Totally nonnegative Grassmannians, Grassmann necklaces, and quiver Grassmannians

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Abstract. Postnikov constructed a cellular decomposition of the totally nonnegative Grassmannians. The poset of cells can be described (in particular) via Grassmann necklaces. We study certain quiver Grassmannians for the cyclic quiver admitting a cellular decomposition, whose cells are naturally labeled by Grassmann necklaces. We show that the posets of cells coincide with the reversed cell posets of the cellular decomposition of the totally nonnegative Grassmannians. We investigate algebro-geometric and combinatorial properties of these quiver Grassmannians. In particular, we describe the irreducible components, study the action of the automorphism groups of the underlying representations, and describe the moment graphs. We also construct a resolution of singularities for each irreducible component; the resolutions are defined as quiver Grassmannians for an extended cyclic quiver.

1 Introduction

Total positivity has a long story, starting in the first half of the 20th century [GK37, Schoe47]. Thanks to Lusztig [Lus94, Lus98a, Lus98b], it became of interest to Lie theorists and combinatorialists because of the relation with canonical bases. More precisely, Lusztig introduced the notion of totally nonnegative (tnn) part of (real generalized) flag varieties. The tnn Grassmannian $Gr(k, n)_{\geq 0}$ is, hence, a special case of these and admits an easy description under the Plücker embedding. Namely, $Gr(k, n)_{\geq 0}$ is the subvariety of the (real) Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$ whose points have Plücker coordinates all of the same sign (i.e., they can be normalized to be all nonnegative).

The tnn Grassmannians attracted a lot of attention due to a large number of applications and various links with other mathematical structures (see [Lam16] and the references therein). In [Pos06], Postnikov constructed a stratification of $Gr(k, n)_{\geq 0}$ by the so-called positroid cells, where each cell is isomorphic to a product of some copies of $\mathbb{R}_{>0}$. There are many nice combinatorial ways to label the positroid cells.

In particular, the cells are in bijection with the so-called Grassmann necklaces: collections $(I_1, \ldots, I_n)$ of k-element subsets of the set $[n] = \{1, \ldots, n\}$ such that

$$I_1 \subset I_2 \cup \{1\}, \; I_2 \subset I_3 \cup \{2\}, \ldots, I_n \subset I_1 \cup \{n\}$$

(a version of this definition leads to the notion of juggling pattern from [KLS13]). It is possible to equip the set of Grassmann necklaces with a combinatorial partial order, which turns out to agree with the reversed positroid cell closure relation (see, e.g., [W05]). The same poset pops up in the study of the positroid varieties [KLS13, KLS14, Lus98a]. These varieties stratify complex Grassmannians, and one can use the tools of algebraic geometry for their study. We note that the positroid varieties are no longer affine cells, and that Postnikov's positroid cells are obtained by intersecting positroid varieties with the tnn Grassmannians.

The notion of Grassmann necklaces has a natural linear algebra analog. Namely, let us consider an n-dimensional complex vector space W with a fixed ordered basis $(e_j)_{j\in[n]}$ and the projections pr_i along the basis vectors of W. Let $X(k, n)$ be the variety of collections of k-dimensional subspaces $V_1, \ldots, V_n$ of W such that

$$\mathrm{pr}_1 V_1 \subset V_2, \; \mathrm{pr}_2 V_2 \subset V_3, \ldots, \mathrm{pr}_n V_n \subset V_1$$

(this variety is denoted by Jugg in [Kn08]). Every Grassmann necklace $\mathcal{I} = (I_a)_{a\in[n]}$ gives rise to a point $p(\mathcal{I}) = p(\mathcal{I})_{a\in[n]}$ in $X(k, n)$, where V_a is spanned by $\{e_j : j \in I_a\}$ for any $a \in [n]$. Our key observation is that $X(k, n)$ is isomorphic to a quiver Grassmannian for the (equioriented) cyclic quiver Δ_n on n vertices (Proposition 4.1): there exists a Δ_n module

$$U_{[n]} \cong \left((M^{(a)} = W)_{a\in[n]}, (M_a = \mathrm{pr}_a)_{a\in[n]} \right)$$

such that

(1.1) $$X(k, n) \cong \mathrm{Gr}_{(k,\ldots,k)}(U_{[n]}).$$

We note that every point of $X(k, n)$ is isomorphic (as a Δ_n-module) to $p(\mathcal{I})$ for some Grassmann necklace $\mathcal{I}$.

In 1992, motivated by the study of quiver representations, Schofield introduced quiver Grassmannians [Scho92]. Since then, they have been widely investigated, also in relation with positivity phenomena in cluster algebra theory (see the survey [CI20] for an account of all progress achieved on quiver Grassmannian in these decades). It is shown in [Re13] that every complex projective variety can be realized as a quiver Grassmannian. Finding a suitable quiver Grassmannian realization, as in our case, is, hence, desirable, as one can exploit all developed quiver techniques on top of the classical algebro-geometric tools. In particular, using (1.1), we prove the following theorem.

Theorem 1 $X(k, n)$ *admits a cellular decomposition, each cell contains exactly one point of the form $p(\mathcal{I})$ for a Grassmann necklace $\mathcal{I}$, and the poset of cells is isomorphic to the reversed poset of the tnn Grassmannian cells.*

In other words, the (complex) topology of $X(k, n)$ is the reverse of the (real) topology of the tnn Grassmannian (in the sense that the two face posets are dual).

The realization (1.1) allows to use the action of the automorphism group $\mathrm{Aut}_{\Delta_n}(U_{[n]})$ for the study of the topological properties of $X(k, n)$. We prove the following theorem.

Theorem 2 *The cells from Theorem 1 coincide with the $\mathrm{Aut}_{\Delta_n}(U_{[n]})$ orbits. $X(k, n)$ is equipped with an $(n + 1)$-dimensional torus action, which preserves the cellular decomposition. The corresponding moment graph has a combinatorial description in terms of Grassmann necklaces.*

Finally, let us list the main algebro-geometric properties of our quiver Grassmannians.

Theorem 3 $X(k, n)$ *has $\binom{n}{k}$ irreducible components, each of them of dimension $k(n - k)$. Each irreducible component of $X(k, n)$ is desingularized by an explicitly given quiver Grassmannian for an extended cyclic quiver $\widetilde{\Delta_n}$.*

There are several reasons why we are interested in the varieties $X(k, n)$. First, $X(k, n)$ serve as the linear algebra analogues of the discrete notion of the Grassmann necklaces. Second, the realization of $X(k, n)$ as quiver Grassmannians shows that they are natural degenerations of the classical complex Grassmannians $\mathrm{Gr}(k, n)$. Finally, the Δ_n representations $U_{[n]}$ are natural analogues of the A_n-modules used to construct the degenerate flag varieties for the equioriented type A quivers [CFR12]. These varieties have been widely investigated in the past 10 years, straddling the intersection of Lie theory, geometry, and combinatorics. Studying the geometry of degenerations via quiver Grassmannians leads to many nice combinatorial structures involving Catalan and Genocchi combinatorics (see, e.g., [CL15, F11, F12, Pue20]).

1.1 Structure of the paper

In Sections 2 and 3, we collect background material on tnn Grassmannians/Grassmann necklaces and quiver Grassmannians, respectively. Our main object $X(k, n)$ is introduced in Section 4, where we describe the key isomorphism (1.1) and construct the cellular decomposition, as well as the irreducible components. In Section 5, we deal with several tori acting on $X(k, n)$, and in the following section, we focus on one of them and give its moment graph. The Poincaré polynomial is determined in Section 7. Section 8 is about resolutions of singularities of the irreducible components. Finally, Appendices A and B are about linear degenerations and the $k = 1$ case, respectively.

2 Totally nonnegative Grassmannians and positroids

In this section, we briefly recall basic facts on the tnn Grassmannians following [Lam16]. We discuss their geometry and combinatorics. In particular, we recall the definition of the Grassmann necklaces and juggling patterns, which provide a bridge to the theory of quiver Grassmannians.

2.1 Totally nonnegative Grassmannians

The tnn Grassmannian $\mathrm{Gr}(k, n)_{\geq 0}$ is the subset of the real Grassmann variety $\mathrm{Gr}(k, n)$ represented by the subspaces whose Plücker coordinates have all the same sign.

Postnikov [Pos06] defined a cellular decomposition of $\mathrm{Gr}(k,n)_{\geq 0}$ as follows. For $L \in \mathrm{Gr}(k,n)$ and a k-element subset $I \subset [n] = \{1, \ldots, n\}$, let $X_I(L)$ be the Ith Plücker coordinate of L. We define

$$\mathcal{M}(L) = \left\{ I \subset \binom{[n]}{k} : X_I(L) \neq 0 \right\}.$$

Then $\mathcal{M}(L)$ is a matroid attached to L. If $L \in \mathrm{Gr}(k,n)_{\geq 0}$, then the matroid $\mathcal{M}(L)$ is called a positroid.

The importance of this notion is explained by the following theorem due to Postnikov [Pos06].

Theorem 2.1 *Let $\mathcal{P}(k,n)$ be the set of positroids. For $\mathcal{M} \in \mathcal{P}(k,n)$, we denote by $\Pi(\mathcal{M}) \subset \mathrm{Gr}(k,n)_{\geq 0}$ the set of subspaces L such that $\mathcal{M}(L) = \mathcal{M}$. Then each stratum $\Pi(\mathcal{M})$, $\mathcal{M} \in \mathcal{P}(k,n)$ is a cell $\mathbb{R}_{>0}^s$ (s depends on the positroid $\mathcal{M}$).*

Hence, one gets a cellular decomposition of the tnn Grassmannian labeled by the positroids. It is thus natural to ask how to label the positroids $\mathcal{P}(k,n)$ and how to compute the dimension of the cell $\Pi(\mathcal{M})$ for $\mathcal{M} \in \mathcal{P}(k,n)$.

Remark 2.2 The cellular decomposition for the tnn part of the flag varieties G/P can be found in [Lus94, Rie06, Rie99]. Postnikov's positroid decomposition agrees with the general construction.

Remark 2.3 It is important that we consider only the matroids corresponding to the points of the tnn Grassmannians. The strata corresponding to the general matroids have much less transparent structure (see [GGMS87] and the discussion in the introduction of [KLS13]).

2.2 Grassmann necklaces

There are several ways to parametrize the elements of $\mathcal{P}(k,n)$. The one providing the bridge between the theory of tnn Grassmannians and quiver Grassmannians for (equioriented) cyclic quivers is the following one (see [Pos06]).

Definition 2.4 A (k,n) Grassmann necklace is a collection $I_1, \ldots, I_n$ of subsets of $[n]$ such that $|I_a| = k$ for all a and $I_a \subset I_{a+1} \cup \{a\}$ for all $a = 1, \ldots, n$. The set of (k,n) Grassmann necklaces is denoted by $\mathcal{GN}_{k,n}$.

We note that for $a = n$ the last condition is understood as $I_n \subset I_1 \cup \{n\}$. In other words, the condition on the sets I_a can be written as $i \in I_a \backslash \{a\}$ implies $i \in I_{a+1}$ (which works for $a = n$ as well). There is a slightly different version of Definition 2.4 (see, e.g., [KLS13]). Namely, given a collection $(I_a)_{a \in [n]} \in \mathcal{GN}_{k,n}$, we define $J_a = \{i - a : i \in I_a\}$, where $i - a$ is understood as an element of $[n]$, which is equal to $a - n$ modulo n. The resulting collections $(J_a)_{a \in [n]}$ are called juggling patterns in [KLS13] (modulo the overall change $j \to n + 1 - j$ of the elements of J_a). Clearly, one can put forward the following definition.

Definition 2.5 A collection $(J_a)_{a=1}^n$, $J_a \in \binom{[n]}{k}$ is called a juggling pattern if $j \in J_a \backslash \{n\}$ implies $j + 1 \in J_{a+1}$.

Remark 2.6 In Theorem 4.10, we show that Grassmann necklaces and juggling patterns naturally parametrize the torus fixed points in a certain quiver Grassmannian for the cyclic quiver. The two combinatorial Definitions 2.4 and 2.5 correspond to two natural choices of ordered basis in the representation space of the quiver (cf. Definition 4.3).

The set of (k, n) Grassmann necklaces can be equipped with a partial order. For two elements $I, J \in \binom{[n]}{k}$ such that $I = (i_1 < \cdots < i_k)$, $J = (j_1 < \cdots < j_k)$, we write $I \leq J$ if $i_u \leq j_u$ for all $u \in [k]$. Now, for a number $a \in [n]$, we consider the rotated order

$$(2.1) \qquad a <_a a + 1 <_a \cdots <_a n <_a 1 <_a \cdots <_a a - 1$$

on the set $[n]$. This order induces the order $<_a$ on the set $\binom{[n]}{k}$. Now, for two (k, n) Grassmann necklaces $\mathfrak{I} = (I_1, \ldots, I_n)$ and $\mathfrak{J} = (J_1, \ldots, J_n)$, we write $\mathfrak{I} \leq \mathfrak{J}$ if $I_a \leq_a J_a$ for all $a \in [n]$.

Example 2.7 Let $k = 1$ and $n = 3$. Given a Grassmann necklace $(I_1 = \{i_1\}, I_2 = \{i_2\}, I_3 = \{i_3\}) \in \mathcal{GN}_{1,3}$, we represent such an element by $i_1 i_2 i_3$. In this case, the Hasse diagram of the poset $(\mathcal{GN}_{1,3}, \leq)$ is the following:
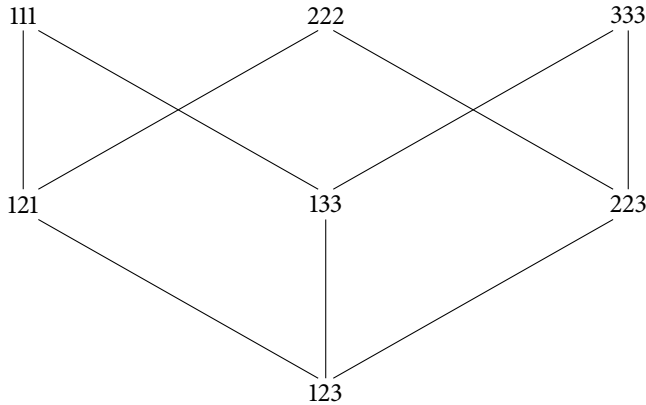
Given a positroid $\mathcal{M} \in \mathcal{P}(k, n)$, we define the corresponding Grassmann necklace $\mathfrak{I}(\mathcal{M})$ by the formula

$$(2.2) \qquad \mathfrak{I}(\mathcal{M})_a = \min{}_a \{J \in \mathcal{M}\},$$

where $\min_a$ is the minimum with respect to the order $\leq_a$.

Proposition 2.8 [Lam16, Theorem 7.12] *The map $\mathcal{M} \mapsto \mathfrak{I}(\mathcal{M})$ is an order reversing bijection between the set of (k, n) positroids ordered by containment and the set of (k, n) Grassmann necklaces.*

2.3 Bounded affine permutations

We recall here briefly the definition of bounded affine permutations and their relation with Grassmann necklaces. More details can be found in [KLS13, Section 3] and [Lam16, Section 6].

Recall that each stratum $\Pi(\mathcal{M})$ is a cell $\mathbb{R}^d_{>0}$. In order to give a formula for the dimension d of the cell, we use one more parametrization via the bounded affine permutations. A (k, n) affine permutation (not yet bounded) is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the following properties:

- $f(i + n) = f(i) + n$ for all $i \in \mathbb{Z}$,
- $\sum_{i=1}^{n} (f(i) - i) = kn$.

In particular, there is a distinguished (k, n) affine permutation id_k given by $\mathrm{id}_k(i) = i + k$. The length of an affine permutation is defined as

$$l(f) = |\{(i, j) \in [n] \times \mathbb{Z} : i < j \text{ and } f(i) > f(j)\}|.$$

We note that the set of $(0, n)$ affine permutations is a group isomorphic to the affine Weyl group W_n of type $A_{n-1}^{(1)}$. For general $k = 1, \ldots, n$, the group W_n acts freely and transitively on the set of (k, n) affine permutations; the action of the permutation $s_i = (i, i+1) \in W_n$, for $i = 0, \ldots, n-1$, permutes the values $f(i + rn)$ and $f(i + rn + 1)$ for all $r \in \mathbb{Z}$. This allows to identify the set of (k, n) affine permutations with W_n by sending $w \in W_n$ to $w.\mathrm{id}_k$. We thus obtain an order $\leq$ on the set of (k, n) affine permutations coming from the Bruhat order on W_n. For example, the unique minimal element is id_k.

A (k, n) bounded affine permutation is a (k, n) affine permutation subject to the extra condition

$$i \leq f(i) \leq i + n \text{ for all } i \in \mathbb{Z}.$$

We denote the set of (k, n) bounded affine permutations by $\mathcal{B}_{k,n}$.

It is shown in [KLS13] that $\mathcal{B}_{k,n}$ is a lower-order ideal in the set of (k, n) affine permutations (unbounded). For $f, g \in \mathcal{B}_{k,n}$, we write $f \leq g$ for the induced order. The Grassmann necklace $\mathcal{I}(f) = (I_1, \ldots, I_n)$ for $f \in \mathcal{B}_{k,n}$ is defined by the formula

$$I_a = \{f(b) : b < a \text{ and } f(b) \geq a\} \mod n.$$

Example 2.9 The Grassmann necklace corresponding to id_k is the one defined by

$$I_a = (a, a + 1, \ldots, a + k - 1) \qquad (a \in [n]).$$

By [Lam16, Theorem 6.2], this defines an order-preserving bijection between the set $\mathcal{B}_{k,n}$ and the set of (k, n) Grassmann necklaces.

In the opposite direction, given a Grassmann necklace $\mathcal{I}$, we define the bounded affine permutation $f = f(\mathcal{I})$ as follows: if $a \notin I_a$, then $f(a) = a$. If $a \in I_a$, then $I_{a+1} = I_a \backslash \{a\} \cup \{b\}$. We define $f(a) = c$, where $b \equiv c \mod n$ and $a < c \leq a + n$.

Remark 2.10 Let $\mathcal{M}(f)$ be the positroid defined by $f \in \mathcal{B}_{k,n}$, that is the one obtained from $\mathcal{I}(f)$ as explained in Section 2.2. Then (see, e.g., [Lam16]):

- $\dim \Pi_{\mathcal{M}(f)} = k(n - k) - l(f)$.
- The closure of the cell $\Pi_{\mathcal{M}(f)}$ contains $\Pi_{\mathcal{M}(g)}$ if and only if $f \geq g$.

Remark 2.11 There is a complex version of the cellular decomposition of the tnn Grassmannians. Namely, one can define the stratification of the complex Grassmann varieties into the positroid varieties [KLS13]. The latter ones are not isomorphic to affine cells in general, but they are irreducible complex projective algebraic varieties with many nice properties.

3 Quiver Grassmannians for cyclic quivers

In this short section, we recall some definitions and results concerning quiver Grassmannians and discuss the equioriented cycle case. Later, we will relate certain quiver Grassmannians for the cycle to tnn Grassmannians.

3.1 Quivers and representations

A finite quiver Q consists of a finite set of vertices Q_0 and a finite set of arrows Q_1. Each $\alpha \in Q_1$ has a unique source and target $i, j \in Q_0$, and we write $(\alpha : i \to j)$. A finite-dimensional Q representation M is a pair of tuples $(M^{(i)})_{i \in Q_0}$ and $(M_\alpha)_{\alpha \in Q_1}$, where each $M^{(i)}$ is a finite-dimensional $\mathbb{C}$-vector space and each M_α is a linear map from $M^{(i)}$ to $M^{(j)}$. The notion of a subrepresentation will be fundamental to us: a tuple of finite-dimensional $\mathbb{C}$-vector spaces $N = (N^{(i)})_{i \in Q_0}$ is a subrepresentation of $M = ((M^{(i)})_{i \in Q_0}, (M_\alpha)_{\alpha \in Q_1})$ if $N^{(i)} \subset M^{(i)}$ for all $i \in Q_0$ and $M_\alpha N^{(i)} \subset N^{(j)}$ for all $\alpha : i \to j \in Q_1$.

A morphism from the Q representation M to the Q representation N is a tuple of linear maps $\varphi = (\varphi_i)_{i \in Q_0} \in \prod_{i \in Q_0} \mathrm{Hom}_{\mathbb{C}}(M^{(i)}, N^{(i)})$ such that:

$$
\begin{array}{ccc}
M^{(i)} & \xrightarrow{\varphi_i} & N^{(i)} \\
M_\alpha \downarrow & \diagdown\diagdown & \downarrow N_\alpha \\
M^{(j)} & \xrightarrow{\varphi_j} & N^{(j)}
\end{array}
$$

By $\mathrm{Hom}_Q(M, N)$, we denote the set of all Q morphisms from M to N. The category of finite-dimensional Q representations over $\mathbb{C}$ is denoted by $\mathrm{rep}_{\mathbb{C}}(Q)$. The dimension vector of $M \in \mathrm{rep}_{\mathbb{C}}(Q)$ is

$$
\mathbf{d} := \left(\dim_{\mathbb{C}} M^{(i)}\right)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}.
$$

Definition 3.1 For $M \in \mathrm{rep}_{\mathbb{C}}(Q)$ and $\mathbf{e} \in \mathbb{Z}^{Q_0}$, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is the variety of all subrepresentations of M whose dimension vector equals $\mathbf{e}$.

3.2 The path algebra of a quiver

A path p in a quiver Q is a concatenation of consecutive arrows. We define the source of a path as the source of its first arrow, and its target is the target of the last arrow. The path algebra $\mathbb{C}Q$ has all paths in Q as basis, and the multiplication $*$ of two paths p and p' is defined via concatenation: if the target of p is the source of p', then $p' * p := p' \circ p$. Otherwise, the product is zero.

Using paths in Q, we can define a set of relations R on the objects of $\mathrm{rep}_\mathbb{C}(Q)$. Let $I \subset \mathbb{C}Q$ be the ideal generated by the relations in R. Then there is an equivalence of categories between $\mathrm{rep}_\mathbb{C}(Q, I)$, i.e., representations satisfying the relations in I (called bounded quiver representations) and $\mathrm{mod}_\mathbb{C}(\mathbb{C}Q/I)$, i.e., modules over the so-called bounded path algebra [Sch14, Theorem 5.4].

The injective bounded representation I_k at the vertex $k \in Q_0$ consists of the vector spaces $V^{(j)}$ for $j \in Q_0$ with a basis indexed by equivalence classes of paths (in $\mathbb{C}Q/I$) from j to k. The linear map V_α along the arrow $(\alpha : i \to j) \in Q_1$ sends a basis element of $V^{(i)}$ indexed by the equivalence class of a path p to a basis element of $V^{(j)}$ which is indexed by another equivalence class with representative p' such that $p = \alpha \circ p'$. A basis element of $V^{(i)}$ is sent to zero by the map along the arrow $(\alpha : i \to j)$ if the paths in the equivalence class indexing this basis element do not factor through α.

3.3 The equioriented Cycle

Let Δ_n denote the equioriented cycle on n vertices, where the orientation is chosen in such a way that $1 \to 2$ is an arrow. Then the set of vertices and the set of arrows are both in bijection with $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. Unless specified differently, we consider all indices of vertices and arrows modulo n. Given a representation $M \in \mathrm{rep}_\mathbb{C}(\Delta_n)$, we write M_{α_i} for $M_{i \to i+1}$ for any $i \in \mathbb{Z}_n$. We call $M \in \mathrm{rep}_\mathbb{C}(\Delta_n)$ nilpotent if all concatenations of the maps of M along cyclic paths vanish beyond a certain length of the paths.

Every point M of the affine variety

$$\mathrm{R}_\mathbf{n}(\Delta_n) := \bigoplus_{i \in \mathbb{Z}_n} \mathrm{Hom}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^n)$$

parametrizes a Δ_n representation of dimension vector $\mathbf{n} := (n, \ldots, n) \in \mathbb{Z}^n$. With a fixed ordered basis $(e_j)_{j \in [n]}$ of $\mathbb{C}^n$, each $g \in \mathbf{G}_\mathbf{n} := \prod_{i \in \mathbb{Z}_n} GL_n(\mathbb{C})$ acts on $\mathrm{R}_\mathbf{n}(\Delta_n)$ via conjugation

$$g.M := \left(g_{i+1} M_{\alpha_i} g_i^{-1} \right)_{i \in \mathbb{Z}_n}.$$

The automorphism group $\mathrm{Aut}_{\Delta_n}(M)$ of a Δ_n representation $M \in \mathrm{R}_\mathbf{n}(\Delta_n)$ is its stabilizer in $\mathbf{G}_\mathbf{n}$.

Let M be a Q representation. A basis of M is a basis B of the underlying vector space $\bigoplus_{i \in Q_0} M^{(i)}$. In the case of $Q = \Delta_n$, we will always pick bases B of M compatible with the $\mathbb{Z}_n$-grading on $\bigoplus_{i \in \mathbb{Z}_n} M^{(i)}$: $B = \bigcup_{i \in \mathbb{Z}_n} B^{(i)}$, where $B^{(i)} = \{w_1^{(i)}, \ldots, w_{d_i}^{(i)}\}$ is a basis for $M^{(i)}$.

Definition 3.2 Let $M \in \mathrm{rep}_\mathbb{C}(\Delta_n)$, and let B be a basis of M. The coefficient quiver $Q(M, B)$ consists of:

(QM0) the vertex set $Q(M, B)_0 = B$,

(QM1) the set of arrows $Q(M, B)_1$, containing $\left(\alpha : w_k^{(i)} \to w_\ell^{(i+1)}\right)$ if and only if the coefficient of $w_\ell^{(i+1)}$ in $M_{\alpha_i} w_k^{(i)}$ is non-zero.

Example 3.3 Let $U(i; n)$ be the Δ_n representation given by

$$U(i; n)^{(j)} = \mathbb{C}, \qquad U(i; n)_{\alpha_j} = \begin{cases} \mathrm{id}_\mathbb{C}, & \text{if } j \neq i, \\ 0, & \text{if } j = i. \end{cases}$$

If we denote by $w^{(j)}$ a generator of $U(i;n)^{(j)}$ (that is, any nonzero element), then the corresponding coefficient quiver is just an equioriented type A_n Dynkin quiver:

$$w^{(i+1)} \to w^{(i+2)} \to \cdots \to w^{(i-1)} \to w^{(i)}.$$

In [LP20], quiver Grassmannians for nilpotent Δ_n-representations are investigated. More precisely, they are equipped with torus actions, and these actions exploited to obtain cellular decompositions and study equivariant cohomology.

Given a fixed basis B of $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n)$, a grading of M is simply a map $\mathrm{wt} : B \to \mathbb{Z}^B$. This induces a $\mathbb{C}^*$ action on the vector spaces of M, defined on the basis B as follows and then extended by linearity:

$$(3.1) \qquad\qquad \lambda.b := \lambda^{\mathrm{wt}(b)} \cdot b \qquad (b \in B, \lambda \in \mathbb{C}^*).$$

With some additional assumptions about the grading (see [LP20, Section 5.1]), the $\mathbb{C}^*$ action extends to the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ with finitely many fixed points

$$\{L_1, \ldots, L_m\} =: \mathrm{Gr}_{\mathbf{e}}(M)^{\mathbb{C}^*}$$

indexed by appropriate subquivers of $Q(M, B)$ (see [CI11, Proposition 1]). Moreover, the $\mathbb{C}^*$ action induces an α-partition of $\mathrm{Gr}_{\mathbf{e}}(M)$ into the attracting sets of the fixed points

$$W_L := \left\{ U \in \mathrm{Gr}_{\mathbf{e}}(M) : \lim_{\lambda \to 0} \lambda.U = L \right\},$$

i.e., there exists a total order on the fixed point set such that $\bigsqcup_{i=1}^{s} W_{L_i}$ is closed in $\mathrm{Gr}_{\mathbf{e}}(M)$ for all $s \in [m]$. All the W_L's are isomorphic to affine spaces by [LP20, Theorem 5.6].

Remark 3.4 In [Pue20], quiver Grassmannians for nilpotent Δ_n representations are equipped with a special $\mathbb{C}^*$ action to prove that they admit a cellular decomposition. Furthermore, the geometry of their irreducible components is studied. The construction of the $\mathbb{C}^*$ action is generalized in [LP20] and embedded into a larger torus which allows to give a combinatorial description of the moment graph for quiver Grassmannians of nilpotent representations of the equioriented cycle. Our distinguished representation $U_{[n]}$ is clearly nilpotent, so that the whole machinery from [LP20, Pue20] is available. The needed constructions are recalled in the following sections for the reader's convenience. Moreover, the specific structure of $U_{[n]}$ allows us to strengthen some of the results from [LP20, Pue20] (see, e.g., Remark 4.13).

4 The main object

In this section, we identify $X(k, n)$ from the introduction with a certain quiver Grassmannian and apply methods from representation theory of quivers to investigate its geometric properties.

For convenience, we start by recalling the definition of $X(k, n)$. Let W be an n-dimensional $\mathbb{C}$-vector space, and let $(e_1, \ldots, e_n)$ be an ordered basis, then

$$X(k, n) = \left\{ (V_i) \in \prod_{i \in \mathbb{Z}_n} \mathrm{Gr}_k(W) \mid \mathrm{pr}_i V_i \subseteq V_{i+1} \right\},$$

where $\mathrm{Gr}_k(W)$ denotes the usual Grassmann variety of k-dimensional subspaces of W, and the projection morphisms are defined as $\mathrm{pr}_i(e_j) = e_j$ for any $j \neq i$ and $\mathrm{pr}_i(e_i) = 0$ for all $i \in \mathbb{Z}_n$.

Moreover, recall $U(i; n)$ from Example 3.3, i.e., the Δ_n representation which is one-dimensional over each vertex $i \in \mathbb{Z}_n$ and the map along the arrow $j \to j + 1$ is the identity for $j \neq i$ and zero for $i \to i + 1$.

The following result tells us that the variety $X(k, n)$ can be realized as a quiver Grassmannian for a very special Δ_n representation. Namely, let

$$U_{[n]} = \bigoplus_{i \in \mathbb{Z}_n} U(i; n).$$

Proposition 4.1   *Let $k, n \in \mathbb{N}$ with $k < n$, and let $\mathbf{k} = (k, \dots, k) \in \mathbb{Z}^n$. Then*

$$X(k, n) \cong \mathrm{Gr}_{\mathbf{k}}(U_{[n]}).$$

Proof   By definition, $U(i; n)$ is isomorphic to the representation V with $V^{(j)} = \mathbb{C}$ for all $j \in \mathbb{Z}_n$ and $V_{\alpha_j} = \mathrm{id}_{\mathbb{C}}$ for all $j \in \mathbb{Z}_n$ with $j \neq i$ and $V_{\alpha_i} = 0$. Hence, the vector spaces of $U_{[n]}$ over the vertices of Δ_n are all n-dimensional, and with a suitable order of the direct summands of $U_{[n]}$, we obtain

$$U_{[n]} \cong M := \left(\left(M^{(i)} = \mathbb{C}^n \right)_{i \in \mathbb{Z}_n}, \left(M_{\alpha_i} = \mathrm{pr}_i \right)_{i \in \mathbb{Z}_n} \right),$$

where pr_i sends the ith basis vector of $\mathbb{C}^n$ to zero and preserves the remaining. This implies that $\mathrm{Gr}_{\mathbf{k}}(U_{[n]}) \cong \mathrm{Gr}_{\mathbf{k}}(M)$ and the desired isomorphism follows from the definition of $X(k, n)$. ∎

Remark 4.2 $U_{[n]}$ is a representation for the bounded quiver Δ_n with the relation that all length n loops vanish. Let I be the ideal of the path algebra $\mathbb{C}\Delta_n$ generated by all paths of length n, then we can view $U_{[n]}$ as a module over the bounded path algebra $\mathbb{C}\Delta_n/I$ (cf. [Pue20, Section 2.2, Proposition 4.1]).

4.1 The automorphism group of $U_{[n]}$

The explicit realization of the group $\mathrm{Aut}_{\Delta_n}(U_{[n]})$ as a subgroup of $\mathbf{G_n}$ depends on the basis of $U_{[n]}$.

There are two special ordered bases which we use throughout this paper.

Definition 4.3

(i) The first ordered basis is compatible with the choice made in the proof of Proposition 4.1: for any $i \in \mathbb{Z}_n$, we set

$$B^{(i)} = \left\{ b_1^{(i)}, \dots, b_n^{(i)} \right\},$$

so that $(U_{[n]})_{\alpha_i}(b_j^{(i)}) = \begin{cases} b_j^{(i+1)}, & \text{if } j \neq i, \\ 0, & \text{if } j = i. \end{cases}$

We will borrow notation from Proposition 4.1 and write pr_i for $(U_{[n]})_{\alpha_i}$ with respect to the above basis. This basis will allow us to relate $X(k, n)$ to Grassmann necklaces.

(ii) By rearranging the previous basis vectors,[1] we get

$$B^{(i)} = \{v_1^{(i)}, \ldots, v_n^{(i)}\},$$

and with respect to this ordered basis, we have

$$(U_{[n]})_{\alpha_i}(v_j^{(i)}) = \begin{cases} v_{j+1}^{(i+1)}, & \text{if } j \neq n, \\ 0, & \text{if } j = n. \end{cases}$$

We denote this morphism by s_1. This basis will allow us to relate $X(k, n)$ to juggling patterns. From now on, we will work with this ordered basis most of the time.

Observe that the choice of a basis corresponds to a certain realization of $U_{[n]}$ as a point in $R_{\mathbf{n}}(\Delta_n)$. If it is clear from the context that we use our preferred ordered basis, we just refer to it as basis from now on.

Remark 4.4 The second realization of $U_{[n]}$ from the definition above leads to the following realization of $X(k, n)$ (juggling patterns style). Let W' be an n-dimensional $\mathbb{C}$-vector space, and let $(v_1, \ldots, v_n)$ be a basis of W', then

$$X(k, n) = \left\{ (V_i) \in \prod_{i \in \mathbb{Z}_n} \mathrm{Gr}_k(W') \mid s_1(V_i) \subseteq V_{i+1} \right\},$$

where $s_1(v_j) = v_{j+1}$ for any $j \neq n$ and $s_1(v_n) = 0$.

If $M \in R_{\mathbf{n}}(\Delta_n)$, then its endomorphism algebra $\mathrm{End}_{\Delta_n}(M)$ is defined as the set of matrix tuples $E = (E_i)_{i \in \mathbb{Z}_n} \prod_{i \in \mathbb{Z}_n} M_n(\mathbb{C})$ such that

$$E_{i+1} M_i = M_i E_i \quad \text{for all } i \in \mathbb{Z}_n.$$

Proposition 4.5 With respect to the basis $\bigcup_{i \in \mathbb{Z}_n} \{v_1^{(i)}, \ldots, v_n^{(i)}\}$, the elements of the endomorphism algebra $\mathrm{End}_{\Delta_n}(U_{[n]})$ are exactly the matrix tuples $E = (E_i)_{i \in \mathbb{Z}_n}$ with

$$E_i = \begin{pmatrix} e_{1,1}^{(i)} & & & & \\ e_{2,1}^{(i)} & e_{1,1}^{(i-1)} & & & \\ \vdots & \vdots & \ddots & & \\ e_{n-1,1}^{(i)} & e_{n-2,1}^{(i-1)} & \cdots & e_{1,1}^{(i-n+2)} & \\ e_{n,1}^{(i)} & e_{n-1,1}^{(i-1)} & \cdots & e_{2,1}^{(i-n-2)} & e_{1,1}^{(i-n+1)} \end{pmatrix},$$

where $e_{k,1}^{(i)} \in \mathbb{C}$ for all $i \in \mathbb{Z}_n$, $k \in [n]$. In particular, $\dim_{\mathbb{C}} \mathrm{End}_{\Delta_n}(U_{[n]}) = n^2$.

Proof By definition of $\mathrm{End}_{\Delta_n}(U_{[n]})$, we have that $(E_i)_{i \in \mathbb{Z}_n}$ if and only if

$$E_{i+1} s_1 = s_1 E_i \quad \text{for all } i \in \mathbb{Z}_n,$$

that is,

(4.1) $$E_{i+1} s_1(v_l^{(i)}) = s_1 E_i(v_l^{(i)}) \quad \text{for all } i, l \in \mathbb{Z}_n.$$

---

[1]More precisely, we reorder any set $\{b_1^{(i)}, \ldots, b_n^{(i)}\}$ decreasingly with respect to the shifted total order $\leq_i$ (2.1).

Let us we write $e_{k,l}^{(i)} := (E_i)_{k,l}$, so that $E_i(v_l^{(i)}) = \sum_{k=1}^{n} e_{k,l}^{(i)} v_k^{(i)}$. It is then easy to see that equations (4.1) are equivalent to

$$e_{k,l}^{(i)} = e_{k+1,l+1}^{(i+1)}, \quad e_{k,n}^{(i)} = 0, \quad e_{n,l}^{(i)} = 0, \qquad \text{for all } k,l \in [n-1].$$

From the previous equations, it follows by induction on $n-l$ that $e_{k,l}^{(i)} = 0$ for any $l > k$, and by induction on l that $e_{k,l}^{(i)} = e_{k+1,l+1}^{(i+1)}$. This implies that the E_i's are of the desired form. ∎

Remark 4.6 We obtain $\operatorname{Aut}_{\Delta_n}(U_{[n]}) \subset \operatorname{End}_{\Delta_n}(U_{[n]})$ by the condition $e_{1,1}^{(i)} \neq 0$ for all $i \in \mathbb{Z}_n$, because $E_{i+1}s_1 = s_1 E_i$ is equivalent to $s_1 = E_{i+1}s_1 E_i^{-1}$ if the matrices E_i are invertible.

Remark 4.7 We will see in Section 5 that the automorphism group of $X(k,n)$ is larger than the automorphism group of $U_{[n]}$.

4.2 Geometric properties of the main object

We prove here, by applying quiver representation theory results, geometric properties of $X(k,n)$.

Proposition 4.8 *Let $k, n \in \mathbb{N}$ with $k < n$, then:*

 (i) $X(k,n)$ *has $\binom{n}{k}$ irreducible components.*
 (ii) $X(k,n)$ *is equidimensional of dimension $k(n-k)$.*

Proof By [Pue20, Lemma 4.10], the irreducible components of $X(k,n)$ are parametrized by the set

$$\left\{ p = (p_i)_{i \in \mathbb{Z}_n} \in \prod_{i \in \mathbb{Z}_n} \{0,1\} \ : \ \sum_{i \in \mathbb{Z}_n} p_i = k \right\},$$

and they are all of dimension $k(n-k)$. The above set is in bijection with $\binom{[n]}{k}$, i.e., the set containing all k-element subsets of $[n] := \{1, \dots, n\}$.

Example 4.9 Let $k = 1$ and $n = 3$. Observe that, in this case,

$$X(1,3) \simeq X := \left\{ \left(\begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} \right) \in \prod_{i \in \mathbb{Z}_3} \mathbb{P}^2 \ \Big|\ \mathrm{pr}_i \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} \in \mathbb{C} \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ c_{i+1} \end{pmatrix}, \ i \in \mathbb{Z}_3 \right\}.$$

The three irreducible components are

$$X_i = \left\{ \left(\begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} \right) \in X \ \Big|\ \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = [b_i^{(i)}] \right\} \qquad (i \in \mathbb{Z}_3),$$

where $[b_i^{(i)}]$, according to Definition 4.3(1), denotes the class of the ith standard basis vector of $\mathbb{C}^3$. For example,

$$X_1 = \left\{ \left(\mathbb{P} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ 0 \\ c_3 \end{bmatrix} \right) \in \prod_{i \in \mathbb{Z}_3} \mathbb{P}^2 \mid a_2 c_3 - a_3 c_2 = 0 \right\}.$$

We see immediately that X_1 is a projective variety of dimension 2.

Theorem 4.10 *For $k, n \in \mathbb{N}$ with $k < n$, the following holds:*

(i) $X(k, n)$ *admits a cellular decomposition.*
(ii) *The cells are naturally labeled by the (k, n) Grassmann necklaces.*

Proof Let $B = \cup_{i \in \mathbb{Z}_n} B^{(i)}$ be the first basis of $U_{[n]}$ from Definition 4.3. We fix the weight function $\mathrm{wt}(b_j^{(i)}) := j$ for all $i \in \mathbb{Z}_n$ and all $j \in [n]$. Hence, by [CI11, Proposition 1], the $\mathbb{C}^*$ fixed points are parametrized by the elements of

$$\left\{ I = (I_j)_{j \in \mathbb{Z}_n} \in \prod_{i \in \mathbb{Z}_n} \binom{[n]}{k} \; : \; I_j \backslash \{j\} \subset I_{j+1} \text{ for all } j \in \mathbb{Z}_n \right\}.$$

This set coincides with the set $\mathcal{GN}_{k,n}$ of Grassmannian necklaces. For a Grassmann necklace $\mathcal{I} = (I_1, \ldots, I_n)$, the ath component $p(\mathcal{I})_a$ of the corresponding $\mathbb{C}^*$ fixed point $p(\mathcal{I})$ is a linear span of the basis vectors $b_i^{(a)}$, $i \in I_a$.

With this $\mathbb{C}^*$ action, not all attracting sets of the fixed points are isomorphic to affine spaces. For this reason, we switch to the second basis of $U_{[n]}$ from Definition 4.3, but keep the same weight function. Observe that now the weight difference along each arrow of the coefficient quiver is one, whereas it was zero for the first choice of a basis of $U_{[n]}$. Now, by [CI11, Proposition 1], the fixed points of the induced $\mathbb{C}^*$ action are exactly the juggling patterns as in Definition 2.5. Using the second basis of $U_{[n]}$ and the $\mathbb{C}^*$ action described above, the first part is a special case of [Pue20, Theorem 4.13]. ∎

Remark 4.11 The two different bases from Definition 4.3 lead to the parametrization of the cells via (k, n) Grassmann necklaces and juggling patterns, respectively. In what follows, for a Grassmann necklace $\mathcal{I}$, we denote the cell containing $p(\mathcal{I})$ by $C(\mathcal{I})$.

For a point $U \in \mathrm{Gr}_\mathbf{k}(M)$, the isomorphism class $\mathcal{S}_U$ in the quiver Grassmannian is called stratum and is irreducible by [CFR12, Lemma 2.4].

Remark 4.12 The closures of the top-dimensional strata are the irreducible components of $X(k, n) = \mathrm{Gr}_\mathbf{k}(U_{[n]})$, and by [Pue20, Lemma 4.10], these strata have the representatives

$$U_J = \bigoplus_{j \in J} U(j; n) \quad \text{for } J \in \binom{[n]}{k}.$$

Remark 4.13 In general, the cellular decomposition as in Theorem 4.10 is a refinement of the stratification based on [CFR12, Lemma 2.4]. It turns out that, for the representation $U_{[n]}$, both decompositions coincide, as discussed below. In particular,

we can assign to each irreducible component of $X(k, n)$ a specific Grassmann necklace in $\mathcal{GN}_{k,n}$.

Theorem 4.14 *For $k, n \in \mathbb{N}$ with $k < n$, the following holds:*

(i) *Two points of $X(k, n)$ belong to the same cell if and only if they are isomorphic as Δ_n modules.*

(ii) *Each cell contains exactly one $\mathbb{C}^*$ fixed point.*

(iii) *A cell equals the $\mathrm{Aut}_{\Delta_n}(U_{[n]})$ orbit of the $\mathbb{C}^*$ fixed point sitting in this cell.*

Proof By construction, each cell contains exactly one $\mathbb{C}^*$ fixed point. It follows from the parametrization of the $\mathbb{C}^*$ fixed points as in the proof of Theorem 4.10 that their corresponding coordinate subrepresentations of $U_{[n]}$ are pairwise nonisomorphic. This implies that the cells are the same as the strata of the fixed points, since in general each stratum in a quiver Grassmannian for a nilpotent representation of Δ_n decomposes into cells of isomorphic $\mathbb{C}^*$ fixed points. As $U_{[n]}$ is an injective bounded Δ_n representation (see Remark 4.2), we can prove analogous to [Re08, Lemma 6.3] that the $\mathrm{Aut}_{\Delta_n}(U_{[n]})$ orbits are exactly the strata (see [Pue19, Lemma 2.28]). ∎

5 Torus actions

Analogous to (3.1), we can construct actions of tori on the vector spaces of M, using multiple weight functions. Once again, these actions extend to the quiver Grassmannians of M only under special assumptions. In this section, we introduce several tori acting via weight tuples and explain whether their actions extend to $X(k, n)$.

Remark 5.1 The "obvious" torus of $\mathrm{Aut}_{\Delta_n}(U_{[n]})$ is only n-dimensional. Observe that this action extends to $X(k, n)$, but has infinitely many one-dimensional orbits in general, whereas we are interested in torus actions whose fixed point set and one-dimensional orbit set are finite.

Let M be a Δ_n representation, and let B be a basis of M. A weight tuple is a collection of integer valued vectors all of the same dimension: $\{\mathbf{wt}(b) = (w_1(b), \ldots, w_r(b))\}_{b \in B}$, where $\mathbf{wt}(b) \in \mathbb{Z}^r$ for some r. Given a weight tuple, we can define an action of a rank r torus $T \simeq (\mathbb{C}^*)^r$ on (the vector space) M by setting

$$(\gamma_1, \ldots, \gamma_r).b = \gamma_1^{w_1(b)} \cdot \ldots \cdot \gamma_d^{w_r(b)} b \qquad (b \in B, (\gamma_1, \ldots, \gamma_r) \in T).$$

All torus actions we deal with are obtained by weight tuples.

5.1 The largest torus acting faithfully

The largest torus acting faithfully on the vector spaces of $U_{[n]} = \bigoplus_{i \in \mathbb{Z}_n} U(i; n)$ is n^2-dimensional, since each vector space of $U_{[n]}$ over $i \in \mathbb{Z}_n$ is n-dimensional and we can act on each basis vector with a different parameter. However, the maximal torus whose action extends to $X(k, n)$ is much smaller:

Lemma 5.2 *The action of $T' := (\mathbb{C}^*)^r$ on $X(k, n)$ with $r \geq 2n$ factors through the faithful action of a rank $2n$ torus on $X(k, n)$.*

Proof Assume that we have a weight tuple such that the corresponding T action extends to each quiver Grassmannian associated with $U_{[n]}$. Then this weight tuple has a fixed weight difference along each arrow of Δ_n, i.e., if there are two arrows $b_1 \to b_1'$ and $b_2 \to b_2'$ in $Q(U_{[n]}, B)$ with the same underlying arrow of Δ_n, then

$$\mathbf{wt}(b_1') - \mathbf{wt}(b_1) = \mathbf{wt}(b_2') - \mathbf{wt}(b_2)$$

(see the proof of [LP20, Lemma 5.10]). The above property of the weight tuples is equivalent to the condition that the grading is constructible from the weights of the predecessor free points in $Q(U_{[n]}, B)$ and the weights of the edges of Δ_n. For $U_{[n]} = \bigoplus_{i \in \mathbb{Z}_n} U(i; n)$ and B being the standard basis for each copy of $\mathbb{C}^n$, the coefficient quiver $Q(U_{[n]}, B)$ has exactly n predecessor free points. This implies the claim since Δ_n has n arrows. Hence, we can choose at most $2n$ independent parameters. ∎

Corollary 5.3 *There exists a faithful $(\mathbb{C}^*)^{2n}$ action on $X(k, n)$.*

This corollary can also be obtained using the connection of $X(k, n)$ with the affine flag varieties, following Knutson–Lam–Speyer [Kn08].

5.2 Skeletal torus action

In [LP20], we introduced an action of $T = (\mathbb{C}^*)^{n+1}$ on quiver Grassmannians for nilpotent representations of the equioriented cycle. We recall the action in the case of our distinguished quiver Grassmannian $X(k, n)$. First of all, we enumerate the connected components (i.e., segments) of the coefficient quiver and, hence, denote by s_j the (unique) segment ending in j, that is corresponding to the indecomposable summand $U(j; n)$. We denote moreover by $b_{j,p}$ the basis vector of $U(j; n)$ corresponding to the pth vertex of s_j (e.g., the starting point of s_j is denoted by $b_{j,1}$, whereas the end vertex is $b_{j,n}$). We define a T action on the underlying vector space to M by setting

(5.1) $\quad (\gamma_0, \gamma_1, \ldots, \gamma_n).b_{j,p} = \gamma_0^p \gamma_j b_{j,p} \qquad (\gamma_0, \gamma_1, \ldots, \gamma_n) \in T, \quad j, p \in [n]$

and then extending by linearity. By [LP20, Lemma 5.10], this induces an action on $X(k, n)$. From now on, we deal with this torus action.

We recall here the definition of skeletality, which is necessary in the next section in order to consider moment graphs.

Definition 5.4 Let T be a torus acting on a complex projective algebraic variety X. The T-action on X is said to be skeletal if the number of T-fixed points and one-dimensional T-orbits in X is finite.

The following proposition summarizes some results of [LP20].

Proposition 5.5 *For $k, n \in \mathbb{N}$ with $k < n$, the following holds:*

(i) *The action of T on $X(k, n)$ is skeletal.*
(ii) *Each cell (from Theorem 4.14) contains exactly one T fixed point.*

Proof T fixed points are the same as $\mathbb{C}^*$ fixed points since $T := (\mathbb{C}^*)^{n+1}$ acts on $X(k, n)$ as in [LP20, Lemma 5.10]. The action is skeletal since the number of T fixed points is finite by [LP20, Theorem 5.12] and the number of one-dimensional T orbits is finite by [LP20, Proposition 6.3]. ∎

6 Moment graph

6.1 The description

If a complex algebraic variety X is acted upon by a torus T via a skeletal action, one can consider the corresponding moment graph. This is usually an unoriented graph, but if X admits a T-stable cellular decomposition (as in our case), one can give the edges an orientation. More precisely, we have the following definition.

Definition 6.1 Let T be an algebraic torus, and let X be a complex projective algebraic T-variety. Assume that X admits a T-stable cellular decomposition where every cell has exactly one fixed point. If the action of T on X is skeletal, then the corresponding moment graph is given by:

- the vertex set is the fixed point set: $\mathcal{V} = X^T$;
- there is an edge $x \to y$ if and only if x and y belong to the same one-dimensional T orbit closure $\overline{\mathcal{O}_{x \to y}}$ and y belongs to the closure of the cell containing x; and
- the label of the edge $x \to y$ is the character $\alpha \in \mathrm{Hom}(T, \mathbb{C}^*)$ the torus acts by on $\mathcal{O}_{x \to y}$.

The label of any edge is only well defined up to a sign, but since this does not play any role in the applications (e.g., computation of equivariant cohomology), we assume that the labels are fixed once and for all, and forget about this ambiguity.

We want to explicitly describe the moment graph corresponding to the torus action on $X(k, n)$ from Section 5.2.

First of all, we need to relate the second basis of Definition 4.3 to the basis $\{b_{j,p}\}_{j,p \in \mathbb{Z}_n}$ of $U_{[n]}$ that we used to define the T action (5.1). It is immediate to see (by induction on $n - p$) that

$$v_p^{(j+p)} = b_{j,p}, \quad \text{for all } j, p \in \mathbb{Z}_n.$$

For the rest of this section, we consider $B = \bigcup_{i \in \mathbb{Z}_n} \{v_1^{(i)}, \dots, v_n^{(i)}\}$. Thus, a successor closed subquiver Q' of dimension $(k, \dots, k)$ of $Q(U_{[n]}, B)$ is a full subquiver whose vertex set is $Q_0' = Q_0'^{(1)} \bigsqcup Q_0'^{(2)} \bigsqcup \dots \bigsqcup Q_0'^{(n)}$, with $Q_0'^{(i)} = \{v_{h_1}^{(i)}, \dots, v_{h_k}^{(i)}\} = Q_0' \cap B^{(i)}$ such that for any $i \in [n]$,

$$Q_0'^{(i+1)} = \begin{cases} \left\{ v_{h_1+1}^{(i+1)}, \dots, v_{h_k+1}^{(i+1)} \right\}, & \text{if } v_n^{(i)} \notin Q_0'^{(i)}, \\ \left\{ v_{h_j+1}^{(i+1)} \,\middle|\, v_{h_j}^{(i)} \in Q_0'^{(i)}, \ h_j \neq n \right\} \cup \left\{ v_h^{(i+1)} \right\} \text{ for some } h \in [n], & \text{otherwise.} \end{cases}$$

By Theorem 4.10, Proposition 5.5, and [CI11, Proposition 1], both Grassmann necklaces and successor closed subquivers of $Q(U_{[n]}, B)$ of dimension $(k, k, \dots, k)$ parametrize the T fixed point set. We write down here the explicit correspondence

$$(6.1) \qquad \psi : \mathrm{SC}_{\mathbf{k}}(U_{[n]}) \to \mathcal{GN}_{k,n} \qquad Q' \mapsto \left(\psi^{(a)}(Q_0'^{(a)}) \right)_{a \in [n]},$$

where $\psi^{(a)}(Q_0'^{(a)}) \in \binom{[n]}{k}$ is the image of $Q_0'^{(a)}$ under the following map:

$$\psi^{(a)} : B^{(a)} \to [n] \quad v_h^{(a)} \mapsto a - h.$$

It is easy to check that ψ is well defined and bijective.

Recall from Section 2.2 that, for any $a \in [n]$, there is a corresponding total order $\leq_a$ on $[n]$. The corresponding partial order on $\mathcal{GN}_{k,n}$ was denoted by $\leq$. For any $a, x, y \in [n]$, we define

$$[x, y]_a = \{z \in [n] \mid x \leq_a z \leq_a y\},$$

and $[x, y)_a = [x, y]_a \setminus \{y\}$.

The following notion allows us to provide a combinatorial description of the moment graph.

Definition 6.2 Let $M \in rep_{\mathbb{C}}(\Delta_n)$, $\mathbf{e} \in \mathbb{Z}_{\geq 0}^n$, and $B = \cup_{i \in \mathbb{Z}_n}\{w_1^{(i)}, \ldots, w_{m_i}^{(i)}\}$, and consider two successor closed subquivers Q', Q'' of $Q(M, B)$. We say that Q'' is obtained from Q' by a mutation if:

- There exists a segment $s_j \subseteq Q(M, B)$ such that

$$s_j \cap Q' = w_{p_1}^{(a)} \to w_{p_2}^{(a+1)} \to \cdots w_{p_l}^{(a+l-1)}$$

  for some $a, l \in \mathbb{Z}_n$ and $\mathbf{p} \in [n]^l$.
- There exist $l', r \in [n]$ with $l' \leq l$ and $p_i + r \leq n$ for any $i \in [l']$ such that

$$Q'' = \left(Q' \setminus \{w_{p_1}^{(a)} \to w_{p_2}^{(a+1)} \to \cdots w_{p_{l'}}^{(a+l'-1)}\}\right) \cup \left\{w_{p_1+r}^{(a)} \to w_{p_2+r}^{(a+1)} \to \cdots w_{p_{l'}+r}^{(a+l'-1)}\right\}.$$

We are now ready to provide a description of the moment graph of the T-action on $X(k, n)$ which only involves Grassmann necklace combinatorics.

Proposition 6.3 *The moment graph $\mathcal{G}$ of $X(k, n)$ has the following form:*

(i) $\mathcal{G}_0 = \mathcal{GN}_{k,n}$.

(ii) $\mathcal{I} \to \mathcal{I}'$ *if and only if $\mathcal{I}' < \mathcal{I}$ and there exist $a, b \in [n]$ such that:*
  - $I_h \neq I_h'$ *if and only if $h \in [a, b]_a$;*
  - *There exist $j, j' \in [n]$ such that $I_h = (I_h' \setminus \{j'\}) \cup \{j\}$ for any $h \in [a, b]_a$.*

(iii) *The label of $\mathcal{I} \to \mathcal{I}'$ as above is $\varepsilon_j - \varepsilon_{j'} - \#[j', j)_a \delta$, where $\delta(\gamma) = \gamma_0$ and $\varepsilon_i(\gamma) = \gamma_i$ for any $i \in [n]$, if $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in T$.*

Proof By [LP20, Theorem 6.13], $\mathcal{G}_0$ can be identified with $SC_{\mathbf{k}}(U_{[n]})$, which in turn we identify with $\mathcal{GN}_{k,n}$ via the bijection ψ. Again, by [LP20, Theorem 6.13], there is an edge $Q' \to Q''$ for $Q', Q'' \in SC_{\mathbf{k}}(M)$ if and only if Q'' is obtained from Q' by a mutation, that is, there exist $a, p, p', l \in [n]$ with $p < p'$ such that:

- $Q' \setminus (Q'' \cap Q') = v_p^{(a)} \to v_{p+1}^{(a+1)} \to \cdots \to v_{p+l-1}^{(a+l-1)}$.
- $Q'' \setminus (Q'' \cap Q') = v_{p'}^{(a)} \to v_{p'+1}^{(a+1)} \to \cdots \to v_{p'+l-1}^{(a+l-1)}$.

Since $v_p^{(a)} \in s_{a-p}$ and $v_p'^{(a)} \in s_{a-p'}$, then

$$Q' \setminus (Q'' \cap Q') \subseteq s_{a-p}, \quad Q'' \setminus (Q'' \cap Q') \subseteq s_{a-p'}.$$

If we set $j := a - p$ and $j' := a - p'$, the corresponding edge is labeled by $\varepsilon_j - \varepsilon_{j'} + (p - p')\delta$.

Now, we want to translate the conditions on Q', Q'' into conditions on the pair of Grassmann necklaces $\psi(Q') = \mathcal{I}, \psi(Q'') = \mathcal{I}'$. Observe that $j' <_a j$ as $p' > p$, and set $b := a + l - 1$. Hence, we have that the above conditions on Q and Q' translate into:

- $I_h = I'_h$ for all $h \notin \{a, a+1, \ldots, a+l-1\} = [a,b]_a$,
- $I_h \neq I'_h$ and $I_h = (I'_h \backslash \{j'\}) \cup \{j\}$ for all $h \in [a,b]_a$,

where the second condition follows from the fact that

$$\psi^{(a)}(v_p^{(a)}) = \psi^{(a+1)}(v_{p+1}^{(a+1)}) = \psi^{(a+l-1)}(v_{p+l-1}^{(a+l-1)}) = a - p = j$$

and

$$\psi^{(a)}(v_{p'}^{(a)}) = \psi^{(a+1)}(v_{p'+1}^{(a+1)}) = \psi^{(a+l-1)}(v_{p'+l-1}^{(a+l-1)}) = a - p' = j'.$$

Notice that $j' <_h j$ for any $h \in [a,b]_a$, and thus $I'_h \leq_h I_h$ for any $h \in [n]$, that is, $I' < I$. Finally, observe that $p - p' = -\#[j',j)_a$.

Remark 6.4 The moment graph described in the above proposition is related to the affine Bruhat graph. More precisely, our moment graph (with the labels omitted) is the subgraph of the Bruhat graph for the affine symmetric group (restricted to the lower-order ideal of bounded affine permutations). We are grateful to the referee for this remark.

Example 6.5 Let $n = 3$ and $k = 1$. We keep the same notation as in Example 2.7 and write $i_1 i_2 i_3$ for the Grassmann necklace $(I_1 = \{i_1\}, I_2 = \{i_2\}, I_3 = \{i_3\}) \in \mathcal{GN}_{1,3}$. Moreover, to shorten notation, we write $\alpha_{i,j} := \varepsilon_i - \varepsilon_j$ for $i, j \in [3]$. Then, by Proposition 6.3, the corresponding moment graph is
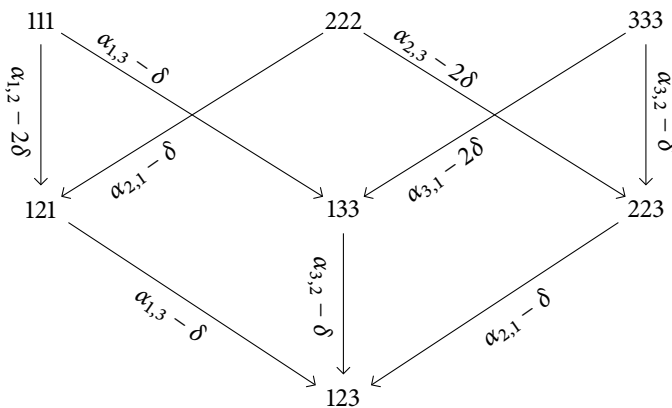
Notice that the underlying graph coincides with the Hasse diagram we found in Example 2.7. This if of course not always the case, as the Hasse diagram is in general missing several of the edges of the moment graph.

6.2 *T*-equivariant cohomology

By [LP20, Corollary 5.13], the moment graph described in the previous result encodes all needed information to compute the (*T*-equivariant) cohomology ring of $X(k, n)$. More precisely, let $R := \mathbb{Q}[\varepsilon_1, \ldots, \varepsilon_n, \delta]$ and consider it as a $\mathbb{Z}$-graded ring with $\deg(\varepsilon_i) = \deg(\delta) = 2$ ($i \in [n]$). Denote by $\alpha(\mathcal{I}, \mathcal{I}')$ the label of the edge $\mathcal{I} \to \mathcal{I}'$. Corollary 5.13 of [LP20] gives immediately the following result.

Corollary 6.6 *There is an isomorphism of ($\mathbb{Z}$-graded) rings:*

$$H_T^*(X(k, n), \mathbb{Q}) \simeq \left\{ (z_\mathcal{I}) \in \bigoplus_{\mathcal{I} \in \mathcal{GN}_{k,n}} R \;\middle|\; \begin{array}{c} z_\mathcal{I} \equiv z_{\mathcal{I}'} \mod \alpha(\mathcal{I}, \mathcal{I}') \\ \forall \text{ edge } \mathcal{I} \to \mathcal{I}' \end{array} \right\}.$$

Moreover, by [LP21, Theorem 3.21], $H_T^\bullet(X(k, n))$ admits a very nice basis as a free module over R, namely a so-called Knutson–Tao basis (cf. [LP21, Definition 3.2]).

Example 6.7 By Corollary 6.6, the *T*-equivariant cohomology of $X(1, 3)$ can be read off from the moment graph from Example 6.5:

$$H_T^*(X(1, 3)) \simeq \left\{ (z_{123}, z_{121}, z_{133}, z_{223}, z_{111}, z_{222}, z_{333}) \;\middle|\; \begin{array}{c} z_{i_1 i_2 i_3} \equiv z_{i_1' i_2' i_3'} \\ \mod \alpha(i_1 i_2 i_3, i_1' i_2' i_3') \\ \forall \text{ edge } i_1 i_2 i_3 \to i_1' i_2' i_3' \end{array} \right\},$$

where all $z_{i_1 i_2 i_3} \in R = \mathbb{Q}[\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta]$. In this case, the Knutson–Tao basis is

$$(1, 1, 1, 1, 1, 1, 1), \quad (0, 0, \alpha_{3,2} - \delta, 0, \alpha_{1,2} - 2\delta, 0, \alpha_{3,2} - \delta),$$

$$(0, \alpha_{1,3} - \delta, 0, 0, \alpha_{1,3} - \delta, \alpha_{2,3} - 2\delta, 0), \quad (0, 0, 0, \alpha_{2,1} - \delta, 0, \alpha_{2,1} - \delta, \alpha_{3,1} - 2\delta),$$

$$(0, 0, 0, 0, (\alpha_{1,2} - 2\delta)(\alpha_{1,3} - \delta), 0, 0), \quad (0, 0, 0, 0, 0, (\alpha_{2,1} - \delta)(\alpha_{2,3} - 2\delta), 0),$$

$$(0, 0, 0, 0, 0, 0, (\alpha_{2,1} - \delta)(\alpha_{3,2} - \delta)).$$

7 Poincaré polynomials

Recall from Sections 3 and 4: $U(i; n)$ denotes the indecomposable n-dimensional Δ_n module terminating at the vertex i. The quiver Grassmannian $X(k, n) = \mathrm{Gr}_\mathbf{k}(U_{[n]})$, where $U_{[n]} = \bigoplus_{i=1}^n U(i; n)$ admits a cellular decomposition. Each cell contains exactly one *T* fixed point $p(\mathcal{I})$, where $\mathcal{I} = (I_1, \ldots, I_n)$ is a (k, n) Grassmann necklace and the ath component $p(\mathcal{I})_a$ of $p(\mathcal{I})$ is a linear span of the basis vectors e_i, $i \in I_a$. The cell containing $p(\mathcal{I})$ is denoted by $C(\mathcal{I})$. We thus obtain the following formula for the Poincaré polynomial of the quiver Grassmannian $X(k, n)$:

$$P_{k,n}(q) = \sum_{\mathcal{I} \in \mathcal{GN}_{k,n}} q^{\dim_\mathbb{C} C(\mathcal{I})}.$$

Remark 7.1 For any Grassmann necklace $\mathcal{I}$, the dimension of the cell $C(\mathcal{I})$ equals the number of outgoing edges from the corresponding vertex in the moment graph of $X(k, n)$ (cf. [LP20, Corollary 6.4]). There is a bijection between these edges and so-called fundamental mutations of the *T* fixed point corresponding to the Grassmann necklace $\mathcal{I}$ [LP20, Theorem 6.13]. These combinatorial moves are essential for the proof of Lemma 7.4.

Example 7.2 Let $n = 3$ and $k = 1$. By the previous remark, the Poincaré polynomial can be read off from the moment graph. Thus, from Example 6.5, we deduce that

$$P_{1,3}(q) = 1 + 3q + 3q^2.$$

We notice that $P_{1,3}(q)$ is dual to the Poincaré polynomial of the $(1, 3)$ tnn Grassmannian, which is $3 + 3q + q^2 = q^2 P_{1,3}(q^{-1})$. This is a general fact (see Theorem 7.7).

Let $\preceq$ be the partial order on $\mathcal{GN}_{k,n}$ coming from the cell closure relation in $X(k, n)$, and recall the partial order $\leq$ on $\mathcal{GN}_{k,n}$ introduced in Section 2.2.

Proposition 7.3 *The partial orders $\preceq$ and $\leq$ on $\mathcal{GN}_{k,n}$ coincide.*

Proof We show that the map ψ from (6.1) is a poset isomorphism, where we also use the notation $\preceq$ to denote the cell closure relation order on $\mathrm{SC}_{\mathbf{k}}(U_{[n]})$.

Since $\preceq$ is in our case generated by the mutations, it follows immediately from the proof of Proposition 6.3 that $Q'' \preceq Q'$ implies $\psi(Q'') \leq \psi(Q')$.

Assume now that $\mathcal{I}, \mathcal{I}' \in \mathcal{GN}_{k,n}$ are such that $\mathcal{I}' \leq \mathcal{I}$, and denote $Q' := \psi^{-1}(\mathcal{I})$, $Q'' := \psi^{-1}(\mathcal{I}')$. For any $a \in [n]$, let

$$\mathbf{h}^{(a)} := (h_1^{(a)} < h_2^{(a)} < \cdots < h_k^{(a)}), \quad \mathbf{l}^{(a)} := (l_1^{(a)} < l_2^{(a)} < \cdots < l_k^{(a)})$$

be such that

$$Q_0'' \cap B^{(a)} = \left\{ v_{h_j^{(a)}}^{(a)} \mid j \in [k] \right\}, \quad Q_0' \cap B^{(a)} = \left\{ v_{l_j^{(a)}}^{(a)} \mid j \in [k] \right\}.$$

By definition of $\leq$ and ψ, we have that

$$h_j^{(a)} \geq l_j^{(a)} \quad \text{for all } j \in [k], \ a \in [n].$$

Thus, we want to show that there exists a sequence of mutations from Q' to Q''. We proceed by induction on $d = n - \#\{a \mid \mathbf{h}^{(a)} = \mathbf{l}^{(a)}\}$.

The base case is $\mathbf{h}^{(a)} = \mathbf{l}^{(a)}$ for all $a \in [n]$, that is, $Q' = Q''$, and there is nothing to be shown. Otherwise, there exists an $\bar{a} \in [n]$ such that $\mathbf{h}^{(\bar{a})} \neq \mathbf{l}^{(\bar{a})}$. We hence find an $r \in [k]$ such that $h_j^{(\bar{a})} = l_j^{(\bar{a})}$ for all $j \in [r + 1, k]$ and $h_r^{(\bar{a})} > l_r^{(\bar{a})}$. To save notation, we write h and l for $h_r^{(\bar{a})}$ and $l_r^{(\bar{a})}$, respectively. Define

$$\bar{c} := \min_{\bar{a}-l+1} \left\{ c \mid v_{l+c-\bar{a}}^{(c)} \in Q_0' \cap B^{(c)} \right\}, \quad \bar{e} := \min_{\bar{a}-l+1} \left\{ e \mid v_{h+e-\bar{a}}^{(e)} \in Q_0' \cap B^{(e)} \right\}.$$

Observe that $v_h^{(\bar{a})} = b_{\bar{a}-h,h}$, so that $v_{h+\bar{c}+t-\bar{a}-1}^{(\bar{c}+t-1)} \notin Q_0' \cap B^{\bar{c}+t-1}$ for any $t \in [\bar{e} - \bar{c}]_{\bar{c}}$. Thus, the following successor-closed subquiver is obtained from Q' by mutation:

$$Q''' := \left(Q' \setminus \left\{ v_{l+\bar{c}-\bar{a}}^{(\bar{c})} \to \cdots \to v_l^{(\bar{a})} \to \cdots \to v_{l+\bar{e}-\bar{a}-1}^{(\bar{e}-1)} \right\} \right)$$
$$\cup \left\{ v_{h+\bar{c}-\bar{a}}^{(\bar{c})} \to \cdots \to v_h^{(\bar{a})} \to \cdots \to v_{h+\bar{e}-\bar{a}-1}^{(\bar{e}-1)} \right\}.$$

Note that, thanks to the choice of $\bar{c}$, we have $Q'' \preceq Q''' \prec Q'$. Let

$$\mathbf{l}'^{(a)} = l_1'^{(a)} < l_2'^{(a)} < \cdots < l_k'^{(a)} \text{ be such that } Q_0''' \cap B^{(a)} = \left\{ v_{l_j'^{(a)}}^{(a)} \mid j \in [k] \right\},$$

then $h_j^{(\overline{a})} = l_j'^{(\overline{a})}$ for any $j \in [r, k]$, and $\mathbf{h}^{(a)} = \mathbf{l}'^{(a)}$ whenever $\mathbf{h}^{(a)} = \mathbf{l}^{(a)}$. If $r = 1$, we are done, otherwise we proceed recursively until we get equality for any $j \in [k]$, and we denote the resulting successor-closed subquiver by $\overline{Q}$. By construction, now,

$$n - \#\{a \mid Q_0'' \cap B^{(a)} = \overline{Q}_0 \cap B^{(a)}\} < d,$$

and we can apply the inductive step to complete the proof. ∎

Lemma 7.4 *Assume that, for two Grassmann necklaces $\mathfrak{I}$ and $\mathfrak{J}$, the closure of the cell $C(\mathfrak{I})$ contains the cell $C(\mathfrak{J})$ and there is no cell C such that*

$$\overline{C(\mathfrak{I})} \supset C, \qquad \overline{C} \supset C(\mathfrak{J}).$$

Then $\dim_{\mathbb{C}} C(\mathfrak{I}) = \dim_{\mathbb{C}} C(\mathfrak{J}) + 1$.

Proof By Propositions 5.5 and 6.3, every cell $C(\mathfrak{I})$ of $X(k, n)$ contains a unique T fixed point $p(\mathfrak{I})$ together with a unique subquiver $S(\mathfrak{I}) \subset Q(U_{[n]}, B)$ consisting of segments $s_1, \ldots, s_n$. Now, the condition that there is no cell between $C(\mathfrak{J})$ and $C(\mathfrak{I})$ implies that there is a corresponding fundamental mutation $\mu : p(\mathfrak{I}) \to p(\mathfrak{J})$ and it does not factor through any other mutation. On the coefficient quivers $S(\mathfrak{I})$ and $S(\mathfrak{J})$, we can view this mutation as cutting exactly one predecessor-closed subquiver of a segment s_{j_1} of $S(\mathfrak{I})$ and adding it to the start of a segment s_{j_2} of $S(\mathfrak{I})$ to obtain the segments s_{j_1}' and s_{j_2}' of $S(\mathfrak{J})$ (see [LP21, Definition 2.15] for more details).

By convention, the index of the element of B corresponding to the starting point of s_{j_1} is larger than the index of the basis element corresponding to the start of s_{j_2}', i.e., mutations are always index increasing and hence we also speak of a downward movement of the subsegment. All the other segments remain unchanged and hence coincide for both points.

The height $h_{\mathfrak{I}}(s_j)$ is defined as the number of the mutations of $p(\mathfrak{I})$ which start at the segment s_j. For $U_{[n]}$, this equals the number of points in $Q(U[n], B)$ which have larger index than the start of s_j and are not contained in $S(\mathfrak{I})$. We sometimes refer to this as counting the holes below the start of s_j (cf. [CFFFR17, Remark 6]).

The starting points of the segments between s_{j_1} and s_{j_2} cannot live over the support of the moved subsegment in Δ_n because otherwise μ would factor through them and this implies that they contribute as holes for the height functions $h_{\mathfrak{I}}(s_{j_1})$ and $h_{\mathfrak{J}}(s_{j_1}')$. Hence, we obtain $h_{\mathfrak{J}}(s_{j_1}') = h_{\mathfrak{I}}(s_{j_1}) - 1$.

Every mutation starting at s_{j_2} can also be started at s_{j_2}', so we immediately obtain $h_{\mathfrak{J}}(s_{j_2}') \geq h_{\mathfrak{I}}(s_{j_2})$. For the other segments, we distinguish two cases: if the segment S_j is between s_{j_1} and s_{j_2}, by the discussion above, its starting point cannot be above the support of the moved segment. Hence, both s_{j_1} and s_{j_2} have marked points in $Q(U[n], B)$ below the starting point of s_j such that they both contribute as zero to the height functions $h_{\mathfrak{I}}(s_j)$ and $h_{\mathfrak{J}}(s_j)$.

In the other case, the start of the segment s_j is either below the points of s_{j_2}' and the height function does not change at all, or above s_{j_1} and in this case the value of the height function stays the same since the role of the points on s_{j_1} is exchanged with the points on s_{j_2}' and vice versa for s_{j_2} and s_{j_1}'. By [LP21, Proposition 3.19], we obtain $\dim_{\mathbb{C}} C(\mathfrak{J}) \leq \dim_{\mathbb{C}} C(\mathfrak{I}) - 1$ and the above computation shows that equality is achieved in our setting. ∎

Theorem 7.5 *The dimension of the cell $C(\mathcal{I})$ is equal to the length of the bounded affine permutation $f(\mathcal{I})$.*

Proof By Proposition 7.3, we know that the poset structure of $\mathcal{GN}_{k,n}$ coming from the closure relations in $X(k, n)$ is isomorphic to the poset $\mathcal{B}_{k,n}$ (see [KLS14, Lam16]). We note that both posets contain a unique minimal element—a zero-dimensional cell corresponding to the collection $(I_j)_j$ with $I_j = \{j, j + 1, \ldots, j + k - 1\}$ (all the numbers are taken mod n) and the length zero element id_k. Since the set $\mathcal{B}_{k,n}$ of bounded affine permutations can be identified with the lower-order ideal in the affine Weyl group, we conclude that if $f > g$ with no element $h \in \mathcal{B}_{k,n}$ in between, one has $l(f) = l(g) + 1$. Now, Lemma 7.4 completes the proof. ∎

Example 7.6 If $\mathcal{I} = \{\{1, 3\}, \{3, 4\}, \{3, 4\}, \{1, 4\}\} \in \mathcal{GN}_{2,4}$, we then have $\dim_{\mathbb{C}}(C(\mathcal{I})) = 2$.

The next result follows immediately from Theorem 7.5 and Remark 2.10.

Theorem 7.7 *The Poincaré polynomial of $X(k, n)$ is dual to the Poincaré polynomial of the tnn Grassmannian.*

For a natural number m, we denote by $[m]$ its q^{-1}-analogue, that is, $[m] := 1 + q^{-1} + \cdots + q^{-(m-1)}$. By combining the previous theorem with [W05, Theorem 4.1], we immediately obtain an explicit formula.

Corollary 7.8 *We have the following formula for the Poincaré polynomial of $X_{k,n}$:*

$$P_{k,n}(q) = q^{k(n-k)} \sum_{i=0}^{k-1} \binom{n}{i} q^{(k-i)^2} \left([i - k]^i [k - i + 1]^{n-i} - [i - k + 1]^i [k - i]^{n-i} \right).$$

Remark 7.9 In contrast to [LP21, Remark 5.14], Theorem 4.14 implies that the closure of every cell in $X(k, n)$ is the union of smaller cells. Moreover, by Proposition 6.3, we obtain it explicitly as

$$\overline{C(\mathcal{I})} = \bigcup_{\mathcal{J} \in \mathcal{GN}_{k,n} \text{ s.t.: } \mathcal{J} \leq \mathcal{I}} C(\mathcal{J}).$$

The moment graph of $\overline{C(\mathcal{I})}$ is the full subgraph of $\mathcal{G}$ from Proposition 6.3 on the vertices $\mathcal{J} \in \mathcal{GN}_{k,n}$ with $\mathcal{J} \leq \mathcal{I}$, and the computation of the corresponding T-equivariant cohomology and the Poincaré polynomial is done in the same way.

8 Resolution of singularities

The goal of this section is to present a construction of a resolution of singularities for the irreducible components of $X(k, n)$ (the simplest example of a singular component is $X_{\{1,2\}}(2, 4)$; see Remark 8.13). In short, an irreducible component of $X(k, n)$ will be desingularized by a quiver Grassmannian for the extended cyclic quiver $\tilde{\Delta}_n$ (see the definition below). For various constructions of desingularizations in the context of quiver Grassmannians, see [CFR13, CFR14, FF13, FFL14, KS14, S17]. In particular, our construction is inspired by the equioriented type A construction from [CFR13, FF13].
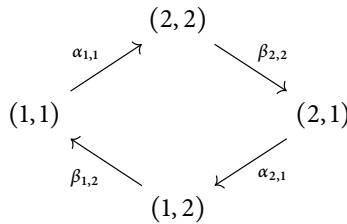
8.1 The extended cyclic quiver and extended representations

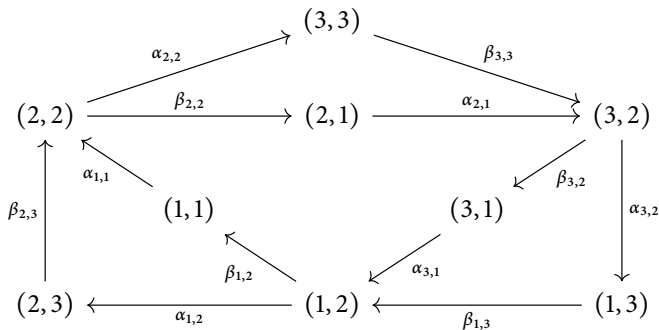Let us denote by $\widetilde{\Delta}_n$ the extended cyclic quiver defined by the following data:

- The vertices of $\widetilde{\Delta}_n$ are of the form (i, r) with $i \in \mathbb{Z}_n$, $r = 1, \ldots, n$.
- There are two types of arrows: $\alpha_{i,r} : (i, r) \to (i + 1, r + 1)$, $r < n$ and $\beta_{i,r} : (i, r) \to (i, r - 1)$, $r > 1$.

Example 8.1

(i) The extended cyclic quiver for $n = 2$ looks as follows:

$$
\begin{array}{ccc}
 & (2,2) & \\
\alpha_{1,1} \nearrow & & \searrow \beta_{2,2} \\
(1,1) & & (2,1) \\
\beta_{1,2} \nwarrow & & \swarrow \alpha_{2,1} \\
 & (1,2) &
\end{array}
$$

(ii) The extended cyclic quiver for $n = 3$ looks as follows:

$$
\begin{array}{ccccc}
 & & (3,3) & & \\
 & \overset{\alpha_{2,2}}{\nearrow} & & \overset{\beta_{3,3}}{\searrow} & \\
(2,2) & \xrightarrow{\beta_{2,2}} & (2,1) & \xrightarrow{\alpha_{2,1}} & (3,2) \\
\end{array}
$$

with arrows $\beta_{2,3}$, $\alpha_{1,1}$, $(1,1)$, $(3,1)$, $\beta_{3,2}$, $\alpha_{3,2}$, and

$$
(2,3) \xleftarrow{\alpha_{1,2}} (1,2) \xleftarrow{\beta_{1,3}} (1,3)
$$

with arrows $\beta_{1,2}$, $\alpha_{3,1}$.

Now, let $M = ((M^{(i)})_{i \in \mathbb{Z}_n}, (M_{\alpha_i})_{i \in \mathbb{Z}_n})$ be a Δ_n representation. We define the extended $\widetilde{\Delta}_n$ module $\widetilde{M}$ as follows. Let $M^{(i,1)} = M^{(i)}$ for $i \in \mathbb{Z}_n$, and let

$$
\widetilde{M}^{(i,r)} = M_{\alpha_{i-1}} \ldots M_{\alpha_{i-r+1}} M^{(i-r+1)} \qquad (r = 2, \ldots, n).
$$

In particular, $\widetilde{M}^{(i,r)} \subset M^{(i)}$ for all r. We also have natural surjections and inclusions

(8.1) $$\widetilde{M}^{(i,n)} \twoheadleftarrow \widetilde{M}^{(i-1,n-1)} \twoheadleftarrow \cdots \twoheadleftarrow \widetilde{M}^{(i-n+1,1)} = M^{(i-n+1)},$$

(8.2) $$\widetilde{M}^{(i,n)} \subset \widetilde{M}^{(i,n-1)} \subset \cdots \subset \widetilde{M}^{(i,1)} = M^{(i)}.$$

We complete the definition of the extended representation $\widetilde{M}$ by letting

$$
\widetilde{M}_{\alpha_{i,r}} : \widetilde{M}^{(i,r)} \to \widetilde{M}^{(i+1,r+1)}, \quad \widetilde{M}_{\beta_{i,r}} : \widetilde{M}^{(i,r)} \to \widetilde{M}^{(i,r-1)}
$$

to be the maps from (8.1) and (8.2), respectively.

Remark 8.2 Let us identify $\widetilde{M}^{(i,1)}$ with $M^{(i)}$. Then, the maps M_{α_i} are recovered as compositions $\widetilde{M}_{\beta_{i+1,2}} \widetilde{M}_{\alpha_{i,1}}$.

Example 8.3 Let $n = 2$, and let $M = U(1; 2) \oplus U(2; 2) \in \text{rep}_{\mathbb{C}}(\Delta_2)$. Then the $\widetilde{\Delta}_2$ module $\widetilde{M}$ looks as follows:

$$
\begin{array}{ccccc}
 & & \mathbb{C} & & \\
 & (10) \nearrow & & \searrow \binom{1}{0} & \\
\mathbb{C}^2 & & & & \mathbb{C}^2 \\
 & \binom{0}{1} \searrow & & \nearrow (01) & \\
 & & \mathbb{C} & &
\end{array}
\quad .
$$
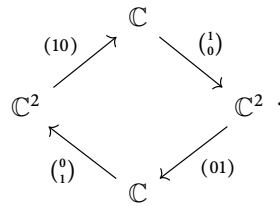
Example 8.4 Let $M = U_{[n]} = \bigoplus_{i=1}^{n} U(i; n)$, so that $\dim_{\mathbb{C}} \widetilde{M}^{(i,r)} = n - r + 1$. To describe the maps between the vector spaces $\widetilde{M}^{(i,r)}$, we use the second basis from Definition 4.3 Then $\widetilde{M}^{(i,r)}$ is identified with the linear span of vectors $v_h^{(i)}$ with $h = r, \ldots, n$, and the maps $\widetilde{M}_{\alpha_{i,r}}$ and $\widetilde{M}_{\beta_{i,r}}$ are given by

$$
\widetilde{M}_{\beta_{i,r}} v_h^{(i)} = v_h^{(i)}, \ \widetilde{M}_{\alpha_{i,r}} v_h^{(i)} = v_{h+1}^{(i+1)}, \text{ if } h < n, \ \widetilde{M}_{\alpha_{i,r}} v_n^{(i)} = 0.
$$

Lemma 8.5 *For a Δ_n module M, the extended representation $\widetilde{M}$ is a representation of the bounded quiver $\widetilde{\Delta}_n$ with the relations $\beta_{i+1,r+1}\alpha_{i,r} = \alpha_{i,r-1}\beta_{i,r}$ for all $i \in \mathbb{Z}_n$ and $r = 2, \ldots, n-1$.*

Proof Obvious from the definition of $\widetilde{M}$. ∎

We say that an element of the path algebra of $\widetilde{\Delta}_n$ is an α path, respectively, β path, if it is given by the concatenation of maps exclusively of type $\widetilde{M}_{\alpha_\bullet}$, respectively, $\widetilde{M}_{\beta_\bullet}$.

Lemma 8.6 *Assume that the length n paths of the path algebra of Δ_n act trivially on a Δ_n module M. Then all the length $2n - 1$ paths from the path algebra of $\widetilde{\Delta}_n$ act trivially on $\widetilde{M}$.*

Proof Let $i, r \in \mathbb{Z}_n$, and let us take $v \in \widetilde{M}^{(i,r)}$. We attach to v a pair (r', r''), where r' is the maximal length of a β path on $\widetilde{\Delta}_n$ starting in (i, r) and where r'' is the maximal length of an α path on $\widetilde{\Delta}_n$ ending in (i, r). It follows immediately from the definition of $\widetilde{M}^{(i,r)}$ that $r' = r - 1$.

Notice that:

- application of a map $\widetilde{M}_{\beta_\bullet}$ to v changes the coordinates (r', r'') to $(r' - 1, r'')$,
- application of a map $\widetilde{M}_{\alpha_\bullet}$ to v changes the coordinates (r', r'') to $(r' + 1, r'' + 1)$, and
- there are no vectors with coordinates (r', r'') with $r' < 0$ or $r'' \geq n$.

For example, $r'' < n$, because any length n path acts trivially on M. Our lemma is equivalent to the statement that one cannot do more than $2n - 1$ steps of the form $(r', r'') \to (r' - 1, r'')$ or (r', r'') to $(r' + 1, r'' + 1)$ starting at $(r - 1, r - 1)$ and staying inside the region $r' \geq 0, r'' < n$.

8.2 Construction of the desingularization

Lemma 8.7 *Let $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n)$, and let $U \subset M$ be an $\mathbf{e}$-dimensional subrepresentation. Then there exists a natural map*

$$(8.3) \qquad\qquad \Psi : \mathrm{Gr}_{\mathbf{dim}\,\widetilde{U}}(\widetilde{M}) \to \mathrm{Gr}_{\mathbf{e}}(M),$$

defined by the following rule: for $N \in \mathrm{Gr}_{\mathbf{dim}\,\widetilde{U}}(\widetilde{M})$, its image $\Psi(N) \in \overline{S}_U$ is given by

$$\Psi(N)^{(i)} = N^{(i,1)}, \; \Psi(N)_{\alpha_i} = N_{\beta_{i+1,2}} N_{\alpha_{i,1}}.$$

Proof By construction, $\Psi(N)$ is an $\mathbf{e}$-dimensional submodule of M (see Remark 8.2). ∎

Recall that the irreducible components of $X(k,n)$ are labeled by the k-element subsets $J \subset \{1, \ldots, n\}$ (see Remark 4.12). We denote the corresponding irreducible component by $X_J(k,n)$. It contains a point p_J which is a summandwise embedding of the representation

$$U_J = \bigoplus_{j \in J} U(j;n)$$

into $U_{[n]} = \bigoplus_{i=1}^{n} U(i;n)$ (special case of [Pue20, Lemma 4.10]).

The component $X_J(k,n)$ is the closure of the $\mathrm{Aut}_{\Delta_n}(U_{[n]})$ orbit $C(p_J)$ of the point p_J. Each point of the cell $C(p_J)$ is isomorphic to U_J as a Δ_n module, and points in different cells are not isomorphic. Moreover, each cell contains exactly one $\mathbb{C}^*$ fixed point (see Theorem 4.14).

Now, let us consider the $\widetilde{\Delta}_n$ module $\widetilde{U}_J$, and let $\mathbf{d}_J = \mathbf{dim}\,\widetilde{U}_J$, so that $\mathbf{d}_J$ is a vector with components $d_J^{(i,r)}$.

Lemma 8.8 *The numbers $d_J^{(i,r)}$ are explicitly given by*

$$d_J^{(i,r)} = \#\left(J \cap \{ i' : \, i \le i' \le i + n - r \} \right).$$

In particular, $d_J^{(i,r)} \ge d_J^{(i,r+1)}$ for any $i \in \mathbb{Z}_n$ and $r \in [n-1]$.

Proof By definition,

$$d_J^{(i,r)} = \dim_{\mathbb{C}} U_{J,\alpha_{i-1}} \ldots U_{J,\alpha_{i-r+1}} U_J^{(i-r+1)}.$$

One easily sees that

$$U_{J,\alpha_{i-1}} \ldots U_{J,\alpha_{i-r+1}} U(i';n)^{(i-r+1)}$$

is nontrivial (i.e., one-dimensional) if and only if $i \le i' \le i + n - r$. ∎

Example 8.9 One has $d_J^{(i,1)} = k$ for all i. The dimensions $d_J^{(i,n)}$ are either zero or one.

Theorem 8.10 *The quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$ is smooth.*

Proof We show that $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$ is isomorphic to a tower of fibrations

$$\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]}) = Y_1 \to Y_2 \to \cdots \to Y_n = \mathrm{pt}$$

such that each map $Y_r \to Y_{r+1}$, $r = 1, \ldots, n-1$ is a fibration with the fiber being a product of classical Grassmann varieties.

A point N of $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$ is a collection of subspaces $N^{(i,r)} \subset \widetilde{U}_{[n]}^{(i,r)}$. We define Y_r as the image of the natural projection map

$$\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]}) \to \prod_{r'=r}^{n} \prod_{i \in \mathbb{Z}_n} \mathrm{Gr}_{d_J^{(i,r')}}(\widetilde{U}_{[n]}^{(i,r')}), \quad N \mapsto (N^{(i,r')})_{\substack{i \in \mathbb{Z}_n \\ r \le r' \le n}}.$$

We start with $r = n$ and then proceed by decreasing induction on r. One has $\dim_{\mathbb{C}} \widetilde{U}_{[n]}^{(i,n)} = 1$, and $d_J^{(i,n)}$ are either 0 or 1. Thus, Y_n is a product of n points (corresponding to $i \in \mathbb{Z}_n$). We also note that

$$U_{\alpha_{i,n-1}} U_{\beta_{i,n}}(N^{(i,n)}) \subset N^{(i+1,n)},$$

because the composition $U_{\alpha_{i,n-1}} U_{\beta_{i,n}}$ vanishes on $\widetilde{U}_{[n]}$. Here and below, we use the notation $U_\gamma = \widetilde{U}_{[n],\gamma}$ for the map corresponding to the edge γ of $\widetilde{\Delta}_n$.

Now, assume that all the subspaces $N^{(i,r')}$ are fixed for $r' > r$. Since N is a subrepresentation, the subspace $N^{(i,r)} \subset \widetilde{U}_{[n]}^{(i,r)}$ has to satisfy the conditions

$$U_{\beta_{i,r+1}} N^{(i,r+1)} \subset N^{(i,r)}, \quad U_{\alpha_{i,r}} N^{(i,r)} \subset N^{(i+1,r+1)}$$

and hence

(8.4) $$U_{\alpha_{i,r}} U_{\beta_{i,r+1}}(N^{(i,r+1)}) \subset N^{(i+1,r+1)}.$$

Since $U_{\beta_{i,r+1}}$ is an embedding, $U_{\alpha_{i,r}}$ is a surjection, and condition (8.4) holds, the choice of $N^{(i,r)}$ as above is equivalent to the choice of a point in the Grassmannian

$$\mathrm{Gr}_{d_J^{(i,r)} - d_J^{(i,r+1)}}\left(\widetilde{U}_{[n]}^{(i,r)} / U_{\beta_{i,r+1}}(N^{(i,r+1)})\right).$$

Proposition 8.11 *The image of the natural map $f_J : \mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]}) \to X(k,n)$ is equal to $X_J(k,n)$ and is a bijection over the open cell $C(p_J)$.*

Proof Let us take a point $M \in C(p_J)$. By Theorem 4.14(i), all the points in the cell are isomorphic as Δ_n modules, and it follows that

$$\dim_{\mathbb{C}} M_{\alpha_{i-r+1}} \ldots M_{\alpha_i} M^{(i)} = d_J^{(i,r)}$$

for all vertices (i,r). Hence, the preimage of M is a single point. Moreover, we claim that $f_J^{-1}(C(p_J))$ is open in $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{M})$. The proof is analogous to the proof of Theorem 8.10: we consider the tower

$$f_J^{-1}(C(p_J)) = Y_1 \cap f_J^{-1}(C(p_J)) \to Y_2 \cap f_J^{-1}(C(p_J)) \to \cdots \to Y_n \cap f_J^{-1}(C(p_J)) = \mathrm{pt}$$

and show by decreasing induction on r that at each step one obtains an open part of Y_r.

Recall that by definition $X_J(k,n)$ coincides with the closure of the cell $C(p_J)$. Therefore,

$$f_J\big(\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})\big) = \overline{f_J(f_J^{-1}C(p_J))} = \overline{C(p_J)} = X_J(k,n),$$
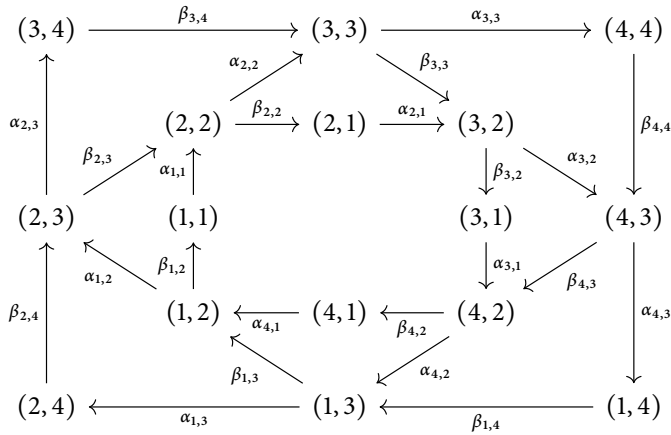
where the first equality is true since $f_J^{-1}(C(p_J))$ is open in $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{M})$. ∎

We summarize the whole picture in the following corollary.

Corollary 8.12 *The quiver Grassmannian* $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$ *desingularizes* $X_J(k,n)$.

8.3 Example

We work out two examples of the desingularization above for $n = 4$ and $k = 2$. The variety $X(2,4)$ has six irreducible components, labeled by the cardinality two sets $J \subset [4]$. Essentially, there are two different cases (up to rotation): $J = \{1,2\}$ and $J = \{1,3\}$. The quiver $\widetilde{\Delta}_4$ looks as follows:

$$
\begin{array}{c}
(3,4) \xrightarrow{\ \beta_{3,4}\ } (3,3) \xrightarrow{\ \alpha_{3,3}\ } (4,4) \\[4pt]
\alpha_{2,3}\uparrow \quad \nearrow^{\alpha_{2,2}} \quad \searrow_{\beta_{3,3}} \quad \downarrow\beta_{4,4} \\[4pt]
(2,2) \xrightarrow{\beta_{2,2}} (2,1) \xrightarrow{\alpha_{2,1}} (3,2) \\[4pt]
\beta_{2,3}\nearrow \quad \uparrow\alpha_{1,1} \qquad \downarrow\beta_{3,2} \quad \nearrow\alpha_{3,2} \\[4pt]
(2,3)\quad (1,1) \qquad\qquad (3,1)\quad (4,3) \\[4pt]
\alpha_{2,4}\uparrow \quad \searrow_{\alpha_{1,2}}\ \uparrow\beta_{1,2} \qquad \downarrow\alpha_{3,1}\ \searrow\beta_{4,3} \quad \downarrow\alpha_{4,3} \\[4pt]
(1,2) \xleftarrow{\alpha_{4,1}} (4,1) \xleftarrow{\beta_{4,2}} (4,2) \\[4pt]
\beta_{2,4}\ \quad \searrow_{\beta_{1,3}} \qquad \swarrow\alpha_{4,2} \\[4pt]
(2,4) \xleftarrow{\alpha_{1,3}} (1,3) \xleftarrow{\beta_{1,4}} (1,4)
\end{array}
$$

Now, let $J = \{1,2\}$. Then the dimension vector $d_J^{(i,r)}$ is given by

$$
\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\[6pt]
& & 1 & & 2 & & 1 \\[6pt]
1 & & 2 & & & 2 & & 1 \\[6pt]
& & 2 & & 2 & & 2 & \\[6pt]
& & 1 & & & 2 & & & 1
\end{array}
$$

Hence, the desingularization $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$ is the tower bundle

$$(8.5) \qquad \mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]}) = Y_1 \to Y_2 \to Y_3 \to Y_4 = pt,$$

where the fiber of the fibration $Y_3 \to Y_4$ is $\mathbb{P}^1$ (the other three factors are points), the fiber of the fibration $Y_2 \to Y_3$ is $\mathbb{P}^1 \times \mathbb{P}^1$ (with two trivial factors), and the fiber of the fibration $Y_1 \to Y_2$ is $\mathbb{P}^1$ (with three trivial factors).

Now, let $J = \{1,3\}$. Then the dimension vector $d_J^{(i,r)}$ is given by

1		1		0	
	1	2	2		
1	2		2	1	
	2	2	1		
0		1		1	

The tower (8.5) is as follows: Y_4 is a point, the fiber of the fibration $Y_3 \to Y_4$ is $\mathbb{P}^1 \times \mathbb{P}^1$ (with two trivial factors), the fiber of the fibration $Y_2 \to Y_3$ is a point (the product of four points), and the fiber of the fibration $Y_1 \to Y_2$ is $\mathbb{P}^1 \times \mathbb{P}^1$ (with two trivial factors).

Remark 8.13 All the irreducible components $X_J(2,4)$ of the variety $X(2,4)$ are isomorphic either to $X_{\{1,2\}}(2,4)$ or to $X_{\{1,3\}}(2,4)$. Using the description of the moment graph from Proposition 6.3, one shows that there are six arrows pointing to the zero-dimensional cell of the irreducible component $X_{\{1,2\}}(2,4)$. Since $\dim \mathrm{Gr}(2,4) = 4$, Lemma 2.7 of [Ca02] implies that $X_{\{1,2\}}(2,4)$ is singular. We note also that one can show that the desingularization map $f_{1,3}$ is one-to-one over $X_{\{1,3\}}(2,4)$ and hence this component is smooth.

8.4 Geometric and combinatorial properties of the desingularization

Lemma 8.14 *The automorphism group of $\widetilde{U}_{[n]}$ satisfies*

$$\mathrm{Aut}_{\widetilde{\Delta}_n}(\widetilde{U}_{[n]}) \cong \mathrm{Aut}_{\Delta_n}(U_{[n]}).$$

Proof Composing $\beta_{i+1,2} \circ \alpha_{i,1}$ for all $i \in \mathbb{Z}_n$, we obtain the same relations on each component $A^{(i,1)}$ of $A \in \mathrm{Aut}_{\widetilde{\Delta}_n}(\widetilde{U}_{[n]})$ as for the component $B^{(i)}$ of $B \in \mathrm{Aut}_{\Delta_n}(U_{[n]})$ (see Proposition 4.5). By construction of $\widetilde{U}_{[n]}$, all other components $A^{(i,r)}$ are the lower diagonal blocks of size $n - r + 1$ in the matrices $A^{(i,1)}$. This gives us the desired isomorphism. ■

Observe that the $G_{\mathbf{n}}$ action on $R_{\mathbf{n}}(\Delta_n)$ preserves the relations satisfied by the representations and the automorphism group of $M \in R_{\mathbf{n}}(\Delta_n)$ is its $G_{\mathbf{n}}$ stabilizer.

Hence, on both sides, we obtain the same groups if we view $U_{[n]}$ and $\widetilde{U}_{[n]}$ as bounded quiver representations.

Lemma 8.15 *The strata in the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$ are exactly the $\mathrm{Aut}_{\widetilde{\Delta}_n}(\widetilde{U}_{[n]})$ orbits.*

Proof The representation $\widetilde{U}_{[n]}$ is an injective bounded $\widetilde{\Delta}_n$ representation for the relations as in Lemmas 8.5 and 8.6. Hence, we can apply [Pue19, Lemma 2.28]. ∎

Remark 8.16 Analogous to Section 5.2, we define an action of $T = (\mathbb{C}^*)^{n+1}$ on the quiver Grassmannians $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$, induced by the fact that the bases for the vector spaces of $\widetilde{U}_{[n]}$ are subsets in the basis for the vector spaces of $U_{[n]}$, and we can hence restrict the weight tuples. In particular, the desingularization map is T equivariant with respect to this action. This is convenient if it comes to the computation of equivariant Euler classes (see [LP20, Lemma 2.1(3)]).

Lemma 8.17 *The $\mathrm{Aut}_{\widetilde{\Delta}_n}(\widetilde{U}_{[n]})$ orbits of the T fixed points in the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$ provide a cellular decomposition.*

Proof The vector space over the vertex (i, r) of $\widetilde{\Delta}_n$ of any subrepresentation in $\mathrm{Gr}_{\mathbf{d}}(\widetilde{U}_{[n]})$ is an element of the Grassmannian of subspaces $\mathrm{Gr}_{d_J^{(i,r)}}(\mathbb{C}^{n-r+1})$. Let $p \in \mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$ be a T fixed point. By $p^{(i,r)} \in \mathrm{Gr}_{d_J^{(i,r)}}(\mathbb{C}^{n-r+1})$, we denote its component over the vertex (i, r) of $\widetilde{\Delta}_n$. Every element $A \in \mathrm{Aut}_{\widetilde{\Delta}_n}(\widetilde{U}_{[n]})$ acts on $p^{(i,r)}$ via the component $A^{(i,r)}$, which is a lower triangular matrix by Proposition 4.5 and Lemma 8.14. Hence, the orbit of $p^{(i,r)}$ in the Grassmannian of subspaces $\mathrm{Gr}_{d_J^{(i,r)}}(\mathbb{C}^{n-r+1})$ is a cell.

The orbit $\mathrm{Aut}_{\widetilde{\Delta}_n}(\widetilde{U}_{[n]}).p$ is the intersection of these cells along the maps of $\widetilde{U}_{[n]}$. Since the maps along arrows $\alpha_{i,r}$ are inclusions and the maps along $\beta_{i,r}$ are projections where the last coordinate is sent to zero, the intersection of the cells in the Grassmannians of subspaces is a cell of $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$.

It remains to show that every $\mathrm{Aut}_{\widetilde{\Delta}_n}(\widetilde{U}_{[n]})$ orbit contains a T fixed point. Locally, every orbit in $\mathrm{Gr}_{d_J^{(i,r)}}(\mathbb{C}^{n-r+1})$ contains a T fixed point. From the explicit shape of the maps of $\widetilde{U}_{[n]}$, it follows that the intersection of these cells also contains a T fixed point. Hence, the stratification of $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$ into the isomorphism classes of subrepresentations is also a cellular decomposition indexed by the T fixed points. ∎

Remark 8.18 Let us consider the $\mathbb{C}^*$ action on $\mathrm{Gr}_{\mathbf{d}_J}(\widetilde{U}_{[n]})$ induced by the $\mathbb{C}^*$ action (3.1) on $X(k, n)$. The cells in the previous lemma coincide with the attracting sets of the fixed points as introduced in Section 3.3 for the cycle.

A Linear degenerations

Analogous to [CFFFR17, Section 2], we construct linear degenerations of the Grassmannian $\mathrm{Gr}_k(n)$ of k-dimensional subspaces of $\mathbb{C}^n$: Recall that $\mathrm{Gr}_k(n)$ is isomorphic

to the quiver Grassmannian for Δ_n with the representation

$$M_{\mathrm{id}} := \left(\left(V^{(i)} := \mathbb{C}^n \right)_{i\in\mathbb{Z}_n}, \left(V_{\alpha_i} := \mathrm{id}_{\mathbb{C}^n} \right)_{i\in\mathbb{Z}_n} \right) \in \mathrm{R_n}(\Delta_n)$$

and dimension vector $\mathbf{k} = (k,\dots,k) \in \mathbb{Z}^n$. This motivates our definition of linear degenerations. Since there are multiple ways of realizing $\mathrm{Gr}_k(n)$ as a quiver Grassmannian, this is of course only one possible definition.

Definition A.1   A linear degeneration of $\mathrm{Gr}_k(n) = \mathrm{Gr}_k(M_{\mathrm{id}})$ is a quiver Grassmannian $\mathrm{Gr_k}(M)$ for $M \in \mathrm{R_n}(\Delta_n)$.

In particular, $X(k,n) = \mathrm{Gr_k}(U_{[n]})$ is a linear degeneration of $\mathrm{Gr}_k(n)$.

The group $G_\mathbf{n}$ acts on $\mathrm{R_n}(\Delta_n)$ such that the isomorphism classes of the Δ_n representations with dimension vector $\mathbf{n} = (n,\dots,n) \in \mathbb{Z}^n$ are exactly the $G_\mathbf{n}$ orbits in $\mathrm{R_n}(\Delta_n)$. Moreover, two quiver Grassmannians are isomorphic if the corresponding quiver representations are isomorphic. Hence, the $G_\mathbf{n}$ orbits in $\mathrm{R_n}(\Delta_n)$ parametrize the isomorphism classes of linear degenerations of $\mathrm{Gr}_k(n)$.

The rank tuple of $M \in \mathrm{R_n}(\Delta_n)$ is

$$\mathbf{r} := \mathbf{r}(M) := \left(r_{i,\ell} := \mathrm{rank}\, M_{i+\ell-1} \circ \cdots \circ M_i \right)_{(i,\ell)\in\mathbb{Z}_n\times\{0,\dots,n^2+1\}}.$$

It is sufficient to consider $\ell \le n^2 + 1$ because the maximal nilpotent representations in $\mathrm{R_n}(\Delta_n)$ are $U(i;n^2)$, i.e., the representations whose coefficient quiver is an equioriented string on n^2 points winding around the cycle and ending over the ith vertex. This implies that the isomorphism classes of linear degenerations are parametrized by the rank tuples of $G_\mathbf{n}$ orbits in $\mathrm{R_n}(\Delta_n)$, since isomorphic representations have the same rank tuple and two points with the same rank tuple are conjugated by [Ke82, p. 32].

We define a partial order of the rank tuples by comparing their entries componentwise. We denote the rank tuples of M_{id} and $U_{[n]}$ by $\mathbf{r}_{\mathrm{id}}$ and $\mathbf{r}_{[n]}$. The entries of $\mathbf{r}_{\mathrm{id}}$ are all equal to n, and the (i,ℓ)th entry of $\mathbf{r}_{[n]}$ equals $\max\{n-\ell,0\}$. In the setting of linear degenerations of the flag variety of type A, the degenerations with rank tuple between the flag and the so-called Feigin degeneration $\mathcal{F}l_{n+1}^a$ are all of the same dimension [CFFFR17, Theorem A].

The construction of the quiver representation $U_{[n]}$ is somehow analogous to the construction of $\mathcal{F}l_{n+1}^a$ [CFR12, Definition 2.5]. However, despite that, there are linear degenerations of $\mathrm{Gr}_k(n)$ with rank tuple between $\mathbf{r}_{\mathrm{id}}$ and $\mathbf{r}_{[n]}$ which are not of dimension $k(n-k)$, as explained in the following example.

Example A.2   We collect some intermediate degenerations which are not of dimension $k(n-k)$. For $M = U(i;n^2)$, the quiver Grassmannian $\mathrm{Gr_k}(M)$ consists of the single point $U(i;kn)$ and its rank tuple is between $\mathbf{r}_{\mathrm{id}}$ and $\mathbf{r}_{[n]}$. For $r \in \mathbb{Z}_{\ge0}$ with $r \le n$, we define

$$M_r := \bigoplus_{i=1}^{n-r} U(i;n) \oplus V_{\Delta_n} \otimes \mathbb{C}^r$$

and compute

$$\dim_\mathbb{C} \mathrm{Gr_k}(M_r) = \max_{\ell\in\{0,1,\dots,r\}} \left\{ k(n-k) + (k-\ell)(\ell-r) - \ell(n-k+\ell-r) \right\}$$

using [Pue20, Proposition 4.4]. This only matches $k(n-k)$ if $r = n$ or $r = 0$, which corresponds exactly to M_{id} and $U_{[n]}$, respectively.

We close by showing that the degeneration of the Grassmannian into $X(k,n)$ is not flat. Let us consider the case $n = 3$, $k = 1$. The classical Grassmannian (the projective plane in this case) is embedded diagonally into $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. In particular, the coordinate ring is triply graded. The component of degree $(1,1,1)$ is of dimension $10 = \dim_{\mathbb{C}} S^3(\mathbb{C}^3)$. We can make it explicit by introducing the coordinates $x_i, y_i, z_i, i = 1,2,3$ corresponding to three projective planes. Then a basis of the $(1,1,1)$ component is formed by monomials

$$x_1 y_1 z_1, \ x_1 y_1 z_2, \ x_1 y_1 z_3, \ x_1 y_2 z_2, \ x_1 y_2 z_3,$$
$$x_1 y_3 z_3, \ x_2 y_2 z_2, \ x_2 y_2 z_3, \ x_2 y_3 z_3, \ x_3 y_3 z_3.$$

Now, let us look at the quiver Grassmannian $X(1,3) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ formed by triples of lines (l_1, l_2, l_3) such that $\mathrm{pr}_1 l_1 \subset l_2$, $\mathrm{pr}_2 l_2 \subset l_3$, and $\mathrm{pr}_3 l_3 \subset l_1$. Then the following 11 elements form a basis of the degree $(1,1,1)$ component:

$$x_1 y_1 z_1, \ x_1 y_2 z_1, \ x_1 y_2 z_3, \ x_2 y_2 z_2, \ x_2 y_2 z_3,$$
$$x_2 y_2 z_1, \ x_2 y_3 z_3, \ x_2 y_3 z_1, \ x_1 y_3 z_3, \ x_1 y_3 z_1, \ x_3 y_3 z_3.$$

In particular, the two weight zero elements $x_1 y_2 z_3$ and $x_2 y_3 z_1$ are linearly independent. In fact, if one takes $l_1 = \mathrm{span}(a,b,c)$ with $(a,b,c) \neq 0$, then $l_2 = \mathrm{span}(0,b,c)$, $l_3 = \mathrm{span}(0,0,c)$, and the value of $x_1 y_2 z_3$ is equal to abc. However, the value of $x_2 y_3 z_1$ is zero.

B The case $k = 1$

In this appendix, we consider the case $k = 1$ and arbitrary n in more details.

One can identify $\mathrm{Gr}(1,n)_{\geq 0} \subset \mathrm{Gr}(1,n) = \mathbb{P}^{n-1}$ with the unit simplex in $\mathbb{R}^n$. If $(t_1 : \cdots : t_n)$ are homogeneous coordinates of a point in $\mathrm{Gr}(1,n)_{\geq 0}$, then we can uniquely rescale in such a way that $\sum_{i=1}^n t_i = 1$. There are $2^n - 1$ faces of the simplex, and the dimension of a face is just its number of nonzero coordinates (i.e., the number of a-dimensional cells is $\binom{n}{a}$).

From the combinatorial point of view, the $k = 1$ Grassmann necklaces are collections $(i_1, \ldots, i_n)$, with $i_a \in [n]$ such that if $i_a \neq a$, then $i_{a+1} = i_a$ (as usual, $i_{n+1} = i_1$). Recall that, by Proposition 4.8, the irreducible components $X_J(1,n)$ of the quiver Grassmannian $X(1,n)$ are labeled by $J = \{j\}$, $j \in [n]$. We use the notation $X_j(1,n) := X_{\{j\}}(1,n)$.

Proposition B.1 *One has:*

(i) *The total number of cells of $X(1,n)$ is equal to $2^n - 1$.*

(ii) *The Poincaré polynomial is given by $(1+q)^n - q^n$.*

(iii) *All irreducible components of $X(1,n)$ are isomorphic. Each irreducible component $X_j(1,n)$ is isomorphic to a height n Bott tower of $\mathbb{P}^1$ fibrations over a point.*

(iv) *The desingularization map is an isomorphism over each irreducible component $X_j(1,n)$.*

Proof The first two claims can be easily deduced from Theorem 7.7 and [W05] (note that the positroids of type $(1, n)$ are just the nonempty subsets of $[n]$). Let us prove the third claim: there are n irreducible components and the rotation group action on $X(k, n)$ induces the transitive action on the components $X_j(1, n)$. Hence, all the irreducible components are isomorphic. Now, let us consider the component $X_n(1, n)$. Recall the second basis of $U_{[n]}$ from Definition 4.3. Let $v_1^{(a)}, \ldots, v_n^{(a)}$ be the corresponding basis of $U_{[n]}^{(a)}$. Recall that we have identified all the spaces $U_{[n]}^{(a)}$ with $\mathbb{C}^n$ in such a way that the map $U_{[n]}^{(a)} \to U_{[n]}^{(a+1)}$ sends $v_i^{(a)}$ to $v_{i+1}^{(a+1)}$ for $i < n$ and v_n is sent to zero. We denote this map by s_1. We also use the notation s_{-1} for the map sending v_1 to zero and $v_i^{(a)}$ to $v_{i-1}^{(a-1)}$ for $i > 1$. Then $X_n(1, n)$ consists of collections $(V_a)_{a=1}^n$ of lines in $\mathbb{C}^n$ such that:

- $V_a \subset V_{a+1} + \mathbb{C} v_n$,
- $V_a \subset \mathrm{span}(v_a, \ldots, v_n)$.

Then one has $V_n = \mathbb{C} v_n$, V_{n-1} is an arbitrary line in the span of v_n and v_{n-1}, V_{n-2} is an arbitrary line in the span of v_n and $s_{-1} V_{n-1}$, and in general V_a is an arbitrary line in the span of v_n and $s_{-1} V_{a+1}$. This construction identifies $X_n(1, k)$ with a Bott tower.

Finally, let us prove claim (iv) for $j = n$. The dimensions $d_n^{(i,r)} := d_{\{n\}}^{(i,r)}$ are easily computed as

$$d_n^{(i,r)} = \begin{cases} 1, & r \le i, \\ 0, & r > i. \end{cases}$$

The quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}_n}(\widetilde{U}_{[n]})$ (which desingularizes the component $X_n(1, n)$) consists of collections of subspaces $(V^{(i,r)})$ such that (in particular) $V^{(i,r)} \subset V^{(i,r-1)}$ and the nontrivial (i.e., one-dimensional) subspaces correspond to $r \le i$. Therefore, for $r \le i$, one has $V^{(i,r)} = V^{(i,1)}$, which means that the map $\mathrm{Gr}_{\mathbf{d}_n}(\widetilde{U}_{[n]}) \to X_n(1, n)$ is one-to-one. ∎

In the $k = 1$ case, also the cell dimensions have a particularly nice behavior. For an element $i_\bullet = (i_1, \ldots, i_n) \in \mathcal{GN}_{1,n}$, we denote by $n_{i_\bullet}$ the number of distinct entries of $i_\bullet$.

Example B.2 If $i_\bullet = (1, 1, \ldots, 1) \in \mathcal{GN}_{1,n}$, then $n_{i_\bullet} = 1$. On the other hand, if $i_\bullet = (1, 2, 3, \ldots, n) \in \mathcal{GN}_{1,n}$, then $n_{i_\bullet} = n$.

The following statement is known to the experts, but we were not able to find it in the literature. Several proofs of it, relying on different languages (combinatorics, quiver Grassmannians, tnn Grassmannian, etc.), are available. Following the referee's suggestion, we give the tnn Grassmannian one.

Lemma B.3 Let $i_\bullet \in \mathcal{GN}_{1,n}$. Then,

$$\dim_{\mathbb{C}} C(i_\bullet) = n - n_{i_\bullet}.$$

Proof We can identify the tnn Grassmannian $\mathrm{Gr}(1, n)_{\ge 0}$ with the unit simplex in $\mathbb{R}^n$. There are $2^n - 1$ many faces of the simplex, and the dimension of a face is just its number of nonzero coordinates. The cardinality of the positroid $\mathcal{M}$ showing up in the right-hand side of (2.2) is exactly the number of nonzero coordinates for a point in

the cell corresponding to $\mathcal{M}$. Now, it is clear that the cardinality of $\mathcal{M}$ coincides with the value of $n_{i_\bullet}$ for the Grassmann necklace from the left-hand side of (2.2).

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