6

Decay of the False Vacuum

In this chapter we give the first example of an application of the methods we have learned so far. We will apply the methods of instantons to the problem of vacuum instability in quantum field theory. We consider a scalar field governed by a Lagrangian of the form

$$L = \int d^3x \, \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - V(\phi(x)). \tag{6.1}$$

The potential $V(\phi(x))$ for $\phi(x) = \phi$, a constant independent of the spacetime coordinates, has the form represented by the graph in Figure 6.1. There are two minima, a global minimum at ϕ_- and a local minimum at ϕ_+. Classically the configurations $\phi(x) = \phi_\pm$ are stable. The energy is given by the functional

$$E = \int d^3x \, \frac{1}{2}\dot{\phi}(x)^2 + \frac{1}{2}\vec{\nabla}\phi(x)\cdot\vec{\nabla}\phi(x) + V(\phi(x)). \tag{6.2}$$

When $\phi(x)$ is a constant the first two terms, which are positive semi-definite, give zero contribution, thus the energy comes solely from the potential term. The potential is minimized and normally adjusted by adding a constant to make it vanish at the global minimum $\phi(x) = \phi_-$, so normally the energy of this classical configuration is zero. At $\phi(x) = \phi_+$ the potential is in a local minimum, however, and then the value of the potential is finite and the total energy is divergent. The divergence is proportional to the volume. However, the physically important quantity is not the total energy but the energy density, which is given directly by the potential. Then the energy density difference between the two classical ground states is finite. ϕ_+ is the false vacuum while ϕ_- is the true vacuum. The false vacuum is unstable while the true vacuum is stable.

We will, however, adjust the zero of the potential not in the normal way but as depicted in Figure 6.1, by adding a constant, so that the energy density of the false vacuum state is zero. Such a redefinition cannot affect the local physics. Then we will calculate the decay of the false vacuum to the true vacuum per

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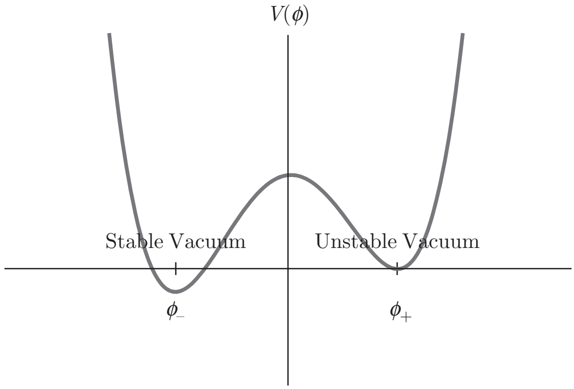

Figure 6.1. The potential giving rise to a false vacuum

unit time and per unit volume, $\frac{\Gamma}{V}$. We will find an expression of the form

$$\frac{\Gamma}{V} = Ae^{-\frac{B}{\hbar}}\left(1+0(\hbar)\right) \tag{6.3}$$

in the semi-classical limit. This form is exactly that which we have seen for decays via tunnelling. B will correspond to the classical action for a critical configuration while A will come from the quantum considerations. We proceed in an analogous fashion to the problem we considered in quantum mechanics. We wish to define the analytic continuation of the matrix element

$$\mathcal{A}.C.\{\langle\phi_+|\,e^{-\frac{\beta\hat{H}}{\hbar}}\,|\phi_+\rangle\} \tag{6.4}$$

from a potential for which the vacuum constructed at ϕ_+ is stable to the potential we are considering. As we have seen, the analytic continuation instructs us on how to deal with Gaussian integrals over fluctuations about a critical configuration which correspond to negative frequencies.

6.1 The Bounce Instanton Solution

Otherwise we proceed in the usual way with the semi-classical analysis of the Euclidean functional integral. We look at

$$\mathcal{N}\int\mathcal{D}\phi e^{-\frac{S_E[\phi(x)]}{\hbar}} \tag{6.5}$$

with the boundary conditions $\phi\left(\tau=\pm\frac{\beta}{2}\right)=\phi_+$. Here

$$S_E[\phi(x)] = \int dt d^3x\left(\frac{1}{2}\partial_\mu\phi(x)\partial_\mu\phi(x)+V(\phi(x)\right) \tag{6.6}$$

with the equation of motion corresponding to

$$\frac{\delta S_E[\phi(x)]}{\delta\phi} = -\partial_\mu\partial_\mu\phi(x)+V'(\phi(x)) = 0. \tag{6.7}$$

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Here we use the Euclidean metric. This equation is exactly the equation of motion for the scalar field in minus the potential. We take the boundary conditions for the case $\beta = \infty$

$$\lim_{\tau \to \pm\infty} \phi(\vec{x}, \tau) = \phi_+ \tag{6.8}$$

and we add the condition

$$\partial_\tau \phi(\vec{\tau}, \tau = 0) = 0, \tag{6.9}$$

which determines the Euclidean time at the classical turning point. This time of the classical turning point is completely at our disposal for the case $\beta = \infty$. The condition that the classical action should be finite gives

$$\lim_{|\vec{x}| \to \infty} \phi(\vec{x}, \tau) = \phi_+. \tag{6.10}$$

We assume a form that is $O(4)$-invariant

$$\phi(\vec{x}, \tau) = \phi\left((|\vec{x}|^2 + \tau^2)^{\frac{1}{2}} \right). \tag{6.11}$$

The equation of motion becomes, with $\rho = (|\vec{x}|^2 + \tau^2)^{\frac{1}{2}}$

$$\frac{d^2}{d\rho^2}\phi + \frac{3}{\rho}\frac{d}{d\rho}\phi - V'(\phi) = 0. \tag{6.12}$$

The action is

$$S_E[\phi] = 2\pi^2 \int_0^\infty d\rho \rho^3 \left(\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 + V(\phi) \right) \tag{6.13}$$

with the boundary conditions $\left.\frac{d\phi}{d\rho}\right|_{\rho=0} = 0$ and $\lim_{\rho\to\infty} \phi(\rho) = \phi_+$. The first condition avoids a singularity at $\rho = 0$ while the second comprises all of the asymptotic boundary conditions.

A rigorous proof of the existence of a solution and that it is the minimum action solution is given by Coleman, Glaser and Martin [34], but we shall be content with the following argument due to Coleman [31]. The equation of motion (6.12) can be interpreted as that for a particle with "position" ϕ moving in "time" ρ. The particle is subject to a force, $-V'(\phi)$, and a frictional force with a "time"-dependent Stokes coefficient of friction $\frac{3}{\rho}$. The equation of motion for a particle in a potential with Stokes coefficient of friction μ is

$$\frac{d^2}{d\rho^2}\phi(\rho) + \mu\frac{d}{d\rho}\phi(\rho) + V'(\phi(\rho)) = 0. \tag{6.14}$$

The solution in the absence of a potential, $V'(\phi(\rho)) = 0$, is simply $\phi(\rho) = a - be^{-\mu\rho}$ for arbitrary constants a, b, with a related to the initial position and b related to the initial velocity. This solution confirms that motion with friction without external forces will come to rest exponentially fast. In the present case μ depends on ρ.

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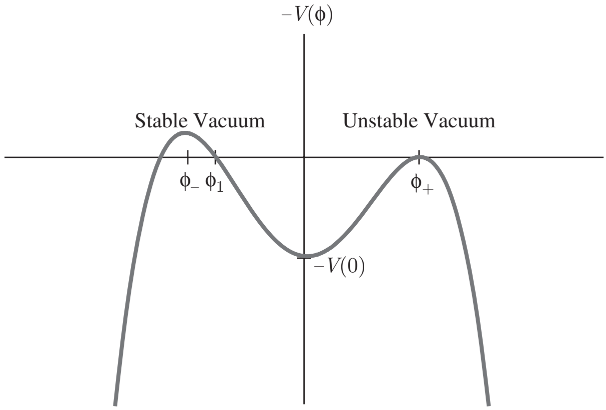

Figure 6.2. The reversed potential and effective dynamical problem

We can prove the existence of a solution satisfying our boundary conditions by the following continuity argument. We must show that there exists an initial point ϕ_0 from which the particle can start at $\rho = 0$ and achieve $\phi = \phi_+$ at $\rho = \infty$. The potential is reversed to give $-V(\phi)$ as depicted in Figure 6.2, and ϕ_1 is defined as the point at which the potential crosses zero. If $\phi_0 > \phi_1$, $\phi(\rho)$ will never reach ϕ_+ even as $\rho \to \infty$ starting with zero velocity. If, however, $\phi_- < \phi_0 < \phi_1$, and ϕ_0 is sufficiently close to ϕ_-, $\phi(\rho)$ will surpass ϕ_+ at some finite time. We can understand this intuitively; if ϕ_0 is arbitrarily close to ϕ_-, the particle will roll off this potential hill arbitrarily slowly. We can make this time so long that the coefficient of friction, $\frac{3}{\rho}$, becomes negligibly small. Then the particle will roll off and eventually climb the hill at ϕ_+ and even surpass ϕ_+ since it is now a conservative system. Indeed, for $\phi(\rho)$ close to ϕ_- we can linearize the equation of motion,

$$\left(\frac{d^2}{d\rho^2}\phi + \frac{3}{\rho}\frac{d}{d\rho}\phi - \omega^2\right)(\phi(\rho) - \phi_-) = 0, \tag{6.15}$$

which has the solution

$$(\phi(\rho) - \phi_-) = 2\,(\phi(0) - \phi_-)\,\frac{I_1(\omega\rho)}{\omega\rho} \tag{6.16}$$

where ω^2 is $V''(\phi_-)$, and $I_1(\omega\rho)$ is the modified Bessel function of the first kind. This implies that for $(\phi(0) - \phi_-)$ sufficiently small, $(\phi(\rho_0) - \phi_-)$ can be kept arbitrarily small such that $\phi(\rho_0) < \phi_1$ so that the potential energy remains positive, where ρ_0 is determined by the condition that the subsequent energy lost to the friction term is negligible. Once the friction term becomes negligible, the system is conservative and, since at ρ_0 the potential energy is positive, the particle will clearly surpass ϕ_+ at finite ρ^f. A measure of the energy lost in the

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friction is obtained by the integral

$$\int_{\rho_0}^{\rho^f} d\rho\, \frac{3}{\rho}\frac{d}{d\rho}\phi < \frac{3}{\rho_0}\int_{\rho_0}^{\rho^f} d\rho\, \frac{d}{d\rho}\phi \approx \frac{3}{\rho_0}\int_{\phi_-}^{\phi_+} d\phi = \frac{3}{\rho_0}(\phi_+ - \phi_-). \tag{6.17}$$

Thus we choose ρ_0 large enough so that this energy is negligible in comparison to the energy scales that drive the dynamics, say $V(0)$:

$$\frac{3}{\rho_0}(\phi_+ - \phi_-) \ll V(0). \tag{6.18}$$

Then finally we conclude that there must exist some intermediate ϕ_0 from which $\phi(\rho)$ will attain ϕ_+ exactly as $\rho \to \infty$. This implies the existence of a solution of the form we desire.

6.2 The Thin-Wall Approximation

We can go much further with the assumption that the energy density difference between the two vacua is small.

$$V(\phi) = U(\phi) + \frac{\epsilon}{2a}(\phi - a) \tag{6.19}$$

with $U(\phi) = U(-\phi)$, $U'(\pm a) = 0$, $U''(\pm a) = \omega^2$ and ϵ is arbitrarily small, as depicted in Figure 6.3. We can calculate the action for the bounce to first order in ϵ. The reversed potential is given in Figure 6.4. At $\rho = 0$ the field is very close to $-a$, it stays there for a very long "time", and then it rolls relatively quickly through the minimum of the reversed potential, up to the hill at $\phi = +a$ since now the friction is negligible. It achieves $\phi = +a$ only as $\rho \to \infty$. The bounce is like a large four-ball of radius R, in Euclidean space, of true vacuum, separated by a thin wall, from the false vacuum without.

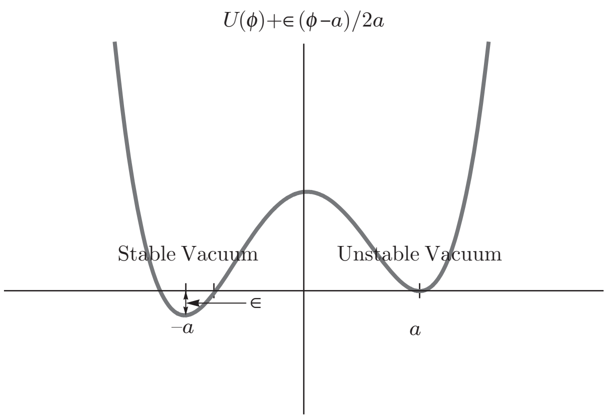

Figure 6.3. The symmetric potential with a small asymmetry

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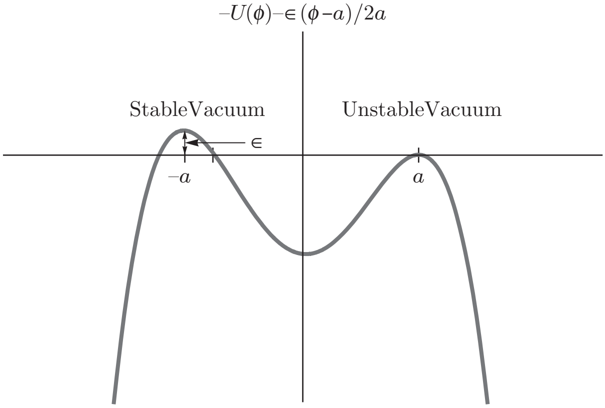

Figure 6.4. The reversed symmetric potential with a small asymmetry

For ρ near R, if we drop the friction term we obtain the equation of motion (to zero order in ϵ)

$$\frac{d^2}{d\rho^2}\phi - U'(\phi) = 0. \tag{6.20}$$

This is exactly the same equation that we have studied in the double-well problem of Chapter (3). The instanton solution interpolates from one well to the other as in Figure 3.3. It is given in this region, which is near the wall, approximately by the equation

$$\rho - R = \int_0^{\tilde{\phi}(\rho)} \frac{d\phi}{\sqrt{2U(\phi(\rho))}}. \tag{6.21}$$

For large $|\rho - R|$, the solution is given by

$$\tilde{\phi}(\rho) = \pm\left(a - \alpha e^{-\omega|\rho-R|}\right). \tag{6.22}$$

For example, for the choice of the potential

$$U(\phi) = \frac{\lambda}{4}\left(\phi^2 - a^2\right)^2 \tag{6.23}$$

the solution is

$$\tilde{\phi}(\rho) = a\tanh\left(\omega\left(\rho - R\right)\right) \tag{6.24}$$

with $\alpha = 2a$ and $\omega^2 = 2\lambda a^2$.

Thus our bounce is given by

$$\phi_{bounce}(\rho) = \begin{cases} -a & 0 < \rho \ll R \\ \tilde{\phi}(\rho) & \rho \approx R \\ a & \rho \gg R \end{cases}. \tag{6.25}$$

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To find R we do a variational calculation in R.

$$S_E[\phi_{bounce}] = 2\pi^2 \int_0^{R-\Delta} d\rho \rho^3 (-\epsilon) + 2\pi^2 \int_{R-\Delta}^{R+\Delta} d\rho \rho^3 \left(\frac{1}{2} \left(\frac{d\tilde{\phi}(\rho)}{d\rho} \right)^2 + U\left(\tilde{\phi}(\rho)\right) \right) +$$
$$+ 2\pi^2 \int_{R+\Delta}^{\infty} d\rho \rho^3 \, (0)$$
$$\approx -\frac{1}{2}\pi^2 R^4 \epsilon + 2\pi^2 R^3 S_1, \quad \text{for} \quad R \gg \Delta, \tag{6.26}$$

where S_1 is the action for the one-dimensional instanton $\tilde{\phi}(\rho)$ calculated in Equation (3.27) which is independent of R (we call it S_1 here to emphasize that it is the one-dimensional instanton action),

$$S_1 \approx \int_{-\infty}^{\infty} dx \left(\frac{1}{2} \frac{d^2}{dx^2} \tilde{\phi}(x) + U\left(\tilde{\phi}(x)\right) \right)$$
$$= \int_{-a}^{a} d\phi \sqrt{2U(\phi)}. \tag{6.27}$$

$S_E(R)$ should be stationary under variations of R,

$$\frac{dS_E(R)}{dR} = -2\pi^2 R^2 \epsilon + 6\pi^2 R^2 S_1 = 0 \tag{6.28}$$

hence

$$R = \frac{3S_1}{\epsilon}. \tag{6.29}$$

This confirms our expectation that $R \to \infty$ as $\epsilon \to \infty$. Finally, the Euclidean action for the bounce is

$$S_E^{bounce} = \frac{1}{2}\pi^2 \left(\frac{3S_1}{\epsilon} \right)^4 \epsilon + 2\pi^2 \left(\frac{3S_1}{\epsilon} \right)^3 S_1 = \frac{27\pi^2 S_1^4}{2\epsilon} \left(1 + o(\epsilon)^3\right). \tag{6.30}$$

6.3 The Fluctuation Determinant

The calculation of the coefficient A of Equation (6.3) is not so straightforward, even approximately. It is given by the determinant of the operator governing small fluctuations about the bounce.

$$\langle \phi_+ | e^{-\frac{\beta \hat{H}}{\hbar}} | \phi_+ \rangle = e^{-\frac{S_E^{bounce}}{\hbar}} \mathcal{N} \det{}^{-\frac{1}{2}} \left(-\frac{d^2}{d\tau^2} - \nabla^2 + V''(\phi_{bounce}) \right). \tag{6.31}$$

When we attempt to evaluate the determinant we encounter the same problems that we have already seen in particle quantum mechanics: non-positive frequencies for the spectrum of Gaussian fluctuations.

Zero modes come from invariance of the action under translations. We can translate in space and Euclidean time which gives us four independent zero modes (we write ϕ_{bounce} as ϕ_b for the sake of brevity)

$$\phi_\mu(\vec{x}, \tau) = N \frac{\partial}{\partial x^\mu} \phi_b(\vec{x}, \tau). \tag{6.32}$$

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Zero modes correspond to continuous degeneracies of the critical point of the Euclidean action. Here they correspond to the arbitrariness of the location of the centre of the bounce in Euclidean $\boldsymbol{R}^4$ which is actually $\mathbb{R}$. We cannot integrate over these directions in the integrations over fluctuations about the bounce; however, we can equivalently integrate over the position of the bounce in $\boldsymbol{R}^4$ which is actually $\mathbb{R}$. This gives a (divergent) factor of βV and a Jacobian corresponding to the change of integration variable from the fluctuation degree of freedom to the coordinate giving the position of the bounce. The Jacobian factor is of the same type as before, indeed,

$$\delta\phi = \frac{1}{N}\frac{\partial}{\partial x_0^\mu}\phi_b\left((x-x_0)^\nu\right)dc^{\,\mu} \tag{6.33}$$

for an infinitesimal change dc_μ of the coefficient of the Gaussian fluctuation along the normalized zero-mode direction, $\frac{1}{N}\frac{\partial}{\partial x_0^\mu}\phi_b\left((x-x_0)^\nu\right)$, while

$$\delta\phi = \frac{\partial}{\partial x_0^\mu}\phi_b\left((x-x_0)^\nu\right)dx_0^\mu. \tag{6.34}$$

Equating the variation in Equations (6.33) and (6.34) gives

$$\frac{dc^\mu}{\sqrt{2\pi\hbar}} = \frac{N}{\sqrt{2\pi\hbar}}dx_0^\mu. \tag{6.35}$$

Now

$$\int d^4x \frac{1}{N^2}\frac{\partial}{\partial x_0^\mu}\phi_b\left((x-x_0)^\nu\right)\frac{\partial}{\partial x_0^\nu}\phi_b\left((x-x_0)^\nu\right) = \frac{\delta_{\mu\nu}}{4N^2}\int d^4x\left(\partial_\lambda\phi_b(x)\partial_\lambda\phi_b(x)\right). \tag{6.36}$$

We can evaluate this integral by using the fact that the action S_E is stationary at the bounce.

$$\begin{aligned}
0 = \frac{d}{d\lambda}S_E\left[\phi_b(\lambda x)\right]\Big|_{\lambda=1} &= \frac{d}{d\lambda}\int d^4x\left(\frac{1}{2}\left(\partial_\mu\phi_b(\lambda x)\partial_\mu\phi_b(\lambda x)\right) + V\left(\phi_b(\lambda x)\right)\right)\Big|_{\lambda=1}\\
&= \frac{d}{d\lambda}\int d^4x\left(\frac{1}{\lambda^2}\frac{1}{2}\left(\partial_\mu\phi_b(x)\partial_\mu\phi_b(x)\right) + \frac{1}{\lambda^4}V\left(\phi_b(x)\right)\right)\Big|_{\lambda=1}\\
&= \int d^4x\left(-2\frac{1}{2}\left(\partial_\mu\phi_b(x)\partial_\mu\phi_b(x)\right) - 4V\left(\phi_b(x)\right)\right)\\
&= -4S_E\left[\phi_b(x)\right] + \int d^4x\left(\partial_\mu\phi_b(x)\partial_\mu\phi_b(x)\right).
\end{aligned} \tag{6.37}$$

Hence

$$\int d^4x\left(\partial_\mu\phi_b(x)\partial_\mu\phi_b(x)\right) = 4S_E\left[\phi_b(x)\right] \tag{6.38}$$

and finally

$$N = \sqrt{S_E\left[\phi_b(x)\right]} \tag{6.39}$$

exactly as in the one-dimensional case. The Jacobian factor becomes $\left(\sqrt{\frac{S_E[\phi_b(x)]}{2\pi\hbar}}\right)^4$, giving the integration over the position of the bounce

$$\frac{\left(S_E\left[\phi_b(x)\right]\right)^2}{4\pi^2\hbar^2}\beta V. \tag{6.40}$$

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We do the same analysis for N well-separated bounces, which are approximate critical points, which gives us

$$\left(\frac{\left(S_E\left[\phi_b(x)\right]\right)^2}{4\pi^2\hbar^2}\right)^N \frac{(\beta V)^N}{N!}, \tag{6.41}$$

where the $N!$ simply indicates that the permutations of the positions of the bounces do not give new configurations. This gives

$$\langle\phi_+|\,e^{-\frac{\beta\hat{H}}{\hbar}}\,|\phi_+\rangle = \mathcal{N}\det{}^{-\frac{1}{2}}\left(-\partial_\mu\partial_\mu + V''(\phi_+)\right)e^{-\left(\beta V\left(e^{-\frac{S_E[\phi_b(x)]}{\hbar}}\right)\frac{(S_E[\phi_b(x)])^2}{4\pi^2\hbar^2}K\right)}, \tag{6.42}$$

where K is now the ratio

$$K = \left(\frac{\det'\left(-\partial_\mu\partial_\mu + V''(\phi_b)\right)}{\det\left(-\partial_\mu\partial_\mu + V''(\phi_+)\right)}\right)^{-\frac{1}{2}} \tag{6.43}$$

and the prime indicates that the zero modes are removed. The normalization constant $\mathcal{N}$ is defined to exactly cancel the free determinant that appears

$$\mathcal{N}\det{}^{-\frac{1}{2}}\left(-\partial_\mu\partial_\mu + V''(\phi_+)\right) = 1 \tag{6.44}$$

This is, not the whole story, because the operator

$$-\partial_\mu\partial_\mu + V''(\phi_b) \tag{6.45}$$

has a negative mode. Again our analysis of meta-stable states in quantum mechanics applies directly. Taking into account the factor of $\frac{1}{2}$ which comes from the analytic continuation and deformation of the contour, we find

$$i\frac{\Gamma}{V} = \frac{\left(S_E\left[\phi_b(x)\right]\right)^2}{4\pi^2\hbar^2}\left(e^{-\frac{S_E[\phi_b(x)]}{\hbar}}\right)\left(\frac{\det'\left(-\partial_\mu\partial_\mu + V''(\phi_b)\right)}{\det\left(-\partial_\mu\partial_\mu + V''(\phi_+)\right)}\right)^{-\frac{1}{2}}. \tag{6.46}$$

The prime still indicates that only the zero modes are removed, the square root of the negative eigenvalue reproduces the imaginary nature and the factor of $\frac{1}{2}$ is taken into account because the lifetime is $\frac{1}{2}$ of the imaginary part. Analysis of the negative modes is left for Section 6.5.

6.4 The Fate of the False Vacuum Continued

We continue our analysis of the decay of the false vacuum by considering the evolution of the field after the tunnelling event. We can obtain some intuition from the WKB analysis of tunnelling in particle quantum mechanics. Consider the decay of a nucleus by α-particle emission. A reasonably successful phenomenological potential has the form of a square well of depth extending to less than zero attached to a short-range drop off potential from the top reaching to zero, as depicted in Figure 6.5. The negative energy levels in the well are stable,

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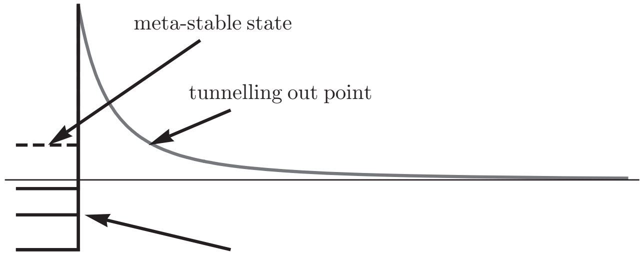

Figure 6.5. A nuclear tunnelling potential

but the positive energy levels are meta-stable and decay by tunnelling. The semi-classical description of the decay process proceeds as follows. The particle stays in the well up to a time, the "transition time", which is a random variable, when it makes a quantum jump to the other side of the barrier. It appears suddenly at the other side at a point, which we call the "tunnelling out point", with the same energy as the meta-stable state within. Subsequently, it continues like a free classical particle until it eventually moves off to infinity.

Quantum mechanics only enters in the calculation of the process of barrier penetration. It allows us to calculate the mean value of the "transition time". In the WKB analysis, the tunnelling out point is the point on the other side of the barrier with equal energy to the energy of the meta-stable state inside, from which, if the particle were released, it would move off to infinity under the classical dynamics. This is the turning point in the usual WKB analysis. We identify this point as the point where all velocities are zero in the bounce solution. We choose this point by the condition

$$\partial_\tau \phi(\vec{x},\tau)|_{\tau=0} = 0. \tag{6.47}$$

This is satisfied by the $O(4)$ symmetric ansatz that we have taken,

$$\partial_\tau \phi(\rho)|_{\tau=0} = \partial_\rho \phi(\rho) \left(\frac{\tau}{\rho} \right) \bigg|_{\tau=0} = 0. \tag{6.48}$$

The field appears at $\tau = 0$ in the state described by $\phi_b(\vec{x},\tau=0)$ and then evolves classically. The WKB analysis should not be taken too literally. It will not be accurate for observations made just after the tunnelling event occurs. It is more correctly an asymptotic description for what happens long after and far away from the tunnelling event.

6.4.1 Minkowski Evolution After the Tunnelling

We continue nevertheless with the initial condition for after the tunnelling event

$$\phi(\vec{x},t=0) = \phi_b(\vec{x},\tau=0), \quad \partial_\tau \phi(\vec{x},\tau)|_{\tau=0} = 0 \tag{6.49}$$

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and then the field evolves according to the classical, now Minkowskian, equation of motion,

$$\left(\frac{d^2}{dt^2}-\nabla^2\right)\phi\left(\vec{x},t\right)+V'\left(\phi\left(\vec{x},t\right)\right)=0. \tag{6.50}$$

At $t=0$, $\phi\left(\vec{x},t=0\right)=\phi_b(\vec{x},\tau=0)$ is exactly a bubble of radius R of true vacuum, separated by a thin wall from the false vacuum without. This is because $\phi_b(\vec{x},\tau)=\phi_b(\sqrt{|\vec{x}|^2+\tau^2})\to\phi(r)$ for $t=0$ with $r=|\vec{x}|$. We can immediately write down the solution to the classical Minkowskian equation of motion for the subsequent evolution of the bubble. Simply

$$\phi\left(\vec{x},t\right)=\phi_b\left(\sqrt{|\vec{x}|^2-t^2}\right). \tag{6.51}$$

In detail for the Minkowskian signature, with $\tilde{\rho}\equiv\sqrt{|\vec{x}|^2-t^2}=\sqrt{-x_\mu x^\mu}$,

$$\begin{aligned}\partial_\mu\partial^\mu\phi(\tilde{\rho})&=\partial_\mu\left(\frac{d}{d\tilde{\rho}}\phi(\tilde{\rho})\partial^\mu\tilde{\rho}\right)\\&=\frac{d^2}{d\tilde{\rho}^2}\phi(\tilde{\rho})\partial_\mu\tilde{\rho}\partial^\mu\tilde{\rho}+\frac{d}{d\tilde{\rho}}\phi(\tilde{\rho})\partial_\mu\partial^\mu\tilde{\rho}.\end{aligned} \tag{6.52}$$

Using $\partial_\mu\tilde{\rho}=-\frac{x_\mu}{\tilde{\rho}}$, $\partial_\mu\tilde{\rho}\partial^\mu\tilde{\rho}=-1$ and hence $\partial_\mu\partial^\mu\tilde{\rho}=-\frac{3}{\tilde{\rho}}$ we get

$$\partial_\mu\partial^\mu\phi(\tilde{\rho})=-\left(\frac{d^2}{d\tilde{\rho}^2}+\frac{3}{\tilde{\rho}}\right)\phi(\tilde{\rho}). \tag{6.53}$$

The Euclidean equation satisfied by $\phi_b\left(\sqrt{|\vec{x}|^2+\tau^2}\right)$ is

$$\left(\frac{d^2}{d\tau^2}+\nabla^2\right)\phi_b\left(\sqrt{|\vec{x}|^2+\tau^2}\right)-V'\left(\phi_b\left(\sqrt{|\vec{x}|^2+\tau^2}\right)\right)=0. \tag{6.54}$$

Since

$$\partial_\mu\partial_\mu\phi(\rho)=\left(\frac{d^2}{d\rho^2}+\frac{3}{\rho}\right)\phi(\rho). \tag{6.55}$$

gives

$$\left(\frac{d^2}{d\rho^2}+\frac{3}{\rho}\right)\phi(\rho)-V'\left(\phi(\rho)\right)=0. \tag{6.56}$$

Thus

$$\left(\frac{d^2}{dt^2}-\nabla^2\right)\phi_b\left(\sqrt{|\vec{x}|^2-t^2}\right)+V'\left(\phi_b\left(\sqrt{|\vec{x}|^2-t^2}\right)\right)=0 \tag{6.57}$$

and it should be noted that this solution is only valid for $|\vec{x}|^2>t^2$, *i.e.* for the exterior of the bubble.

Then the $O(4)$ invariance of the Euclidean solution is replaced by the $O(3,1)$ invariance of the Minkowskian regime. This implies that the evolution of the bubble appears the same to all Lorentz observers. When the bubble is nucleated, the wall of the bubble is at $r\approx R$, and then it follows the hyperbola, $\tilde{\rho}^2=r^2-t^2=R^2$. This is because the functional form of $\phi(\tilde{\rho})$ describes the wall for all $\tilde{\rho}\approx R^2$. This means that the bubble grows with a speed which approaches

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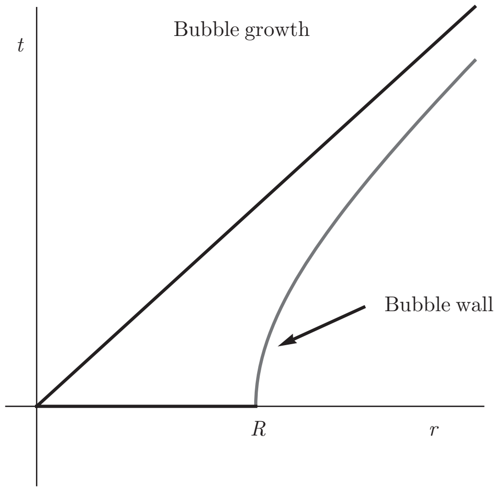

Figure 6.6. The growth of the bubble wall after tunnelling

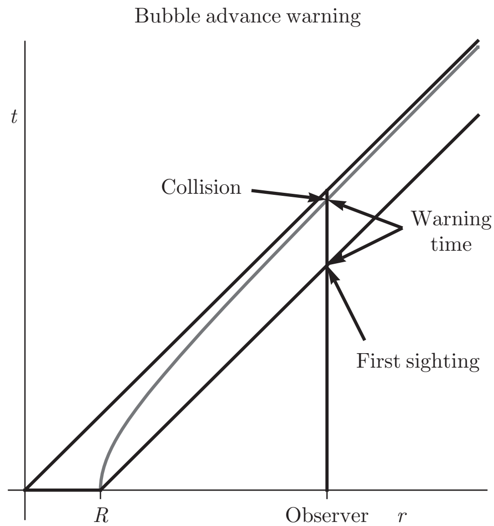

Figure 6.7. Collision warning time with the growth of the bubble wall

the speed of light asymptotically, as depicted in Figure 6.6. How quickly the growth approaches c depends on R. If R is a microscopic number, like $10^{-10} \to 10^{-30}$ as we would expect, the bubble grows with the speed of light almost instantaneously. If a bubble is coming towards us, the warning time we have is given by the projection of the forward light cone from the creation point to our world line (vertical), as depicted in Figure 6.7. The time this gives us in warning, T, is essentially the time it takes light to travel the distance R, as long as the

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observer is far from the creation point relative to R. For R micro-physical, T is also microphysical. After the bubble hits us, quoting directly from Coleman [31]: "We are dead. All constants of nature inside the bubble are different. We cannot function biologically or even chemically". But, paraphrasing, as further pointed out by Coleman, this is no cause for concern, since for $R \sim 10^{-15}$ metres, $T \sim 3 \times 10^{-8}$ seconds, this is much less time than the time it takes for a single neuron to fire. If such a bubble is coming towards us, we won't know what hit us.

6.4.2 Energetics

The energy carried by the wall of the bubble is exactly all the energy gained by converting a sphere of radius R of false vacuum into true vacuum. The energy in the wall per unit area is

$$\begin{aligned}
\mathcal{E} &= \frac{1}{4\pi R^2}\int_{|r|\approx R} d^3x \left(\frac{1}{2}\left(\vec{\nabla}\phi_b\right)^2 + V\left(\phi_b\right)\right) \\
&= \frac{1}{R^2}\int_{R-\Delta}^{R+\Delta} dr r^2 \left(\frac{1}{2}\left(\vec{\nabla}\phi_b\right)^2 + V\left(\phi_b\right)\right) \\
&\approx \int_{-\infty}^{\infty} dr \left(\frac{1}{2}\left(\vec{\nabla}\phi_b\right)^2 + V\left(\phi_b\right)\right) = S_1.
\end{aligned} \tag{6.58}$$

Now, in time, the wall follows the hyperbola $r^2 - t^2 = R^2$, hence the energy in the wall always stays in the wall. After some time, the element of area will have a velocity v. Energy per unit area just transforms as the zero component of a Lorentz vector,

$$S_1 \to \frac{S_1}{\sqrt{1-v^2}}. \tag{6.59}$$

So at such a time the energy in the wall is

$$\mathcal{E} = 4\pi r^2 \frac{S_1}{\sqrt{1-v^2}} \tag{6.60}$$

with

$$v = \frac{dr}{dt} = \frac{d}{dt}\sqrt{R^2+t^2} = \frac{t}{\sqrt{R^2+t^2}} = \sqrt{\frac{r^2-R^2}{r^2}} = \sqrt{1-\frac{R^2}{r^2}}. \tag{6.61}$$

Thus $\sqrt{1-v^2} = \sqrt{1-\left(1-\frac{R^2}{r^2}\right)} = \frac{R}{r}$, and

$$\mathcal{E} = 4\pi r^2 S_1 \frac{r}{R} = \frac{4}{3}\pi r^3 \left(\frac{3S_1}{R}\right) = \frac{4}{3}\pi r^3 \epsilon. \tag{6.62}$$

(In the thin-wall approximation, we have $R = \frac{3S_1}{\epsilon}$.) This is exactly the energy obtained from the conversion of a ball of radius r of false vacuum into true vacuum. Hence all the energy goes into the wall. Inside the bubble is just the tranquil, true vacuum. There is no boiling, roiling, hot plasma of excitations.

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6.5 Technical Details

We complete this chapter with some technical points which we have left unaddressed.

6.5.1 Exactly One Negative Mode

We have assumed that there was exactly one negative energy mode to the operator governing small fluctuations

$$\left(-\partial_\mu\partial_\mu + V''(\phi_b)\right)\phi_n = \lambda_n\phi_n. \tag{6.63}$$

We can prove this in the thin-wall approximation. $O(4)$ invariance means that we can expand in the scalar spherical harmonics in four dimensions

$$\phi_{n,j}(\rho,\Omega) = \frac{1}{\rho^{\frac{3}{2}}}\chi_{n,j}(\rho)\, Y_{j,m,m'}(\Omega), \tag{6.64}$$

where $Y_{j,m,m'}(\Omega)$ transforms according to the representation D^{jj} of $SO(4) = SO(3)\times SO(3)$, with m and m' independently going from $-j$ to j. These are the eigenfunctions of the transverse Laplacian in four dimensions. Then to zero order in ϵ,

$$\left(-\frac{d^2}{d\rho^2} + \frac{8j(j+1)+3}{4\rho^2} + U''(\phi_b(\rho))\right)\chi_{n,j}(\rho) = \lambda_{n,j}\chi_{n,j}(\rho) \tag{6.65}$$

for the resulting radial equation. This is analogous to the Schrödinger equation for a particle in a radial potential in three dimensions.

The zero modes

$$\frac{1}{\sqrt{S_E^{bounce}}}\partial_\mu\phi_b(\rho) \tag{6.66}$$

transform according to the $j=\frac{1}{2}$ representation. ($\frac{1}{2}+\frac{1}{2} = 1+0$ for the three-dimensional rotation subgroup.) Since $\phi_b(\rho)$ is an increasing function, it starts at ϕ_- and increases to ϕ_+ at $\rho=\infty$, the zero modes have no nodes. Hence they are the modes of lowest "energy" for $j=\frac{1}{2}$. For $j>\frac{1}{2}$ the Hamiltonian is simply greater than for $j=\frac{1}{2}$, hence all modes have energy greater than zero. Thus the negative modes can only arise in the sector with $j=0$. There must be at least one negative mode since the Hamiltonian is simply smaller for $j=0$. In the thin-wall limit, $U''(\phi_b(\rho))$ has the form given in Figure 6.8 where $\omega^2 = U''(\phi_\pm)$. This is because $\phi_b(\rho)$ starts at ϕ_- at $\rho=0$ and stays so until about $\rho=R$ where it interpolates relatively quickly to ϕ_+, and then stays essentially constant until $\rho=\infty$. The zero modes, corresponding to derivatives of $\phi_b(\rho)$, hence have support localized at the wall. The negative energy modes must also be localized there. Thus we approximate the equation near $\rho\approx R$ by replacing in the centrifugal term $\rho\to R$. This yields the equation

$$\left(-\frac{d^2}{d\rho^2} + \frac{8j(j+1)+3}{4R^2} + U''(\phi_b(\rho))\right)\chi_{n,j}(\rho) = \lambda_{n,j}\chi_{n,j}(\rho). \tag{6.67}$$

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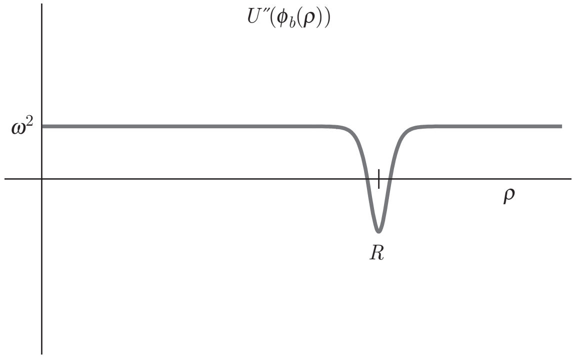

Figure 6.8. The potential for the small fluctuations about a thin-wall bubble

Clearly

$$\lambda_{n,j} = \lambda_n + \frac{8j(j+1)+3}{4R^2} \tag{6.68}$$

with λ_n, ordered to be increasing with n, evidently independent of j. For $R \to \infty$, λ_n are the eigenvalues of the one-dimensional operator

$$\left(-\frac{d^2}{dx^2} + U''(f(x))\right), \tag{6.69}$$

where $f(x) = \phi_b(x)$ with $x \in [-\infty, \infty]$, *i.e.* we can neglect the effect of the boundary at $\rho = 0$. We already know that for $j = \frac{1}{2}$ the minimum eigenvalue is zero, thus

$$\lambda_0 \to -\frac{8j(j+1)+3}{4R^2}\bigg|_{j=\frac{1}{2}} = -\frac{8 \cdot \frac{1}{2} \cdot \frac{3}{2} + 3}{4R^2} = -\frac{9}{4R^2}. \tag{6.70}$$

This gives

$$\lambda_{0,0} = -\frac{9}{4R^2} + \frac{3}{4R^2} = -\frac{3}{2R^2}, \tag{6.71}$$

which is negative. All other eigenvalues for $j = \frac{1}{2}$ are positive, for all R. This implies that all the other λ_n are greater than zero, since

$$\lim_{R\to\infty}\left(\lambda_n + \frac{8 \cdot \frac{1}{2} \cdot \frac{3}{2} + 3}{4R^2}\right) = \lim_{R\to\infty}(\lambda_n) > 0 \quad \text{for} \quad n > 0. \tag{6.72}$$

Thus also for $j = 0$

$$\lambda_n + \frac{3}{4R^2} > 0, \quad \text{for} \quad n > 0, \tag{6.73}$$

for R large, hence there are no other negative eigenvalues.

In the limit $\epsilon \to 0$ we obtain the double-well potential depicted in Figure 6.9. There are no bounce-type solutions for this potential. Our solution just becomes a ball of true vacuum of infinite radius, $R = \frac{3S_1}{\epsilon} \to \infty$. There exist only the solutions

$$\phi = \phi_- \quad \text{or} \quad \phi = \phi_+ \tag{6.74}$$

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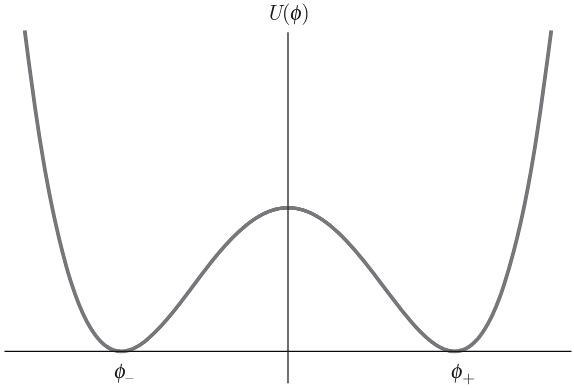

Figure 6.9. The symmetric double-well potential $U(\phi)$

to the Euclidean equation of motion. This is different from the case of particle quantum mechanics, where there are tunnelling-type solutions between the two wells. This difference is completely consistent with our understanding of quantum field theory in a potential with two symmetric wells of the same depth. In such a theory there is spontaneous symmetry breaking. The two vacua, constructed above each well, correspond to inequivalent representations of the quantum field. They cannot exist in the same Hilbert space, and hence there is no tunnelling between them.

6.5.2 Fluctuation Determinant and Renormalization

The determinant that we must compute is

$$\kappa \equiv \det\left(-\partial_\mu\partial_\mu + V''(\phi_b)\right) = e^{\ln\left(\det\left(-\partial_\mu\partial_\mu+V''(\phi_b)\right)\right)} = e^{\mathrm{tr}\ln\left(-\partial_\mu\partial_\mu+V''(\phi_b)\right)}. \tag{6.75}$$

We expand about $\phi=\phi_+$, then $V''(\phi_+)\approx\omega^2$, then we have

$$\begin{aligned}\kappa &= e^{\mathrm{tr}\ln\left(-\partial_\mu\partial_\mu+\omega^2+\left(V''(\phi_b)-\omega^2\right)\right)}\\ &= e^{\mathrm{tr}\ln\left(\left(-\partial_\mu\partial_\mu+\omega^2\right)\left(1+\left(-\partial_\mu\partial_\mu+\omega^2\right)^{-1}\left(V''(\phi_b)-\omega^2\right)\right)\right)}\\ &= e^{\mathrm{tr}\ln\left(\left(-\partial_\mu\partial_\mu+\omega^2\right)+\mathrm{tr}\ln\left(1+\left(-\partial_\mu\partial_\mu+\omega^2\right)^{-1}\left(V''(\phi_b)-\omega^2\right)\right)\right)}\\ &= \kappa_0 e^{\mathrm{tr}\left(\left(-\partial_\mu\partial_\mu+\omega^2\right)^{-1}\left(V''(\phi_b)-\omega^2\right)-\frac{1}{2}\left(\left(-\partial_\mu\partial_\mu+\omega^2\right)^{-1}\left(V''(\phi_b)-\omega^2\right)\right)^2+\cdots\right)}\end{aligned} \tag{6.76}$$

where $\kappa_0=\det\left(-\partial_\mu\partial_\mu+\omega^2\right)$. The free determinant will be absorbed in the definition of the factor $K=(\kappa/\kappa_0)^{-\frac{1}{2}}$ of Equation (6.43).

The first two terms in this expansion are infinite; however, all the rest are finite. $V''(\phi_b)-\omega^2$ is exponentially small for $\rho\gg R$, so we may Fourier transform it to obtain

$$\tilde{f}(k_\mu) = \int \frac{d^4x}{(2\pi)^4} e^{-ik_\mu x_\mu}\left(V''(\phi_b)-\omega^2\right). \tag{6.77}$$

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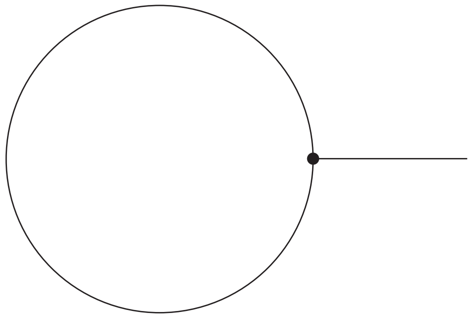

Figure 6.10. Feynman diagram for the first term in the expansion of Equation (6.76)

$\tilde{f}(k_\mu)$ its Fourier transform is then also exponentially small for large k_μ. Then

$$
\begin{aligned}
&\operatorname{tr}\left(\left(-\partial_\mu\partial_\mu+\omega^2\right)^{-1}\left(V''\left(\phi_b\right)-\omega^2\right)\right)\\
&=\int d^4xd^4y\langle x|\frac{1}{-\partial_\mu\partial_\mu+\omega^2}|y\rangle\langle y|V''\left(\phi_b\right)-\omega^2|x\rangle\\
&=\int d^4xd^4y\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik_\mu(x_\mu-y_\mu)}}{k^2+\omega^2}\left(V''\left(\phi_b(x)\right)-\omega^2\right)\delta(x-y)\\
&=\int\frac{d^4k}{(2\pi)^4}\int d^4xd^4y\int d^4q\,\frac{e^{ik_\mu(x_\mu-y_\mu)}e^{iq_\mu x_\mu}}{k^2+\omega^2}\tilde{f}(q_\mu)\delta(x-y)\\
&=\int\frac{d^4k}{(2\pi)^4}\int d^4q\int d^4x\frac{e^{iq_\mu x_\mu}\tilde{f}(q_\mu)}{k^2+\omega^2}\\
&=\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\omega^2}\left(\int d^4q\delta(q_\mu)\tilde{f}(q_\mu)\right).
\end{aligned}
\tag{6.78}
$$

The integral over d^4k is divergent, and can be represented by the diagram given in Figure 6.10. The infinity arising here must be absorbed via a non-trivial renormalization of the theory. The next term is

$$
\begin{aligned}
&\operatorname{tr}\left(\frac{1}{2}\left(\left(-\partial_\mu\partial_\mu+\omega^2\right)^{-1}\left(V''\left(\phi_b\right)-\omega^2\right)\right)^2\right)\\
&=\frac{1}{2}\int d^4xd^4y\langle x|\frac{1}{-\partial_\mu\partial_\mu+\omega^2}|y\rangle\langle y|\frac{1}{-\partial_\mu\partial_\mu+\omega^2}|x\rangle\times\\
&\quad\left(V''\left(\phi_b(y)\right)-\omega^2\right)\left(V''\left(\phi_b(x)\right)-\omega^2\right)\\
&=\frac{1}{2}\int d^4xd^4y\int\frac{d^4kd^4ld^4pd^4q}{(2\pi)^8}\frac{e^{ik_\mu(x-y)_\mu+il_\mu(y-x)_\mu+iq_\mu y_\mu+ip_\mu x_\mu}}{\left(k^2+\omega^2\right)\left(l^2+\omega^2\right)}\tilde{f}(q_\mu)\tilde{f}(p_\mu)\\
&=\frac{1}{2}\int d^4pd^4l\frac{1}{\left((l-p)^2+\omega^2\right)}\frac{1}{\left(l^2+\omega^2\right)}\tilde{f}(p_\mu)\tilde{f}(-p_\mu),
\end{aligned}
\tag{6.79}
$$

where integrating over x and y obtains two delta functions in momentum, and then integrating over k and q eliminates these two variables. The integrals can be represented diagrammatically as depicted in Figure 6.11. The integration over

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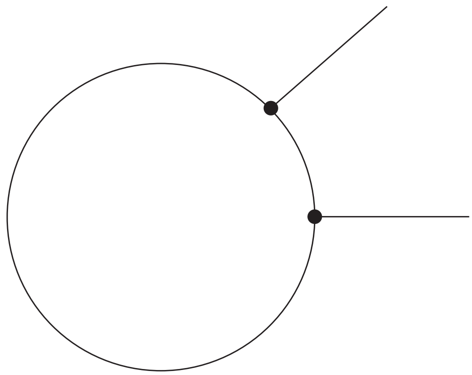

Figure 6.11. Feynman diagram for the second term in the expansion of Equation (6.76)

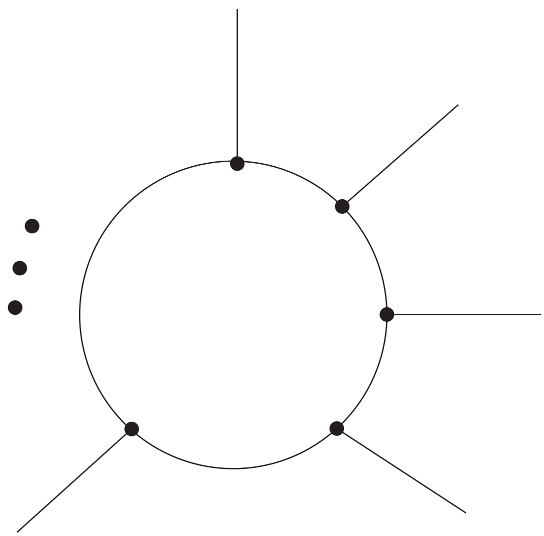

Figure 6.12. General Feynman diagram of the expansion of Equation (6.76)

l is divergent and also requires a non-trivial renormalization of the theory.

In general we get a diagram of the form given in Figure 6.12. It corresponds to the integral

$$\int d^4 l \int \frac{d^4 p_1 \cdots d^4 p_N}{(2\pi)^{4(N-1)}} \frac{\delta(p_1 + p_2 + \cdots + p_N)\tilde{f}(p_{1\mu}) \cdots \tilde{f}(p_{N\mu})}{(l^2 + \omega^2)\left((l - p_N)^2 + \omega^2\right) \cdots \left((l - \sum_{i=2}^{N} p_i)^2 + \omega^2\right)}. \tag{6.80}$$

It is only the integration over l which can cause problems, the $f(p_{i\mu})$ are exponentially decreasing for $p_{i\mu} \to \infty$. For three or more insertions the integral

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is finite

$$\int d^4l \frac{1}{(l^2+\omega^2)((l-p_1)^2+\omega^2)((l-(p_1+p_2))^2+\omega^2)} \sim \int \frac{dl}{l^2}. \tag{6.81}$$

The solution of the problem of how to remove the divergences is by adding a set of (an infinite number of) counter-terms to the action, which will cancel the infinities arising from the integrations. It is a property of a renormalizable field theory that all such counter-terms can be reabsorbed into a multiplicative redefinition of the coupling constants and fields of the original theory. This means that the counter-terms correspond to terms which are of the same form as those already present.

$$S_{bare}(\phi) = S^R(\phi) + \hbar S^1(\phi) + \cdots, \tag{6.82}$$

where $S^R(\phi)$ is finite, but $S^1(\phi)$ is not and the higher terms are not. This implies a change in the bounce, which will also be of the form

$$\phi_b = \phi_b^R + \hbar\phi^1 + \cdots, \tag{6.83}$$

where ϕ_b^R is the same function as ϕ_b but now of the renormalized parameters. Now

$$\begin{aligned} S_{bare}\left(\phi_b^R + \hbar\phi^1 + \cdots\right) &= S^R\left(\phi_b^R\right) + \hbar S^1\left(\phi_b^R\right) + \frac{\delta S^R(\phi)}{\delta\phi}|_{\phi_b^R}\hbar\phi^1 + o(\hbar^2) \\ &= S^R\left(\phi_b^R\right) + \hbar S^1\left(\phi_b^R\right) + o(\hbar^2), \end{aligned} \tag{6.84}$$

where the third term in the first equality vanishes by the equations of motion. Then

$$\frac{\Gamma}{V} = \frac{\left(S^R\left(\phi_b^R\right)\right)^2}{4\pi^2\hbar^2} e^{-\frac{S^R\left(\phi_b^R\right)+\hbar S^1\left(\phi_b^R\right)+\cdots}{\hbar}} \left(\frac{\det'\left(-\partial^2 + V^{R''}\left(\phi_b^R\right)\right)}{\det\left(-\partial^2 + V^{R''}\left(\phi_+^R\right)\right)}\right)^{-\frac{1}{2}} \tag{6.85}$$

with the stipulation that

$$e^{-\frac{\hbar S^1\left(\phi_b^R\right)+\cdots}{\hbar}} \left(\frac{\det'\left(-\partial^2 + V^{R''}\left(\phi_b^R\right)\right)}{\det\left(-\partial^2 + V^{R''}\left(\phi_+^R\right)\right)}\right)^{-\frac{1}{2}} \tag{6.86}$$

be finite. We choose $S^1(\phi_b)$ so that we cancel the two divergent terms in the expansion of the determinant. This can be made even clearer by ensuring that the bare action to $o(\hbar)$ vanish at the renormalized unstable vacuum value ϕ_+^R. This requires

$$S^R\left(\phi_+^R\right) + \hbar S^1\left(\phi_+^R\right) = \hbar S^1\left(\phi_+^R\right) = 0 \tag{6.87}$$

since by definition $S^R\left(\phi_+^R\right) = 0$. We can achieve this by subtracting the constant $\hbar S^1\left(\phi_+^R\right)$ from the bare action in Equation (6.82), giving

$$S_{bare}(\phi) = S^R(\phi) + \hbar\left(S^1(\phi) - S^1\left(\phi_+^R\right)\right) + \cdots. \tag{6.88}$$

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This change implies the condition that

$$e^{-\frac{\hbar\left(S^1\left(\phi_b^R\right)-S^1\left(\phi_+^R\right)\right)+\cdots}{\hbar}}\left(\frac{\det'\left(-\partial^2+V^{R''}\left(\phi_b^R\right)\right)}{\det\left(-\partial^2+V^{R''}\left(\phi_+^R\right)\right)}\right)^{-\frac{1}{2}} \tag{6.89}$$

be free of infinities. We see that one factor of counter-terms matches with each determinant, ensuring the independent renormalizability.

In a renormalizable theory, such as ϕ^4 theory, it is possible to prove that it can be done keeping $S^1(\phi)$ of the same form as $S_{bare}(\phi)$. In the general case, it is clear that the infinities can be cancelled; however, it is not clear that it can be done keeping the same functional form of the bare Lagrangian. Continuing the perturbative expansion of the functional integral beyond the Gaussian approximation will yield higher loop corrections and infinities, for which it will be necessary to add further counter-terms, written as $\hbar^2 S^2(\phi)+\cdots$. These again, for a renormalizable theory will be of the same form as the bare Lagrangian. We will not belabour the point any further.

One final avenue for controlling the determinant is to decompose it into angular momentum eigen-sectors using

$$-\partial_j^2+V''(\phi_b)=\frac{d^2}{d\rho^2}+\frac{8j(j+1)+3}{4\rho^2}+V''(\phi_b(\rho)) \tag{6.90}$$

in the angular momentum j sector. The multiplicity of the spherical harmonics of order j is $(2j+1)^2$. Then

$$\frac{\det'\left(-\partial^2+V''(\phi_b)\right)}{\det\left(-\partial^2+\omega^2\right)}=e^{\sum'^{\infty}_{j=0,\frac{1}{2},1,\cdots}\left(\operatorname{tr}\ln\left(\frac{-\partial_j^2+V''(\phi_b)}{-\partial_j^2+\omega^2}\right)^{(2j+1)^2}-\text{counter terms}\right)}. \tag{6.91}$$

Each term is a one-dimensional determinant which we know in principle how to calculate. It is finite. The infinities reappear after the summation over j.

6.6 Gravitational Corrections: Coleman–De Luccia

In this section we will consider gravitational corrections to vacuum decay. This is eminently reasonable as the application of these methods will be to situations where gravity is important, such as the evolution of the universe, where we invoke Lorentz invariance. The relevance of gravitational effects to vacuum decay in condensed matter systems may not be so important. However, in cosmological applications, the consideration of gravitational effects is clearly indicated. This analysis was first done by Coleman and De Luccia [33], and we will follow their presentation closely.

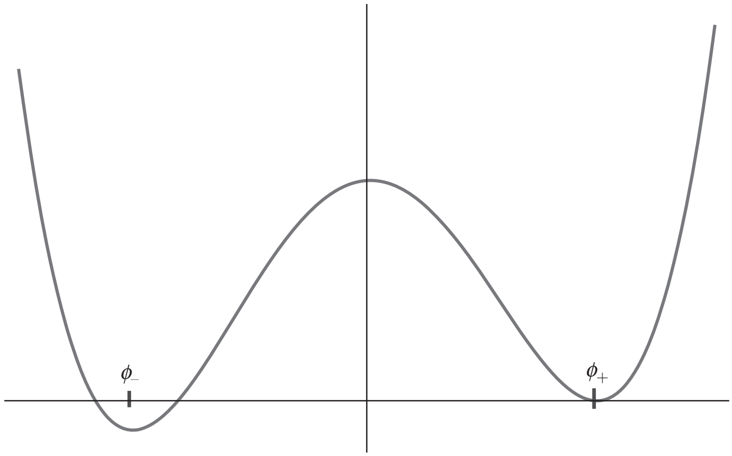

Figure 6.13. The potential with a small asymmetry

For simplicity, we consider a single scalar field with the Euclidean action

$$S_E[\phi] = \int d^4x \left(\left(\frac{1}{2} \partial_\mu \phi \right)^2 + V(\phi) \right), \tag{6.92}$$

which is valid with the absence of gravity. The potential $V(\phi)$ will be as in Figure 6.13, with a true minimum at ϕ_- and a false minimum at ϕ_+; however, we will not assume that the potential is symmetric under reflection $\phi \to -\phi$. We will further assume that the value of the potential at each minimum is very small, proportional to a parameter ϵ. Thus

$$V(\phi) = V_0(\phi) + o(\epsilon), \tag{6.93}$$

where $V_0(\phi_\pm) = 0$.

Adding gravitational corrections may seem pointless at microscopic scales, but for other scales they can be very important. Indeed, if a bubble of radius Λ of false vacuum is converted to a true vacuum, an energy in the amount $E = \epsilon 4\pi\Lambda^3/3$ will be released, and this energy will gravitate in the usual Newton–Einstein fashion. The Schwarzschild radius of the gravitating energy will be $2GE$. This radius will be equal to the radius of the bubble when $\Lambda = 2GE = 2G\epsilon 4\pi\Lambda^3/3$. This gives

$$\Lambda = (8\pi G\epsilon/3)^{-1/2}. \tag{6.94}$$

For energy densities of the order of $\epsilon \approx (1\, GeV)^4$ this gives a radius of about 0.8 kilometres. Thus the gravitational effects of vacuum decay occur at scales which are neither microscopic nor cosmological, but right in the scales of planetary and terrestrial physics. It might well be that gravitational effects in vacuum decay are very relevant.

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Adding the gravitational interaction, the action changes to

$$S_E[\phi, g_{\mu\nu}] = \int d^4x\sqrt{g}\mathcal{L}_E = \int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi) + \frac{1}{16\pi G}R\right), \tag{6.95}$$

where $g_{\mu\nu}$ is the spacetime metric, $g^{\mu\nu}$ its inverse, g is the determinant of the metric and R is the curvature scalar. We note that in Euclidean spacetime, the determinant of the metric g is positive. Adjusting the zero of the potential $V(\phi) \to V(\phi) - V_0$, V_0 a constant, corresponds to adding $\sqrt{g}V_0$ to the action, which is exactly the same as modifying or adding a cosmological constant. Thus the gravitational spacetime inside the bubble and outside the bubble will necessarily be quite different, with different values of the cosmological constant. This makes perfect sense with our understanding that gravitation is sensitive to and couples to the total energy in a system, including the vacuum energy density. Thus we have to specify the cosmological constant of our initial false vacuum, of which we are going to compute the decay. The cosmological constant being exceptionally small at the present time, we will consider two cases of potential interest. First we will consider the possibility that we are living in a false vacuum with zero cosmological constant and this false vacuum decays to a true vacuum of negative cosmological constant, *i.e.* $V(\phi_+) = 0$. Second, we will consider that a false vacuum with a finite, positive cosmological constant decays to the true vacuum without cosmological constant where we live, *i.e.* $V(\phi_-) = 0$.

6.6.1 Gravitational Bounce

We assume that the bounce in the presence of gravity will have maximal symmetry, $O(4)$ symmetry. The metric, remember that we are now in Euclidean spacetime, then must be of the form

$$ds^2 = d\xi^2 + \rho(\xi)^2 d\Omega^2, \tag{6.96}$$

where $d\Omega^2$ is the metric on the three-sphere S^3, and ξ is the Euclidean radial coordinate and corresponds to the proper radial distance along a radial trajectory. $\rho(\xi)$ is the radius of curvature of each concentric S^3 that foliate the space. $d\Omega^2$ can be expressed in a number of coordinates, for example the analogue of spherical polar coordinates in $\mathbf{R}^4$, or in a more sophisticated manner in terms of left invariant 1-forms on the group manifold of $SU(2)$ which is exactly S^3. But we will not need this part of the metric explicitly and hence we will not exhibit it, as we will assume everything is spherically symmetric and hence independent of the angular degrees of freedom.

We can then compute the Euclidean equations of motion. These for the scalar field are

$$\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) - \sqrt{g}V'(\phi) = 0. \tag{6.97}$$

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Using the rather simple form for the metric and the assumption that our field ϕ does not depend on the angular coordinates, we find the equation simplifies to

$$\partial_\xi\left(\sqrt{g}g^{\xi\xi}\partial_\xi\phi\right)-\sqrt{g}V'(\phi)=0. \tag{6.98}$$

Furthermore, $g^{\xi\xi}=1$ and $\sqrt{g}=\rho^3(\xi)\sqrt{g_\Omega}$ where g_Ω is the determinant of the metric of the angular coordinates, which is just the metric on a unit three-sphere. g_Ω depends explicitly on the angular coordinates but it does not depend on ξ. Since the only derivative that appears in the equation of motion is with respect to ξ, g_Ω simply factors out of both terms and then can be cancelled. This gives

$$\begin{aligned}0&=\partial_\xi\left(\rho^3(\xi)\partial_\xi\phi\right)-\rho^3(\xi)V'(\phi)\\&=\rho^3(\xi)\partial_\xi^2\phi+3\rho^2(\xi)\partial_\xi\rho\partial_\xi\phi-\rho^3(\xi)V'(\phi).\end{aligned} \tag{6.99}$$

Dividing through by ρ^3 yields

$$\partial_\xi^2\phi+\frac{3\partial_\xi\rho}{\rho}\partial_\xi\phi=V'(\phi). \tag{6.100}$$

This field equation is augmented by the Einstein equation $G_{\mu\nu}=-8\pi GT_{\mu\nu}$. The sign in this equation is convention-dependent, corresponding to the definition of the curvature tensor, the signature of the metric and the definition of the Ricci tensor. We will use the sign convention in Coleman–De Luccia [33], which is not our favourite convention, but we will stick with it to be close to the original paper. The Einstein equation yields only one net equation,

$$G_{\xi\xi}=-8\pi GT_{\xi\xi}. \tag{6.101}$$

The other components, which are just the diagonal spatial components, are either trivial identities or equivalent to this equation. The energy momentum tensor of the scalar field is

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\mathcal{L}_E. \tag{6.102}$$

To obtain the Einstein equation, Equation (6.101), one has to compute the Ricci curvature through the Christoffel symbols, which is straightforward but somewhat tedious. We will not spell out the details here; with the use of symbolic manipulation software, the calculation is actually trivial. We find that there is only one independent equation,

$$(\partial_\xi\rho)^2=1+\frac{1}{3}8\pi G\rho^2\left(\frac{1}{2}(\partial_\xi\phi)^2-V(\phi)\right). \tag{6.103}$$

It makes perfect sense that there are only two independent equations of motion, as there are only two independent fields, ρ and ϕ. The two equations of motion can be obtained from an effective one-dimensional Euclidean action

$$S_E[\phi,\rho]=2\pi^2\int d\xi\left(\rho^3\left(\frac{1}{2}(\partial_\xi\phi)^2+V(\phi)\right)+\frac{3}{8\pi G}\left(\rho^2\partial_\xi^2\rho+\rho(\partial_\xi\rho)^2-\rho\right)\right). \tag{6.104}$$

The equation of motion for ϕ is straightforward, that for ρ appears only after self-consistently using the derivative of Equation (6.103) to eliminate the second-order derivative in its usual equation of motion.

We solve Equation (6.100) in the approximation that the first derivative term is negligible, and the assumption that the potential term can be approximated by a function $V(\phi) = V_0(\phi) + o(\epsilon)$ with the condition that $V'(\phi_\pm) = 0$ and $V_0(\phi_+) = V_0(\phi_-)$. This latter assumption is very reasonable if the actual potential is obtained from a small perturbation of a degenerate double-well potential. We do not assume that the double well is symmetric, however, just that the minima have the same value for the potential. Then Equation (6.100) becomes

$$\partial_\xi^2 \phi = V_0'(\phi), \tag{6.105}$$

which admits an immediate first integral

$$\frac{1}{2}\left(\partial_\xi \phi\right)^2 = V_0(\phi) + C, \tag{6.106}$$

where C is the integration constant. C is determined by the value of V_0 at ϕ_+, as we are looking for a solution that interpolates from ϕ_- at the initial value of ξ, which is normally taken to be zero, to ϕ_+ as $\xi \to \infty$. Thus

$$\frac{1}{2}\left(\partial_\xi \phi\right)^2 = V_0(\phi) - V_0(\phi_+). \tag{6.107}$$

This equation can be easily integrated as

$$\int_{(\phi_+ + \phi_-)/2}^{\phi} d\phi \sqrt{2\left(V_0 - V_0(\phi_+)\right)} = \int_{\bar{\xi}}^{\xi} d\xi = \xi - \bar{\xi}, \tag{6.108}$$

where $\bar{\xi}$ is the value at which the field is mid-way between ϕ_+ and ϕ_-, which can be taken as the position of the wall. In principle, then, we should solve for ϕ which is implicitly defined by this equation. This will not be done explicitly and, continuing implicitly, once we have ϕ, we can solve Equation (6.103) for ρ. To solve this first-order differential equation requires the specification of one integration constant, we choose that as

$$\bar{\rho} = \rho(\bar{\xi}), \tag{6.109}$$

which is the radius of curvature of the wall. We do not need to have ϕ or ρ explicitly, if all we want is the value of the action for the bounce. This will depend on $\bar{\rho}$; however, we can determine $\bar{\rho}$ by imposing that the action be stationary with respect to variations of $\bar{\rho}$.

We start with the Euclidean action, Equation (6.104), and integrate by parts on the two-derivative term to bring it all in terms of single derivatives. We will only be calculating the action relative to its value for the false vacuum, thus it is calculated in a limiting fashion as the difference of two terms which separately do not make sense and diverge in principle, but the difference is finite. Thus the surface term is irrelevant as we will do the same to the action without the bounce instanton with just the false vacuum. This gives

$$S_E = 4\pi^2 \int d\xi \left(\rho^3 \left(\frac{1}{2}\phi'^2 + V \right) - \frac{3}{8\pi G} \left(\rho\rho'^2 + \rho \right) \right) \tag{6.110}$$

and then we eliminate ρ' with Equation (6.103). This gives the rather compact expression

$$S_E = 4\pi^2 \int d\xi \left(\rho^3 V - \frac{3\rho}{8\pi G} \right) = -\frac{12\pi^2}{8\pi G} \int d\xi \rho \left(1 - \frac{8\pi G}{3}\rho^2 V \right). \tag{6.111}$$

Now we use the thin-wall approximation, *i.e.* we assume that the bounce instanton will be much like the same in the absence of gravity, and for $\epsilon \to 0$, it will be of the form of a thin-wall bubble. We will justify the thin-wall approximation after the analysis. Outside the bubble the bounce configuration is entirely in the false vacuum and we are comparing the bounce action to the action of exactly the false vacuum, thus the contribution to the action is zero

$$S_{E,\text{ outside}} = 0. \tag{6.112}$$

Within the wall, we can put $\rho = \bar{\rho}$, and $V \to V_0$ up to $o(\epsilon)$ terms, giving

$$S_{E,\text{ wall}} = 4\pi^2 \bar{\rho}^2 \int d\xi \left(V_0(\phi) - V_0(\phi_+) \right) = 2\pi^2 \bar{\rho}^3 S_1, \tag{6.113}$$

where S_1 was defined by Equation (6.27) in the absence of gravity. Finally, for the inside of the bubble, $\phi = \phi_\pm$ is a constant, for both cases when we are computing the action for the bounce or for the false vacuum, thus we have from Equation (6.103)

$$d\xi = d\rho \left(1 - \frac{8\pi G}{3}\rho^2 V(\phi_\pm) \right)^{-1/2}. \tag{6.114}$$

Thus choosing ϕ_- for the bounce and ϕ_+ for the false vacuum we have

$$\begin{aligned} S_{E,\text{ inside}} &= -\frac{12\pi^2}{8\pi G} \int_0^{\bar{\rho}} \rho d\rho \left(\left(1 - \frac{8\pi G}{3}\rho^2 V(\phi_-) \right)^{1/2} - \left(1 - \frac{8\pi G}{3}\rho^2 V(\phi_+) \right)^{1/2} \right) \\ &= \frac{12\pi^2}{(8\pi G)^2} \left(\frac{1}{V(\phi_-)} \left(\left(1 - \frac{8\pi G}{3}\bar{\rho}^2 V(\phi_-) \right)^{3/2} - 1 \right) \right. \\ &\quad \left. - \frac{1}{V(\phi_+)} \left(\left(1 - \frac{8\pi G}{3}\bar{\rho}^2 V(\phi_+) \right)^{3/2} - 1 \right) \right) \end{aligned} \tag{6.115}$$

where,

$$S_E = S_{E,\text{ outside}} + S_{E,\text{ wall}} + S_{E,\text{ inside}}. \tag{6.116}$$

This yields an unwieldy expression; however, for the cases which interest us, it is quite simple.

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Firstly, for the case $\phi_+ = \epsilon, \phi_- = 0$, the case where we are living in a spacetime after the formation of a bubble, we have the simple expression (after taking the limit $V(\phi_-) \to 0$ in the action $S_{E,\text{ inside}}$)

$$S_E = 2\pi^2\bar{\rho}^3 S_1 + \frac{12\pi^2}{(8\pi G)^2}\left(-4\pi G\bar{\rho}^2 - \frac{1}{\epsilon}\left(\left(1 - \frac{8\pi G}{3}\bar{\rho}^2\epsilon\right)^{3/2} - 1\right)\right). \quad (6.117)$$

Then setting the derivative with respect to $\bar{\rho}$ to vanish, gives

$$\frac{dS_E}{d\bar{\rho}} = 0 = 6\pi^2\bar{\rho}^2 S_1 + \frac{12\pi^2}{8\pi G}\bar{\rho}\left(-1 + \left(1 - \frac{8\pi G}{3}\bar{\rho}^2\epsilon\right)^{1/2}\right), \quad (6.118)$$

which is easily solved as

$$\bar{\rho} = \frac{12 S_1}{4\epsilon + 24\pi G S_1^2} \equiv \frac{\bar{\rho}_0}{1 + (\bar{\rho}_0/2\Lambda)^2}, \quad (6.119)$$

where $\bar{\rho}_0 = 3S_1/\epsilon$, which is the bubble radius in the absence of gravity, and $\Lambda = \sqrt{3/(8\pi G\epsilon)}$, the radius at which the Schwarzschild radius of the energy from converting a false vacuum to a true vacuum is equal to the bubble radius as defined in Equation (6.94). Evaluating the action at the value of $\bar{\rho}$ yields

$$S_E = \frac{1}{\left(1 + (\bar{\rho}_0/2\Lambda)^2\right)^2}\frac{27\pi^2 S_1^4}{2\epsilon^3} = \frac{S_E^0}{\left(1 + (\bar{\rho}_0/2\Lambda)^2\right)^2}, \quad (6.120)$$

where S_E^0 is the action of the bounce in the absence of gravity. We can obtain this formula by brute force replacement for $\bar{\rho}$; however, we can minimize the algebra by noting the Euclidean action, as a function of $\bar{\rho}$, has the form

$$S_E = \alpha\bar{\rho}^3 - \beta\bar{\rho}^2 + \gamma - \delta\left(1 - \zeta\bar{\rho}^2\right)^{3/2} \quad (6.121)$$

with $\alpha = 2\pi^2 S_1,\ \beta = 3\pi/4G,\ \gamma = 3/16G^2\epsilon,\ \delta = 8\pi G\epsilon/3$ and $\bar{\rho} = (3S_1/\epsilon)/(1 + (\bar{\rho}_0/2\Lambda)^2)$ and the above definitions of $\bar{\rho}_0$ and Λ. The action is stationary at $\bar{\rho}$ hence

$$3\alpha\bar{\rho}^2 - 2\beta\bar{\rho} - 3\delta\left(1 - \zeta\bar{\rho}^2\right)^{1/2}(-\zeta\bar{\rho}) = 0. \quad (6.122)$$

Then factoring out by 3, multiplying by $\bar{\rho}$ and adding and subtracting terms we can reconstruct S_E

$$3\left(\alpha\bar{\rho}^3 - \beta\bar{\rho}^2 + \gamma - \delta\left(1 - \zeta\bar{\rho}^2\right)^{1/2}\left(1 - \zeta\bar{\rho}^2\right) + \delta\left(1 - \zeta\bar{\rho}^2\right)^{1/2} - \gamma + \frac{\beta}{3}\bar{\rho}^2\right) = 0 \quad (6.123)$$

so then we get

$$S_E = \gamma - \frac{\beta}{3}\bar{\rho}^2 - \delta\left(1 - \zeta\bar{\rho}^2\right)^{1/2}. \quad (6.124)$$

From the derivative, Equation (6.122), we can easily find

$$\delta\left(1 - \zeta\bar{\rho}^2\right)^{1/2} = \frac{2\beta}{3\zeta} - \frac{\alpha\bar{\rho}}{\zeta} \quad (6.125)$$

and then we have

$$S_E = \gamma - \frac{\beta}{3}\bar{\rho}^2 - \frac{2\beta}{3\zeta} + \frac{\alpha\bar{\rho}}{\zeta}, \tag{6.126}$$

which now is straightforward to evaluate, yielding Equation (6.120).

For the second case, $V(\phi_+) = 0$, $V(\phi_-) = -\epsilon$, where we are now living in a false vacuum that may decay at any moment, we obtain with similar algebra

$$\bar{\rho} = \frac{\bar{\rho}_0}{1 - (\bar{\rho}/2\Lambda)^2} \tag{6.127}$$

while

$$S_E = \frac{S_E^0}{\left(1 - (\bar{\rho}/2\Lambda)^2\right)^2}. \tag{6.128}$$

For the thin-wall approximation to be valid, we required that the radius of the bubble was much larger than the length scale over which ϕ changed significantly. The friction term, $(3/\rho)(d\phi/d\rho)$, was neglected in Equation (6.12) as the factor $(3/\rho) \sim (3/\bar{\rho}) \approx 0$. Now in the presence of gravitation we have a different friction term, $(3\partial_\xi\rho)/\rho)$, which is given by Equation (6.103)

$$\frac{1}{\rho^2}\left(\frac{d^2\rho}{d\xi^2}\right)^2 = \frac{1}{\rho^2} + \frac{8\pi G}{3}\left(\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 - V\right). \tag{6.129}$$

The first term is the same as without gravity and small if $\bar{\rho}$ is large. The second term vanishes on one side of the wall, is constant and of $o(\epsilon)$ on the other, and over the wall it interpolates between these two values. From Equation (6.107) it is to lowest order a constant, $-V_0(\phi_+)$, which in our two cases is of $o(\epsilon)$ plus corrections which are also of $o(\epsilon)$. Hence we lose nothing by replacing it with ϵ. This turns the second term into $1/\Lambda^2$. Hence the two terms which control the size of $\frac{1}{\rho^2}\left(\frac{d^2\rho}{d\xi^2}\right)^2$ are negligible, justifying self-consistently the thin-wall approximation, if $\bar{\rho}$ and Λ are large compared to the variation of ϕ. The variation of ϕ is from ϕ_+ to ϕ_-, over the thickness of the wall. This thickness is determined by the masses and coupling constants that are in V_0 which are not taken to be remarkable, *i.e.* neither very large nor very small. Thus the wall thickness will be independent of ϵ and hence the variation of ϕ is of $o(1)$. Thus self-consistently, for small ϵ, we can impose that $\bar{\rho}$ and Λ are large compared to the variation of ϕ, and the thin-wall approximation is justified. It is important to note that this puts no constraint on $\bar{\rho}_0/\Lambda$ which governs the difference in the solutions Equations (6.119), (6.120), (6.127), (6.128) with gravitation and those without, Equations (6.29), (6.30), for $\bar{\rho}$ and S_E above. Thus $\bar{\rho}_0/\Lambda$ can be taken as large as we want. Although this may not be phenomenologically relevant, it is interesting to consider the possibility.

In the first case with $\phi_+ = \epsilon, \phi_- = 0$ we see that the effect of gravitation is to increase the probability of vacuum decay, as the denominator in Equation (6.120) is greater than 1 and hence reduces S_E. Gravitation also diminishes the bubble

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radius. For the second case, $V(\phi_+) = 0$, $V(\phi_-) = -\epsilon$, the effects of gravitation are in the opposite direction, making it harder for the vacuum to decay as the denominator in Equation (6.128) is less than 1 and can even vanish, hence increasing S_E to arbitrarily large values. In this case, the bubble radius is increased by gravity, in the limiting case, pushing it to infinite radius at a finite value of $\bar{\rho}_0/\Lambda = S_1\sqrt{24\pi G/\epsilon}$. Thus for fixed S_1 and ϵ but for increasing G, we reach a point when the bubble has infinite radius and its action is infinite, completely suppressing vacuum decay. Thus gravitation totally suppresses vacuum decay for $\bar{\rho}_0 = 2\Lambda$, which means

$$\epsilon = 6\pi G S_1^2. \tag{6.130}$$

An explanation of the quenching of vacuum decay is because of energy conservation. If we calculate the energy of a bubble of radius $\bar{\rho}$ first in the absence of gravitation, we have the volume term and the surface term

$$E = -\frac{4\pi}{3}\epsilon\bar{\rho}^2 + 4\pi S_1\bar{\rho}^2. \tag{6.131}$$

In this (second) case of interest, $V(\phi_+) = 0$, $V(\phi_-) = -\epsilon$, thus we are living in a false vacuum of zero-energy density and the true vacuum has negative energy density. Then using the expression $\bar{\rho}_0 = 3S_1/\epsilon$ we have

$$E = \frac{4\pi}{3}\epsilon\bar{\rho}^2(\bar{\rho}_0 - \bar{\rho}), \tag{6.132}$$

thus we see that the energy vanishes for the bubble, which is expected as the energy before the bubble materialized was zero. Then the effects of gravitation can be taken into account, imposing energy conservation. If gravitation increases the total energy of the bubble, then the bubble must grow in size to compensate and if the gravitation decreases the energy it must shrink. In the case at hand, evidently the bubble must grow.

The gravitational contribution to the energy has two terms. First, the ordinary Newtonian potential energy, which is computed by integrating the gravitational field squared over all space

$$E_{\text{Newton}} = -\frac{\epsilon\pi\bar{\rho}_0^5}{15\Lambda^2}. \tag{6.133}$$

This follows from the straightforward calculation of the Newtonian energy of the gravitational field inside a sphere with negative mass density $-\epsilon$. That energy is

$$E_{\text{Newton}} = \frac{1}{2}\int d^3x(-\epsilon)\Phi(\vec{x}), \tag{6.134}$$

where the gravitational potential satisfies

$$\nabla^2\Phi(\vec{x}) = 4\pi G(-\epsilon). \tag{6.135}$$

Then the energy is given by

$$E_{\text{Newton}} = \frac{1}{2}\int d^3x \frac{\nabla^2\Phi}{4\pi G}\Phi = -\frac{1}{8\pi G}\int d^3x\, |\vec{g}|^2, \tag{6.136}$$

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where $\vec{g} = -\vec{\nabla}\Phi$ is the gravitational field. Applying Gauss' law to

$$\vec{\nabla} \cdot \vec{g} = -4\pi G(-\epsilon) \tag{6.137}$$

yields

$$\vec{g} = \frac{4\pi G}{3}\epsilon \vec{r} \tag{6.138}$$

for the interior of the bubble. The gravitational field vanishes in the exterior. The integral Equation (6.136) quickly yields the result, Equation (6.133). The second contribution comes because the existence of the energy distorts the geometry correcting the volume of the bubble and hence correcting the volume term in the energy. From Equation (6.114) we can write the volume element of the bubble

$$4\pi\rho^2 d\xi = 4\pi\rho^2 d\rho\left(1 - \frac{1}{2}\frac{\rho^2}{\Lambda^2}\right) + o(G^2). \tag{6.139}$$

Then integrating over the bubble, the energy density $-\epsilon$ yields a change

$$E_{\text{geom}} = \frac{2\pi\epsilon\bar{\rho}_0^5}{5\Lambda^2} \tag{6.140}$$

giving a total change

$$E_{\text{grav}} = \frac{\pi\epsilon\bar{\rho}_0^5}{3\Lambda^2}. \tag{6.141}$$

Thus the change in energy is positive, which means that, with gravitation, the radius of the bubble must increase. It appears that for finite values of the couplings and parameters, when $\bar{\rho}_0 = 2\Lambda$, the bubble size becomes infinite. Increasing the gravitational coupling then gives no solution, *i.e.* the false vacuum becomes stable.

Once the bubble has materialized through quantum tunnelling, we can describe its subsequent evolution essentially classically. For Minkowski space-like separated points with respect to the centre of the bubble, all we have to do is analytically continue the solution back to Minkowski time. Thus for flat space we had $\rho^2 = \vec{x}\cdot\vec{x} + \tau^2 \to \vec{x}\cdot\vec{x} - t^2$. However, we must continue both the solution and the metric back to Minkowski time. Thus an $O(4)$-invariant Euclidean manifold becomes a $O(3,1)$-invariant Minkowskian manifold. The metric starts as

$$ds^2 = -d\xi^2 - (\rho(\xi))^2 d\Omega^2, \tag{6.142}$$

the negative definite metric being chosen as we wish to continue to a metric of signature $(+,-,-,-)$ where $d\Omega^2$ becomes the metric on a unit hyperboloid with space-like normal once continued to Minkowski spacetime. For this region $\phi = \phi(\rho)$ the solution that we have implicitly assumed to exist (although we have not been required to find it explicitly) and for a thin wall, the bubble wall is always at $\rho = \bar{\rho}$ and lies in this region. If we are outside a materializing bubble, then this is all we have to know about the manifold. It is possible to describe further the evolution of the bubble for the two cases that we have considered;

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however, we will not continue the discussion further, it no longer requires the methods of instantons. We recommend the reader to consult the original article of Coleman and De Luccia [33].

6.7 Induced Vacuum Decay

We continue our study of the decay of the false vacuum precipitated by the existence of topological defects in that vacuum [79, 85]; we restrict our attention to the example of the decay of a "false cosmic string". Such a topological soliton corresponds to a topologically stable, non-trivial configuration inside a spacetime that is in the false vacuum. We will not worry about gravitational corrections. Topological solitons exist when the vacuum is degenerate and, generically, we have spontaneous symmetry breaking.

6.7.1 Cosmic String Decay

Cosmic strings occur in a spontaneously broken $U(1)$ gauge theory, a generalized Abelian Higgs model [61]. This model contains a complex scalar field interacting with an Abelian gauge field, hence scalar electrodynamics. However, we consider the inverse from the usual case, the potential for the complex scalar field ϕ, has a local minimum at a non-zero value $\phi^2 = a^2$, where the symmetry is broken, while the true minimum occurs at vanishing scalar field, $\phi = 0$. The scalar field potential is considered an effective potential, we do not worry about renormalizability. We assume the energy density splitting between the false vacuum and the true vacuum is very small. The spontaneously broken vacuum is the false vacuum.

In a scenario where from a high-temperature, unbroken symmetry phase the theory passes through an intermediate phase of spontaneous symmetry breaking, it is generic that there will be topological defects trapped in the symmetry-broken vacuum. Furthermore, the system could be trapped in the spontaneously broken phase, even though, as the temperature cools, the true vacuum returns to the unbroken symmetry phase. For the complex scalar field, its phase $e^{i\theta}$, can wrap the origin an integer number of times so that $\Delta\theta = 2n\pi$, as we go around a given line in three-dimensional space. The line can be infinite or form a closed loop. Corresponding to the given line there must exist a line of zeros of the scalar field, where the scalar field vanishes and corresponds to the true vacuum. The corresponding minimum energy configuration (when the roles of the false vacuum and true vacuum are reversed) is called a cosmic string, alternatively a Nielsen–Olesen string [96] or a vortex string [3]. In the scenario that we have described, the true vacuum lies at the regions of vanishing scalar field, thus the interior of the cosmic string is in the true vacuum while the exterior is in the false vacuum. It is already interesting that such classically stable configurations actually exist. Such strings must be unstable to quantum mechanical tunnelling decay. In this section we show how to calculate the amplitude for this decay, in the thin-wall limit.

In [85], the decay of vortices in the strictly two-spatial-dimensional context was considered. There, the vortex was classically stable at a given radius R_0. Through quantum tunnelling, the vortex could tunnel to a larger vortex of radius R_1, which was no longer classically stable. Dynamically the interior of the vortex was at the true vacuum, thus energetically lower by the energy density splitting multiplied by the area of the vortex. The vacuum energy behaves as $\sim -\epsilon R^2$, while the magnetic field energy behaves like $\sim 1/R^2$ and the energy in the wall behaves like $\sim R$. Thus the energy functional has the form

$$E = \alpha/R^2 + \beta R - \epsilon R^2. \tag{6.143}$$

For sufficiently small ϵ, this energy functional is dominated by the first two terms. It is infinitely high for a small radius due to the magnetic energy, and will diminish to a local minimum when the linear wall energy begins to become important. This occurs at a radius R_0, well before the quadratic area energy, due to the energy splitting between the false vacuum and the true vacuum becoming important, when ϵ is sufficiently small. Clearly, though, for large enough radius of the thin-wall string configuration, the energy splitting will be the most important term, and a thin-walled vortex configuration of sufficiently large radius will be unstable to expanding to infinite radius. However, a vortex of radius R_0 will be classically stable and only susceptible to decay via quantum tunnelling. The amplitude for such tunnelling, in the semi-classical approximation in the strictly two-dimensional context, has been calculated in [85].

Here we consider the model in a 3+1-dimensional setting. The vortex can be continued along the third additional dimension as a string, called a cosmic string. The interior of the string contains a large magnetic flux distributed over a region of the true vacuum. It is separated by a thin wall from the outside, where the scalar field is in the false vacuum. The analysis of the decay of two-dimensional vortices cannot directly apply to the decay of the cosmic string, as the cosmic string must maintain continuity along its length. Thus the radius of the string at a given position cannot spontaneously make the quantum tunnelling transition to the larger iso-energetic radius, called R_1, as it is continuously connected to the rest of the string. The whole string could, in principle, spontaneously tunnel to the fat string along its whole length, but the probability of such a transition is strictly zero for an infinite string, and correspondingly small for a closed string loop. Here we will describe the tunnelling transition to a state that corresponds to a spontaneously formed bulge in the putatively unstable thin cosmic string.

6.7.2 Energetics and Dynamics of the Thin, False String

6.7.2.1 Set-up We consider the Abelian Higgs model (spontaneously broken scalar electrodynamics) with a modified scalar potential as in [85] but now generalized to 3+1 dimensions. The Lagrangian density of the model has the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi^*\phi), \tag{6.144}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi = (\partial_\mu + eA_\mu)\phi$. The potential is a sixth-order polynomial in ϕ [79, 111], written

$$V(\phi^*\phi) = \lambda(|\phi|^2 - \epsilon v^2)(|\phi|^2 - v^2)^2. \tag{6.145}$$

Note that the Lagrangian is no longer renormalizable in 3+1 dimensions; however, the understanding is that it is an effective theory obtained from a well-defined renormalizable fundamental Lagrangian. The fields ϕ and A_μ, the vacuum expectation value v have mass dimension 1, the charge e is dimensionless and λ has mass dimension 2 since it is the coupling constant of the sixth-order scalar potential. The potential energy density of the false vacuum $|\phi| = v$ vanishes, while that of the true vacuum has $V(0) = -\lambda v^6\epsilon$. We rescale as

$$\phi \to v\phi \quad A_\mu \to vA_\mu \quad e \to \lambda^{1/2}ve \quad x \to x/(v^2\lambda^{1/2}) \tag{6.146}$$

so that all fields, constants and the spacetime coordinates become dimensionless, then the Lagrangian density is still given by Equation (6.144) where now the potential is

$$V(\phi^*\phi) = (|\phi|^2 - \epsilon)(|\phi|^2 - 1)^2. \tag{6.147}$$

and there is an overall factor of $1/(\lambda v^2)$ in the action.

Initially, the cosmic string will be independent of z, the coordinate along its length, and will correspond to a tube of radius R with a trapped magnetic flux in the true vacuum inside, separated by a thin wall from the false vacuum outside. R will vary in Euclidean time τ and in z to yield an instanton solution. Thus we promote R to a field $R \to R(z,\tau)$. Hence we will look for axially symmetric solutions for ϕ and A_μ in cylindrical coordinates (r, θ, z, τ). We use the following ansatz for a vortex of winding number n:

$$\phi(r,\theta,z,\tau) = f(r,R(z,\tau))e^{in\theta}, \qquad A_i(r,\theta,z,\tau) = -\frac{n}{e}\frac{\varepsilon^{ij}r_j}{r^2}a(r,R(z,\tau)), \tag{6.148}$$

where ε^{ij} is the two-dimensional Levi–Civita symbol. This ansatz is somewhat simplistic; it is clear that if the radius of the cosmic string swells out at some range of z, the magnetic flux will dilute and hence through the (Euclidean) Maxwell's equations some "electric" fields will be generated. In three-dimensional, source-free Euclidean electrodynamics, there is no distinct electric field, the Maxwell equations simply say that the three-dimensional magnetic field is a divergence-free and rotation-free vector field that satisfies superconductor boundary conditions at the location of the wall. It is clear that the correct form of the electromagnetic fields will not simply be a diluted magnetic field that always points along the length of the cosmic string as with our ansatz; however, the correction will not give a major contribution and we will neglect it. Indeed, the induced fields will always be smaller by a power of $1/c^2$ when the usual units are used.

The Euclidean action functional for the cosmic string then has the form

$$S_E[A_\mu,\phi] = \frac{1}{\lambda v^2}\int d^4x \left[\sum_i \left(\frac{1}{2}F_{0i}F_{0i} + \frac{1}{2}F_{i3}F_{i3}\right) + \frac{1}{2}F_{03}F_{03} + \sum_{ij}\frac{1}{4}F_{ij}F_{ij} \right.$$
$$\left. + (\partial_\tau\phi)^*(\partial_\tau\phi) + (\partial_z\phi)^*(\partial_z\phi) + \sum_i D_i(\phi)^*(D_i\phi) + V(\phi^*\phi)\right] \quad (6.149)$$

where i,j take values just over the two transverse directions and we have already incorporated that $A_0 = A_3 = 0$.

Substituting Equations (6.147) and (6.148) into Equation (6.149), we obtain

$$S_E = \frac{2\pi}{\lambda v^2}\int dz d\tau \int_0^\infty dr\, r \left[\frac{n^2\dot{a}^2}{2e^2r^2} + \frac{n^2a'^2}{2e^2r^2} + \frac{n^2(\partial_r a)^2}{2e^2r^2} + \dot{f}^2 + f'^2 + (\partial_r f)^2 \right.$$
$$\left. + \frac{n^2}{r^2}(1-a)^2f^2 + (f^2-\epsilon)(f^2-1)^2\right], \quad (6.150)$$

where the dot and prime denote differentiation with respect to τ and z, respectively. Then $\dot{a} = \left(\frac{\partial a(r,R)}{\partial R}\right)\dot{R}$ and $a' = \left(\frac{\partial a(r,R)}{\partial R}\right)R'$, and likewise for f, hence the action becomes

$$S_E = \frac{2\pi}{\lambda v^2}\int dz d\tau \int_0^\infty dr\, r \left[\frac{n^2\left(\left(\frac{\partial a(r,R)}{\partial R}\right)\dot{R}\right)^2}{2e^2r^2} + \frac{n^2\left(\left(\frac{\partial a(r,R)}{\partial R}\right)R'\right)^2}{2e^2r^2} + \frac{n^2(\partial_r a)^2}{2e^2r^2}\right.$$
$$+ \left(\frac{\partial f(r,R)}{\partial R}\dot{R}\right)^2 + \left(\frac{\partial f(r,R)}{\partial R}R'\right)^2$$
$$\left. +(\partial_r f)^2 + \frac{n^2}{r^2}(1-a)^2f^2 + (f^2-\epsilon)(f^2-1)^2\right]$$
$$= \frac{2\pi}{\lambda v^2}\int dz \int_0^\infty dr\, r \left[\left(\frac{n^2}{2e^2r^2}\left(\frac{\partial a(r,R)}{\partial R}\right)^2 + \left(\frac{\partial f(r,R)}{\partial R}\right)^2\right)(\dot{R}^2 + R'^2)\right.$$
$$\left. + \frac{n^2(\partial_r a)^2}{2e^2r^2} + (\partial_r f)^2 + \frac{n^2}{r^2}(1-a)^2f^2 + (f^2-\epsilon)(f^2-1)^2\right]. \quad (6.151)$$

We note the two- (Euclidean) dimensional, rotationally invariant form $(\dot{R}^2 + R'^2)$ which appears in the kinetic term. This allows us to make the $O(2)$ symmetric ansatz for the instanton, and the easy continuation of the solution to Minkowski time, to a relativistically invariant $O(1,1)$ solution, once the tunnelling transition has been completed.

In the thin-wall limit, the Euclidean action can be evaluated essentially analytically, up to corrections which are smaller by at least one power of $1/R$. The method of evaluation is identical to that in [85] and we shall not give the details here; we get

$$S_E = \frac{1}{\lambda v^2}\int d^2x \frac{1}{2}M(R(z,\tau))(\dot{R}^2 + R'^2) + E(R(z,\tau)) - E(R_0), \tag{6.152}$$

where

$$M(R) = \left[\frac{2\pi n^2}{e^2 R^2} + \pi R\right] \tag{6.153}$$

$$E(R) = \frac{n^2\Phi^2}{2\pi R^2} + \pi R - \epsilon\pi R^2. \tag{6.154}$$

Φ is the total magnetic flux and R_0 is the classically stable thin tube string radius.

6.7.3 Instantons and the Bulge

6.7.3.1 Tunnelling Instanton We look for an instanton solution that is $O(2)$ symmetric. The appropriate ansatz is

$$R(z,\tau) = R(\sqrt{z^2+\tau^2}) = R(\rho) \tag{6.155}$$

with the imposed boundary condition that $R(\infty) = R_0$. It is useful to understand what this ansatz means. We expect that the solution will be localized in Euclidean two space, say around the origin. Far from the origin, the solution will be $R = R_0$. Thus if we go to $\tau = -\infty$, the string will be in its dormant, thin state, all at $R = R_0$. As Euclidean time progresses, at some Euclidean time $\tau = -R_1$ a small bulge, an increase in the radius, will start to form at $z = 0$. This bulge will then increase dramatically, until at $\tau = 0$ it will be distributed over a region of the original string of length $2R_1$, the factor of 2 because the radius of the $O(2)$ symmetric bubble is R_1 in both directions. Then the bubble will "bounce" back and shrink and the string will return to its original radius. An alternative description is in terms of the creation of a soliton–anti-soliton pair. The instanton solution will describe the transition from a string of radius R_0 at $\tau = -\infty$, to a point in $\tau = -R_1$ at $z = 0$ when a soliton–anti-soliton pair starts to be created. The configuration then develops a bulge which forms when the pair separates to a radius which again has to be R_1 because of $O(2)$ invariance and which is the bounce point of the instanton along the z-axis at $\tau = 0$. Finally the subsequent Euclidean time evolution continues in a manner which is just the (Euclidean) time reversal of evolution leading up to the bounce point configuration, until a simple cosmic string of radius R_0 is re-established for $\tau \geq R_1$ and all z, *i.e.* for $\rho \geq R_1$. The action functional is given by

$$S_E = \frac{2\pi}{\lambda v^2}\int d\rho\, \rho \left[\frac{1}{2}M(R(\rho))\left(\frac{\partial R(\rho)}{\partial \rho}\right)^2 + E(R(\rho)) - E(R_0)\right]. \tag{6.156}$$

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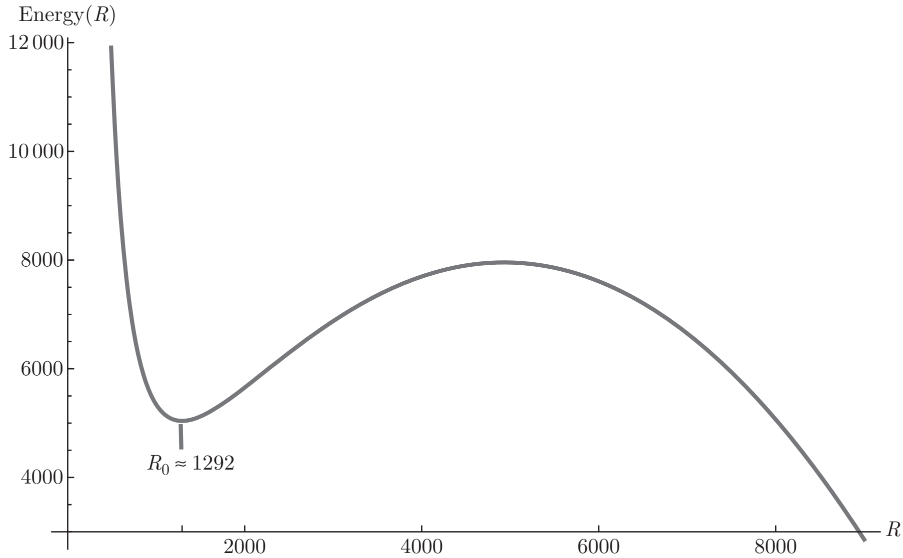

Figure 6.14. The energy as a function of R, for $n = 100, e = .005$ and $\epsilon = .001$

The instanton equation of motion is

$$\frac{d}{d\rho}\left(\rho M(R)\frac{dR}{d\rho}\right) - \frac{1}{2}\rho M'(R)\left(\frac{dR}{d\rho}\right)^2 - \rho E'(R) = 0 \tag{6.157}$$

with the boundary condition that $R(\infty) = R_0$, and we look for a solution that has $R \approx R_1$ near $\rho = 0$, where R_1 is the large radius for which the string is approximately iso-energetic with the string of radius R_0. The solution necessarily "bounces" at $\tau = 0$ since $\partial R(\rho)/\partial\tau|_{\tau=0} = R'(\rho)(\tau/\rho)|_{\tau=0} = 0$. (The potential singularity at $\rho = 0$ is not there since a smooth configuration requires $R'(\rho)|_{\rho=0} = 0$.)

The equation of motion is better cast as an essentially conservative, dynamical system with a "time"-dependent mass and the potential given by the inversion of the energy function as pictured in Figure 6.14, but in the presence of a "time"-dependent friction where ρ plays the role of time:

$$\frac{d}{d\rho}\left(M(R)\frac{dR}{d\rho}\right) - \frac{1}{2}M'(R)\left(\frac{dR}{d\rho}\right)^2 - E'(R) = -\frac{1}{\rho}\left(M(R)\frac{dR}{d\rho}\right). \tag{6.158}$$

As the equation is "time"-dependent, there is no analytic trick to evaluating the bounce configuration and the corresponding action. The solution must be found numerically, which starts with a given $R \approx R_1$ at $\rho = 0$ and achieves $R = R_0$ for $\rho > \rho_0$. We can be confident of the existence of a solution by showing the existence of an initial condition that gives an overshoot and another initial condition that gives an undershoot, as pioneered by Coleman [32, 23]. If we start at the origin at $\rho=0$ high enough on the far right side of the (inverted) energy functional pictured in Figure 6.14, the equation of motion, Equation (6.158), will cause the radius R to slide down the potential and then roll up the hill towards $R=R_0$. If we start too far up to the right, we will roll over the maximum at $R=R_0$, while if we do not start high enough we will never make it to the top of the hill at $R=R_0$. The right-hand side of Equation (6.158) acts as a "time"-dependent friction, which becomes negligible as $\rho\to\infty$, and once it is negligible, the motion is effectively conservative. It is not unrealistic to believe that there will be a correct initial point that will give exactly the solution that we desire, that as $\rho\to\infty$, $R(\rho)\to R_0$. We find the solution exists using numerical integration. For the parameter choice $n=100, e=.005$ and $\epsilon=.001$, if we start at $R\approx 11,506.4096$, we generate the profile function $R(\rho)$ in Figure 6.15. Actually, numerically integrating to $\rho\approx 80,000$ the function falls back to the minimum of the inverted energy functional Equation (6.14). On the other hand, if we increase the starting point by .0001, the numerical solution overshoots the maximum at $R=R_0$. Hence we have numerically implemented the overshoot/undershoot criterion of [32, 23].

The cosmic string emerges with a bulge described by the function numerically evaluated and represented in Figure 6.15 which corresponds to $R(z,\tau=0)$. A three-dimensional depiction of the bounce point is given in Figure 6.16. One should imagine the radius $R(z)$ along the cosmic string to be R_0 to the left, then bulging out to the the large radius as described by the mirror image of the function in Figure 6.15 and then returning to R_0 according to the function in Figure 6.15. This radius function has argument $\rho=\sqrt{z^2+\tau^2}$. Due to the Lorentz invariance of the original action, the subsequent Minkowski time evolution is given by $R(\rho)\to R(\sqrt{z^2-t^2})$, which is only valid for $z^2-t^2\geq 0$. Fixed $\rho^2=z^2-t^2$ describes a space-like hyperbola that asymptotes to the light cone. The value of the function $R(\rho)$ therefore remains constant along this hyperbola. This means that the point at which the string has attained the large radius moves away from $z\approx 0$ to $z\to\infty$ at essentially the speed of light. The other side moves towards $z\to-\infty$. Thus the soliton–anti-soliton pair separates quickly, moving at essentially the speed of light, leaving behind a fat cosmic string, which is subsequently classically unstable to expand and fill all space.

The rate at which the classical fat string expands depends on the actual value of ϵ. Once the string radius is large enough, its boundary wall is completely analogous to a domain wall that separates a true vacuum from a false vacuum. The true vacuum exerts a constant pressure on the wall, and it accelerates into the region of false vacuum. Obviously, if there is nothing to retard its expansion, it will accelerate to move at a velocity that eventually approaches the speed of light. The only effects retarding the velocity increase are the inertia and possible radiation. Radiation should be negligible as there are no massless fields in the exterior and there are no accelerating charges. The acceleration,

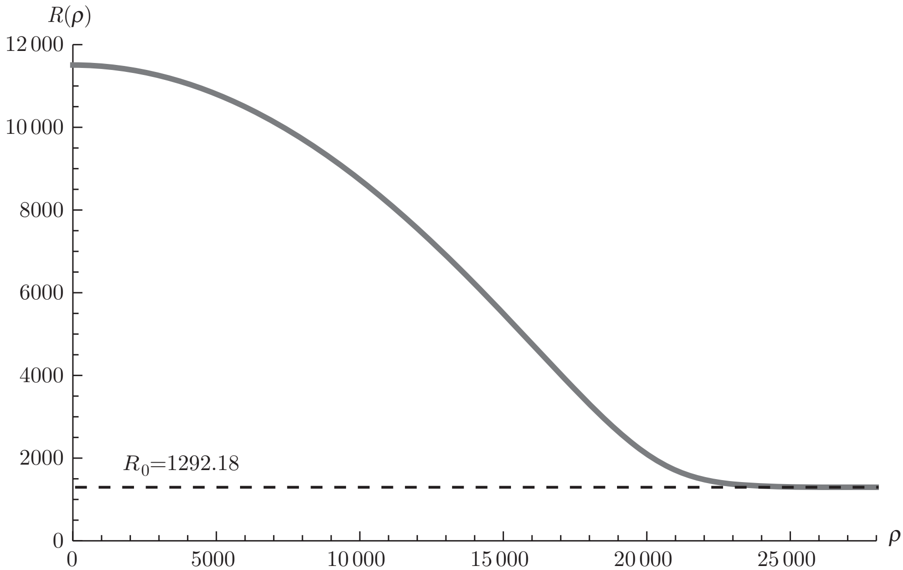

Figure 6.15. The radius as a function of ρ

a, is proportional to pressure divided by the mass per unit area. The pressure is simply the energy density difference, $p = \epsilon$. The mass per unit area can be obtained from Equation (6.153). Here the contribution to the mass per unit length from the wall is simply πR. Thus the mass per unit area, μ, is obtained from $\pi R \times L = \mu 2\pi R \times L$ for a given length L, which gives $\mu = 1/2$. Then we have

$$a \approx \epsilon/\mu = 2\epsilon. \tag{6.159}$$

Thus it is clear that this acceleration can be arbitrarily small, for small ϵ, and it is possible to imagine that once the tunnelling transition has occurred the fat cosmic string will exist and be identifiable for a long time.

6.7.4 Tunnelling Amplitude

It is difficult to say too much about the tunnelling amplitude or the decay rate per unit volume analytically in the parameters of the model. The numerical solution we have obtained for some rather uninspired choices of the parameters gives rise to the profile of the instanton given in Figure 6.15. This numerical solution could then be inserted into the Euclidean action to determine its numerical value, call it $S_0(\epsilon)$. It seems difficult to extract any analytical dependence on ϵ; however, it is reasonable to expect that as $\epsilon \to 0$ the tunnelling barrier, as can be seen in Figure 6.14, will get bigger and bigger and hence the tunnelling amplitude will vanish. On the other hand, there should exist a limiting value, call it ϵ_c, where the tunnelling barrier disappears at the so-called dissociation point [126, 81, 80],

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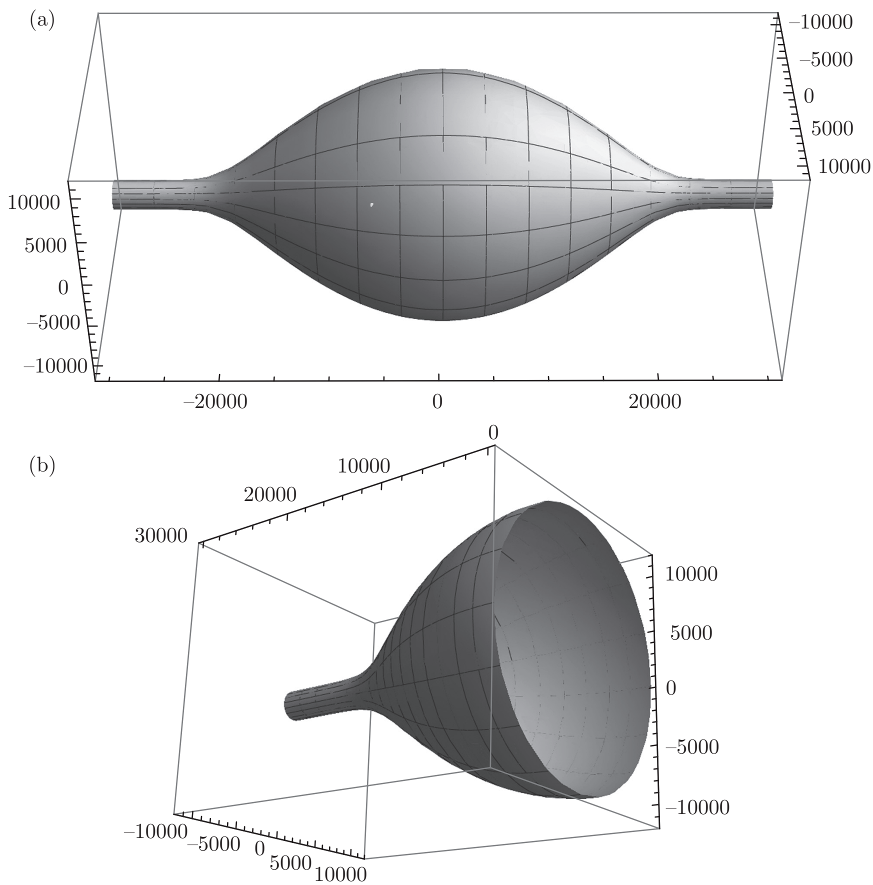

Figure 6.16. (a) Cosmic string profile at the bounce point. (b) Cut away of the cosmic string profile at bounce point

such that as $\epsilon \to \epsilon_c$, the action of the instanton will vanish, analogous to what was found in [85]. In general, the decay rate per unit length of the cosmic string will be of the form

$$\Gamma = A^{\text{c.s.}} \left(\frac{S_0(\epsilon)}{2\pi} \right) e^{-S_0(\epsilon)}, \tag{6.160}$$

where $A^{\text{c.s.}}$ is the determinantal factor excluding the zero modes and $\left(\frac{S_0(\epsilon)}{2\pi}\right)$ is the correction obtained after taking into account the two zero modes of the bulge instanton. These correspond to invariance under Euclidean time translation and spatial translation along the cosmic string [32, 23]. In general, there will be a length L of cosmic string per volume L^3. For a second-order phase transition to the meta-stable vacuum, L is the correlation length at the temperature of the transition which satisfies $L^{-1} \approx \lambda v^2 T_c$ [70, 69, 130, 129]. For first-order transitions, it is not clear what the density of cosmic strings will be. We will keep L as a parameter, but we expect that it is microscopic. Then in a large volume Ω, we will have a total length NL of cosmic string, where $N = \Omega/L^3$. Thus the decay rate for the volume Ω will be

$$\Gamma \times (NL) = \Gamma\left(\frac{\Omega}{L^2}\right) = A^{\text{c.s.}}\left(\frac{S_0(\epsilon)}{2\pi}\right) e^{-S_0(\epsilon)}\frac{\Omega}{L^2} \tag{6.161}$$

or the decay rate per unit volume will be

$$\frac{\Gamma \times (NL)}{\Omega} = \frac{\Gamma}{L^2} = \frac{A^{\text{c.s.}}\left(\frac{S_0(\epsilon)}{2\pi}\right) e^{-S_0(\epsilon)}}{L^2}. \tag{6.162}$$

A comparable calculation with point-like defects [85] would give a decay rate per unit volume of the form

$$\frac{\Gamma^{\text{point like}}}{L^3} = \frac{A^{\text{point like}}\left(\frac{S_0^{\text{point like}}(\epsilon)}{2\pi}\right)^{3/2} e^{-S_0^{\text{point like}}(\epsilon)}}{L^3} \tag{6.163}$$

and the corresponding decay rate from vacuum bubbles (without topological defects) [32, 23] would be

$$\Gamma^{\text{vac. bubble}} = A^{\text{vac. bubble}}\left(\frac{S_0^{\text{vac. bubble}}(\epsilon)}{2\pi}\right)^2 e^{-S_0^{\text{vac. bubble}}(\epsilon)}. \tag{6.164}$$

Since the length scale L is expected to be microscopic, we would then find that the number of defects in a macroscopic volume (*i.e.* universe) could be incredibly large, suggesting that the decay rate from topological defects would dominate over the decay rate obtained from simple vacuum bubbles [32, 23]. Of course the details depend on the actual values of the Euclidean action and the determinantal factor that is obtained in each case.

There are many instances where the vacuum can be meta-stable. The symmetry-broken vacuum can be meta-stable. Such solutions for the vacuum can be important for cosmology, and for the case of supersymmetry breaking see [1, 47] and the many references therein. In string cosmology, the inflationary scenario that has been obtained in [67] also gives rise to a vacuum that is meta-stable, and it must necessarily be long-lived to have cosmological relevance.

In a condensed matter context, symmetry-breaking ground states are also of great importance. For example, there are two types of superconductors [7]. The cosmic string is called a vortex-line solution in this context, and it is relevant to type II superconductors. The vortex line contains an unbroken symmetry region that carries a net magnetic flux, surrounded by a region of broken symmetry. If the temperature is raised, the true vacuum becomes the unbroken vacuum, and it is possible that the system exists in a superheated state where the false vacuum is meta-stable [41]. This technique has actually been used to construct detectors for particle physics [11, 105]. Our analysis might even describe the decay of vortex lines in superfluid liquid Helium III [86]. The decay of all of these meta-stable states could be described through the tunnelling transition mediated by instantons in the manner that we have computed. For appropriate limiting values of the parameters, for example when $\epsilon \to \epsilon_c$, the suppression of tunnelling is absent, and the existence of vortex lines or cosmic strings could cause the decay of the meta-stable vacuum without bound.