Canad. Math. Bull. Vol. **26** (2), 1983

L^p SPACES FROM MATRIX MEASURES: A CORRECTION AND THEIR INTERPOLATION

BY
PATRICK J. BROWNE* AND CLAUDE COSTA

ABSTRACT. We discuss the construction of the spaces $L^p(\mu_{ij})$, $1\le p\le\infty$, where $\{\mu_{ij}\}$ is an $n\times n$ positive matrix measure, correct a mistake in the literature concerning those spaces and develop an interpolation theory for them.

This paper has two aims. Firstly we shall correct an error which appears in [2] wherein the theory of the matrix measure function spaces $L^p(\mu_{ij})$, $p\ge 1$, is presented and secondly we shall develop an interpolation theory for these spaces.

1. **The Spaces** $L^p(\mu_{ij})$. We commence with a brief outline of the construction of the spaces $L^p(\mu_{ij})$. Most of the details can be found in [2] and [4].

DEFINITION 1. Let $\{\mu_{ij}\}$, $1\le i,\ j\le n$, be a family of complex valued set functions defined on the bounded Borel subsets of the real line. The family $\{\mu_{ij}\}$ will be called an $n\times n$ *positive matrix measure* if

(i) the matrix $\{\mu_{ij}(e)\}$ is Hermitian and positive semidefinite for each bounded Borel set e, and

(ii)
$$\mu_{ij}\left(\bigcup_{m=1}^{\infty}\right)e_m=\sum_{m=1}^{\infty}\mu_{ij}(e_m),\qquad 1\le i,j\le n,$$

for each sequence $\{e_m\}$ of pairwise disjoint Borel sets with bounded union.

DEFINITION 2. Let $\{\mu_{ij}\}$ be an $n\times n$ positive matrix measure defined on the bounded Borel sets of the real line and let ν be a non-negative regular σ-finite Borel measure with respect to which each μ_{ij} is absolutely continuous. Let the matrix of densities $M=\{m_{ij}\}$ be defined by the equations

$$\mu_{ij}(S)=\int_S m_{ij}(t)\,d\nu(t),\qquad 1\le i,j\le n,$$

Received by the editors May 21, 1981 and in revised form, January 8, 1982 and May 7, 1982.
AMS Classification 46E30.
* Research supported in part by a grant from the NSERC of Canada.

https://doi.org/10.4153/CMB-1983-022-x Published online by Cambridge University Press

where S is any bounded Borel set. For $1 \le p < \infty$, the space $L_0^p(\mu_{ij})$ is defined to be the space of all n-tuples of Borel functions $F(t) = (F_1(t), \ldots, F_n(t))$ such that

$$\|F\| = \left[\int_{-\infty}^{\infty} [F^*(t)M(t)F(t)]^{p/2}\, d\nu(t)\right]^{1/p} < \infty.$$

Here, and throughout, we write

$$F^*(t)M(t)F(t) = \sum_{i,j=1}^{n} F_i(t) m_{ij}(t) \overline{F_j(t)}.$$

If D denotes the subspace of $L_0^p(\mu_{ij})$ consisting of those F with $\|F\| = 0$ we define $L^p(\mu_{ij})$ to be the quotient space $L^p(\mu_{ij})/D$.

The space $L^\infty(\mu_{ij})$ is defined as usual using essential suprema in place of integrals.

It is easy to check that the $L^p(\mu_{ij})$, $p \ge 1$ are normed linear spaces.

DEFINITION 3. If $S \subseteq \mathbb{R}$ is a Borel set, $k \ge 1$ is an integer and ν is a non-negative regular σ-finite Borel measure, the space $L^p(C^k, S, \nu)$ is the space of (equivalence classes of) complex k-vector valued functions G on S normed by

$$\|G\| = \left[\int_S \left[\sum_{i=1}^{k} |G_i(t)|^2\right]^{p/2} d\nu(t)\right]^{1/p}, \qquad 1 \le p < \infty,$$

$$\|G\| = \nu\text{-ess}\sup_{t \in S} \left[\sum_{i=1}^{k} |G_i(t)|^2\right]^{1/2}, \qquad p = \infty.$$

DEFINITION 4. Given n normed linear spaces $X_1, \ldots, X_n$, we define $l^p(X_i)$, $1 \le p \le \infty$ to be the space $\oplus_{i=1}^n X_i$ with

$$\|(F_1, \ldots, F_n)\| = \left(\sum_{i=1}^{n} \|F_i\|^p\right)^{1/p}, \qquad 1 \le p < \infty$$

$$= \sup_{1 \le i \le n} \|F_i\|, \qquad p = \infty.$$

THEOREM 1. *Let $\{\mu_{ij}\}$ be an $n \times n$ positive matrix measure on $\mathbb{R}$ and let ν be a non-negative regular σ-finite Borel measure with respect to which each μ_{ij} is absolutely continuous. Then there exists a collection of pairwise disjoint ν-measureable sets $S_1, \ldots, S_n$*

$$L^p(\mu_{ij}) \cong l^p(L^p(C^i, S_i, \nu)). \qquad 1 \le p \le \infty.$$

Here $\cong$ denotes isometric isomorphism.

The details of the proof of this result can be found in [2]; it is sufficient for us to note that the isometric isomorphism is uniquely determined by the matrix measure $\{\mu_{ij}\}$ and has the same analytic form for each value of p.

We close this introduction by giving an example to show that Theorem 1 of [2] which claims that the space $L^p(\mu_{ij})$ is independent of the measure ν used to define it, is correct only for the case $p=2$. A proof for the case $p=2$ can be found in [4].

Let μ be a positive (one by one matrix) measure defined on the bounded Borel sets of the real line and let ν be a non-negative regular σ-finite Borel measure with respect to which μ is absolutely continuous. Let the density M be defined by the equation

$$\mu(S)=\int_S M(t)\,d\nu(t)$$

where S is any bounded Borel set. Then using our definition of norm of a function we obtain

$$\|F\|_\nu^p=\int_{-\infty}^{\infty} (M(t)|F(t)|^2)^{p/2}\,d\nu(t).$$

Let $a\in\mathbb{R}^+$ and define $\tilde{\nu}=a\nu$. Then $\tilde{\nu}$ is a non-negative regular σ-finite Borel measure with respect to which μ is absolutely continuous. $\tilde{\nu}$ has a density $\tilde{M}$ which satisfies $M(t)=a\tilde{M}(t)$. Then we now obtain

$$\begin{aligned}\|F\|_{\tilde{\nu}}^p &=\int_{-\infty}^{\infty} (M(t)\,|F(t)|^2)^{p/2}\,d\tilde{\nu}(t)\\ &=\int_{-\infty}^{\infty} (M(t)\,|F(t)|^2)^{p/2}\,a^{1-p/2}\,d\nu(t)\\ &\neq\|F\|_\nu\end{aligned}$$

unless $p=2$. Hence our norm is not ν-independent. Of course in this simple example the spaces $L^p(\mu_{ij})$ constructed with ν and $\tilde{\nu}$ are easily seen to be isomorphic.

The fault with the proof of Theorem 1 of [2] lies in the fact that the function G defined towards the end of the proof is ν-dependent when $p\neq 2$.

2. **Interpolation theory, some preparatory theorems.** Our standard reference for results from the theory of interpolation spaces will be [1] and we shall follow precisely the notation and terminology used there.

We present a brief survey of the material needed.

Definition 5. Two Banach spaces A_0, A_1 are said to form a *compatible couple* if there is a Hausdorff topological vector space $\mathcal{A}$ so that A_0, A_1 are subspaces of $\mathcal{A}$. In $A_0\cap A_1$ and A_0+A_1 we use the norms

$$\|a\|_{A_0\cap A_1}=\max(\|a\|_{A_0}, \|a\|_{A_1}),$$

$$\|a\|_{A_0+A_1}=\inf_{a=a_0+a_1}(\|a_0\|_{A_0}+\|a_1\|_{A_1}).$$

https://doi.org/10.4153/CMB-1983-022-x Published online by Cambridge University Press

A Banach space A will be called an *intermediate space* with respect to (A_0, A_1) if

$$A_0 \cap A_1 \subset A \subset A_0 + A_1$$

with continuous inclusions.

DEFINITION 6. An intermediate space A will be called an *interpolation space* with respect to (A_0, A_1) if for any linear map T with $T: A_0 + A_1 \to A_0 + A_1$ and $T: A_i \to A_i (i = 0, 1)$ continuously with respect to the appropriate norm in each case, we have $T: A \to A$ continuously.

There exist many ways of constructing interpolation spaces; the different interpolation methods are often referred to as *interpolation functors*. If A is an interpolation space with respect to (A_0, A_1) obtained by some method which we call F, we write, using functor notation, $A = F(A_0, A_1)$.

The next two definitions present the so-called "real interpolation method" and "complex interpolation method".

DEFINITION 7. For $a \in A_0 + A_1$ and $t > 0$ we define

$$k(t, a) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

If $0 < \theta < 1$, $1 \leq q < \infty$ or if $0 \leq \theta \leq 1$, $q = \infty$, the space $(A_0, A_1)_{\theta,q}$ is defined to consist of all $a \in A_0 + A_1$ for which

$$\|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} k(t, a))^q \, dt/t\right)^{1/q} < \infty.$$

For $q = \infty$, we use an essential supremum in place of the integral. It can be shown that the space $(A_0, A_1)_{\theta,q}$ is an interpolation space with respect to (A_0, A_1). The interpolation functor we just described is called the "real interpolation method".

DEFINITION 8. Let $\mathscr{F}(A_0, A_1)$ be the Banach space of all functions f with values in $A_0 + A_1$ which have the following properties:

(i) f is bounded and continuous on the strip $S = \{z \mid 0 \leq Re z \leq 1\}$ and analytic on the interior of S.
(ii) f is such that the functions $t \to f(it)$, $t \to f(1 + it)$ are continuous from $\mathbb{R}$ to A_0, A_1 respectively and tend to zero as $|t| \to \infty$.

A norm on $\mathscr{F}(A_0, A_1)$ is provided by:

$$\|f\| = \max(\sup \|f(t)\|_A, \sup \|f(1 + it)\|_A)$$

The space which consists of all $a \in A_0 + A_1$ for which $a = f(\theta)$ for some $f \in \mathscr{F}(A_0, A_1)$ and whose norm is given by

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathscr{F}} : f(\theta) = a, f \in \mathscr{F}(A_0, A_1)\}$$

can be shown to be an interpolation space with respect (A_0, A_1). This interpolation functor is called the "complex interpolation method".

Again, full details of these constructions can be found in [1]. We also have, from this source,

DEFINITION 9. Two compatible couples (A_0, A_1), $(\tilde{A}_0, \tilde{A}_1)$ are said to be *isometrically isomorphic* if there is a bijection $\xi:(A_0, A_1) \to (\tilde{A}_0 + \tilde{A}_1)$ such that the restriction of ξ to A_0 maps A_0 isometrically onto $\tilde{A}_0$, while the restriction of ξ to A_1 maps A_1 isometrically onto $\tilde{A}_1$.

THEOREM 2. *Let (A_0, A_1), $(\tilde{A}_0, \tilde{A}_1)$ be two isometrically isomorphic compatible couples. Then $(A_0, A_1)_{\theta,q}$ is isometrically isomorphic to $(\tilde{A}_0, \tilde{A}_1)_{\theta,q}$ and $(A_0, A_1)_{[\theta]}$ is isometrically isomorphic to $(\tilde{A}_0, \tilde{A}_1)_{[\theta]}$.*

Proof. Note that for $a \in A_0 + A_1$ we have

$$k(t, a, A_0 + A_1) = k(t, a, \tilde{A}_0 + \tilde{A}_1)$$

which yields

$$\|a\|_{(A_0,\, A_1)_{\theta,q}} = \|\xi a\|_{(\tilde{A}_0,\, \tilde{A}_1)_{\theta,q}},$$

so proving the first claim. Further if $f \in \mathscr{F}(A_0, A_1)$ then $\xi f \in \mathscr{F}(\tilde{A}_0, \tilde{A}_1)$ and $\|f\|_{\mathscr{F}(A_0,A_1)} = \|\xi f\|\ \mathscr{F}(\tilde{A}_0, \tilde{A}_1)$. In fact ξ generates an isometric isomorphism between $\mathscr{F}(A_0, A_1)$ and $\mathscr{F}(\tilde{A}_0, \tilde{A}_1)$. The second result now follows easily.

We now give, through the next lemma, a recharacterization of the space $L^p(\mu_{ij})$.

In the notation of Theorem 1, let $S = \bigcup_{i=1}^n S_i$ and let $(\mathscr{T}, \tilde{\nu})$ be the measure space obtained by taking for $\mathscr{T}$ the disjoint union of n copies of S each carrying a copy of its measure ν.

We note that the spaces $l^p(L^p(\mathbb{C}^i, S_i, \nu))$ described in Section 1 are closed subspaces of the spaces $l^p(L^p(\mathbb{C}^n, S_i, \nu))$; in fact these subspaces are uniformly complemented for we can obtain a projection on $l^p(L^p(\mathbb{C}^n, S_i, \nu))$ whose range is $l^p(L^p(\mathbb{C}^i, S_i, \nu))$.

To obtain this projection for a given $F = (F_1, \ldots,, F_n)$ in $l^p(L^p(\mathbb{C}^n, S_i, \nu))$, map each F_i in $L^p(\mathbb{C}^n, S_i, \nu)$ to PF_i in $L^p(\mathbb{C}^i, S_i, \nu)$ where P is the orthogonal projection of $\mathbb{C}^n$ onto $\mathbb{C}^i$.

LEMMA 1. *If $\mathbb{C}^n$ is given the p-norm*

$$\|\boldsymbol{\alpha}\| = \left(\sum_{k=1}^n |\alpha_i|^p\right)^{1/p}, \qquad \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n),$$

then the spaces $l^p(L^p(\mathbb{C}^n, S_i, \nu))$ are isometrically isomorphic to the spaces $L^p(\mathbb{C}, \mathscr{T}, \tilde{\nu})$.

Proof. $\mathscr{T}$ can be written as $\mathscr{T} = \dot{\bigcup}_{i=1}^n S^i$, the S^i being disjoint copies of

S, $1 \le j \le n$. Each S^j can in turn be written $S^j = \dot{\bigcup}_{i=1}^n (S^j)_i$, the $(S^j)_i$ being the corresponding disjoint copies of S_i, $1 \le i, j \le n$.

Given F in $l^p(L^p(\mathbb{C}^n, S_i, \nu))$ then $F = (F_1, \ldots, F_n)$ where $F_j = (F_{j1}, \ldots, F_{jn})$ with $F_{ji} : (S^j)_i \to \mathbb{C}$. We now define

$$G(\tau) = F_{ji}(t) \quad \text{when} \quad \tau = t \in (S^j)_i.$$

It is easy to check that the correspondence between F and G is an isometric isomorphism.

We should also note that this construction has the same form for all values of p.

We close this section with a result on interpolation of complemented subspaces.

THEOREM 3. [**5**, Theorem 1, p. 118] *Let* (A_0, A_1) *be a compatible couple. Let* B *be a complemented subspace of* $A_0 + A_1$ *whose projection* P *belongs to* $L((A_0, A_1), (A_0, A_1))$. *Let* F *be an arbitrary interpolation functor. Then* $(A_0 \cap B, A_1 \cap B)$ *is also a compatible couple and*

$$F(A_0 \cap B, A_1 \cap B) = F(A_0, A_1) \cap B.$$

3. **Interpolation of the spaces** $L^p(\mu_{ij})$. We now apply our preparatory theorems to the interpolation of the spaces $L^p(\mu_{ij})$. We present here a representative result. A collection of further theorems may be found in [3].

THEOREM 4. *Let* $1 \le p_0, p_1 < \infty, 0 < \theta < 1$ *and*

$$1/p = (1-\theta)/p_0 + \theta/p_1$$

Then

$$(L^{p_0}(\mu_{ij}), L^{p_1}(\mu_{ij}))_{\theta,p} \cong L^p(\mu_{ij}) \qquad (\textit{equivalent norms}),$$

and

$$(L^{p_0}(\mu_{ij}), L^{p_1}(\mu_{ij}))_{[\theta]} \cong L^p(\mu_{ij}), \qquad (\textit{equivalent norms}).$$

Proof. It is clear that $(L^{p_0}(\mu_{ij}), L^{p_1}(\mu_{ij}))$ is a compatible couple. From our previous sections we have

$$L^{p_0}(\mu_{ij}) \cong l^{p_0}(L^{p_0}(\mathbb{C}^i, S_i, \nu)) \subseteq l^{p_0}(L^{p_0}(\mathbb{C}^n, S_i, \nu))$$
$$\cong L^{p_0}(\mathbb{C}, \mathcal{T}, \tilde{\nu}) \quad \text{(equivalent norms)}$$

wish similar results holding with p_1 in place of p_0. We further know from [1, Chapter 5, Section 5.1 and Section 5.2] that

$$(L^{p_0}(\mathbb{C}, \mathcal{T}, \tilde{\nu}), L^{p_1}(\mathbb{C}, \mathcal{T}, \tilde{\nu}))_{\theta,p} \cong L^p(\mathbb{C}, S, \tilde{\nu}) \qquad \text{(equivalent norms)},$$
$$(L^p(\mathbb{C}, \mathcal{T}, \tilde{\nu}), L^{p_1}(\mathbb{C}, \mathcal{T}, \tilde{\nu}))_{[\theta]} \cong L^p(\mathbb{C}, S, \tilde{\nu}), \qquad \text{(equivalent norms)}.$$

The result is now immediate using Theorem 3 and Lemma 1.

https://doi.org/10.4153/CMB-1983-022-x Published online by Cambridge University Press

In our original version of this paper, Theorem 4 was proved only for the real method. We are grateful to a referee for suggesting the method used here to cover the complex method as well.

References

1. J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, New York, 1976.

2. P. Binding and P. J. Browne, L^p spaces from matrix measures, *Canad. Math. Bull.*, **18**, (1975), 19–26.

3. C. Costa, *Interpolation of Matrix Measure Function Spaces*, M.Sc. Thesis, University of Calgary, 1981.

4. N. Dunford and J. T. Schwartz, *Linear Operators, Part II*, Wiley-Interscience, New York, 1963.

5. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North Holland, New York, 1978.

Department of Mathematics and Statistics
The University of Calgary
Calgary, Alberta, Canada

https://doi.org/10.4153/CMB-1983-022-x Published online by Cambridge University Press