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Torsion theories and coherent rings

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Chase has given several characterizations of a right coherent ring, among which are: every direct product of copies of the ring is left-flat; and every finitely generated submodule of a free right module is finitely related. We extend his results to obtain conditions for the ring of quotients of a ring with respect to a torsion theory to be coherent.

We use *ring* to mean an associative ring with identity. Our reference for torsion theories is Stenström [2].

In [1], Chase proves that for a ring A, the following statements are equivalent:

(1) every direct product of copies of A is a flat left A-module;

(2) the class C of flat left A-modules is closed under the formation of direct products;

(3) every finitely generated submodule of a free right A-module is finitely related;

(4) every finitely generated right ideal of A is finitely related.

Bourbaki has called a ring with these properties *right coherent*. In order to relate these results to torsion theories, let A be a ring and t a (hereditary) torsion radical on right A-modules ([2], p. 8).

In place of the class C it is now appropriate to consider the narrower class C_t of *t-flat* left A-modules: N is *t-flat* if it is flat, and in addition

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$$T \otimes_A N = 0$$

for all right torsion modules T ; this last condition is equivalent to

$$JN = N$$

for all dense right ideals J (by *dense*, we mean that A/J is a torsion module).

Having narrowed the class C to C_t we find that we must broaden the concept *finitely related* to *t-finitely related*: a finitely generated right A-module M is *t-finitely related* if there exists an exact sequence

$$0 \to K \to F \to M \to 0$$

in which F is free of finite rank and K contains a finitely generated dense submodule K' .

Our first theorem is then:

THEOREM 1. *If the torsion theory defined by* t *satisfies:*

(i) every dense right ideal of A *contains a finitely generated dense right ideal,*

(ii) every finitely generated right ideal of A *is* *t-finitely related,*

then the class C_t *is closed under the formation of direct products.*

Our second theorem is:

THEOREM 2. *If the torsion theory defined by* t *is perfect* ([2], p. 74), *and* B *is the ring of quotients of* A *in this theory* ([2], p. 35), *then the following statements are equivalent:*

(a) the ring B *is coherent;*

(b) every finitely generated submodule of a free right A*-module is t-finitely related;*

(c) every finitely generated right ideal of A *is* *t-finitely related;*

(d) the class C_t *is closed under the formation of direct products;*

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(e) every direct product of copies of B is a flat left A-module.

Note that we recover Chase's original results by taking t to be the trivial torsion radical $(t(M) = 0$ for every module $M)$.

Proof of Theorem 1. Let $\{N_\sigma\}$ be a set of t-flat left A-modules, and put $N = \prod_\sigma N_\sigma$. We must show that N is t-flat, that is,

(1) $JN = N$ for every dense right ideal J ,

(2) N is flat.

To prove (1), let J be a dense right ideal. By condition *(i)* of Theorem 1, J contains a finitely generated dense right ideal J' , and it is clearly enough to show that $J'N = N$.

Since each module N_σ is t-flat, we have $J'N_\sigma = N_\sigma$. Let $\langle \xi_\sigma \rangle \in N$. Each ξ_σ may be written

$$\xi_\sigma = \sum_{i=1}^{m} u_i a_{\sigma i} ,$$

where $\{u_i\}_1^m$ is a generating set for J' , and $a_{\sigma i} \in N$. But then

$$\langle \xi_\sigma \rangle = \sum_{i=1}^{m} u_i \langle a_{\sigma i} \rangle \in J'N ,$$

so that $J'N = N$ as required.

To prove (2), we show that if J is any finitely generated right ideal then the map $\mu : J \otimes_A N \to N$ given by $\mu(c \otimes x) = cx$ is injective. Therefore suppose that

$$\mu\left(\sum_k c_k \otimes \langle x_{\sigma k} \rangle\right) = \sum_k c_k \langle x_{\sigma k} \rangle = 0 , \tag{2.1}$$

where $c_k \in J$ and $\langle x_{\sigma k} \rangle \in N$. We must show that

$$\sum_k c_k \otimes \langle x_{\sigma k} \rangle = 0 \text{ in } J \otimes_A N . \tag{2.2}$$

By condition *(ii)* of Theorem 1, there exists an exact sequence of right A-modules

$$0 \to K \xrightarrow{f} F \xrightarrow{g} J \to 0$$

in which F is free of finite rank and K contains a finitely generated dense submodule K'. Since each N_σ is t-flat, the inclusion $K' \subset K$ induces isomorphisms

$$K' \otimes_A N_\sigma \to K \otimes_A N_\sigma .$$

We therefore obtain exact sequences

(2.3) $$0 \to K' \otimes_A N_\sigma \xrightarrow{f_\sigma} F \otimes_A N_\sigma \xrightarrow{g_\sigma} J \otimes_A N_\sigma \to 0 ,$$

where the maps f_σ and g_σ are defined in the obvious way.

Let $\{u_i\}_1^n$ be a generating set for K', and $\{v_j\}_1^m$ be a free base for F. The elements

$$w_j = g(v_j) \quad (j = 1, \ldots, m)$$

therefore generate J. We also need to express each element $f(u_i)$ in terms of the base $\{v_j\}_1^m$:

(2.4) $$f(u_i) = \sum_{j=1}^{m} v_j l_{ji} ,$$

where $l_{ji} \in A$.

Now to prove (2.2). From (2.1) we obtain

$$\sum_k c_k x_{\sigma k} = 0 ;$$

and since N_σ is flat, this yields

$$\sum_k c_k \otimes x_{\sigma k} = 0 \quad \text{in} \quad J \otimes_A N_\sigma .$$

Writing

$$c_k = \sum_{j=1}^{m} w_j t_{jk} \quad (t_{jk} \in A) ,$$

(as we may, since the elements w_j generate J) we obtain

$$0 = \sum_k c_k \otimes x_{\sigma k} = \sum_{j,k} w_j t_{jk} \otimes x_{\sigma k} = \sum_{j,k} w_j \otimes t_{jk} x_{\sigma k}$$
$$= g_\sigma\left(\sum_{j,k} v_j \otimes t_{jk} x_{\sigma k}\right) ;$$

and since (2.3) is exact, $K' \otimes_A N$ contains an element $\sum_{s=1}^{n} u_s \otimes y_{\sigma s}$ such that

$$f_\sigma\left(\sum_{s=1}^{n} u_s \otimes y_{\sigma s}\right) = \sum_{j,k} v_j \otimes t_{jk} x_{\sigma k} .$$

Combining this with (2.4), we obtain

$$\sum_{j,s} v_j \otimes l_{js} y_{\sigma s} = \sum_{j,k} v_j \otimes t_{jk} x_{\sigma k} ;$$

and since $\{v_j\}_1^m$ is a free base for F , we must have

$$\sum_{s=1}^{n} l_{js} y_{\sigma s} = \sum_k t_{jk} x_{\sigma k} . \tag{2.5}$$

(2.2) now follows easily:

$$\sum_k c_k \otimes \langle x_{\sigma k} \rangle = \sum_{j,k} w_j t_{jk} \otimes \langle x_{\sigma k} \rangle = \sum_j w_j \otimes \left\langle \sum_k t_{jk} x_{\sigma k} \right\rangle$$
$$= \sum_j w_j \otimes \left\langle \sum_s l_{js} y_{\sigma s} \right\rangle \text{ by (2.5)}$$
$$= \sum_{j,s} w_j l_{js} \otimes \langle y_{\sigma s} \rangle = gf(u_s) \otimes \langle y_{\sigma s} \rangle$$
$$= 0 .$$

The proof of Theorem 1 is now complete.

Proof of Theorem 2. We shall prove that *(a)* ⇒ *(b)* ⇒ *(c)* ⇒ *(d)* ⇒ *(e)* ⇒ *(a)*. Indeed, *(b)* ⇒ *(c)* is trivial; and *(d)* ⇒ *(e)* is trivial, for B is a flat left A-module since the torsion theory is assumed to be perfect ([2], p. 73). *(c)* ⇒ *(d)* follows from Theorem 1 since a perfect torsion theory always satisfies condition *(i)* of that theorem ([2], p. 73). We are therefore reduced to proving:

(e) ⇒ *(a)*. We need the following result ([2], p. 85). Let B be the ring of quotients of a ring A with respect to a perfect torsion theory, and $\phi : A \to B$ the associated ring homomorphism. If M is a right B-module and N a left B-module, then

$$\text{(3)} \qquad \operatorname{Tor}_1^A(M, N) \simeq \operatorname{Tor}_1^B(M, N)$$

where, to define $\operatorname{Tor}_1^A(M, N)$, we interpret M, N as A-modules via ϕ.

To prove *(e)* ⇒ *(a)*, observe that *(e)* implies that any direct product N of copies of B is a flat left A-module. (3) now shows that N is also a flat left B-module. We therefore conclude from Chase's results that B is coherent.

(a) ⇒ *(b)*. Assume that B is coherent, and let

$$\text{(4)} \qquad 0 \to K \to F \to M \to 0 ,$$

be any exact sequence of right A-modules in which F is free of finite rank and M is a submodule of a free module. We shall prove *(b)* by showing that K contains a finitely generated dense submodule. Applying $\otimes_A B$ to (4), we obtain the exact sequence of right B-modules

$$0 \to K \otimes_A B \to F \otimes_A B \to M \otimes_A B \to 0$$

in which $F \otimes_A B$ is free of finite rank and $M \otimes_A B$ is isomorphic to a submodule of a free module. From the coherence of B we deduce that $K \otimes_A B$ is finitely generated. As generators we may evidently choose elements

$$u_i \otimes 1 \quad (u_i \in K , \quad i = 1, \ldots, m) .$$

If K' is the A-submodule of K generated by the elements $\{u_i\}_1^m$ it follows that $K' \otimes_A B = K \otimes_A B$, or that $K/K' \otimes_A B = 0$. Since $\ker(M \to M \otimes_A B) = t(M)$ for a perfect torsion theory ([2], p. 73), we deduce that K/K' is a torsion module.

K' is therefore the required finitely generated dense submodule of K; and the proof of Theorem 2 is complete.

References

[1] Stephen U. Chase, "Direct products of modules", *Trans. Amer. Math. Soc.* **97** (1960), 457-473.

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