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CONSTRUCTION OF QUASIGROUPS SATISFYING THE IDENTITY $X(XY) = YX$

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1. **Introduction.** In [4], A. Sade defines the *singular direct product* for quasigroups. In this paper we use the singular direct product to construct quasigroups satisfying the identity $x(xy)=yx$. In particular, we show that the singular direct product preserves the identity $x(xy)=yx$.

DEFINITION. Let $(V, \odot)$ be an idempotent quasigroup, and $(Q, \circ)$ a quasigroup containing a subquasigroup P. Let $P'=Q\backslash P$ and let $(P', \otimes)$ be a quasigroup, where $\otimes$ is not necessarily related to $\circ$. On the set $P \cup (P' \times V)$ define the quasigroup $\oplus$ denoted by $V \times Q(P, P'_{\otimes})$ as follows:

(1) $p_1 \oplus p_2 = p_1 \circ p_2$ if $p_1, p_2 \in P$;

(2) $p \oplus (p', v) = (p \circ p', v)$ where $p \in P$, $p' \in P'$, $v \in V$;

(3) $(p', v) \oplus p = (p' \circ p, v)$ where $p \in P$, $p' \in P'$, $v \in V$;

(4) $(p'_1, v) \oplus (p'_2, v) = p'_1 \circ p'_2$ if $p'_1 \circ p'_2 \in P$
$\qquad = (p'_1 \circ p'_2, v)$ if $p'_1 \circ p'_2 \in P'$;

(5) $(p'_1, v_1) \oplus (p'_2, v_2) = (p'_1 \otimes p'_2, v_1 \odot v_2)$, $v_1 \neq v_2$.

The quasigroup $V \times Q(P, P'_{\otimes})$ so constructed is called the *singular direct product* of V and Q, [4].

2. **Construction of quasigroups satisfying** $x(xy)=yx$. Let $(V, \odot)$ and $(Q, \circ)$ be quasigroups satisfying $x(xy)=yx$ and let $(P, \circ)$ be a subquasigroup of $(Q, \circ)$. If $P'=Q\backslash P$ and there is a quasigroup of order $|P'|$ satisfying $x(xy)=yx$, then we can define on P' an operation $\otimes$ so that $(P', \otimes)$ satisfies this identity. Now take the singular direct product $V \times Q(P, P'_{\otimes})$.

THEOREM. *$V \times Q(P, P'_{\otimes})$ satisfies the identity $x(xy)=yx$.*

Proof. Let $a, b \in P \cup (P' \times V)$. We have five cases to consider.

(1) $\underline{a, b \in P}$. $a \oplus (a \oplus b) = a \oplus (a \circ b) = a \circ (a \circ b) = b \circ a = b \oplus a$.

(2) $\underline{a \in P,\ b \in (P' \times V)}$. Set $a=p$ and $b=(p', v)$. Then

$$\begin{aligned} a \oplus (a \oplus b) &= p \oplus (p \oplus (p', v)) = p \oplus (p \circ p', v) \\ &= (p \circ (p \circ p'), v) = (p' \circ p, v) \\ &= (p', v) \oplus p = b \oplus a. \end{aligned}$$

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(3) $\underline{a \in (P' \times V),\ b \in P}$. This is analogous to (2).

(4) $\underline{a = (p_1', v),\ b = (p_2', v)}$. In this case we have two subcases.

(i) $\underline{p_1' \circ p_2' \in P}$. Then

$$\begin{aligned} a \oplus (a \oplus b) &= (p_1', v) \oplus (p_1' \circ p_2') = (p_1' \circ (p_1' \circ p_2'), v) \\ &= (p_2' \circ p_1', v) = (p_2', v) \oplus (p_1', v) = b \oplus a, \end{aligned}$$

since if $p_1' \circ p_2' \in P$, then $p_1' \circ (p_1' \circ p_2') \in P'$.

(ii) $\underline{p_1' \circ p_2' \in P'}$. Then

$$a \oplus (a \oplus b) = (p_1', v) \oplus (p_1' \circ p_2', v).$$

If $p_1' \circ (p_1' \circ p_2') \in P$, then

$$a \oplus (a \oplus b) = P_1' \circ (p_1' \circ p_2') = p_2' \circ p_1' = (p_2', v) \oplus (p_1', v).$$

If $p_1' \circ (p_1' \circ p_2') \in P'$, then

$$a \oplus (a \oplus b) = (p_1' \circ (p_1' \circ p_2'), v) = (p_2' \circ p_1', v) = (p_2', v) \oplus (p_1', v).$$

In either case we have $a \oplus (a \oplus b) = b \oplus a$.

(5) $\underline{a = (p_1', v_1),\ b = (p_2', v_2),\ v_1 \neq v_2}$. Then

$$a \oplus (a \oplus b) = (p_1', v_1) \oplus (p_1' \otimes p_2', v_1 \odot v_2).$$

Since $v_1 \odot v_2 \neq v_1$ and $p_1' \otimes p_2' \in P'$,

$$\begin{aligned} (p_1', v_1) \oplus (p_1' \otimes p_2', v_1 \odot v_2) &= (p_1' \otimes (p_1' \otimes p_2')), (v_1 \odot (v_1 \odot v_2)) \\ &= (p_2' \otimes p_1', v_2 \odot v_1) = (p_2', v_2) \oplus (p_1', v_1) = b \oplus a. \end{aligned}$$

Hence in all cases $a \oplus (a \oplus b) = b \oplus a$ which completes the proof.

COROLLARY. *If there are quasigroups V, Q, P, and P', of orders v, q, p, p' respectively, satisfying $x(xy) = yx$ and such that P is a subquasigroup of Q and $q = p + p'$, then there is a quasigroup of order $v(q-p)+p$ satisfying $x(xy) = yx$.*

3. REMARKS. It is known that there are quasigroups of order $4^k \cdot m$ satisfying this identity, where $k \geq 0$ and every prime divisor of m for which 5 is a quadratic nonresidue appears with even multiplicity [3]. Also $12k+1$, $12k+4$, $20k+1$, and $20k+5$ (see [1], [2], and [5]). Since there are quasigroups of orders 4 and 5 satisfying $x(xy) = yx$ the above procedure produces a quasigroup of order $17 = 4(5-1) + 1$ satisfying this identity. So far as the author can tell this is the first quasigroup of order 17 produced satisfying this identity. So that the construction given in this note does enlarge the known class of quasigroups satisfying this identity.

4. **Problems.** The problem of determining the spectrum of the identity $x(xy) = yx$ was raised by S. K. Stein [6]. The results obtained in this note should aid in this

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investigation. Another problem worth looking at is the classification of quasigroup identities preserved under the singular direct product.

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