $Z_1 \in M(Y_1)$, there exists $Z_2 \supset Z_1$ with $Z_2 \in M(Y_2)$. Then A is closed.*

Proof. Let C in E and $x \in C$ be given, and suppose $y \in \text{cl}\, A(C, x)$. Let $Y \in N(y)$. Then $Y = y + V$ for some $V \in N(0)$. Choose $U \in N(0)$ such that $U + U \subset V$, and choose $u \in U$ such that $\bar{y} := y + u \in A(C, x)$. Then $\# \ X \in N(x)$, $\$ \ \lambda > 0$ there exist $W \in J(X)$ and $Z_1 \in M(\bar{y} + U)$, $\#' \ x' \in W \cap C$, $\$' \ t \in (0, \lambda)$, there exists $y' \in Z_1$ with $x' + ty' \in C$. Now $y' \in Y$ and there exists $Z_2 \in M(Y)$ with $Z_2 \supset Z_1$, so y must actually be in $A(C, x)$. Therefore $A(C, x)$ is a closed set and A is closed.

https://doi.org/10.4153/CJM-1988-029-6 Published online by Cambridge University Press

COROLLARY 2.4. *The tangent cones T, T^*, k, K, D, and D^* are closed.*

THEOREM 2.5. *Suppose A is a q-cone in whose definition $* = \exists$ and M is such that if $Y_1 \subset Y_2$ and $Z_2 \in M(Y_2)$, there exists $Z_1 \in M(Y_1)$ with $Z_1 \subset Z_2$. Then A is open.*

Proof. Let C in E, x in C and $y \in A(C, x)$ be given. There exists $Y \in N(y)$, $\#\ X \in N(x)$, $\$\ \lambda > 0$, there exist $W \in J(X)$ and $Z \in M(Y)$, $\#'\ x' \in W \cap C$, $\$'\ t \in (0, \lambda)$, $x' + tZ \subset C$. Again $Y = y + V$ for some V in $N(0)$, and there exists $U \in N(0)$ such that $U + U \subset V$. Let $\bar{u} \in U$, and call $\bar{y} := y + \bar{u}$. By hypothesis, there exists $Z' \subset Z$ with $Z' \in M(\bar{y} + U)$. Thus $\#\ X \in N(x)$, $\$\ \lambda > 0$, there exists W in $J(X)$, $\#'\ x' \in W \cap C$, $\$'\ t \in (0, \lambda)$, for all $y' \in Z'$,

$$x' + ty' \in x' + tZ' \subset x' + tZ \subset C.$$

Therefore $y + U \in A(C, x)$ and A is open.

COROLLARY 2.6. *The tangent cones IT, IT^*, Ik, IK, ID, and ID^* are open.*

We next investigate the relationship between A and IA. The key fact is inclusion (2.1) below.

THEOREM 2.7. *Let A be a q-cone in whose definition $* = \forall$ and $M(Y) = Y$. Then*

$$A + IT \subset IA. \tag{2.1}$$

Proof. Let C in E and x in C be given, and let $u \in A(C, x)$ and $v \in IT(C, x)$. Call $w := u + v$. By the definition of IT, there exist neighborhoods $X_1 \in N(x)$ and $V \in N(0)$ and a number $\lambda_1 > 0$ such that

$$(X_1 \cap C) + (0, \lambda_1)(v + V) \subset C.$$

We may choose neighborhoods $X_2 \in N(x)$ and $U \in N(0)$ and a number $\lambda_2 \in (0, \lambda_1)$ such that $U - U \subset V$ and $X_2 + [0, \lambda_2)(u + U) \subset X_1$. Since $u \in A(C, x)$, $\#\ X \in N(x)$ with $X \subset X_2$, $\$\ \lambda \in (0, \lambda_2)$, $\exists\ W \in J(X)$, $\#'\ x' \in W \cap C$, $\$'\ t \in (0, \lambda)$, there exists $u_1 \in U$ with

$$x' + t(u + u_1) \in X_1 \cap C.$$

Now let $u_2 \in U$ be given. Then $\#\ X \in N(x)$ with $X \subset X_2$, $\$\ \lambda \in (0, \lambda_2)$, $\exists\ W \in J(X)$, $\#'\ x' \in W \cap C$, $\$'\ t \in (0, \lambda)$,

$$\begin{aligned} x' + t(w + u_2) &= x' + t(u + u_1) + t(v + u_2 - u_1) \\ &\in (X_1 \cap C) + t(v + V) \subset C. \end{aligned}$$

Hence $w \in IA(C, x)$, and (2.1) holds.

COROLLARY 2.8. *Let A be a q-cone in whose definition $* = \forall$ and $M(Y) = Y$, and suppose $IT(C, x)$ is nonempty. Then*

https://doi.org/10.4153/CJM-1988-029-6 Published online by Cambridge University Press

$$A(C, x) = \text{cl } RA(C, x) = \text{cl } IA(C, x) \quad \textit{and}$$
$$\text{int } A(C, x) = \text{int } RA(C, x) = IA(C, x).$$

Proof. By Theorem 2.3, A is closed, and since A contains IA, A contains the closure of IA. Conversely, let $y \in A(C, x)$ and $v \in IT(C, x)$. For any $t > 0$, $tv \in IT(C, x)$ by Proposition 2.2 and $y + tv \in IA(C, x)$ by (2.1). Thus $y \in \text{cl } IA(C, x)$, and it follows that

$$A(C, x) = \text{cl } RA(C, x) = \text{cl } IA(C, x).$$

To establish the second assertion, first note that IA is open by Theorem 2.5, so that IA is contained in int A and int RA. Now let $y \in \text{int } A(C, x)$. Then for $t > 0$ sufficiently small,

$$y - tv \in A(C, x),$$

and by (2.1),

$$y = y - tv + tv \in IA(C, x).$$

Hence

$$\text{int } A(C, x) = \text{int } RA(C, x) = IA(C, x).$$

Remark 2.9. (a) The $A := T$ case of (2.1) is proved in [**27**, **12**], and the $A := k$ case is proved in [**21**]. If $A := K$ or k, IT can be replaced by larger open cones in (2.1) [**23**, **41**].

(b) Results like Theorem 2.7 are very useful in the proofs of subdifferential calculus formulae [**23**, **44**, **45**].

(c) If $IT(C, x)$ is nonempty, C is said to be *epi-Lipschitzian* at x. This class of sets is studied in [**25-28**, **7**, **12**].

A large class of q-cones shares the isotonicity property of K. We now give a simple criterion for isotonicity.

THEOREM 2.10. *Let A be a q-cone in whose definition $\# = \forall$. Then A is isotone.*

Proof. Let C_1, C_2 be subsets of E such that $C_1 \subset C_2$, and let $x \in C_1$. Let $y \in A(C_1, x)$ be given. Then $*\ Y \in N(y)$, for all X in $N(x)$, $\$\ \lambda > 0$, there exist $W \in J(X)$, $Z \in M(Y)$ and $x' \in W \cap C_1 \subset W \cap C_2$, $\$'\ t \in (0, \lambda)$, $*'\ y' \in Z$, $x' + ty' \in C_1 \subset C_2$. Therefore $y \in A(C_2, x)$ and A is isotone.

COROLLARY 2.11. *The tangent cones A, RA, and IA are isotone for $A := D, D^*, K$, and k.*

The other tangent cones in the principal series are not isotone (see Table 6.1). On the other hand, T, RT, and IT are convex [**13**, **25**, **27**], T^* is convex for all locally closed subsets of a Banach space [**33**], and RT^* and IT^* are convex for all locally closed subsets of a l.c.s. [**40**]. The question of the existence of tangent cones satisfying both properties is discussed in [**42**] and Section 4.

Many q-cones, including all those in the principal series, are local approximations in the sense that if $x \in C$ and X is any neighborhood of x, then

(2.2) $\quad A(C \cap X, x) = A(C, x).$

We conclude this section by demonstrating this fundamental q-cone property. We treat separately the cases $\# = \forall$ and $\# = \exists$ in Theorems 2.12 and 2.13.

THEOREM 2.12. *Let A be a q-cone in whose definition $\# = \forall$. Then* (2.2) *holds for all neighborhoods X of x.*

Proof. By Theorem 2.10, A is isotone, so the inclusion

$$A(C \cap X, x) \subset A(C, x)$$

holds for all $X \in N(x)$. To establish the opposite inclusion, let $y \in A(C, x)$ and X in $N(x)$ be given. There exist X_1 in $N(x)$, $\lambda_1 > 0$, and Y_1 in $N(y)$ such that

$$X_1 + [0, \lambda_1)Y_1 \subset X.$$

Then $* \ Y \in N(y)$ with $Y \subset Y_1$, for all $X_2 \in N(x)$ with $X_2 \subset X_1$, $\$ \ \lambda \in (0, \lambda_1)$, there exist W in $J(X_2)$, Z in $M(Y)$, and $x' \in W \cap C = W \cap C \cap X$, $\$' \ t \in (0, \lambda)$, $*' \ y' \in Z$, $x' + ty' \in C \cap X$. Therefore

$$y \in A(C \cap X, x)$$

and (2.2) holds.

THEOREM 2.13. *Let A be a q-cone in whose definition $\# = \exists$. Suppose that for all X and X_1 in $N(x)$ and W in $J(X_1)$, the set $W \cap X$ is in $J(X \cap X_1)$. Then* (2.2) *holds for all neighborhoods X of x.*

Proof. Let X be a neighborhood of x. The proof that

$$A(C, x) \subset A(C \cap X, x)$$

is analogous to that of Theorem 2.12. To establish the reverse inclusion, let $y \in A(C \cap X, x)$. Then $* \ Y \in N(y)$, there exists X_1 in $N(x)$, $\$ \ \lambda > 0$, there exist W in $J(X_1)$ and Z in $M(Y)$ such that for all $x' \in W \cap C \cap X$, $\$' \ t \in (0, \lambda)$, $*' \ y' \in Y$, $x' + ty' \in C$. By hypothesis, $W \cap X$ is in $J(X \cap X_1)$, and we conclude that $y \in A(C, x)$. Therefore (2.2) holds.

It should be noted that not all conical approximants satisfy (2.2). In particular,

$$\operatorname{cone}(C - x) := \bigcup_{\lambda \geqq 0} \lambda(C - x)$$

certainly does not satisfy (2.2) for all nonempty sets C and neighborhoods X of x.

3. Additional tangent cone properties. Which parts of the definition of a q-cone will give it the various properties mentioned in Section 1? In this section, we continue our study of this question. In particular, we focus on the behavior of q-cones on convex sets, sets of the form $f^{-1}(0)$ for strictly differentiable functions f, and intersections, unions, and products of sets. We also give conditions sufficient for the convexity of a q-cone.

We begin by investigating which q-cones A satisfy the inclusion

$$A(C_1, x) \cap A(C_2, x) \subset A(C_1 \cap C_2, x) \tag{3.1}$$

for C_1 and C_2 in E and $x \in C_1 \cap C_2$. This inclusion is essential in the proofs of subdifferential calculus rules and necessary optimality conditions [**20**, **47**, **23**, **17**, **42**, **45**].

THEOREM 3.1. *Let A be a q-cone in whose definition $\$ = \exists$, $* = \exists$, and $\# = \exists$. Suppose that J and M are such that $W_1 \in J(X_1)$, $W_2 \in J(X_2)$ implies*

$$W_1 \cap W_2 \in J(X_1 \cap X_2);$$

and $Z_1 \in M(Y_1)$, $Z_2 \in M(Y_2)$ implies

$$Z_1 \cap Z_2 \in M(Y_1 \cap Y_2)$$

for all X_1, X_2 in $N(x)$ and Y_1, Y_2 in $N(y)$.

Then (3.1) *holds for all nonempty subsets C_1 and C_2 of E with $x \in C_1 \cap C_2$.*

Proof. Suppose $y \in A(C_1, x) \cap A(C_2, x)$. There exist $Y_i \in N(y)$, $X_i \in N(x)$, $W_i \in J(X_i)$, $Z_i \in M(Y_i)$, and $\lambda_i > 0$ such that

$$(C_i \cap W_i) + (0, \lambda_i)Z_i \subset C_i \quad \text{for } i = 1, 2.$$

Let

$$X := X_1 \cap X_2 \in N(x) \quad \text{and} \quad Y := Y_1 \cap Y_2 \in N(y).$$

By hypothesis, $W_1 \cap W_2 \in J(X)$ and $Z_1 \cap Z_2 \in M(Y)$, so the inclusion

$$C_1 \cap C_2 \cap W_1 \cap W_2 + (0, \min(\lambda_1, \lambda_2))(Z_1 \cap Z_2) \subset C_1 \cap C_2$$

implies that $y \in A(C_1 \cap C_2, x)$.

Therefore (3.1) holds for all nonempty C_1, C_2 with x in $C_1 \cap C_2$.

COROLLARY 3.2. *The q-cones IT, Ik, RT, and Rk satisfy* (3.1) *for all nonempty C_1, C_2 with x in $C_1 \cap C_2$. Equality holds in* (3.1) *for the isotone tangent cones Ik and Rk.*

The q-cones listed in Corollary 3.2 are the only ones in the principal series that always satisfy (3.1). Counterexamples for the others are given in Table 6.1. However, (3.1) also holds with $A := T$, k for a large class of sets, including for example those satisfying

(3.2) $\quad T(C_1, x) \cap IT(C_2, x) \neq \emptyset.$

The proof of these facts relies upon (2.1) and the inclusions

(3.3) $\quad T(C_1, x) \cap IT(C_2, x) \subset T(C_1 \cap C_2, x)$

(3.4) $\quad k(C_1, x) \cap Ik(C_2, x) \subset k(C_1 \cap C_2, x),$

which hold for all nonempty C_1, C_2 and x in $C_1 \cap C_2$ [**23**]. Similarly, one can show that

(3.5) $\quad K(C_1, x) \cap k(C_2, x) \subset K(C_1 \cap C_2, x)$

whenever condition (3.2) holds.

We next establish which q-cones satisfy

(3.6) $\quad A(C_1 \cup C_2, x) = A(C_1, x) \cup A(C_2, x).$

THEOREM 3.3. *Let A be a q-cone in whose definition $J(X) = x$, $* = \forall$, and $\$ = \forall$. Suppose for all Y_1, Y_2 in $N(y)$ that each set in $M(Y_1 \cap Y_2)$ can be expressed as $Z_1 \cap Z_2$ for some $Z_1 \in M(Y_1)$ and $Z_2 \in M(Y_2)$. Then A satisfies* (3.6) *for all nonempty C_1, C_2 with $x \in C_1 \cup C_2$.*

Proof. Since A is isotone by Theorem 2.10,

$$A(C_1, x) \cup A(C_2, x) \subset A(C_1 \cup C_2, x)$$

holds. To establish the opposite inclusion, suppose y is not contained in the left hand side of (3.6). Then there exist $Y_i \in N(y)$ and $\lambda_i > 0$ such that for all $Z_i \in M(Y_i)$, $y' \in Z_i$ and $t \in (0, \lambda)$,

$$x + ty' \in E \backslash C_i \quad \text{for } i = 1, 2.$$

Now define $Y := Y_1 \cap Y_2 \in N(y)$, and let Z be in $M(Y)$. By hypothesis, there exist $Z_i \in M(Y_i)$, $i = 1, 2$, with $Z = Z_1 \cap Z_2$. Then for all $y' \in Z$ and $t \in (0, \min(\lambda_i, \lambda_2))$,

$$x + ty' \in E \backslash (C_1 \cup C_2).$$

Hence $y \notin A(C_1 \cup C_2, x)$ and (3.6) holds.

COROLLARY 3.4. *The q-cones K and RK satisfy* (3.6) *for all nonempty C_1, C_2 with x in $C_1 \cup C_2$.*

The contingent cone and RK are the only ones in the principal series that always satisfy (3.6). Counterexamples for the others are listed in Table 6.1.

There is a simple criterion for determining when a q-cone is product-preserving, as we now demonstrate.

THEOREM 3.5. *Suppose E_1, E_2 are* l.c.s. *are $E_1 \times E_2$ is endowed with the product topology. Let $x \in C_1 \subset E_1$ and $x_2 \in C_2 \subset E_2$. Suppose A is a q-cone in whose definition J and M satisfy the following conditions:*

(3.7) $\quad J(X_1 \times X_2) \subset J(X_1) \times J(X_2)$

for all $X_1 \in N(x_1)$, $X_2 \in N(x_2)$.

(3.8) $\quad M(Y_1 \times Y_2) \subset M(Y_1) \times M(Y_2)$

for all $Y_1 \in N(y_1)$, $Y_2 \in N(y_2)$, $(y_1, y_2) \in E_1 \times E_2$.
Then

(3.9) $\quad A(C_1 \times C_2, (x_1, x_2)) \subset A(C_1, x_1) \times A(C_2, x_2)$.

Moreover, if $\$ = \exists$ *and equality holds in* (3.7) *and* (3.8), *then equality holds in* (3.9).

Proof. The proof of (3.9) under conditions (3.7) and (3.8) is a routine matter. Suppose $\$ = \exists$ in the definition of A and equality holds in (3.7) and (3.8). Let $y_1 \in A(C_1, x_1)$ and $y_2 \in A(C_2, x_2)$. Then $*\ Y_i \in N(y_i)$, $\#\ X_i \in N(x_i)$, there exist $\lambda_i > 0$, $W_i \in J(X_i)$, and $Z_i \in M(Y_i)$, $\#'\ x_i' \in C_i \cap W_i$, for all $t \in (0, \lambda_i)$, $*'\ y_i' \in Z_i$, $x_i' + ty_i' \in C_i$, $i = 1, 2$. Define $\lambda := \min(\lambda_i, \lambda_2)$, $W := W_i \times W_2$, $Z := Z_1 \times Z_2$, $Y := Y_1 \times Y_2$, and $X := X_1 \times X_2$. By hypothesis, $W \in J(X)$ and $Z \in M(Y)$. Then

$$*\ Y \in N((y_1, y_2)), \quad \#\ X \in N((x_1, x_2)),$$

there exist $W \in J(X)$ and $Z \in M(Y)$, $\#'(x_i', x_2') \in (C_1 \times C_2) \cap W$, for all $t \in (0, \lambda)$, $*'(y_1', y_2') \in Y \cap Z$,

$$(x_1', x_2') + t(y_1', y_2') \in C_1 \times C_2.$$

Hence $(y_1, y_2) \in A(C_1 \times C_2, (x_1, x_2))$ and equality holds in (3.9).

COROLLARY 3.6. *All tangent cones in the principal series satisfy* (3.9). *The q-cones T, RT, IT, k, Rk, Ik, D, RD, and ID are product-preserving.*

If $\$:= \forall$ in the definition of a q-cone A, that cone will generally not be product-preserving (see **[6]** and Table 6.1). However, such q-cones can team up with product-preserving q-cones A' to produce inclusions of the form

(3.10) $\quad A(C_1, x_1) \times A'(C_2, x_2) \subset A(C_1 \times C_2, (x_1, x_2))$.

We give below a general result along these lines. Its proof is straightforward and is left to the reader.

THEOREM 3.7. *Suppose* E_1, E_2 *are* l.c.s. *and* $E_1 \times E_2$ *is endowed with the product topology. Suppose that A and A′ are q-cones such that* $A' \subset A$, *the definition of A′ contains* $\$:= \exists$, *and J and M are identical in the definitions of A and A′. Assume that equality holds in* (3.7) *and* (3.8). *Then for all* $x_1 \in C_1 \subset E_1$ *and* $x_2 \in C_2 \subset E_2$, (3.10) *holds. In particular,* $A := K$, *RK, IK,* T^*, RT^*, IT^*, D^*, RD^*, ID^* *can be paired, respectively, with* $A' := k$, *Rk, Ik, T, RT, IT, D, RD, ID in* (3.10).

Inclusion (3.10) is important in the proofs of subdifferential calculus formulae [**42**, **46**].

Much is known about q-cones $A(C, x)$ for convex C. For example, if C is a nonempty convex subset of E and $x \in C$, the q-cones $A := RK, Rk$ satisfy

(3.11) $\quad A(C, x) = \text{cone}(C - x).$

Similarly, it can readily be demonstrated that for $A := T, K,$

(3.12) $\quad A(C, x) = \text{cl cone}(C - x).$

It follows that all tangent cones A with $T \subset A \subset K$, including k and T^*, also satisfy (3.12). In addition, Corollary 2.8 tells us that if C is convex and epi-Lipschitzian at x, we at least have

(3.13) $\quad \text{cl } A(C, x) = \text{cl cone}(C - x)$

for $A := RT, IT, RT^*, IT^*, IK$, and Ik. This property is noteworthy in that tangent cones satisfying (3.13) generate subgradients that for convex functions coincide with the ordinary subgradient (see Section 5).

However, the q-cones with $\# = \forall$ and $J(X) = X$ do not even satisfy (3.13). The following example indicates that these cones are not very effective approximants to convex sets.

Example 3.8. Let $E := (-\infty, \infty)$, $C := [0, \infty)$ and $x := 0$. Let $s > 0$, $\delta > 0$ be given, let $y \in (-\infty, \infty)$, and define $X := [-\delta, \delta]$ and $Y := [y - s, y + s]$. Then if $y \geqq s$, $\delta + (0, r)Y \subset C$ for any $r > 0$, and if $y < s$,

$$\delta + (0, -\delta/(y - s))Y \subset C.$$

Letting $x' := \delta$ in the definition of ID, we conclude that $y \in ID(C, 0)$ and $ID(C, 0) = (-\infty, \infty)$. Since $ID \subset A$ for all q-cones A with $\# = \forall$ and $J(X) = X$, we conclude that $A(\mathrm{C}, 0) = (-\infty, \infty)$ for all such q-cones, including $A := D, RD, D^*, RD^*, ID^*$. Therefore none of these q-cones satisfy (3.13) for convex sets C and $x \in C$.

Remark 3.9. It can easily be shown, in fact, that ID and ID^* coincide and can take on only two possible values, E and the empty set.

Definition 3.10 [**26**]. Let E_1, E_2 be l.c.s. The function $h:E_1 \to E_2$ is *strictly differentiable* (in the "full limit sense") at $x \in E$, if there is a continuous linear mapping $\nabla h(x):E_1 \to E_2$ such that

$$\lim_{\substack{x' \to x \\ t \to 0^+ \\ y' \to y}} t^{-1}(h(x' + ty') - h(x')) = \nabla h(x)y \quad \text{for all } y \in E.$$

We next examine the behavior of q-cones on sets of the form $h^{-1}(0)$, where $h:E_1 \to E_2$ is strictly differentiable. The following result is valid for all q-cones.

PROPOSITION 3.11. *Suppose A is a q-cone and $h:E_1 \to E_2$ is strictly differentiable at $x \in h^{-1}(0)$. Then*

(3.14) $A(h^{-1}(0), x) \subset \nabla h(x)^{-1}(0)$.

Proof. We first observe that it suffices to show (3.14) for $A := D^*$, since all q-cones are contained in D^*. Let $y \in D^*(h^{-1}(0), x)$ be given. Then for all $Y \in N(y)$, $X \in N(x)$, and $\lambda > 0$, there exist $x' \in X \cap h^{-1}(0)$, $t \in (0, \lambda)$, and $y' \in Y$ with

$$h(x' + ty') = 0$$

and thus also

$$t^{-1}(h(x' + ty') - h(x')) = 0.$$

Since h is strictly differentiable at x, $\nabla h(x)y = 0$ and (3.14) holds for $A := D^*$.

The reverse inclusion holds much less often, as the example below suggests.

Example 3.12. Consider $h:\mathbf{R}^2 \to \mathbf{R}$ defined by

$$h(x_1, x_2) = x_2 - x_1^3,$$

and let $x = (0, 0)$. Then

$$\nabla h(x)^{-1}(0) = \{(x_1, x_2):x_2 = 0\},$$

while

$$RA(h^{-1}(0), x) = \{(0, 0)\} \quad \text{and} \quad IA(h^{-1}(0), x) = \emptyset$$

for $A := T, T^*, k, K, D$, and D^*.

It is well-known, however, that the reverse inclusion holds for the Clarke tangent cone if $h:\mathbf{R}^n \to \mathbf{R}^m$ and $\nabla h(x)$ has rank m (see for example [**6**, **46**]). We can thus deduce the following result.

THEOREM 3.13. *Suppose A is a q-cone with $* = \forall$ and $M(Y) = Y$, and suppose $h:\mathbf{R}^n \to \mathbf{R}^m$ is strictly differentiable at $x \in h^{-1}(0)$ with $\nabla h(x)$ of rank m. Then*

(3.15) $A(h^{-1}(0), x) = \nabla h(x)^{-1}(0)$.

Proof. If A is a q-cone with $* = \forall$ and $M(Y) = Y$, then $T \subset A$. Equation (3.15) then follows from the fact that

$$\nabla h(x)^{-1}(0) = T(h^{-1}(0), x),$$

as cited in the paragraph above.

COROLLARY 3.14. *Let $h:\mathbf{R}^n \to \mathbf{R}^m$ be strictly differentiable at $x \in h^{-1}(0)$ with $\nabla h(x)$ of rank m. Then* (3.15) *holds for $A := T^*, k, K, D$, and D^*.*

It was mentioned in the introduction that the Clarke tangent cone T is convex (proofs are given in [**12**, **13**, **21**, **25**, **38**]). Inclusion (2.1) with $A := T$ implies that IT is convex, and RT is also convex [**28**]. These are the only convex tangent cones in the principal series (Table 6.1, [**17**, Proposition 1]) except for ID and ID^*, which are trivially convex since they always equal E or $\emptyset$.

We now give conditions sufficient for the convexity of a q-cone.

THEOREM 3.15. *Suppose A is a q-cone in whose definition* $* = \forall$, $\# = \exists$, *and* $\$ = \exists$, *and assume the following conditions hold*:

(a) $M(\alpha Y) = \alpha M(Y)$ *for all* $\alpha > 0$ *and* $Y \in N(y)$.

(b) *If* $Y_1 \in N(y_1)$, $Y_2 \in N(y_2)$, $Y \in N(y_1 + y_2)$ *with* $Y_1 + Y_2 \subset Y$, $Z_1 \in M(Y_1)$ *and* $Z_2 \in M(Y_2)$, *there exists* $Z \in M(Y)$ *with* $Z_1 + Z_2 \subset Z$.

(c) $J(X_1 \cap X_2) = J(X_1) \cap J(X_2)$ *for all* X_1, X_2.

(d) *If* $X' \in N(x)$, $W' \in J(X')$, *and* $Y \in N(y')$, *there exist* $X \in N(x)$, $W \in J(X)$, *and* $\lambda > 0$ *such that* $W + (0, \lambda)Y \subset W'$.

Then A is convex.

Proof. Let $y_1, y_2 \in A(C, x)$. By (a) and Proposition 2.2, $A(C, x)$ is a cone, so it suffices to show that $y := y_1 + y_2$ is an element of $A(C, x)$. Let $Y \in N(y)$ be given, and let $Y_1 \in N(y_1)$, $Y_2 \in N(y_2)$ be such that $Y_1 + Y_2 \subset Y$. Since $y_i \in A(C, x)$, there exist $X_i \in N(x)$, $\lambda_i > 0$, $W_i \in J(X_i)$, and $Z_i \in M(Y_i)$ such that for all $x' \in W_i \cap C$ and $t \in (0, \lambda_i)$,

$$(x' + tZ_i) \cap C \neq \emptyset \quad \text{for } i = 1, 2.$$

By (c) and (d), choose $\lambda \in (0, \min(\lambda_1, \lambda_2))$, $X \subset X_1 \cap X_2$, $X \in N(x)$, and $W \in J(X)$, $W \subset W_1 \cap W_2$ such that

$$W + (0, \lambda)Y_1 \subset W_2.$$

Now let $x' \in C \cap W$, $t \in (0, \lambda)$. Then $x' \in C \cap W_1$, so

$$(x' + tZ_1) \cap C \neq \emptyset.$$

Pick $x'' \in (x' + tZ_1) \cap C \subset C \cap W_2$. Then $(x'' + tZ_2) \cap C \neq \emptyset$. By (b), choose $Z \in M(Y)$ with $Z_1 + Z_2 \subset Z$. Then

$$x'' + tZ_2 \subset x' + tZ_1 + tZ_2 \subset x' + tZ.$$

Hence $(x' + tZ) \cap C \neq \emptyset$, and $y \in A(C, x)$. Therefore $A(C, x)$ is convex.

COROLLARY 3.16 [**8**]. *The q-cones T and RT are convex, as is F, the q-cone obtained from the definition of T by replacing $M(Y) = Y$ with $M(Y)$ equal to the class of all nonempty compact subsets of Y.*

The results of Sections 2 and 3 are summarized in Tables 6.1 and 6.2.

TABLE 6.1

The principal series of tangent cones

$A(C, x) = \{y{:}{*}\ Y \in N(y),\ \#\ X \in N(x),\ \$\ \lambda > 0,\ \exists\ W \in J(X),$
$\exists\ Z \in M(Y),\ \#'\ x' \in W \cap C,\ \$'\ t \in (0, \lambda),\ *'\ y' \in Z,\ x' + ty' \in C\}$

NAME	*	#	$	J	M	cl/op	(3.13)	(3.15)	ISO	Conv	$\cap$	$\cup$	$\times$
T	∀	∃	∃	X	Y	cl	+	+	0 $E1$	+	+(3.2) $E3$	0 $E11$	+
RT	–	∃	∃	X	y	n	$+eL$	0 3.12	0 $E1$	+	+	0 $E3$	+
IT	∃	∃	∃	X	Y	op	$+eL$	0 3.12	0 $E1$	+	+	0 $E3$	+
T^*	∀	∃	∀	X	Y	cl	+	+	0 $E1$	+cl$B$ $E5$	+cl$B$(3.2) $E3$	0 $E11$	+cl$B$ $E5$
RT^*	–	∃	∀	X	y	n	$+eL$	0 3.12	0 $E1$	+cl $E7$	+cl $E3$	0 $E11$	+cl $E6$
IT^*	∃	∃	∀	X	Y	op	$+eL$	0 3.12	0 $E1$	+cl $E5$	+cl $E5$	0 $E3$	+cl $E5$
k	∀	–	∃	x	Y	cl	+	+	+	0 $E2$	+(3.2) $E3$	0 $E9$	+
Rk	–	–	∃	x	y	n	+	0 3.12	+	0 $E2$	+	0 $E3$	+
Ik	∃	–	∃	x	Y	op	$+eL$	0 3.12	+	0 $E2$	+	0 $E3$	+
K	∀	–	∀	x	Y	cl	+	+	+	0 $E2$	0 $E3$	+	0 $E8$
RK	–	–	∀	x	y	n	+	0 3.12	+	0 $E2$	0 $E3$	+	0 $E8$
IK	∃	–	∀	x	Y	op	$+eL$	0 3.12	+	0 $E2$	0 $E10$	0 $E3$	0 $E10$
D	∀	∀	∃	X	Y	cl	0 3.8	+	+	0 $E4$	0 $E3$	0 $E9$	+
RD	–	∀	∃	X	y	n	0 3.8	0 3.12	+	0 $E4$	0 $E5$	0 $E3$	+
ID $=ID^*$	∃	∀	∃	X	Y	op	0 3.8	0 3.12	+	+ triv.	0 $E5$	0 $E3$	+
D^*	∀	∀	∀	X	Y	cl	0 3.8	+	+	0 $E8$	0 $E3$	0 $E4$	0 $E8$
RD^*	–	∀	∀	X	y	n	0 3.8	0 3.12	+	0 $E8$	0 $E3$	0 $E4$	0 $E8$

https://doi.org/10.4153/CJM-1988-029-6 Published online by Cambridge University Press

Abbreviations used in Table 6.1

Symbol	Meaning
–	Either ∀ or ∃
+	A has that property
0	A does not have that property
cl	A is closed
op	A is open
n	A is neither closed nor open
3.8	See Example 3.8
3.12	See Example 3.12
$+eL$	A has that property if C is epi-Lipschitzian at x
+cl	A has that property if C is closed
$+\mathrm{cl}B$	A has that property if C is closed in a Banach space
+(3.2)	A has that property if condition (3.2) is satisfied
$+\mathrm{cl}B$(3.2)	A has that property for closed sets in a Banach space if (3.2) is satisfied
(3.13)	Does A have property (3.13)?
(3.15)	Does A have property (3.15)?
ISO	Is A isotone?
conv	Is A convex?
$\cap$	Does A have intersection property (3.1)?
$\cup$	Does A preserve unions (3.6)?
$\times$	Does A preserve products?
triv.	A is trivially convex, since it equals either E or $\emptyset$

Examples cited in Table 6.1

$E1$: $C_1 = \{(x, y) | x = 0\}$, $C_2 = \{(x, y) | x = 0 \text{ or } y = 0\}$, $x = (0, 0)$.

$E2$: $C = \{(x, y) | y \geqq -x\}$, $x = (0, 0)$.

$E3$: $C_1 = \{\text{rational numbers}\}$.
$C_2 = \{\text{irrational numbers}\} \cup \{0\}$, $x = 0$.

$E4$: $C_1 = \{(x, y) | x = 0\}$, $C_2 = \{(x, y) | y = 0\}$, $x = (0, 0)$.
$C = C_1 \cup C_2$.

$E5$: $C_1 = \bigcup_{n=0}^{\infty} (2^{-4n-1}, 2^{-4n}) \cup \{0\}$.

$C_2 = \bigcup_{n=0}^{\infty} (2^{-4n-3}, 2^{-4n-2}) \cup \{0\}$.

$C = C_1 \times C_2$, $x = (0, 0)$, $x_1 = 0$, $x_2 = 0$.

$E6$: $C_1 = \{\text{rational numbers}\} = C_2$, $x_1 = 0$, $x_2 = 0$.

$E7$: $C = \{\text{rational numbers}\} \times \{\text{rational numbers}\}$, $x = (0, 0)$.

$E8$: $C_1 = \{2^{-2n} | n = 0, 1, 2, \dots\} \cup \{0\}$, $x_1 = 0$.
$C_2 = \{2^{-2n+1} | n = 0, 1, 2, \dots\} \cup \{0\}$, $x = 0$.
$C = C_1 \times C_2$, $x = (0, 0)$.

Examples cited in Table 6.1 (concluded)

$E9$: $C_1 = \{1/m | 2^{2n} \leqq m < 2^{2n+1}, n = 0, 1, 2, \dots\} \cup \{0\}$.
$C_2 = \{1/m | 2^{2n-1} \leqq m < 2^{2n}, n = 0, 1, 2, \dots\} \cup \{0\}$, $x_1 = 0$, $x_2 = 0$.

$E10$: $C_1 = [0, \infty)\backslash\{1/m | 2^{2n} \leqq m < 2^{2n+1}, n = 0, 1, 2, \dots\} \cup \{0\}$.
$C_2 = [0, \infty)\backslash\{1/m | 2^{2n-1} \leqq m < 2^{2n}, n = 0, 1, 2, \dots\} \cup \{0\}$, $x_1 = 0$, $x_2 = 0$.

$E11$: $C_1 = \bigcup_{n=0}^{\infty} (2^{-2n-1}, 2^{-2n}] \cup \{0\}$.

$C_2 = \bigcup_{n=1}^{\infty} (2^{-2n}, 2^{-2n+1}] \cup \{0\}$, $x = 0$.

TABLE 6.2

Summary of results of Sections 2 and 3

*	#	$	J	M	Imply	by Prop. or Thm.
–	–	–	–	all	A is a cone	2.2
$\forall$	–	–	–	all	A is closed	2.3
$\exists$	–	–	–	all	A is open	2.5
–	$\forall$	–	–	–	A is isotone	2.10
$\exists$	$\exists$	$\exists$	either	all	A has property (3.1)	3.1
$\forall$	–	$\forall$	x	all	A has property (3.6)	3.3
–	–	$\exists$	either	all	A preserves products	3.5
$\forall$	$\exists$	$\exists$	X	all	A is convex	3.15

Symbols used in Table 6.2

Symbol	Meaning
–	No restriction.
either	The hypothesis is satisfied for $J(X) = X$ and $J(X) = x$.
all	The hypothesis is satisfied for $M(Y) = Y$, $M(Y) = y$, and $M(Y)$ equal the class of nonempty compact subsets of Y.

4. Tangent cone impossibility theorems. Given a list of desirable tangent cone properties, does there exist a nontrivial tangent cone possessing all of them? Our study of the quantificational tangent cones sheds some light on this question. For example, to construct a q-cone that is closed, isotone, and product-preserving, Theorems 2.3, 2.10, and 3.5 tell us that we may take $* = \forall$, $\# = \forall$, and $\$ = \exists$. Two such tangent cones are k and D. On the other hand, no nontrivial tangent cone in the principal series is both isotone and convex, and none preserves both unions and intersections. In this section, we show that certain combinations of properties that are incompatible for q-cones in the principal series are indeed impossible to attain more generally.

https://doi.org/10.4153/CJM-1988-029-6 Published online by Cambridge University Press

We begin with the fact, previously mentioned in [**42**], that there is no "sensible" convex subcone of the contingent cone that is isotone. Throughout this section, we assume that E is finite-dimensional.

THEOREM 4.1 [**42**]. *Let E have dimension >1. There is no mapping $A:2^E \times E \to 2^E$ having all of the following properties*:

(1) *A is isotone.*

(2) *A is convex.*

(3) *$A \subset K$.*

(4) *$A(C, x) \supset C$ for every subspace C of E and $x \in C$.*

Proof. Define

$$C_1 := \{ (x, y): x = 0\} \quad \text{and} \quad C_2 := \{ (x, y) \mid y = 0\},$$

and let $x_0 := (0, 0)$. If A has properties (1), (3), and (4), then

$$A(C_1 \cup C_2, x_0) = C_1 \cup C_2,$$

so that A is not convex.

Note that if one of the conditions (1) through (4) in Theorem 4.1 is removed, there exist tangent cones satisfying the remaining three. Here are some examples:

(a) K satisfies (1), (3) and (4).

(b) T satisfies (2), (3) and (4).

(c) The cone cl(co K), called the *pseudotangent cone* [**15**], satisfies (1), (2), and (4).

(d) $$A(C, x) = \begin{cases} \{0\} & \text{if } x \in C \\ \emptyset & \text{otherwise} \end{cases}$$

satisfies (1), (2), and (3).

The example used to establish Theorem 4.1 also demonstrates a related impossibility theorem.

THEOREM 4.2. *Let E have dimension >1. There is no mapping $A:2^E \times E \to 2^E$ that satisfies all of the following conditions*:

(1) *A preserves unions.*

(2) *A is convex.*

(3) *$A(C, x) = C$ whenever C is a subspace of E and $x \in C$.*

Again, if any of the conditions in Theorem 4.2 is removed, there are tangent cones fulfilling the remaining ones:

(a) K satisfies (1) and (3).

(b) T satisfies (2) and (3).

(c) $A(C, x) := \{0\}$ satisfies (1) and (2).

Definition 4.3. A mapping $A:2^E \times E \to E$ is *homogeneous* if

$$A(\alpha C, \alpha x) = \alpha A(C, x)$$

for all nonempty subsets C of E, $x \in C$, and $\alpha > 0$.

A large class of q-cones are homogeneous, including all those in the principal series [**40**, Proposition 1.3.3].

There are no tangent cones in the principal series that preserve unions and also satisfy the intersection inclusion (3.1). We next show that these two properties are incompatible, at least for homogeneous tangent cones.

THEOREM 4.4. *There is no mapping* $A:2^E \times E \to 2^E$ *satisfying all of the following conditions*:

(1) A *is a homogeneous tangent cone.*

(2) $A(E, x) = E$ *for all* $x \in E$, *and* $A(\{0\}, 0) = \{0\}$.

(3) $A(C_1 \cup C_2, x) \subset A(C_1, x) \cup A(C_2, x)$ *for all nonempty* C_1, C_2 *in* E *and* $x \in C_1 \cap C_2$.

(4) $A(C_1, x) \cap A(C_2, x) \subset A(C_1 \cap C_2, x)$ *for all nonempty* C_1, C_2 *in* E *and* $x \in C_1 \cap C_2$.

Proof. If $E := \mathbf{R}$, let

$$C_1 := \bigcup_n \{x:2^{2n} < |x| \leqq 2^{2n+1}\} \cup \{0\},$$

and define $C_2 := 2C_1$. Then

$$A(C_2, 0) = A(2C_1, 0) = 2A(C_1, 0)$$

since A is homogeneous. If A satisfies (1), it follows that $A(C_1, 0) = A(C_2, 0)$. If A also satisfies (4) and (2), $A(C_i, 0) \subset \{0\}$ for $i = 1, 2$. However, $C_1 \cup C_2 = E$, so A cannot satisfy (3). In general, define C_1 and C_2 to be subsets of E with $C_1 \cap C_2 = \{0\}$, $C_2 = 2C_1$, and $C_1 \cup C_2 = E$.

Examples of mappings satisfying three of the four conditions of Theorem 4.4 are given below:

(a) $A(C, x) := \{0\}$ satisfies (1), (3), and (4).

(b) $A(C, x) := C$ satisfies (2), (3), and (4).

(c) K satisfies (1), (2) and (3).

(d) RT satisfies (1), (2), and (4).

The example used in the proof of Theorem 4.4 also establishes a related impossibility theorem that indicates that union-preserving tangent cones must be relatively "large".

THEOREM 4.5. *There is no homogeneous mapping* $A:2^E \times E \to 2^E$ *satisfying all of the following conditions*:

(1) $A(C_1 \cup C_2, x) \subset A(C_1, x) \cup A(C_2, x)$ *for all nonempty subsets* C_1 *and* C_2 *of* E *and* $x \in C_1 \cap C_2$.

(2) $A(E, x) = E$ *for all* $x \in E$.

(3) $A \subset k$.

Proof. If $E := \mathbf{R}$, define C_1 and C_2 as in the proof of Theorem 4.4. Then

$$k(C_1, 0) = k(C_2, 0) = \{0\},$$

so if A satisfies (3) and (1), $A(E, 0) \subset \{0\}$, contradicting (2). In general, choose C_1 and C_2 so that $C_1 \cup C_2 = E$ but

$$k(C_1, 0) = k(C_2, 0) = \{0\}.$$

Observe that K satisfies conditions (1) and (2) of Theorem 4.5, k satisfies (2) and (3), and

$$A(C, x) = \begin{cases} C & \text{if } x \in C \\ \emptyset & \text{otherwise} \end{cases}$$

satisfies (1) and (2).

It is also true that tangent cones satisfying (3.1) must be relatively "small", as our next result indicates.

THEOREM 4.6. *There is no mapping* $A:2^E \times E \to 2^E$ *satisfying all of the following conditions*:

(1) $A(C_1, x) \cap A(C_2, x) \subset A(C_1 \cap C_2, x)$ *for all nonempty subsets* C_1 *and* C_2 *of* E *and* $x \in C_1 \cap C_2$.

(2) $A(\{0\}, 0) = \{0\}$.

(3) $T \subset A$.

Proof. Let C_1 and C_2 be two dense subsets of E whose union is E and intersection is $\{0\}$. For example, if $E := \mathbf{R}$, C_1 could be the set of rational numbers and C_2 the union of $\{0\}$ and the irrationals. Then $T(C_1, 0) = T(C_2, 0) = E$, so $A(C_1 \cap C_2, 0) = E$ if A satisfies (1) and (3). Such an A does not satisfy (2).

We observe that RT satisfies (1) and (2), T satisfies (2) and (3), and $A(C, x) := E$ satisfies (1) and (3).

Inclusion (3.1) is an important one in nonsmooth analysis, and a number of tangent cones that do not always satisfy this inclusion at least fulfill it for large classes of sets. Our final impossibility theorem gives one restriction on how large such classes of sets can be.

THEOREM 4.7. *If* E *has dimension* >1, *there is no mapping* $A:2^E \times E \to 2^E$ *satisfying both of the following conditions*:

(1) A *satisfies* (3.15).

(2) *Inclusion* (3.1) *holds whenever* $x \in C_1 \cap C_2$, C_1 *and* C_2 *are closed, and*

$$\text{ri } A(C_1, x) \cap \text{ri } A(C_2, x) \neq \emptyset. \tag{4.1}$$

Proof. Define $h_i:\mathbf{R}^2 \to \mathbf{R}$ by

$$h_i(x, y) := y + (-1)^i x^2$$

https://doi.org/10.4153/CJM-1988-029-6 Published online by Cambridge University Press

and $C_i = h_i^{-1}(0)$ for $i = 1, 2$, and let $x_0 := (0, 0)$. The functions h_i have $\nabla h_i(x_0)$ of full rank, so if A has property (1),

$$A(C_1, x_0) = A(C_2, x_0) = \{ (x, y) \,|\, y = 0\}.$$

If A also satisfies (2), then

$$A(\{x_0\}, x_0) = A(C_1 \cap C_2, x_0) \supset \{ (x, y) \,|\, y = 0\}.$$

Now define $f:\mathbf{R}^2 \to \mathbf{R}^2$ by $f(x, y) := (x, y)$. Then $\nabla f^{-1}(0) = \{x_0\}$, and by (3.15),

$$A(\{x_0\}, x_0) = \{x_0\},$$

a contradiction.

COROLLARY 4.8. *There is no q-cone A satisfying both of the following conditions*:

(1) $T \subset A$.

(2) *Condition* (2) *of Theorem* 4.7.

Proof. As was mentioned in the proof of Theorem 3.13, if A satisfies condition (1), then A satisfies condition (1) of Theorem 4.7.

If A is a tangent cone satisfying (3.15), it follows from Theorem 4.7 that a condition stronger than (4.1) will have to be imposed in order to guarantee inclusion (3.1). In particular, this applies to the tangent cones T, k, and K, as well as the prototangent and quasi-strict tangent cone of [**23**]. The question of what conditions will guarantee (3.1) for these tangent cones is discussed in [**23**, **42**, **46**].

It is hoped that Table 6.1 may serve as a helpful starting point in the search for further tangent cone impossibility theorems.

5. Directional derivatives and subgradients. One of the most fruitful applications of tangent cones has been in the definition of directional derivatives and subgradients via tangent cones of epigraphs.

Definition 5.1. Let A be a tangent cone, and let $f:E \to \overline{\mathbf{R}}$ be finite at $x \in E$. The *A directional derivative* of f at x with respect to y is defined by

(5.1) $$f^A(x; y) := \inf\{r:(y, r) \in A(f, x)\}.$$

The *A subgradient* of f at x is the set

$$\partial^A f(x) := \{x' \in E':\langle y, x'\rangle \leqq f^A(x; y) \text{ for all } y \in E\}.$$

These concepts are especially useful when A is a q-cone, in which case $f^A(x; y)$ has an explicit expression as a generalized limit of difference quotients. The cones $A := RT, T, IT, RK, K, IK, Rk, k, Ik$, in particular, have been widely studied in recent years [**1**, **2**, **7**, **12-14**, **16**, **22**, **23**, **26-28**,

31, **32**, **40-46**]. In this section, we formulate an expression for a general q-cone directional derivative and examine the simplifications in this expression that become possible for continuous functions and for locally Lipschitzian functions.

We begin by observing that the inclusion

$$(5.2)\qquad A(f, x) \subset \operatorname{epi} f^A(x; \cdot)$$

is always valid, and that if A is closed, we have

$$(5.3)\qquad A(f, x) = \operatorname{epi} f^A(x; \cdot).$$

If A is a closed tangent cone, (5.3) tells us immediately that $f^A(x; \cdot)$ is lower semicontinuous (l.s.c.) and positively homogeneous. If in addition A is convex, $f^A(x; \cdot)$ is sublinear. Equation (5.3) and the properties of A can often be used in establishing calculus rules for f^A [**42**, **45**, **46**].

Although (5.3) is not true for all q-cones, a slight modification of (5.3) will be valid in general.

Definition 5.2. Let A be a q-cone in whose definition equality is satisfied in (3.7) for all products of neighborhoods in $E \times R$. Let $f:E \to \overline{\mathbf{R}}$ be finite at $x \in E$. We define

$$\bar{A}(f, x) := \{ (y, d) \in E \times R : \forall\, \delta > 0, (y, d') \in A(f, x) \text{ for some } d' \in [d, d + \delta) \}.$$

We observe that $A(f, x) \subset \bar{A}(f, x)$ is true in general, and that $A(f, x) = \bar{A}(f, x)$, if $* = \forall$ and $M(Y) = Y$. The particular case $\overline{IT}$ is used in [**31**, **32**] in studies of the calculus of subgradients of vector-valued functions and tangent cones of compositions of multifunctions. The significance of $\bar{A}$ lies in the fact that (5.3) becomes true in general if A is replaced by $\bar{A}$. We omit the straightforward proof of this fact, stated as Proposition 5.3. A proof is given in [**43**] for the case in which $J(X) = x$.

PROPOSITION 5.3 (cf. [**14**, Theorem 3.1]). *Let A be a q-cone in whose definition equality is satisfied in* (3.7) *for all products of neighborhoods in $E \times R$. Let $f:E \to \bar{R}$ be finite at $x \in E$. Then*

$$(5.4)\qquad \bar{A}(f, x) = \operatorname{epi} f^A(x; \cdot).$$

Equation (5.4) is the key to the derivation of a more explicit expression for f^A.

THEOREM 5.4. *Under the hypotheses of Proposition* 5.3,

$$(5.5)\qquad f^A(x; y) = \mathop{\square}_{Y \in N(y)} \mathop{\Delta}_{\substack{X \in N(x)\\ u>0}} \mathop{\&}_{\lambda>0} \inf_{\substack{W \in J(X)\\ \times J((f(x)-u, f(x)+u))}} \inf_{Z \in M(Y)} \mathop{\Delta'}_{(x',r) \in \operatorname{epi} f \cap W} \mathop{\&'}_{t \in (0,\lambda)} \mathop{\square'}_{y' \in Z} (f(x' + ty') - r)/t,$$

where

$$\Box := \begin{cases} \sup & \text{if } * = \forall \\ \inf & \text{if } * = \exists \end{cases}, \quad \Delta := \begin{cases} \sup & \text{if } \# = \forall \\ \inf & \text{if } \# = \exists \end{cases},$$

$$\& := \begin{cases} \sup & \text{if } \$ = \forall \\ \inf & \text{if } \$ = \exists \end{cases}, \text{ and}$$

$\Box', \Delta', \&' \in \{\sup, \inf\}, \Box' \neq \Box, \Delta' \neq \Delta, \&' \neq \&.$

Proof. Denote by S the right hand side of (5.5). Then $S \leqq d$ if and only if for all $\delta > 0$, $*\ Y \in N(y)$, $\#\ X \in N(x)$, $\#\ u > 0$, $\$\ \lambda > 0$, there exist

$$W \in J(X) \times J((f(x) - u, f(x) + u)) \text{ and } Z \in M(Y),$$
$$\#'(x', r) \in \operatorname{epi} f \cap W, \$'\ t \in (0, \lambda), *'\ y' \in Z,$$
$$(f(x' + ty') - r)/t \leqq d + \delta.$$

Now

$$(f(x' + ty') - r)/t \leqq d + \delta$$

if and only if there exists $d' \leqq d + \delta$ such that

$$(x', r) + t(y', d') \in \operatorname{epi} f.$$

In other words, $S \leqq d$ if and only if $(y, d) \in \bar{A}(f, x)$. By Proposition 5.3, $S \leqq d$ if and only if $f^A(x; y) \leqq d$, and so (5.5) holds.

Special cases of (5.5) include a number of familiar directional derivatives. For example,

$$f^{Rk}(x; y) = \inf_{\lambda > 0} \sup_{t \in (0,\lambda)} (f(x + ty) - f(x))/t$$

and

$$f^{RK}(x; y) = \sup_{\lambda > 0} \inf_{t \in (0,\lambda)} (f(x + ty) - f(x))/t$$

are simply Dini directional derivatives, and

$$f^{RT}(x; y) = \inf_{\substack{X \in N(x) \\ u > 0 \\ \lambda > 0}} \sup_{\substack{(x',r) \in X \times (f(x) - u, f(x) + u) \\ (x',r) \in \operatorname{epi} f \\ t \in (0,\lambda)}} (f(x' + ty) - r)/t$$

reduces to the Clarke generalized derivative $f^0(x; y)$ [**12**] when f is locally Lipschitzian (Corollary 5.6).

We now examine f^A for several special classes of functions, beginning with the case in which f is convex and finite at x. By (3.11), both $f^{RK}(x; y) = f^{RK}(x; y)$ coincide with the directional derivative

https://doi.org/10.4153/CJM-1988-029-6 Published online by Cambridge University Press

$$f'(x, y) := \lim_{t \to 0^+} (f(x + ty) - f(x))/t.$$

Further, since (3.12) holds for $T \subset A \subset K$,

$$f^T(x; \cdot) = f^{T^*}(x; \cdot) = f^k(x; \cdot) = f^K(x; \cdot) = \text{cl } f'(x; \cdot).$$

This means that for f convex and $A := T, T^*, k, K, Rk, RK$, $\partial^A f(x) = \partial f(x)$, since

$$\partial f(x) = \{x' \in E' | \langle y, x' \rangle \leqq f'(x; y) \text{ for all } y \in E\}$$
$$= \{x' \in E' | \langle y, x' \rangle \leqq \text{cl } f'(x; y) \text{ for all } y \in E\}.$$

If in addition f is *directionally Lipschitzian* at x (i.e., epi f is epi-Lipschitzian at $(x, f(x))$), we have by Corollary 2.8 that

$$\text{cl } f^A(x; \cdot) = \text{cl } f'(x; \cdot)$$

for $A := IT, RT, IT^*, RT^*, IK, Ik$, and $\partial^A f(x) = \partial f(x)$ for all these tangent cones also. In summary, tangent cones that satisfy (3.13) for convex functions are those whose associated subgradient sets coincide with the subgradient of convex analysis for convex functions.

Another important special case is that in which $f: E \to \bar{\mathbf{R}}$ is directionally Lipschitzian at $x \in E$ [**7**, **12**, **26**]. In this case, Corollary 2.8 tells us that

$$f^A(x; y) = \text{cl } f^{RA}(x; y) = \text{cl } f^{IA}(x; y) \quad \text{and}$$
$$\partial^A f(x) = \partial^{RA} f(x) = \partial^{IA} f(x)$$

for $A := T, T^*, k, K, D, D^*$.

The following simplification in the expression for f^A is possible when f is continuous at x.

PROPOSITION 5.5. *Let A be a q-cone in whose definition $\# = \exists$ and $J(X) = X$. Let $f: E \to \bar{R}$ be continuous at $x \in E$. Then*

$$(5.6) \quad f^A(x; y) = \underset{Y \in N(y)}{\square} \ \inf_{X \in N(x)} \ \underset{\lambda > 0}{\&} \ \inf_{Z \in M(Y)}$$
$$\sup_{x' \in X} \ \underset{t \in (0,\lambda)}{\&'} \ \underset{y' \in Z}{\square'} (f(x' + ty') - f(x'))/t.$$

Proof. Denote the right hand side of (5.6) by S, and suppose $f^A(x; y) \leqq d$. Then for all $\delta > 0$, $*\ Y \in N(y)$, there exist $X \in N(x)$ and $u > 0$, $\$\ \lambda > 0$, there exists $Z \in M(Y)$, for all

$$(x', r) \in (X \times (f(x) - u, f(x) + u)) \cap \text{epi } f,$$
$$\$'\ t \in (0, \lambda), *'\ y' \in Z, (f(x' + ty') - r)/t \leqq d + \delta.$$

Since f is continuous, we may assume, by choosing a smaller X if necessary, that

$$f(x') \in (f(x) - u, f(x) + u) \quad \text{for all } x' \in X.$$

https://doi.org/10.4153/CJM-1988-029-6 Published online by Cambridge University Press

Then for any $y' \in Z$ and $t \in (0, \lambda)$, (x', r) satisfies

$$\text{“}x' \in X, r \in (f(x) - u, f(x) + u),$$
$$f(x') \leqq r \text{ implies } (f(x' + ty') - r)/t \leqq d + \delta\text{”}$$

if and only if x' satisfies

$$\text{“}x' \in X \text{ implies } (f(x' + ty') - f(x'))/t \leqq d + \delta\text{”}.$$

Thus $f^A(x; y) \leqq d$ if and only if $S \leqq d$, and (5.6) holds.

COROLLARY 5.6. *Let $f:E \to \mathbf{R}$ be continuous at x. Then*

$$f^{RT}(x; y) = \inf_{\substack{X \in N(x) \\ \lambda > 0}} \sup_{\substack{x' \in X \\ t \in (0,\lambda)}} (f(x' + ty) - f(x'))/t$$

and

$$f^T(x; y) = \sup_{Y \in N(y)} \inf_{\substack{X \in N(x) \\ \lambda > 0}} \sup_{\substack{x' \in X \\ t \in (0,\lambda)}} \inf_{y' \in Y} (f(x' + ty') - f(x'))/t.$$

Definition 5.7. Let E be a normed space. The function $f:E \to \overline{\mathbf{R}}$ is said to be *locally Lipschitzian* near $x \in E$ if there exist $X \in N(x)$ and a constant $M > 0$ such that

$$|f(y) - f(y')| \leqq K\|y - y'\| \quad \text{whenever } y', y \in X.$$

A further simplification can be made in (5.5) if f is locally Lipschitzian near x. In fact, $f^A(x; \cdot) = f^{RA}(x; \cdot) = f^{IA}(x; \cdot)$ will often hold in this case.

THEOREM 5.8. *Let E be a normed space, and suppose $f:E \to \bar{R}$ is locally Lipschitzian near $x \in E$. Let A be a q-cone with $* = \forall$, $\# = \exists$, $M(Y) = Y$, and either $J(X) = X$ or $J(X) = x$. Then*

$$f^A(x; y) = f^{IA}(x; y) = f^{RA}(x; y)$$
$$= \inf_{X \in N(x)} \mathop{\&}_{\lambda > 0} \sup_{x' \in X \cap J(X)} \mathop{\&'}_{t \in (0,\lambda)} (f(x' + ty) - f(x'))/t.$$

Proof. Since $IA \subset RA \subset A$, we know that

$$f^A(x; \cdot) \leqq f^{RA}(x; \cdot) \leqq f^{IA}(x; \cdot).$$

It remains to show that $f^{IA}(x; \cdot) \leqq f^A(x; \cdot)$. Suppose $f^A(x; y) \leqq d$. Then for all $\delta > 0$ and $Y \in N(y)$, there exists $X \in N(x)$, \$ $\lambda > 0$, for all $x' \in J(X)$, \$' $t \in (0, \lambda)$, there exists $y' \in Y$ with

$$(f(x' + ty') - f(x'))/t \leqq d + \delta/2.$$

(For the case $J(X) = X$, this characterization of “$f^A(x; y) \leqq d$” relies on Proposition 5.5.) Since f is locally Lipschitzian near x, there exist $X_1 \in N(x)$ and $M > 0$ such that for all $x', x'' \in X_1$,

$$|f(x') - f(x'')| \leqq M\|x' - x''\|.$$

Choose $X_2 \in N(x)$, $\lambda_1 > 0$, and $Y_1 \in N(y)$ with

$$X_2 + [0, \lambda_1)Y_1 \subset X_1 \quad \text{and} \quad \|y' - y''\| \leqq \delta/2M$$

for all $y', y'' \in Y_1$.

Now let $\delta > 0$ and $Y \in N(y)$ be given, and let $y' \in Y \cap Y_1$. There exists $X \in N(x)$ with $X \subset X_2$, $\$\ \lambda \in (0, \lambda_1)$, for all $x' \in J(X)$, $\$'\ t \in (0, \lambda)$, there exists $y'' \in Y \cap Y_1$ with

$$\begin{aligned}(f(x' + ty') - f(x'))/t &\leqq (f(x' + ty'') + t\delta/2 - f(x'))/t \\ &\leqq (f(x' + ty'') - f(x'))/t + \delta/2 \\ &\leqq d + \delta.\end{aligned}$$

Thus $f^A(x; y) \leqq d$ implies $f^{IA}(x; y) \leqq d$, and it follows that

$$f^A(x; y) = f^{IA}(x; y) = f^{RA}(x; y).$$

That all three equal the given expression follows from Proposition 5.5.

COROLLARY 5.9. *Let E be a normed space, and suppose $f:E \to \bar{\mathbf{R}}$ is locally Lipschitzian near $x \in E$. Then*

$$f^A(x; \cdot) = f^{IA}(x; \cdot) = f^{RA}(x; \cdot)$$

for $A := T, T^*, k, K$.

Many of these directional derivatives and subgradients also coincide when $f:E \to \bar{\mathbf{R}}$ is strictly differentiable at x. In that case (see [**12**, **26**, **38**]),

$$f^A(x; \cdot) = f^{RA}(x; \cdot) = f^{IA}(x; \cdot) \quad \text{and}$$

$$\partial f(x) = \partial^{RA}f(x) = \partial^{IA}f(x) = \{\nabla f(x)\}$$

for $A := T, T^*, k, K$.

For a tangent cone to be useful in optimization, it must at least satisfy the necessary condition $0 \in \partial^A f(x)$ whenever $x \in E$ is a local minimizer of $f:E \to \bar{\mathbf{R}}$. It is well known that K satisfies this condition (see for example [**22**]). An immediate consequence is the following fact.

PROPOSITION 5.10. *Suppose $f:E \to \bar{R}$ has a local minimum at $x \in E$. Then $0 \in \partial^A f(x)$ for all $A \subset K$. In particular, this necessary condition holds for A, RA, and IA where $A := K, k, T^*, T$.*

Example 5.11. This necessary condition is not valid for q-cones with $\# = \forall$ and $J(X) = X$. For instance, consider $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = x$. It is not hard to verify that $ID(f, 0) = \mathbf{R}^2$, so that $f^{ID}(0; y) = -\infty$ for all $y \in \mathbf{R}$ and $\partial^{ID}f(0) = \emptyset$. Since $ID \subset A$ for all q-cones A with $\# = \forall$ and $J(X) = X$, $\partial^A f(0) = \emptyset$ for all such q-cones.

https://doi.org/10.4153/CJM-1988-029-6 Published online by Cambridge University Press

The calculus of f^T, f^k, and f^K has been widely studied in recent years. For details of this calculus and applications to optimization, see for example [**1**, **2**, **13**, **23**, **26**, **41**, **42**, **44**, **46**].

6. Conclusions. The results of Sections 2 and 3 are summarized in Tables 6.1 and 6.2. The properties of the tangent cones of the principal series are compiled in Table 6.1. In Table 6.2, the connections between the properties of a q-cone and the various parts of its definition are listed. Certain combinations of properties seem to be impossible to achieve; some of these combinations are discussed in Section 4.

A working knowledge of the various tangent cones can aid in the choice of the "right" tangent cone for a given application. Theorems in nonsmooth analysis can often be strengthened by the use of an appropriate tangent cone. For example, substitution of k for K in the definition of proper efficiency in [**4**] immediately strengthens the results of that paper. The directional derivative f^{Rk} can be replaced by f^{RK} to extend the applicability of the mean value theorems of [**30**]. Similarly, f^K can be used in place of f^{RK} in [**29**] to strengthen and simplify the proof of some of the exact penalty results presented there [**41**]. In [**26**], f^k could be introduced to improve the conditions for equality in the formulas of the generalized subdifferential calculus [**46**].

It is hoped that this paper will aid the reader's understanding of the capabilities and limitations of nonsmooth analysis.

Acknowledgment. I am very grateful to J. M. Borwein, whose stimulating questions were the starting point of this paper and whose advice and encouragement were invaluable at every stage of its completion. I am also grateful for the financial support of the Killam Trust.

REFERENCES

1. J.-P. Aubin, *Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions*, Math. Anal. Appl., Advances in Math., Suppl. Studies *7A* (Academic Press, New York, 1981).

2. J.-P. Aubin and I. Ekeland, *Applied nonlinear analysis* (Wiley, New York, 1984).

3. M. S. Bazaraa, J. J. Goode, M. F. Nashed and C. M. Shetty, *Nonlinear programming without differentiability in Banach spaces: Necessary and sufficient constraint qualifications*, Applicable Analysis *5* (1976), 165-173.

4. J. M. Borwein, *Proper efficient points for maximizations with respect to cones*, SIAM J. Cont. Opt. *15* (1977), 57-63.

5. ——— *Weak tangent cones and optimization in a Banach space*, SIAM J. Cont. Opt. *16* (1978), 512-522.

6. ——— *Stability and regular points of inequality systems*, Journal of Optimization Theory and Applications *48* (1986), 9-52.

7. J. M. Borwein and H. M. Strojwas, *Directionally Lipschitzian mappings on Baire spaces*, Can. J. Math. *36* (1984), 95-130.

8. ——— *Tangential approximations*, Nonlinear Analysis: Theory, Methods, and Applications *9* (1985), 1247-1266.

9. ——— *Proximal analysis and boundaries of closed sets in Banach space, Part I: Theory*, Can. J. Math. *38* (1986), 431-452.

10. ——— *Proximal analysis and boundaries of closed sets in Banach space, Part II: Applications*, Can. J. Math. *39* (1987), 428-472.

11. ——— *The hypertangent cone study*, submitted.

12. F. H. Clarke, *Optimization and nonsmooth analysis* (Wiley, New York, 1983).

13. S. Dolecki, *Tangency and differentiation: some applications of convergence theory*, Annali di Matematica pura ed applicata *130* (1982), 223-255.

14. K.-H. Elster and J. Thierfelder, *The general concept of cone approximations in nondifferentiable optimization*, in *Nondifferentiable optimization: Motivations and applications*, Lecture Notes in Economics and Mathematical Systems *255* (Springer-Verlag, Berlin, 1985).

15. M. Guignard, *Generalized Kuhn-Tucker conditions for mathematical programming in a Banach space*, SIAM J. Control *7* (1969), 232-241.

16. J.-B. Hiriart-Urruty, *Tangent cones generalized gradients and mathematical programming in Banach spaces*, Mathematics of Operations Research *4* (1979), 79-97.

17. A. D. Ioffe, *On the theory of subdifferential*, in *Fermat Days 85: Mathematics for optimization* (North Holland, Amsterdam, 1986).

18. S. S. Kutateladze, *Infinitesimal tangent cones*, Siberian Math. J. *26* (1985), 833-840.

19. ——— *Nonstandard analysis of tangent cones*, Soviet Math. Dokl. *32* (1985), 437-439.

20. D. H. Martin, R. J. Gardner and G. G. Watkins, *Indicating cones and the intersection principle for tangential approximants in abstract multiplier rules*, Journal of Optimization Theory and Applications *33* (1981), 515-537.

21. D. H. Martin and G. G. Watkins, *Cores of tangent cones and Clarke's tangent cone*, Mathematics of Operations Research *10* (1985), 565-575.

22. J.-P. Penot, *Calcul sous-differentiel et optimisation*, Journal of Functional Analysis *27* (1978), 248-276.

23. ——— *Variations on the theme of nonsmooth analysis: another subdifferential*, in *Nondifferentiable optimization: Motivations and applications*, Lecture Notes in Economics and Mathematical Systems *255* (Springer-Verlag, Berlin, 1985).

24. R. T. Rockafellar, *Convex analysis* (Princeton University Press, Princetown, N.J., 1970).

25. ——— *Clarke's tangent cone and the boundaries of closed sets in* $\mathbf{R}^n$, Nonlinear Analysis: Theory, Methods, and Applications *3* (1979), 145-154.

26. ——— *Directionally Lipschitzian functions and subdifferential calculus*, Proc. London Math. Soc. *39* (1979), 331-355.

27. ——— *Generalized directional derivatives and subgradients of nonconvex functions*, Can. J. Math. *32* (1980), 157-180.

28. ——— *The theory of subgradients and its applications to problems of optimization: Convex and nonconvex functions* (Helderman Verlag, Berlin, 1981).

29. E. Rosenberg, *Exact penalties and stability in locally Lipschitz programming*, Mathematical Programming *30* (1984), 340-356.

30. M. Studniarski, *Mean value theorems and sufficient optimality conditions for nonsmooth functions*, J. Math. Anal. Appl. *111* (1985), 313-326.

31. L. Thibault, *Subdifferentials of nonconvex vector-valued functions*, J. Math. Anal. Appl. *86* (1982), 319-344.

32. ——— *Tangent cones and quasi-interiorly tangent cones to multifunctions*, Trans. Am. Math. Soc. *277* (1983), 601-621.

33. J. S. Treiman, *Characterization of Clarke's tangent and normal cones in finite and infinite dimensions*, Nonlinear Analysis: Theory, Methods, and Applications *7* (1983), 771-783.

34. ——— *Shrinking generalized gradients*, to appear in Nonlinear Analysis: Theory, Methods, and Applications.

35. C. Ursescu, *Tangent sets' calculus and necessary conditions for extremality*, SIAM J. Cont. Opt. *20* (1982), 563-574.

36. M. Vlach, *On necessary conditions of optimality in linear spaces*, Commentationes Mathematicae Universitatis Carolinae *11* (1970), 501-513.

37. ——— *On the cones of tangents*, Methods of Operations Research *37* (1980), 251-256.

38. ——— *Approximation operators in optimization theory*, Zeitschrift fur O.R. *25* (1981), 15-23.

39. ——— *Closures and neighborhoods properties induced by tangential approximations*, in *Selected topics in operations research and mathematical economics* (Springer-Verlag, Berlin, 1984), 119-127.

40. D. E. Ward, *Tangent cones, generalized subdifferential calculus, and optimization*, Ph.D. thesis, Dalhousie University, (1984).

41. ——— *Exact penalties and sufficient conditions for optimality in nonsmooth optimization*, Journal of Optimization Theory and Applications *57* (1988), 485-499.

42. ——— *Convex subcones of the contingent cone in nonsmooth calculus and optimization*, Trans. Am. Math. Soc. *302* (1987), 661-682.

43. ——— *Isotone tangent cones and nonsmooth optimization*, Optimization *18* (1987), 769-783.

44. ——— *Subdifferential calculus and optimality conditions in nonsmooth mathematical programming*, to appear in J. of Information and Optimization Sciences.

45. ——— *Which subgradients have sum formulas?*, to appear in Nonlinear Analysis.

46. D. E. Ward and J. M. Borwein, *Nonsmooth calculus in finite dimensions*, SIAM J. Cont. Opt. *25* (1987), 1312-1340.

47. G. G. Watkins, *Nonsmooth Milyutin-Dubovitskii theory and Clarke's tangent cone*, Mathematics of Operations Research *11* (1986), 70-80.

Miami University,
Oxford, Ohio