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MONADIC INTUITIONISTIC AND MODAL LOGICS ADMITTING PROVABILITY INTERPRETATIONS

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Abstract. The Gödel translation provides an embedding of the intuitionistic logic IPC into the modal logic Grz, which then embeds into the modal logic GL via the splitting translation. Combined with Solovay's theorem that GL is the modal logic of the provability predicate of Peano Arithmetic PA, both IPC and Grz admit provability interpretations. When attempting to 'lift' these results to the monadic extensions MIPC, MGrz, and MGL of these logics, the same techniques no longer work. Following a conjecture made by Esakia, we add an appropriate version of Casari's formula to these monadic extensions (denoted by a '+'), obtaining that the Gödel translation embeds $\mathsf{M^+IPC}$ into $\mathsf{M^+Grz}$ and the splitting translation embeds $\mathsf{M^+Grz}$ into MGL. As proven by Japaridze, Solovay's result extends to the monadic system MGL, which leads us to a provability interpretation of both $\mathsf{M^+IPC}$ and $\mathsf{M^+Grz}$.

§1. Introduction.

1.1. Propositional case. The intuitionistic point of view identifies truth with provability. This has resulted in the well-known Brouwer–Heyting–Kolmogorov interpretation (BHK-interpretation for short), which is considered as the intended, albeit informal interpretation of intuitionistic logic (see, e.g., [2] and the references therein). Gödel [27] took the first step in making this interpretation more formal by introducing a modal calculus of classical provability and translating the intuitionistic propositional calculus IPC into it. Gödel's modal calculus turned out to be equivalent to Lewis' well-known modal logic S4, and the translation became known as the *Gödel translation*. We recall that it is defined as follows:

- $p^t = \Box p$ for a propositional letter p;
- $(\varphi \vee \psi)^t = \varphi^t \vee \psi^t$;
- $(\varphi \wedge \psi)^t = \varphi^t \wedge \psi^t$;
- $(\varphi \to \psi)^t = \Box(\varphi^t \to \psi^t)$;
- $(\neg\varphi)^t = \Box(\neg\varphi^t)$.

McKinsey and Tarski [33] proved that this translation is full and faithful; that is,

$$\mathsf{IPC} \vdash \varphi \text{ iff } \mathsf{S4} \vdash \varphi^t.$$

There are many other normal extensions of S4, called *modal companions* of IPC, in which IPC is embedded fully and faithfully. Esakia [14] showed that the largest

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such companion is *Grzegorczyk's logic* Grz, which is the normal extension of $\mathsf{S4}$ with the *Grzegorczyk axiom*

$$\mathsf{grz} =: \Box(\Box(p \to \Box p) \to p) \to p.$$

Thus, we have

$$\mathsf{IPC} \vdash \varphi \text{ iff } \mathsf{Grz} \vdash \varphi^t.$$

Goldblatt [28], Boolos [7], and Kuznetsov and Muravitsky [32] showed that the splitting translation embeds Grz into the *Gödel–Löb logic* GL which is the normal extension of the least normal modal logic K with the axiom

$$\mathsf{gl} := \Box(\Box p \to p) \to \Box p.$$

We recall that the splitting translation is defined by "splitting boxes" in formulas (see, e.g., [8, p. 8]); that is, for a modal formula φ, let $\Box^+\varphi$ be the abbreviation of the formula $\varphi \wedge \Box\varphi$. Then the *splitting translation* is defined by letting φ^s be the result of replacing all occurrences of $\Box$ in φ by $\Box^+$. We then have

$$\mathsf{Grz} \vdash \varphi \text{ iff } \mathsf{GL} \vdash \varphi^s.$$

Combining these results yields

$$\mathsf{IPC} \vdash \varphi \text{ iff } \mathsf{Grz} \vdash \varphi^t \text{ iff } \mathsf{GL} \vdash (\varphi^t)^s.$$

Finally, by Solovay's theorem [40], GL can be thought of as the modal logic of the provability predicate of Peano Arithmetic PA. Thus, both IPC and Grz admit provability interpretations. That IPC admits such an interpretation is especially important in relation to the BHK-interpretation.

1.2. Predicate case. The Gödel translation extends to the predicate case by setting

- $(\forall x \varphi)^t = \Box\forall x(\varphi^t)$,
- $(\exists x \varphi)^t = \exists x(\varphi^t)$.

Let IQC be the intuitionistic predicate calculus and let $\mathsf{QS4}$ be the predicate $\mathsf{S4}$ (the definitions of predicate intuitionistic and modal logics can, for example, be found in [26]). Then

$$\mathsf{IQC} \vdash \varphi \text{ iff } \mathsf{QS4} \vdash \varphi^t,$$

so the extension of the Gödel translation to the predicate case remains full and faithful (see, e.g., [39]). Let QGrz be the predicate Grz and let QGL be the predicate GL. In [26, p. 157] it is attributed to Pankratyev that the Gödel translation of IQC into QGrz remains full and faithful. However, these are the only positive results in the predicate case since Montagna [34] showed that Solovay's theorem no longer holds for QGL. Moreover, the splitting translation does not embed QGrz fully and faithfully into QGL (see below).

1.3. Monadic case. In view of the above, Esakia [17] suggested to study these translations for the monadic (one-variable) fragments of IQC, QGrz, and QGL. The monadic fragment of IQC was introduced by Prior [38] under the name of MIPC. The monadic fragment of $\mathsf{QS4}$ was studied by Fischer-Servi [22], and the monadic

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fragments of QGrz and QGL by Esakia [17]. We denote these fragments by MS4, MGrz, and MGL, respectively.

Fischer-Servi [22] proved that the Gödel translation embeds MIPC into MS4 fully and faithfully. As we will see, the Gödel translation also embeds MIPC fully and faithfully into MGrz. Japaridze [30, 31] proved that Solovay's result extends to MGL. Therefore, to complete the picture, it would be sufficient to show that the splitting translation embeds MGrz into MGL fully and faithfully. However, as was observed by Esakia, this is no longer true. To see this, we recall Casari's predicate formula

$$\mathsf{Cas}: \quad \forall x((P(x) \to \forall y P(y)) \to \forall y P(y)) \to \forall x P(x).$$

Let $\mathsf{IQ^+C}$ be the predicate logic obtained from IQC by postulating Casari's formula as a new axiom. It was pointed out by Esakia [18] (see also [19, Section 7]) that $\mathsf{IQ^+C}$ is equivalent to the predicate logic obtained from IQC by postulating the following modified version of universal generalization:

$$\frac{(P(a) \to \forall x P(x)) \to P(a)}{\forall x P(x)}. \tag{+}$$

Among other things, Esakia observed that $\mathsf{IQ^+C}$ is a conservative extension of IPC and that Kuroda's formula $\forall x \neg\neg P(x) \to \neg\neg\forall x P(x)$ is provable in $\mathsf{IQ^+C}$. In this regard it is worthwhile to point out that Heyting considered it "one of the most striking features of intuitionistic logic" that Kuroda's formula is not derivable in IQC; see, e.g., [18, p. 27].

We consider the monadic version of Casari's formula

$$\mathsf{MCas}: \quad \forall((p \to \forall p) \to \forall p) \to \forall p.$$

Using the same notation for the Gödel and splitting translations in the monadic setting, we have that $\mathsf{MGL} \vdash ((\mathsf{MCas})^t)^s$ but $\mathsf{MGrz} \nvdash (\mathsf{MCas})^t$ (see Theorems 3.9 and 3.10). This yields that MGrz does not embed into MGL faithfully.

Let

$$\mathsf{M^+IPC} = \mathsf{MIPC} + \mathsf{MCas}$$

be the extension of MIPC by MCas, and let

$$\mathsf{M^+Grz} = \mathsf{MGrz} + (\mathsf{MCas})^t$$

be the extension of MGrz by $(\mathsf{MCas})^t$. Esakia claimed that the translations

$$\mathsf{IPC} \to \mathsf{Grz} \to \mathsf{GL}$$

are lifted to

$$\mathsf{M^+IPC} \to \mathsf{M^+Grz} \to \mathsf{MGL}.$$

Verifying this claim will be our main goal, which together with Japaridze's result [30, 31] on arithmetical completeness of MGL yields the desired provability interpretations of $\mathsf{M^+IPC}$ and $\mathsf{M^+Grz}$. As we point out in Remark 5.19, $\mathsf{M^+IPC}$ axiomatizes the monadic fragment of $\mathsf{IQ^+C}$. Thus, it is the monadic fragment of the amended intuitionistic calculus $\mathsf{IQ^+C}$, and not of IQC, that admits a provability

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interpretation. It would be of interest to investigate philosophical consequences of this result in connection with the BHK-interpretation.

1.4. Main contribution and organization. Our main result is the following theorem.

THEOREM. $\mathsf{M^+IPC} \vdash \varphi$ *iff* $\mathsf{M^+Grz} \vdash \varphi^t$ *iff* $\mathsf{MGL} \vdash (\varphi^t)^s$.

We will prove the theorem semantically. The most challenging part of our argument is in establishing the finite model property for $\mathsf{M^+IPC}$ and $\mathsf{M^+Grz}$ (see Sections 5 and 6). That MGL also has the finite model property was proven by Japaridze [30]. In fact, our technique of proving the finite model property for $\mathsf{M^+Grz}$ can be adapted to provide an alternative proof of Japaridze's result for MGL (see [9]). Once the finite model property of these logics is established, the standard argument yields our main result (see the proof of Theorem 4.12 for details).

The paper is organized as follows. Section 2 provides a brief overview of monadic logics and their corresponding algebraic and relational semantics. Section 3 discusses the Gödel and splitting translations in the monadic setting. In Section 4 we investigate how the addition of the adapted variations of Casari's formula affect the semantics. Finally, in Sections 5 and 6 we establish the finite model property for $\mathsf{M^+IPC}$ and $\mathsf{M^+Grz}$, respectively, using a modified selective filtration, which allows us to conclude the main result stated above.

We use the following as our standard references: [12] for intuitionistic and modal propositional logics, [26] for intuitionistic and modal predicate logics, and [25] for intuitionistic modal logics and classical bi-modal systems.

§2. Monadic logics.

In this section we recall the notion of monadic intuitionistic and modal logics and discuss their algebraic and frame-based semantics.

2.1. Monadic intuitionistic logic. The *monadic intuitionistic propositional calculus* MIPC was defined by Prior [38] and it was shown by Bull [11] that MIPC axiomatizes the monadic fragment of the predicate intuitionistic logic. To define MIPC, let $\mathcal{L}$ be the language of propositional intuitionistic logic, and let $\mathcal{L}_{\forall\exists}$ be the enrichment of $\mathcal{L}$ with the *quantifier modalities* $\forall$ and $\exists$.[1]

DEFINITION 2.1. MIPC is the smallest set of $\mathcal{L}_{\forall\exists}$-formulas containing

- all axioms of IPC,
- the $\mathsf{S4}$-axioms for $\forall$,[2]
- the $\mathsf{S5}$-axioms for $\exists$,[3]
- the connecting axioms $\exists p \to \forall\exists p$ and $\exists\forall p \to \forall p$,

and closed under the inference rules of substitution, modus ponens, and $\forall$-necessitation $\frac{\varphi}{\forall\varphi}$.

REMARK 2.2. The non-symmetric feature of intuitionistic quantifiers is captured in the fact that while $\exists$ is an $\mathsf{S5}$-modality, $\forall$ is merely an $\mathsf{S4}$-modality, and the

[1] $\Box$ and $\Diamond$ are also frequently used in place of $\forall$ and $\exists$.

[2] $\forall p \to p$, $\forall p \to \forall\forall p$, and $\forall(p \wedge q) \leftrightarrow (\forall p \wedge \forall q)$.

[3] $p \to \exists p$, $\exists\exists p \to \exists p$, $\exists(p \vee q) \leftrightarrow (\exists p \vee \exists q)$, and $\exists(\exists p \wedge q) \leftrightarrow (\exists p \wedge \exists q)$.

$\forall$-counterpart $\forall(\forall p \vee q) \leftrightarrow (\forall p \vee \forall q)$ of $\exists(\exists p \wedge q) \leftrightarrow (\exists p \wedge \exists q)$ is not provable in MIPC.

Algebraic semantics for MIPC is given by monadic Heyting algebras [3, 35].

DEFINITION 2.3. A *monadic Heyting algebra* is a triple $(H, \forall, \exists)$ where

- H is a Heyting algebra,
- $\forall : H \to H$ is an S4-operator,[4]
- $\exists : H \to H$ is an S5-operator,[5]
- $\exists a \leq \forall \exists a$ and $\exists \forall a \leq \forall a$.

REMARK 2.4. This in particular implies that the fixpoints of $\forall$ and $\exists$ are equal and form a Heyting subalgebra of H. In fact, every monadic Heyting algebra can be represented as a pair (H, H_0) where H_0 is a Heyting subalgebra of H and the inclusion has both the right $(\forall)$ and left $(\exists)$ adjoint.

As usual, propositional letters of $\mathcal{L}_{\forall\exists}$ are evaluated as elements of H, the connectives as the corresponding operations of H, and the quantifier modalities as the corresponding modal operators of H. The standard Lindenbaum–Tarski construction then yields:

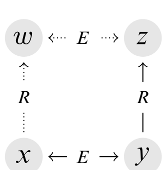
THEOREM 2.5. MIPC $\vdash \varphi \Leftrightarrow \mathfrak{H} \vDash \varphi$ *for each monadic Heyting algebra* $\mathfrak{H}$.

Kripke semantics for MIPC is an extension of Kripke semantics for IPC, and was developed in [17, 24, 36].

DEFINITION 2.6. An MIPC-*frame* is a triple $\mathfrak{F} = (W, R, E)$ where (W, R) is an IPC-frame[6] and E is an equivalence relation on W satisfying

$$(R \circ E)(x) \subseteq (E \circ R)(x) \text{ for all } x \in W;$$

that is, if xEy and yRz, then there is $w \in W$ such that xRw and wEz:



We refer to this condition as *commutativity*. We will sometimes refer to R as a 'vertical relation', and to E as a 'horizontal relation', as depicted in the diagram above.

Valuations on MIPC-frames are defined as for IPC-frames; that is, a valuation v on $\mathfrak{F} = (W, R, E)$ is an assignment of R-upsets of $\mathfrak{F}$ to propositional letters.[7] As usual, the truth relation in $\mathfrak{F}$ is defined by induction. The clauses for the connectives $\wedge, \vee, \to, \neg$ are the same as for IPC-frames:

[4] $\forall a \leq a$, $\forall a \leq \forall\forall a$, $\forall(a \wedge b) = \forall a \wedge \forall b$, and $\forall 1 = 1$.

[5] $a \leq \exists a$, $\exists\exists a \leq \exists a$, $\exists(a \vee b) = \exists a \vee \exists b$, $\exists 0 = 0$, and $\exists(\exists a \wedge b) = \exists a \wedge \exists b$.

[6] A nonempty partially ordered set.

[7] Recall that $U \subseteq W$ is an *R-upset* if $u \in U$ and uRv imply $v \in U$.

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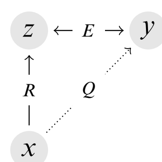
$$
\begin{array}{lll}
w \vDash_v p & \text{iff} & w \in v(p); \\
w \vDash_v \varphi \wedge \psi & \text{iff} & w \vDash_v \varphi \text{ and } w \vDash_v \psi; \\
w \vDash_v \varphi \vee \psi & \text{iff} & w \vDash_v \varphi \text{ or } w \vDash_v \psi; \\
w \vDash_v \varphi \to \psi & \text{iff} & (\text{for all } v)(wRv \text{ and } v \vDash_v \varphi \text{ imply } v \vDash_v \psi); \\
w \vDash_v \neg\varphi & \text{iff} & (\text{for all } v)(wRv \text{ implies } v \nvDash_v \varphi).
\end{array}
$$

To extend this to the truth relation for quantifier modalities, we first define a new relation Q on W as the composition $E \circ R$; that is, xQy iff there is $z \in W$ such that xRz and zEy:



Then Q is a quasi-order (reflexive and transitive) and $\forall, \exists$ are interpreted in $\mathfrak{F}$ as follows:

$$
\begin{array}{lll}
w \vDash_v \forall\varphi & \text{iff} & (\text{for all } v)(wQv \text{ implies } v \vDash_v \varphi); \\
w \vDash_v \exists\varphi & \text{iff} & (\text{there exists } v)(wEv \text{ and } v \vDash_v \varphi).
\end{array}
$$

Sometimes we also write $(\mathfrak{F}, w) \vDash_v \varphi$ to emphasize the underlying frame $\mathfrak{F}$ or simply $w \vDash \varphi$ in case $\mathfrak{F}$ and v are clear from the context.

There is a close connection between algebraic and relational semantics for MIPC. To see this, let $\mathfrak{F} = (W, R, E)$ be an MIPC-frame. For $x \in W$, let

$$Q(x) = \{y \in W \mid xQy\} \text{ and } E(x) = \{y \in W \mid xEy\}.$$

Set $\mathfrak{F}^+ = (\mathsf{Up}(\mathfrak{F}), \forall, \exists)$ where $\mathsf{Up}(\mathfrak{F})$ is the Heyting algebra of R-upsets of $\mathfrak{F}$, and for $U \in \mathsf{Up}(\mathfrak{F})$,

$$\forall U = \{x \in W \mid Q(x) \subseteq U\} \text{ and } \exists U = \{x \in W \mid E(x) \cap U \neq \varnothing\}.$$

Then $\mathfrak{F}^+$ is a monadic Heyting algebra, and every monadic Heyting algebra is represented as a subalgebra of such. To see this, for a monadic Heyting algebra $\mathfrak{H} = (H, \forall, \exists)$, let W be the set of prime filters of H, let R be the inclusion, and let E be defined by $\eta E \zeta$ iff $\eta \cap H_0 = \zeta \cap H_0$, where we recall that H_0 is the fixpoint subalgebra of H (see Remark 2.4). Then $\mathfrak{H}_+ := (W, R, E)$ is an MIPC-frame (where $\eta Q \zeta$ iff $\eta \cap H_0 \subseteq \zeta \cap H_0$) and there is an embedding $e : \mathfrak{H} \to (\mathfrak{H}_+)^+$ given by

$$e(a) = \{\eta \in \mathfrak{H}_+ \mid a \in \eta\}.$$

In general, the embedding e is not onto, so to recognize the e-image of H in the Heyting algebra of upsets, we introduce the concept of a descriptive MIPC-frame. One way to do this is to introduce topology on an MIPC-frame.

We recall that a topological space is a *Stone space* if it is compact Hausdorff and zero-dimensional.[8] A relation R on a Stone space W is *continuous* if (i)

[8] A topological space is *zero-dimensional* if clopen (closed and open) sets form a basis for the topology.

$R(x)$ is closed for each $x \in W$ and (ii) U clopen implies $R^{-1}(U)$ is clopen, where

$$R^{-1}(U) = \{x \in W \mid xRu \text{ for some } u \in U\}.$$

DEFINITION 2.7. An MIPC-frame $\mathfrak{F} = (W, R, E)$ is a *descriptive* MIPC-*frame* if

- W is a Stone space,
- R and Q are continuous relations,
- if A is a clopen R-upset, then $E(A)$ is a clopen R-upset.

REMARK 2.8. This does not imply that A clopen implies $E(A)$ is clopen; see [4, p. 32]. However, we do have that A closed implies $E(A)$ is closed; see [4, Lemma 7].

As follows from Esakia's representation of Heyting algebras [13], for a Heyting algebra H, there is a Stone topology on the set W of prime filters of H generated by the basis

$$\{e(a) \setminus e(b) \mid a, b \in H\},$$

the inclusion relation R on W is continuous, and e is a Heyting isomorphism from H onto the Heyting algebra of clopen R-upsets of W.

By [4, Theorem 13], if $\mathfrak{H} = (H, \forall, \exists)$ is a monadic Heyting algebra, then (W, R, E) is a descriptive MIPC-frame, which we denote by $\mathfrak{H}_*$, and e is an isomorphism from $\mathfrak{H}$ onto the monadic Heyting algebra $(\mathfrak{H}_*)^*$ of clopen R-upsets of $\mathfrak{H}_*$. Thus, every monadic Heyting algebra can be thought of as the algebra of clopen R-upsets of some descriptive MIPC-frame. This representation together with Theorem 2.5 yields:

THEOREM 2.9. MIPC $\vdash \varphi \Leftrightarrow \mathfrak{F} \vDash \varphi$ *for each descriptive* MIPC-*frame* $\mathfrak{F}$.

If the descriptive MIPC-frame is finite, then the topology is discrete, and hence finite descriptive MIPC-frames are simply finite MIPC-frames. It is well known that MIPC has the finite model property (FMP):

THEOREM 2.10. MIPC $\vdash \varphi \Leftrightarrow \mathfrak{F} \vDash \varphi$ *for each finite* MIPC-*frame* $\mathfrak{F}$.

This was first proved by Bull [10] using algebraic semantics. Bull's proof contained a gap, which was later filled by Fischer-Servi [23] and Ono [36] independently of each other. For a more frame-theoretic proof, using the technique of selective filtration, see [25, Section 10.3].

We finish Section 2.1 by recalling an important property of descriptive MIPC-frames, which will be useful later on.

DEFINITION 2.11. Let $\mathfrak{F} = (W, R, E)$ be a descriptive MIPC-frame and let $A \subseteq W$.

1. We say $x \in A$ is R-*maximal* in A if xRy and $y \in A$ imply $x = y$.
2. The R-*maximum* of A is the set of all R-maximal points of A, i.e.,

$$\max A = \{x \in A \mid xRy \text{ and } y \in A \text{ imply } x = y\}.$$

The next lemma states that every point in the E-saturation of clopen A sees a point that is maximal in the E-saturation of A. The proof follows from the result of Fine [21] and Esakia [16, 20] that can be phrased as follows: If A is a closed subset of a descriptive IPC-frame, then for each $x \in A$ there is $y \in \max A$ such that xRy. Since

A clopen implies that $E(A)$ is closed (see Remark 2.8), the proof is a consequence of the Fine-Esakia lemma.

LEMMA 2.12. *Let* $\mathfrak{F} = (W, R, E)$ *be a descriptive* MIPC-*frame. For each clopen* A *and* $x \in E(A)$, *there is* $y \in \max E(A)$ *such that* xRy.

2.2. Monadic modal logics. Let $\mathcal{ML}$ be the basic propositional modal language (with one modality $\Box$). As usual, the least normal modal logic will be denoted by K, and normal modal logics are normal extensions of K.

Let $\mathcal{ML}_\forall$ be the bimodal language which enriches $\mathcal{ML}$ with the modality $\forall$. We use the abbreviation $\exists\varphi$ for $\neg\forall\neg\varphi$.

DEFINITION 2.13.

1. The *monadic* K is the least set of $\mathcal{ML}_\forall$-formulas containing
   - the K-axiom for $\Box$,[9]
   - the S5-axioms for $\forall$,[10]
   - the bridge axiom $\Box\forall p \to \forall\Box p$,

   and closed under $\forall$-necessitation $\frac{\varphi}{\forall\varphi}$ as well as under the usual rules of substitution, modus ponens, and $\Box$-necessitation. We denote the monadic K by MK.
2. A *normal extension* of MK is an extension of MK which is closed under the above rules of inference. We call normal extensions of MK *normal monadic modal logics* or simply *mm-logics*.
3. Let L be a normal modal logic (in $\mathcal{ML}$). The *least monadic extension* ML of L is the smallest mm-logic containing $\mathsf{MK} \cup \mathsf{L}$.

REMARK 2.14.

1. Monadic modal logics are bimodal logics in the language with two modalities $\Box, \forall$, where $\forall$ is an S5-modality. They correspond to expanding relativized products discussed in [25, Section 9].
2. The formula $\forall\Box p \to \Box\forall p$, which is the converse of the bridge axiom, and is the monadic version of Barcan's formula, is not provable in MK.

Algebraic semantics for monadic modal logics is given by monadic modal algebras.

DEFINITION 2.15. A *monadic modal algebra* or simply an *mm-algebra* is a triple $(B, \Box, \forall)$ where

- $(B, \Box)$ is a modal algebra,[11]
- $(B, \forall)$ is an S5-algebra,[12]
- $\Box\forall a \le \forall\Box a$.

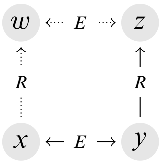
REMARK 2.16. As with monadic Heyting algebras, the $\forall$-fixpoints of an mm-algebra $(B, \Box, \forall)$ form a subalgebra of the modal algebra $(B, \Box)$, and each

[9] $\Box(p \to q) \to (\Box p \to \Box q)$.
[10] $\forall p \to p$, $\forall p \to \forall\forall p$, $\neg\forall p \to \forall\neg\forall p$, and $\forall(p \to q) \to (\forall p \to \forall q)$.
[11] B is a boolean algebra and $\Box : B \to B$ satisfies $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$.
[12] $(B, \forall)$ is a modal algebra satisfying $\forall a \le a$, $\forall a \le \forall\forall a$, and $\neg\forall a \le \forall\neg\forall a$.

mm-algebra is represented as a pair (B, B_0) of modal algebras such that the embedding of B_0 into B has a right adjoint $(\forall)$.

Kripke semantics for mm-logics is given by the augmented Kripke frames of Esakia [17].

DEFINITION 2.17. An *augmented Kripke frame* is a triple $\mathfrak{F} = (W, R, E)$ where (W, R) is a Kripke frame[13] and E is an equivalence relation on W satisfying *commutativity*, i.e., $(R \circ E)(x) \subseteq (E \circ R)(x)$ for all $x \in W$; that is, if xEy and yRz, then there is $w \in W$ such that xRw and wEz:



As with MIPC-frames, we may refer to R as a 'vertical relation', and to E as a 'horizontal relation', as depicted in the diagram above.

Valuations on augmented Kripke frames are defined analogously to Kripke frames; that is, a valuation v on an augmented Kripke frame $\mathfrak{F} = (W, R, E)$ assigns subsets of W to propositional letters. The truth relation clauses for the connectives $\vee, \neg$, the modality $\Box$, and its dual $\Diamond$ are defined as for Kripke frames:

$x \vDash_v p$	iff	$x \in v(p)$;
$x \vDash_v \psi \vee \chi$	iff	$x \vDash_v \psi$ or $x \vDash_v \chi$;
$x \vDash_v \neg\psi$	iff	$x \nvDash_v \psi$;
$x \vDash_v \Box\psi$	iff	(for all $y \in W)(xRy \Rightarrow y \vDash_v \psi)$;
$x \vDash_v \Diamond\psi$	iff	(there exists $y \in W)(xRy$ and $y \vDash_v \psi)$.

The modality $\forall$ and its dual $\exists$ are interpreted via the relation E as follows:

$x \vDash_v \forall\varphi$	iff	(for all $y \in W)(xEy \Rightarrow y \vDash_v \varphi)$;
$x \vDash_v \exists\varphi$	iff	(there exists $y \in W)(xEy$ and $y \vDash_v \varphi)$.

We also use the notation $(\mathfrak{F}, w) \vDash_v \varphi$ or $w \vDash \varphi$.

As in the case of MIPC, there is a close connection between algebraic and relational semantics for mm-logics. For an augmented Kripke frame $\mathfrak{F} = (W, R, E)$, set $\mathfrak{F}^+ = (\wp(\mathfrak{F}), \Box, \forall)$ where $\wp(\mathfrak{F})$ is the powerset of $\mathfrak{F}$, and for $U \in \mathsf{Up}(\mathfrak{F})$,

$$\Box U = \{x \in W \mid R(x) \subseteq U\} \text{ and } \forall U = \{x \in W \mid E(x) \subseteq U\}.$$

Then $\mathfrak{F}^+$ is an mm-algebra, and every mm-algebra is represented as a subalgebra of such. To see this, for an mm-algebra $\mathfrak{B} = (B, \Box, \forall)$, let W be the set of ultrafilters of B, and let R and E be defined by

$$\eta R \zeta \text{ iff } \Box a \in \eta \text{ implies } a \in \zeta, \text{ and } \eta E \zeta \text{ iff } \eta \cap B_0 = \zeta \cap B_0.$$

---

[13] W is nonempty and R is a binary relation on W.

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Then $\mathfrak{B}_+ = (W, R, E)$ is an augmented Kripke frame and there is an embedding $e : \mathfrak{B} \to (\mathfrak{B}_+)^+$ given by

$$e(a) = \{\eta \in \mathfrak{B}_+ \mid a \in \eta\}.$$

In general, the embedding e is not onto, so to recognize the e-image of $\mathfrak{B}$ in the powerset, we introduce the concept of a descriptive augmented Kripke frame. As in the case of MIPC, we do this by introducing topology on augmented Kripke frames.

DEFINITION 2.18. An augmented Kripke frame $\mathfrak{F} = (W, R, E)$ is a *descriptive augmented Kripke frame* if W is a Stone space and R and E are continuous relations.

As follows from the representation of modal algebras, for a modal algebra B, there is a Stone topology on the set W of ultrafilters of B generated by the basis $\{e(a) \mid a \in B\}$, the relation R on W is continuous, and e is a modal isomorphism from B onto the modal algebra of clopen subsets of W.

If $\mathfrak{B} = (B, \Box, \forall)$ is an mm-algebra, then (W, R, E) is a descriptive augmented Kripke frame, which we denote by $\mathfrak{B}_*$, and e is an isomorphism from $\mathfrak{B}$ onto the mm-algebra $(\mathfrak{B}_*)^*$ of clopen subsets of $\mathfrak{B}_*$. Thus, every mm-algebra can be thought of as the algebra of clopen subsets of some descriptive augmented Kripke frame.

2.3. MS4, MGrz, and MGL. We next focus on the least monadic extension MS4 of the modal logic S4.

DEFINITION 2.19.

1. The *monadic* S4 is the least monadic extension MS4 of the modal logic S4.
2. An MS4-*algebra* is an mm-algebra $(B, \Box, \forall)$ such that $(B, \Box)$ is an S4-algebra.
3. An MS4-*frame* is an augmented Kripke frame $\mathfrak{F} = (W, R, E)$ such that (W, R) is an S4-frame.
4. A *descriptive* MS4-*frame* is a descriptive augmented Kripke frame $\mathfrak{F} = (W, R, E)$ such that (W, R, E) is an MS4-frame.

As in the case of MIPC, we have the following standard completeness results:

THEOREM 2.20.

1. $\mathsf{MS4} \vdash \varphi \Leftrightarrow \mathfrak{B} \vDash \varphi$ *for each* MS4-*algebra* $\mathfrak{B}$.
2. $\mathsf{MS4} \vdash \varphi \Leftrightarrow \mathfrak{F} \vDash \varphi$ *for each descriptive* MS4-*frame* $\mathfrak{F}$.

We also have that MS4 has the finite model property (see [6] and the references therein).

THEOREM 2.21. $\mathsf{MS4} \vdash \varphi \Leftrightarrow \mathfrak{F} \vDash \varphi$ *for each finite* MS4-*frame* $\mathfrak{F}$.

Let $\mathfrak{F} = (W, R, E)$ be a descriptive MS4-frame and $A \subseteq W$. The R-*maximal* points of A and the R-*maximum* of A are defined as in Definition 2.11. In the context of MS4-frames, we also need the notion of quasi-R-maximal points.

DEFINITION 2.22. Let $\mathfrak{F} = (W, R, E)$ be a descriptive MS4-frame and $A \subseteq W$.

1. We say $x \in A$ is *quasi-R-maximal* in A if xRy and $y \in A$ imply yRx.
2. The *quasi-R-maximum* of A is the set of all quasi-R-maximal points of A, i.e.,

$$\text{qmax}\, A = \{x \in A \mid xRy \text{ and } y \in A \text{ imply } yRx\}.$$

Note that $\max A \subseteq \text{qmax}\, A$ as R is reflexive, but not conversely. The following lemma is a consequence of the Fine-Esakia lemma [16, 20, 21] for descriptive S4-frames.

LEMMA 2.23. *Let $\mathfrak{F} = (W, R, E)$ be a descriptive* MS4*-frame. For each closed $A \subseteq W$ we have $A \subseteq R^{-1}\text{qmax}\, A$.*

DEFINITION 2.24.

1. The *monadic* Grz is the least monadic extension MGrz of Grzegorczyk's logic Grz.
2. An MGrz-*algebra* is an mm-algebra $(B, \Box, \forall)$ such that $(B, \Box)$ is a Grz-algebra.
3. An MGrz-*frame* is an augmented Kripke frame $\mathfrak{F} = (W, R, E)$ such that (W, R) is a Grz-frame.
4. A *descriptive* MGrz-*frame* is a descriptive S4-frame $\mathfrak{F} = (W, R, E)$ validating Grzegorczyk's axiom grz.

Again, we have the following standard completeness results:

THEOREM 2.25.

1. MGrz $\vdash \varphi \Leftrightarrow \mathfrak{B} \vDash \varphi$ *for each* MGrz-*algebra* $\mathfrak{B}$.
2. MGrz $\vdash \varphi \Leftrightarrow \mathfrak{F} \vDash \varphi$ *for each descriptive* MGrz-*frame* $\mathfrak{F}$.

It is well known that an S4-frame $\mathfrak{F} = (W, R)$ is a Grz-frame iff R is a *Noetherian partial order*; that is, a partial order with no infinite ascending chains (of distinct points). Thus, if $\mathfrak{F}$ is finite, then $\mathfrak{F}$ is a Grz-frame iff R is a partial order.

It is a result of Esakia that a descriptive S4-frame $\mathfrak{F} = (W, R)$ is a descriptive Grz-frame iff for each clopen $A \subseteq W$ the R-maximal and quasi-R-maximal points of A coincide. These results clearly hold for MGrz as well.

LEMMA 2.26. [15]

1. *Let $\mathfrak{F} = (W, R, E)$ be a descriptive* MS4*-frame. Then $\mathfrak{F} \vDash$* grz *iff for each clopen A we have* $\text{qmax}\, A = \max A$.
2. *Let $\mathfrak{F} = (W, R, E)$ be a descriptive* MGrz*-frame. For each clopen A we have $A \subseteq R^{-1} \max A$.*

DEFINITION 2.27.

1. The *monadic* GL is the least monadic extension MGL of the Gödel–Löb logic GL.
2. An MGL-*algebra* is an mm-algebra $(B, \Box, \forall)$ such that $(B, \Box)$ is a GL-algebra.
3. An MGL-*frame* is an augmented Kripke frame $\mathfrak{F} = (W, R, E)$ such that (W, R) is a GL-frame.
4. A *descriptive* MGL-*frame* is a descriptive augmented Kripke frame $\mathfrak{F} = (W, R, E)$ validating gl.

As before, we have the following standard completeness results:

THEOREM 2.28.

1. MGL $\vdash \varphi \Leftrightarrow \mathfrak{B} \vDash \varphi$ *for each* MGL-*algebra* $\mathfrak{B}$.
2. MGL $\vdash \varphi \Leftrightarrow \mathfrak{F} \vDash \varphi$ *for each descriptive* MGL-*frame* $\mathfrak{F}$.

It is well known that a Kripke frame $\mathfrak{F} = (W, R)$ is a GL-frame iff R is transitive and *dually well founded* (no infinite ascending chains). Call R a *strict partial order* if R is irreflexive and transitive. If W is finite, then $\mathfrak{F}$ is a GL-frame iff R is a strict partial order.

A characterization of descriptive GL-frames was originally established by Esakia and given in [1]. It generalizes directly to descriptive MGL-frames. For a transitive frame $\mathfrak{F} = (W, R)$ and $A \subseteq W$, define the *irreflexive maximum* of A by

$$\mu(A) = \{x \in A \mid R(x) \cap A = \varnothing\}.$$

LEMMA 2.29. [1] *Let $\mathfrak{F} = (W, R, E)$ be a descriptive augmented Kripke frame. Then $\mathfrak{F}$ is a descriptive* MGL*-frame iff $\mathfrak{F}$ is transitive and $A \subseteq \mu(A) \cup R^{-1}\mu(A)$ for each clopen A.*

Thus, a descriptive augmented Kripke frame is a descriptive MGL-frame iff it is transitive and each point in a clopen set is either in the irreflexive maximum of the clopen or sees a point in the irreflexive maximum. It was observed by Japaridze that MGL has the finite model property.

THEOREM 2.30. [30] MGL $\vdash \varphi \Leftrightarrow \mathfrak{F} \vDash \varphi$ *for all finite* MGL*-frames* $\mathfrak{F}$.

§3. The Gödel and splitting translations in the monadic setting. In this section we discuss the Gödel and splitting translations in the monadic setting. While the Gödel translation embeds MIPC fully and faithfully into MGrz, the splitting translation from MGrz into MGL does not yield a faithful embedding.

3.1. Gödel translation. The Gödel translation extends to the monadic setting by defining

$$(\forall\varphi)^t = \Box\forall\varphi^t,$$
$$(\exists\varphi)^t = \exists\varphi^t.$$

Using algebraic semantics, Fisher-Servi [22, 23] proved that this provides a full and faithful embedding of MIPC into MS4. The proof also yields a full and faithful embedding of MIPC into MGrz. Below we give an alternate proof of this result, using relational semantics. The proof extends a semantical proof that IPC $\vdash \varphi$ iff S4 $\vdash \varphi^t$ as given, e.g., in [12, pp. 96–97].

For notational simplicity, we abbreviate the formula $\Box\forall\psi$ as $\blacksquare\psi$ and the formula $\Diamond\exists\psi$ as $\blacklozenge\psi$. Observe that this keeps the duality between box and diamond since $\blacksquare\psi = \Box\forall\psi = \neg\Diamond\neg\neg\exists\neg\psi$, which is provably equivalent to $\neg\Diamond\exists\neg\psi = \neg\blacklozenge\neg\psi$.

REMARK 3.1. The modalities $\blacksquare$, $\blacklozenge$ are S4-modalities which can be modeled using the relation $Q = E \circ R$, i.e., we have

$$w \vDash \blacksquare\varphi \quad \text{iff} \quad (\text{for all } v)(wQv \text{ implies } v \vDash \varphi);$$
$$w \vDash \blacklozenge\varphi \quad \text{iff} \quad (\text{there exists } v)(wQv \text{ and } v \vDash \varphi).$$

Using this notation, the ∀-step in the Gödel translation is

$$(\forall\varphi)^t = \blacksquare\varphi^t.$$

For an MS4-frame $\mathfrak{F} = (W, R, E)$ define an equivalence relation $\sim$ on $\mathfrak{F}$ by

$$x \sim y \text{ iff } xRy \text{ and } yRx.$$

Let $[x]$ be the equivalence class of x, and let $W_\sim = W/{\sim}$ be the set of all equivalence classes. Define $R_\sim$ and $E_\sim$ on $W_\sim$ by

$$\begin{aligned} [x]R_\sim[y] \quad &\text{iff} \quad xRy; \\ [x]E_\sim[y] \quad &\text{iff} \quad xQy \text{ and } yQx. \end{aligned}$$

That $E_\sim$ is well defined follows from $R \circ Q \circ R \subseteq Q$ which is true by commutativity in $\mathfrak{F}$ and transitivity of R. Let $\mathfrak{F}_\sim = (W_\sim, R_\sim, E_\sim)$. Set $Q_\sim = E_\sim \circ R_\sim$.

Lemma 3.2. *Let $\mathfrak{F} = (W, R, E)$ be an* MS4*-frame and $x, y \in W$.*

1. *xEy implies $[x]E_\sim[y]$;*
2. *xQy iff $[x]Q_\sim[y]$.*

Proof. (1) If xEy, then xQy and yQx, so $[x]E_\sim[y]$ by definition of $E_\sim$.

(2) Suppose that xQy. Then there is y' with xRy' and $y'Ey$. Therefore, $[x]R_\sim[y']$ by definition of $R_\sim$ and $[y']E_\sim[y]$ by (1). Thus, $[x]Q_\sim[y]$. Conversely, if $[x]Q_\sim[y]$, then there is $[y']$ with $[x]R_\sim[y']$ and $[y']E_\sim[y]$. By the definitions of $R_\sim$ and $E_\sim$, we have xRy' and $y'Qy$. Thus, xQy. ⊣

Lemma 3.3. *$\mathfrak{F}_\sim$ is an* MIPC*-frame.*

Proof. It is well known (and easy to verify) that $R_\sim$ is a partial order (see, e.g., [12, p. 68]). Transitivity and reflexivity of $E_\sim$ easily follow from transitivity and reflexivity of Q, and $E_\sim$ is symmetric by definition. To see that $\mathfrak{F}_\sim$ satisfies commutativity, let $[x], [y], [z] \in W_\sim$ with $[x]E_\sim[y]$ and $[y]R_\sim[z]$. Then xQy and yRz, so xQz. Therefore, there is z' with xRz' and $z'Ez$. From xRz' it follows that $[x]R[z']$, and $z'Ez$ implies $[z']E_\sim[z]$ by Lemma 3.2(1). Thus, $\mathfrak{F}_\sim$ satisfies commutativity. ⊣

Given a valuation v on $\mathfrak{F}$, define a valuation $v_\sim$ on $\mathfrak{F}_\sim$ by

$$v_\sim(p) = \{[x] \mid x \in v(\Box p)\}.$$

Clearly $v_\sim(p)$ is an upset. We call $\mathfrak{F}_\sim$ the *skeleton* of $\mathfrak{F}$ and $(\mathfrak{F}_\sim, v_\sim)$ the *skeleton* of $(\mathfrak{F}, v)$.

Conversely, given an MIPC-frame $\mathfrak{F}$, we regard it as an MS4-frame. In addition, if $\mathfrak{F}$ is finite, then we regard it as a finite MGrz-frame. If v is a valuation on the MIPC-frame $\mathfrak{F}$, then we regard it as a valuation on the MGrz-frame $\mathfrak{F}$.

The following lemma describes how the above frame transformations behave with respect to the Gödel translation. It is proved by induction on the complexity of φ.

Lemma 3.4. *Let φ be a formula of $\mathcal{L}_{\forall\exists}$.*

1. *For an* MIPC*-frame $\mathfrak{F}$ with a valuation v and every $x \in \mathfrak{F}$ we have*

$$(\mathfrak{F}, x) \vDash_v \varphi \Leftrightarrow (\mathfrak{F}, x) \vDash_v \varphi^t.$$

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2. *For an* MS4*-frame* $\mathfrak{F}$ *with a valuation* v *and every* $x \in \mathfrak{F}$, *we have*

$$(\mathfrak{F}, x) \vDash_v \varphi^t \Leftrightarrow (\mathfrak{F}_\sim, [x]) \models_{v_\sim} \varphi.$$

Proof. If $\mathfrak{F}$ is an MIPC-frame, then $\mathfrak{F}_\sim$ is isomorphic to $\mathfrak{F}$. Therefore, (1) follows from (2). To prove (2), by [12, Lemma 3.81], it is sufficient to only consider the case for the modalities $\forall$ and $\exists$. Let $\varphi = \forall\psi$. Then

$$\begin{aligned}
[x] \vDash \forall\psi \quad &\Leftrightarrow \quad (\text{for all } [y])([x]Q_\sim[y] \Rightarrow [y] \vDash \psi)\\
&\Leftrightarrow \quad (\text{for all } [y])([x]Q_\sim[y] \Rightarrow y \vDash \psi^t) \quad (\text{Inductive Hypothesis})\\
&\Leftrightarrow \quad (\text{for all } y)(xQy \Rightarrow y \vDash \psi^t) \quad (\text{Lemma 3.2(2)})\\
&\Leftrightarrow \quad x \vDash \blacksquare\psi^t\\
&\Leftrightarrow \quad x \vDash (\forall\psi)^t.
\end{aligned}$$

Next let $\varphi = \exists\psi$. If $x \models (\exists\psi)^t$, then there is y with xEy and $y \models \psi^t$. Therefore, $[y] \models \psi$ by the inductive hypothesis, and $[x]E_\sim[y]$ by Lemma 3.2(1). Thus, $[x] \models \exists\psi$. Conversely, suppose that $[x] \models \exists\psi$. Then there is $[y]$ with $[x]E_\sim[y]$ and $[y] \models \psi$. Therefore, yQx by the definition of $E_\sim$. Thus, there is x' with yRx' and $x'Ex$. By the definition of $R_\sim$, we have $[y]R_\sim[x']$. So $[x'] \models \psi$ by persistence in $\mathfrak{F}_\sim$. Consequently, $x' \models \psi^t$ by the inductive hypothesis, so $x \models \exists\psi^t$, and hence $x \models (\exists\psi)^t$. ⊣

Theorem 3.5. MIPC $\vdash \varphi$ *iff* MS4 $\vdash \varphi^t$ *iff* MGrz $\vdash \varphi^t$.

Proof. Suppose that MIPC $\nvdash \varphi$. Since MIPC has the FMP (Theorem 2.10), there exists a finite MIPC-frame $\mathfrak{F}$, a valuation v on $\mathfrak{F}$, and $x \in \mathfrak{F}$ such that $x \not\models_v \varphi$. Viewing $\mathfrak{F}$ as an MGrz-frame, $x \not\models_v \varphi^t$ by Lemma 3.4(1). Therefore, MGrz $\nvdash \varphi^t$. Since MS4 $\subseteq$ MGrz, it follows that MS4 $\nvdash \varphi^t$.

Conversely, if MGrz $\nvdash \varphi^t$, then MS4 $\nvdash \varphi^t$. By the FMP for MS4 (see Theorem 2.21), there is a finite MS4-frame $\mathfrak{F}$, a valuation v on $\mathfrak{F}$, and $x \in \mathfrak{F}$ such that $(\mathfrak{F}, x) \not\models_v \varphi^t$. By Lemma 3.4(2), $(\mathfrak{F}_\sim, [x]) \not\models_{v_\sim} \varphi$. Thus, MIPC $\nvdash \varphi$. ⊣

3.2. Splitting translation. Next we discuss the splitting translation in the monadic setting. The key here is Esakia's observation that the splitting translation does not yield a faithful embedding of MGrz into MGL. Since this result is unpublished, we give a proof of it.

Definition 3.6. Let $\mathfrak{F} = (W, R, E)$ be an augmented Kripke frame (modal or intuitionistic), and let $x \in W$.

1. An *E-cluster* (or *cluster*) is a subset of W of the form

$$E(x) = \{w \in W \mid xEw\}$$

(it is the equivalence class of $x \in W$ with respect to E).
2. We say that the E-cluster $E(x)$ is *dirty* if there are $u, v \in E(x)$ with $u \neq v$ and uRv.
3. We say that the cluster is *clean* otherwise; that is, $u, v \in E(x)$ and uRv imply $u = v$:

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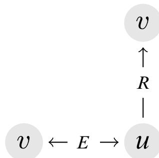
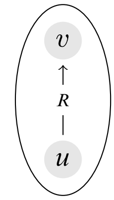


Dirty cluster

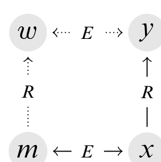


Dirty cluster—alternate depiction
(oval represents E-cluster)

Descriptive MGL-frames have the property that clusters in the irreflexive maximum of an E-saturated clopen are clean.

LEMMA 3.7. *Let $\mathfrak{F} = (W, R, E)$ be a descriptive* MGL*-frame. For clopen A and $m \in \mu(E(A))$ we have that $E(m)$ is clean.*

PROOF. Suppose there exist clopen A and $m \in \mu(E(A))$ with $E(m)$ dirty. Then there are $x, y \in E(m)$ with xRy, xEy, and $x \neq y$. By commutativity, there is w such that mRw and wEy, as shown below:



Since $y \in E(A)$ we have $w \in E(A)$. But this contradicts $R(m) \cap E(A) = \varnothing$. Thus, we cannot have a dirty cluster in $\mu(E(A))$. ⊣

As a consequence of Lemma 3.7, we obtain:

LEMMA 3.8. *Finite* MGL*-frames are finite strict partial orders in which all clusters are clean.*

We next show that the splitting translation of the Gödel translation of the monadic version of Casari's formula

$$\mathsf{MCas}: \quad \forall((p \to \forall p) \to \forall p) \to \forall p$$

is provable in MGL.

Since $\Box\forall p \leftrightarrow \Box\forall\Box p$ is provable in MS4, it is straightforward to check that $(\mathsf{MCas})^t$ is provably equivalent to $\Box\forall\big(\Box(\Box p \to \Box\forall p) \to \Box\forall p\big) \to \Box\forall p$. Using the notation $\blacksquare$ introduced above, we have that $(\mathsf{MCas})^t$ is:

$$\mathsf{M_\Box Cas}: \quad \blacksquare\big(\Box(\Box p \to \blacksquare p) \to \blacksquare p\big) \to \blacksquare p.$$

We use $\blacksquare^+\psi$ to abbreviate $\forall\psi \wedge \Box\forall\psi = \Box^+\forall\psi$. Then the splitting translation of $\mathsf{M_\Box Cas}$ is

$$(\mathsf{M_\Box Cas})^s = \blacksquare^+\big(\Box^+(\Box^+ p \to \blacksquare^+ p) \to \blacksquare^+ p\big) \to \blacksquare^+ p.$$

THEOREM 3.9. $\mathsf{MGL} \vdash (\mathsf{M_\Box Cas})^s$.

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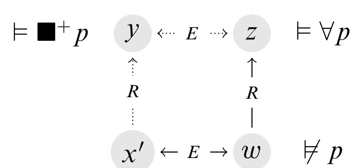
PROOF. Suppose $\mathfrak{F} = (W, R, E)$ is a descriptive MGL-frame. We will prove that $\mathfrak{F} \vDash (\mathsf{M}_\Box\mathsf{Cas})^s$. Let v be a valuation on $\mathfrak{F}$, $x \in \mathfrak{F}$, and $x \nvDash_v \blacksquare^+ p$. We show that $x \nvDash \blacksquare^+(\Box^+(\Box^+ p \to \blacksquare^+ p) \to \blacksquare^+ p)$. Let $A = W \setminus v(\blacksquare^+ p)$. Then $x \in A$ and so by Lemma 2.29, $x \in \mu(A) \cup R^{-1}\mu(A)$. If $x \in R^{-1}\mu(A)$, then there is $x' \in \mu(A)$ with xRx'. If $x \in \mu(A)$, we let $x' = x$. From $x' \in \mu(A)$ it follows that $x' \in A$, so $x' \nvDash \blacksquare^+ p = \forall p \wedge \Box\forall p$. We show that $x' \nvDash \forall p$. If $x' \nvDash \Box\forall p$, then there is y with $x'Ry$ and $y \nvDash \forall p$. Therefore, $y \nvDash \blacksquare^+ p$, so $y \in A$. But this contradicts $x' \in \mu(A)$. Thus, $x' \nvDash \forall p$. So there is w with wEx' and $w \nvDash p$. We show that

$$w \nvDash \Box^+(\Box^+ p \to \blacksquare^+ p) \to \blacksquare^+ p.$$

Since $w \nvDash p$, we have $w \nvDash p \wedge \Box p$, so $w \nvDash \Box^+ p$, and hence $w \vDash \Box^+ p \to \blacksquare^+ p$. Let wRz. By commutativity, there is y such that $x'Ry$ and yEz. Since $x' \in \mu(A)$, we have $y \notin A$. Therefore, $y \vDash \blacksquare^+ p$, so $y \vDash \forall p$, and hence $z \vDash \forall p$:

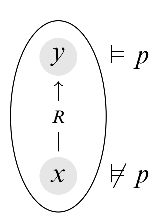


In fact, if zRt, then wRt by transitivity, and so by the same reasoning as above we have $t \vDash \forall p$. It follows that $z \vDash \Box\forall p$, and so $z \vDash \blacksquare^+ p$. Thus, $z \vDash \Box^+ p \to \blacksquare^+ p$, and hence $w \vDash \Box(\Box^+ p \to \blacksquare^+ p)$. This together with $w \vDash \Box^+ p \to \blacksquare^+ p$ yields $w \vDash \Box^+(\Box^+ p \to \blacksquare^+ p)$. Since $w \nvDash \blacksquare^+ p$, we obtain $w \nvDash \Box^+(\Box^+ p \to \blacksquare^+ p) \to \blacksquare^+ p$.

If $x = x'$, then xEw, and so $x \nvDash \forall(\Box^+(\Box^+ p \to \blacksquare^+ p) \to \blacksquare^+ p)$. Otherwise, xRx' and $x'Ew$ imply xQw, so $x \nvDash \blacksquare(\Box^+(\Box^+ p \to \blacksquare^+ p) \to \blacksquare^+ p)$. Thus, in either case, $x \nvDash \blacksquare^+(\Box^+(\Box^+ p \to \blacksquare^+ p) \to \blacksquare^+ p)$ as desired. This yields $x \vDash (\mathsf{M}_\Box\mathsf{Cas})^s$. Since x was arbitrary, $\mathfrak{F} \vDash (\mathsf{M}_\Box\mathsf{Cas})^s$. Because $\mathfrak{F}$ is an arbitrary descriptive MGL-frame, by Theorem 2.28(2), $\mathsf{MGL} \vdash (\mathsf{M}_\Box\mathsf{Cas})^s$. ⊣

THEOREM 3.10. $\mathsf{MGrz} \nvdash \mathsf{M}_\Box\mathsf{Cas}$.

PROOF. Consider the MGrz-frame $\mathfrak{F} = (W, R, E)$ where $W = \{x, y\}$, $R = \{(x, x), (y, y), (x, y)\}$, and $E = W^2 = \{(x, x), (y, y), (x, y), (y, x)\}$, as shown below:



The arrow represents the nontrivial R-relation and the circle represents that both points are in the same E-equivalence class. It is easy to see that this is an MGrz-frame. Let v be a valuation on $\mathfrak{F}$ with $v(p) = \{y\}$. We first show that

$$x \vDash \blacksquare(\Box(\Box p \to \blacksquare p) \to \blacksquare p).$$

Note that $x \nvDash \blacksquare p$ and $y \nvDash \blacksquare p$, but since $y \vDash p$ and y only sees itself (with respect to R), we have $y \vDash \Box p$. Therefore, $y \nvDash \Box p \to \blacksquare p$, so $x \nvDash \Box(\Box p \to \blacksquare p)$, and hence

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$x \vDash \Box(\Box p \to \blacksquare p) \to \blacksquare p$. Moreover, since $y \nvDash \Box(\Box p \to \blacksquare p)$, we have $y \vDash \Box(\Box p \to \blacksquare p) \to \blacksquare p$, and hence $x \vDash \blacksquare(\Box(\Box p \to \blacksquare p) \to \blacksquare p)$. However, $x \nvDash \blacksquare p$ as xQx and $x \nvDash p$. Thus, $x \nvDash \blacksquare(\Box(\Box p \to \blacksquare p) \to \blacksquare p) \to \blacksquare p$, so $\mathfrak{F} \nvDash_{\mathsf{MGrz}} \mathsf{M_\Box Cas}$, and hence $\mathsf{MGrz} \nvdash \mathsf{M_\Box Cas}$. ⊣

As an immediate consequence of Theorems 3.9 and 3.10, we obtain:

Corollary 3.11 (Esakia). *The splitting translation does not embed* MGrz *into* MGL *faithfully.*

§4. The logics $\mathsf{M^+IPC}$ **and** $\mathsf{M^+Grz}$**.** In the previous section we saw that the splitting translation does not embed MGrz into MGL faithfully. In fact, the Gödel translation of MCas is not provable in MGrz, but the splitting translation of the Gödel translation of MCas is provable in MGL. To repair this disbalance, Esakia suggested to strengthen MIPC with MCas and MGrz with the Gödel translation of MCas. This is what we do in this section.

4.1. $\mathsf{M^+IPC}$**.**

Definition 4.1. The logic $\mathsf{M^+IPC}$ is defined as the extension of MIPC by MCas:

$$\mathsf{M^+IPC} = \mathsf{MIPC} + \mathsf{MCas}.$$

Recall from Definition 3.6 that a cluster of an MIPC-frame is called clean if no distinct points in the cluster are R-related. The following semantic characterization of $\mathsf{M^+IPC}$-frames was established by Esakia. For a proof see [5, Lemma 38]. It states that a descriptive MIPC-frame is a descriptive $\mathsf{M^+IPC}$-frame iff the cluster of each point in the R-maximum of the E-saturation of a clopen set is clean.

Lemma 4.2. [5, Lemma 38] *Let* $\mathfrak{F} = (W, R, E)$ *be a descriptive* MIPC-*frame. Then* $\mathfrak{F} \vDash \mathsf{MCas}$ *iff for each clopen* A, *if* $m \in \max E(A)$, *then* $E(m)$ *is clean.*

Remark 4.3. The condition in [5, Lemma 38] is that $\mathfrak{F} \vDash \mathsf{MCas}$ iff for each clopen A we have $A \subseteq Q^{-1}(\max A \cap \max Q^{-1}A)$. But, as discussed after the proof of [5, Lemma 38], this statement is equivalent to the statement in Lemma 4.2.

As a consequence of Lemma 4.2, we obtain:

Lemma 4.4. *Finite* $\mathsf{M^+IPC}$-*frames are finite* MIPC-*frames in which all clusters are clean.*

4.2. $\mathsf{M^+Grz}$**.**

Definition 4.5. The logic $\mathsf{M^+Grz}$ is the extension of MGrz by $\mathsf{M_\Box Cas}$:

$$\mathsf{M^+Grz} = \mathsf{MGrz} + \mathsf{M_\Box Cas}.$$

Remark 4.6. As we pointed out in the previous section, $\mathsf{M_\Box Cas}$ is provably equivalent to the Gödel translation of MCas.

In order to obtain a semantic characterization of $\mathsf{M^+Grz}$, which is an analogue of Lemma 4.2, we require the following lemma.

Lemma 4.7. *Let* $\mathfrak{F} = (W, R, E)$ *be a descriptive* MGrz-*frame,* $A \subseteq W$ *clopen, and* $y \in \max E(A)$. *If* $E(y)$ *is clean, then*:

1. $E(y) \subseteq \max E(A)$.
2. *For all* $z \in W$, *from* yRz *and* zRy *it follows that* $y = z$.

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PROOF. (1) Let $z \in E(y)$ and $w \in E(A)$ with zRw. By commutativity, there is w' with yRw' and $w'Ew$. Therefore, $w' \in E(A)$. Since $y \in \max E(A)$, we have $y = w'$. Thus, $z, w \in E(y)$ and zRw. As $E(y)$ is clean, $z = w$. This shows that $z \in \max E(A)$.

(2) Suppose yRz and zRy. From $y \in E(A)$ and yRy, we have $y \in R^{-1}E(A)$. We show that $y \in \operatorname{qmax} R^{-1}E(A)$. Let yRw and $w \in R^{-1}E(A)$, so wRu for some $u \in E(A)$. Then yRu by transitivity, and $y \in \max E(A)$ implies $y = u$, hence wRy, and so $y \in \operatorname{qmax} R^{-1}E(A)$. By Lemma 2.26(1), this means $y \in \max R^{-1}E(A)$. Since zRy, we have $z \in R^{-1}E(A)$, so yRz implies $z = y$. ⊣

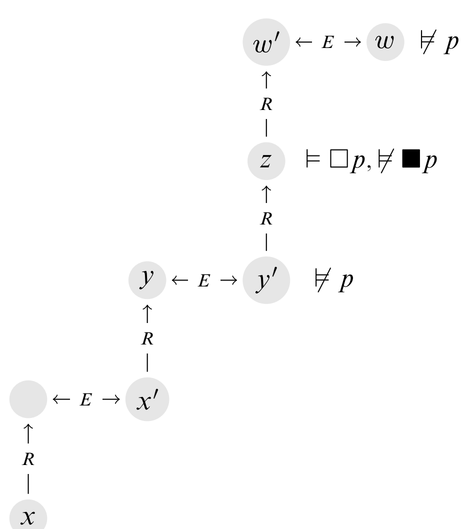
We now have the necessary machinery to prove a semantic characterization of $\mathsf{M}^+\mathsf{Grz}$, which states that a descriptive MGrz-frame is a descriptive $\mathsf{M}^+\mathsf{Grz}$-frame iff the cluster of every point in the maximum of the E-saturation of a clopen set is clean.

LEMMA 4.8. *Let $\mathfrak{F} = (W, R, E)$ be a descriptive* MGrz*-frame. Then $\mathfrak{F} \vDash \mathsf{M}_\Box\mathsf{Cas}$ iff for each clopen A and $m \in \max E(A)$ we have that $E(m)$ is clean.*

PROOF. First suppose $\mathfrak{F} \nvDash \mathsf{M}_\Box\mathsf{Cas}$. Then there is $x \in W$ such that

$$x \nvDash \blacksquare(\Box(\Box p \to \blacksquare p) \to \blacksquare p) \to \blacksquare p,$$

and hence $x \vDash \blacksquare(\Box(\Box p \to \blacksquare p) \to \blacksquare p)$ but $x \nvDash \blacksquare p$. Since $x \nvDash \blacksquare p$, there is $x' \in W$ such that xQx' and $x' \nvDash p$. Let $A = \{w \in W \mid w \nvDash p\}$. Then $x' \in A$, and as $x'Ex'$, we have $x' \in E(A)$. Because A is clopen, so is $E(A)$. By Lemma 2.26(2), there is $y \in \max E(A)$ with $x'Ry$. If $E(y)$ is dirty, then we are done. So assume that $E(y)$ is clean. We show that this leads to a contradiction. Since $y \in E(A)$, there is $y' \in A$ with yEy'. By Lemma 4.7(1), $y' \in \max E(A)$. As xQy' and $x \vDash \blacksquare(\Box(\Box p \to \blacksquare p) \to \blacksquare p)$, we have $y' \vDash \Box(\Box p \to \blacksquare p) \to \blacksquare p$. Because $y' \in A$, we have $y' \nvDash p$ and since $y'Qy'$, we have $y' \nvDash \blacksquare p$, so we must have $y' \nvDash \Box(\Box p \to \blacksquare p)$. Thus, there is $z \in W$ such that $y'Rz$ and $z \nvDash \Box p \to \blacksquare p$, which means $z \vDash \Box p$ but $z \nvDash \blacksquare p$. Because $z \nvDash \blacksquare p$, there exist $w', w \in W$ such that $zRw'Ew$ and $w \nvDash p$ (see the diagram below):



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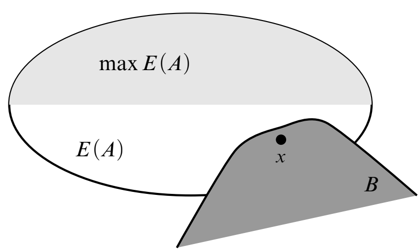
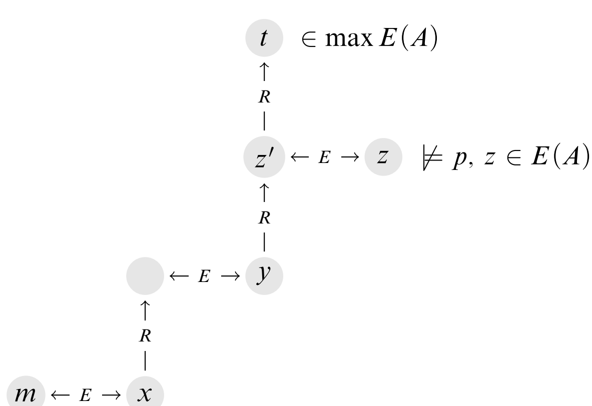
Now, since $w \not\vDash p$, we have $w \in A$ and hence $w' \in E(A)$. Thus, $y'Rw'$ and $w' \in E(A)$, so by R-maximality of y' in $E(A)$, we must have $y' = w'$. But then $y'Rz$ and zRy', and so by Lemma 4.7(2), $y' = z$. This, however, is a contradiction since $z \vDash \Box p$, hence $z \vDash p$, whereas $y' \not\vDash p$.

For the converse, suppose that A is clopen and $m \in \max E(A)$ with $E(m)$ dirty. First observe that since m is maximal in $E(A)$, from mQt it follows that $t \in E(m)$ for all $t \in E(A)$. Indeed, if mQt for $t \in E(A)$, then there is t' with mRt' and $t'Et$. Since $t' \in E(A)$, we have $t' = m$ by maximality of m in $E(A)$. Thus, $t \in E(m)$.

Now, since $E(m)$ is dirty, there are $x, x' \in E(m)$ with xRx' and $x \neq x'$. In particular, $x \notin \max E(A)$. Since $E(A)$ is clopen, $\max E(A)$ is closed (see, e.g., [20, Section III.2]). Thus, we can find clopen B such that $x \in B$ but $B \cap \max E(A) = \varnothing$, as shown below:



Choose a valuation v with $v(p) = W \setminus (B \cap E(A))$. Note that v is well-defined as B and $E(A)$ are clopen. We aim to show that $x \vDash \blacksquare(\Box(\Box p \to \blacksquare p) \to \blacksquare p)$ but $x \not\vDash \blacksquare p$. Since $x \in B \cap E(A)$, we have $x \not\models p$. This implies that $x \not\models \blacksquare p$ because xQx. To finish the argument it suffices to show that $y \models \Box(\Box p \to \blacksquare p) \to \blacksquare p$ for all y with xQy. So let xQy and assume that $y \not\models \blacksquare p$. Then there is z with yQz and $z \not\models p$. Therefore, $z \in B \cap E(A)$ and there is z' with yRz' and $z'Ez$. Clearly $z' \in E(A)$. By Lemma 2.26(2), there is $t \in \max E(A)$ with $z'Rt$:



Since $t \in \max E(A)$, we have $t \notin B$, so $t \models p$ and if tRv for $t \neq v$, then $v \notin E(A)$ by maximality of t, so $v \models p$. Thus, $t \models \Box p$. On the other hand, xQy, yRz', and

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$z'Rt$ imply mQt. As we saw above, this means $t \in E(m)$, and so tEx. Since $x \not\models p$, we have $t \not\models \blacksquare p$. This yields that $t \not\models \Box p \to \blacksquare p$, so $y \not\models \Box(\Box p \to \blacksquare p)$, and hence $y \models \Box(\Box p \to \blacksquare p) \to \blacksquare p$ as desired. ⊣

As a consequence of Lemma 4.8, we obtain:

LEMMA 4.9. *Finite* $\mathsf{M^+Grz}$*-frames are finite* MGrz*-frames in which all clusters are clean.*

4.3. The translations $\mathsf{M^+IPC} \to \mathsf{M^+Grz} \to \mathsf{MGL}$**.** The remaining part of the paper establishes the finite model property for the logics $\mathsf{M^+IPC}$ and $\mathsf{M^+Grz}$. We finish this section by explaining how a proof of Esakia's claim is then obtained.

Let R be a binary relation. We recall that the *irreflexive reduction* of R, denoted R^i, is defined by

$$aR^ib \text{ iff } aRb \text{ and } a \neq b;$$

and the *reflexive closure* of R, denoted R^r, is defined by

$$aR^rb \text{ iff } aRb \text{ or } a = b.$$

For an augmented Kripke frame $\mathfrak{F} = (W, R, E)$, let $\mathfrak{F}^i = (W, R^i, E)$ and $\mathfrak{F}^r = (W, R^r, E)$. Following the terminology of [12, pp. 98–99], we call $\mathfrak{F}^i$ the *irreflexive reduction* and $\mathfrak{F}^r$ the *reflexive closure* of $\mathfrak{F}$.

LEMMA 4.10.

1. *If* $\mathfrak{F}$ *is a finite* $\mathsf{M^+Grz}$*-frame, then* $\mathfrak{F}^i$ *is a finite* MGL*-frame.*
2. *If* $\mathfrak{F}$ *is a finite* MGL*-frame, then* $\mathfrak{F}^r$ *is a finite* $\mathsf{M^+Grz}$*-frame.*

PROOF. Since finite $\mathsf{M^+Grz}$-frames are finite partial orders with clean clusters (Lemma 4.9) and finite MGL-frames are finite strict partial orders with clean clusters (Lemma 3.8), this is an immediate consequence of [12, pp. 98–99]. ⊣

LEMMA 4.11. *Let* φ *be a formula of* $\mathcal{ML}_\forall$.

1. *For a finite* $\mathsf{M^+Grz}$*-frame* $\mathfrak{F}$*, a valuation* v *on* $\mathfrak{F}$*, and every* $x \in \mathfrak{F}$ *we have*

$$(\mathfrak{F}, x) \vDash_v \varphi \Leftrightarrow (\mathfrak{F}^i, x) \vDash_v \varphi^s.$$

2. *For a finite* MGL*-frame* $\mathfrak{F}$*, a valuation* v *on* $\mathfrak{F}$*, and every* $x \in \mathfrak{F}$ *we have*

$$(\mathfrak{F}, x) \vDash_v \varphi^s \Leftrightarrow (\mathfrak{F}^r, x) \models_v \varphi.$$

PROOF. The proof is an immediate consequence of [12, pp. 98–99] since the quantifier modalities are not changed by the translation $(-)^s$, nor is the relation E altered going from $\mathfrak{F}$ to $\mathfrak{F}^i$ or $\mathfrak{F}^r$. ⊣

We are ready to provide a proof of Esakia's claim.

THEOREM 4.12. $\mathsf{M^+IPC} \vdash \varphi$ *iff* $\mathsf{M^+Grz} \vdash \varphi^t$ *iff* $\mathsf{MGL} \vdash (\varphi^t)^s$.

PROOF. The first equivalence is proved exactly as Theorem 3.5 using the fact that finite $\mathsf{M^+IPC}$-frames and finite $\mathsf{M^+Grz}$-frames coincide.

For the second equivalence, suppose $\mathsf{MGL} \nvdash (\varphi^t)^s$. Since MGL has the FMP, there exist a finite MGL-frame $\mathfrak{F}$, a valuation v on $\mathfrak{F}$, and $x \in \mathfrak{F}$ such that $(\mathfrak{F}, x) \nvDash_v (\varphi^t)^s$.

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By Lemma 4.11(2), $(\mathfrak{F}^r, x) \not\models_v \varphi^t$. By Lemma 4.10(2), $\mathfrak{F}^r$ is an $\mathsf{M^+Grz}$-frame. Thus, $\mathsf{M^+Grz} \nvdash \varphi^t$. For the converse, let $\mathsf{M^+Grz} \nvdash \varphi^t$. Since $\mathsf{M^+Grz}$ has the FMP, there exist a finite $\mathsf{M^+Grz}$-frame $\mathfrak{F}$, a valuation v on $\mathfrak{F}$, and $x \in \mathfrak{F}$ such that $(\mathfrak{F}, x) \not\models_v \varphi^t$. By Lemma 4.11(1), $(\mathfrak{F}^i, x) \not\models_v (\varphi^t)^s$. By Lemma 4.10(1), $\mathfrak{F}^i$ is an MGL-frame. Consequently, $\mathsf{MGL} \nvdash (\varphi^t)^s$. ⊣

This concludes lifting the original correspondences given by Goldblatt, Boolos, and Kuznetsov and Muravitsky from the propositional setting to the monadic setting, verifying Esakia's claim. Combining this with Japaridze's result of arithmetical completeness for MGL yields provability interpretations of $\mathsf{M^+IPC}$ and $\mathsf{M^+Grz}$.

§5. The finite model property of $\mathsf{M^+IPC}$. This section is dedicated to the proof of the finite model property of $\mathsf{M^+IPC}$. We do this by modifying the selective filtration technique originally developed by Grefe [29] to prove the finite model property of Fischer Servi's intuitionistic modal logic FS. In [25, Section 10.3] it was used to give an alternative proof of the finite model property of MIPC.

We start by collecting some properties of descriptive $\mathsf{M^+IPC}$-frames that will be useful in what follows. We first give the $\mathsf{M^+IPC}$-version of Lemma 4.7(1).

LEMMA 5.1. *Let $\mathfrak{F} = (W, R, E)$ be a descriptive $\mathsf{M^+IPC}$-frame, $A \subseteq W$ clopen, $y \in \max E(A)$, and $E(y)$ clean. Then $E(y) \subseteq \max E(A)$.*

PROOF. If $E(y) \not\subseteq \max E(A)$, then there are distinct $t \in E(y)$ and $u \in E(A)$ with tRu. By commutativity, there is u' with yRu' and $u'Eu$. Therefore, $u' \in E(A)$, so by maximality of y in $E(A)$ we have $y = u'$. This implies that tEu, contradicting that $E(y)$ is a clean cluster. ⊣

We say a point x is *maximal with respect to a formula* ψ if $x \not\models \psi$ and for each y with xRy and $x \neq y$ we have $y \models \psi$ (that is, x refutes ψ and every point strictly above x validates ψ).

LEMMA 5.2. *Let $\mathfrak{F} = (W, R, E)$ be a descriptive $\mathsf{M^+IPC}$-frame, $t \in W$, and v a valuation on $\mathfrak{F}$.*

1. *Let $A \subseteq W$ be clopen. If $t \in E(A)$, then there is $x \in \max E(A)$ such that tRx and $E(x)$ is clean.*
2. *If $t \not\models \forall\varphi$, then there is x such that tRx, x is maximal with respect to $\forall\varphi$, and $E(x)$ is clean.*
3. *Let $A \subseteq W$ be clopen. If $t \in A$, then there is $x \in A \cap \max E(A)$ such that tQx and $E(x)$ is clean.*
4. *If $t \not\models \varphi$, then there is x such that tQx, x is maximal with respect to φ, and $E(x)$ is clean.*

PROOF. (1) Let $t \in E(A)$. By Lemma 2.12, there is $x \in \max E(A)$ such that tRx. By Lemma 4.2, $E(x)$ is clean.

(2) Suppose that $t \not\models \forall\varphi$. Let $A = W \setminus v(\forall\varphi)$. Then A is clopen, $E(A) = A$, and $t \in E(A)$. By (1), there is $x \in \max E(A)$ such that tRx and $E(x)$ is clean. Since $E(A) = A$, it immediately follows that x is maximal with respect to $\forall\varphi$.

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(3) Let $t \in A$. Then $t \in E(A)$. By (1), there is $x' \in \max E(A)$ such that tRx' and $E(x')$ is clean. Since $x' \in E(A)$, there is $x \in A$ with $x'Ex$. Therefore, tQx, and because $E(x')$ is clean, we have that $x \in \max E(A)$ by Lemma 5.1.

(4) Suppose that $t \not\models \varphi$. Let $A = W \setminus v(\varphi)$. Then A is clopen and $t \in A$. By (3), there is $x \in A \cap \max E(A)$ such that tQx and $E(x)$ is clean. Since $x \in A$, we also have $x \in \max A$. But the latter means that x is maximal with respect to φ. Thus, x is as desired. ⊣

5.1. The construction. We start with a formula φ, a descriptive $\mathsf{M}^+\mathsf{IPC}$-frame $\mathfrak{F} = (W, R, E)$, and a valuation v on $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. By modifying the construction in [25, Section 10.3], we will construct a sequence of finite $\mathsf{M}^+\mathsf{IPC}$-frames $\mathfrak{F}_h = (W_h, R_h, E_h)$ such that $\mathfrak{F}_h \subseteq \mathfrak{F}_{h+1}$ for all $h < \omega$. For each point $t \in W_h$ that we select, we will be creating a copy of some original point in W. We give each added point a new name, say t, and let $\widehat{t}$ denote the original point in W that t was copied from and will behave similar to. Thus, it is possible to have two different points x_1 and x_2 in our new frame, where $\widehat{x_1} = \widehat{x_2}$. The main difference between our construction and the construction given in [25, Section 10.3] will be seen in the $\to$-step, which requires a more careful selection of new points.

To start the construction, let $\mathfrak{F}_0 = (W_0, R_0, E_0)$ where

$$W_0 = \{t_0\}, \quad R_0 = W_0^2, \quad E_0 = W_0^2,$$

and $\widehat{t_0}$ is a point in W such that $\widehat{t_0}$ is from a clean cluster and is maximal with respect to φ. The existence of such $\widehat{t_0}$ follows from Lemma 5.2(4). Moreover, let $W_{-1}^{\forall H} = \varnothing$.

Let $\mathsf{Sub}(\varphi)$ be the set of subformulas of φ, and let (W', R', E') be any of our frames in the construction. To each $t \in W'$ we associate the following subsets of $\mathsf{Sub}(\varphi)$:

$$\begin{aligned}
\Sigma^{\exists}(t) &= \{\exists\delta \in \mathsf{Sub}(\varphi) \mid \widehat{t} \models \exists\delta\},\\
\Sigma^{\forall H}(t) &= \{\forall\beta \in \mathsf{Sub}(\varphi) \mid \widehat{t} \text{ is maximal wrt } \forall\beta\},\\
\Sigma^{\forall V}(t) &= \{\forall\gamma \in \mathsf{Sub}(\varphi) \mid \widehat{t} \not\models \forall\gamma \text{ but is not maximal wrt } \forall\gamma\},\\
\Sigma^{\to}(t) &= \{\alpha \to \sigma \in \mathsf{Sub}(\varphi) \mid \widehat{t} \not\models \alpha \to \sigma \text{ and } \widehat{t} \not\models \alpha\}.
\end{aligned}$$

These are precisely the subformulas of φ whose truth-value at $\widehat{t}$ is relevant for constructing our countermodel. Note that if $\alpha \to \sigma \in \Sigma^{\to}(t)$, then $\widehat{t}$ is not maximal with respect to $\alpha \to \sigma$.

Suppose $\mathfrak{F}_{h-1} = (W_{h-1}, R_{h-1}, E_{h-1})$ has already been constructed so that $\mathfrak{F}_{h-1}$ is a finite $\mathsf{M}^+\mathsf{IPC}$-frame and $E(\widehat{w})$ is a clean cluster for each $w \in W_{h-1}$. We construct $\mathfrak{F}_h$ by applying the four steps described below. They are designed to add the necessary witnesses required by the formulas in the sets $\Sigma^{\exists}(t)$, $\Sigma^{\forall H}(t)$, $\Sigma^{\forall V}(t)$, and $\Sigma^{\to}(t)$, respectively. In the $\exists$-step we ensure that for each formula in $\Sigma^{\exists}(t)$ the point t has an E-successor that witnesses the existential statement. In the $\forall H$-step we ensure that for each formula in $\Sigma^{\forall H}(t)$ the point t has an E-successor that witnesses the refutation of the universal statement. In the vertical steps $\forall V$ and $\to$ we make sure that t has the necessary R-successors that are maximal with respect to the formulas in $\Sigma^{\forall V}(t)$ and $\Sigma^{\to}(t)$, respectively. In each step of the construction we add also points to witness commutativity. Note that the first three of the following four steps are only done once per cluster. This is enough since all points of a cluster in $\mathfrak{F}$ agree on refuting an $\forall$- or $\exists$-formula and points from a clean cluster agree whether such a refutation is maximal.

Roughly speaking, points are added to the construction in the following order: In the first round the cluster of the starting point t_0 is built by adding points for formulas in $\Sigma^{\exists}(t_0)$ and $\Sigma^{\forall H}(t_0)$. After this, no more points are added to this cluster. We call this the 'bottom cluster' of our frame. The first round of the construction proceeds by adding vertical witnesses for each formula in $\Sigma^{\forall V}(t_0)$ and closing each such cluster by adding points for commutativity. The first round then finishes by adding for each point t in the 'bottom cluster' vertical witnesses for the formulas in $\Sigma^{\to}(t)$ and closing under commutativity. In the next round all these newly built clusters will possibly be enlarged in the horizontal steps and then new vertical witnesses will be added in the $\forall V$- and $\to$-steps.

$\exists$-*step* (*Horizontal*): Let $W_h^{\exists} = W_{h-1}$, $R_h^{\exists} = R_{h-1}$, and $E_h^{\exists} = E_{h-1}$. For each $E_h^{\exists}(t) \subseteq W_h^{\exists} \backslash W_{h-1}^{\forall H}$, if $\exists\delta \in \Sigma^{\exists}(t)$ but there is no $s \in W_h^{\exists}$ already such that $tE_h^{\exists}s$ and $\widehat{s} \vDash \delta$, then we add a point s to $W_h^{\exists}$ with $\widehat{s} \models \delta$ and $\widehat{t}E\widehat{s}$. Such a point $\widehat{s}$ exists in W since $\widehat{t} \models \exists\delta$. We then add the ordered pairs (s, s) to $R_h^{\exists}$, the ordered pairs (t, s) to $E_h^{\exists}$, and generate the least equivalence relation.

$\forall H$-*step* (*Horizontal*): Let $W_h^{\forall H} = W_h^{\exists}$, $R_h^{\forall H} = R_h^{\exists}$, and $E_h^{\forall H} = E_h^{\exists}$. For each $E_h^{\forall H}(t) \subseteq W_h^{\forall H} \backslash W_{h-1}^{\forall H}$, if $\forall\beta \in \Sigma^{\forall H}(t)$ but there is no $s \in W_h^{\forall H}$ already such that $tE_h^{\forall H}s$ and $\widehat{s} \nvDash \beta$, then we add a point s to $W_h^{\forall H}$ with $\widehat{s} \not\models \beta$ and $\widehat{t}E\widehat{s}$. Such a point $\widehat{s}$ exists in W since $\widehat{t}$ is maximal with respect to $\forall\beta$. We then add the ordered pairs (s, s) to $R_h^{\forall H}$, the ordered pairs (t, s) to $E_h^{\forall H}$, and generate the least equivalence relation.

$\forall V$-*step* (*Vertical*): Let $W_h^{\forall V} = W_h^{\forall H}$, $R_h^{\forall V} = R_h^{\forall H}$, and $E_h^{\forall V} = E_h^{\forall H}$. For each $E_h^{\forall V}(t) \subseteq W_h^{\forall V} \backslash W_{h-1}^{\forall H}$, consider $\forall\gamma \in \Sigma^{\forall V}(t)$. Since $\widehat{t} \nvDash \forall\gamma$, we can pick a point $\widehat{s} \in W$ as in Lemma 5.2(2). We add the point s to $W_h^{\forall V}$ and (t, s) to $R_h^{\forall V}$.

Since W satisfies commutativity, for each $w \in E_h^{\forall V}(t)$, there is $z_w \in W$ such that $\widehat{w}Rz_w$ and $z_w E\widehat{s}$. To ensure commutativity is satisfied in our new frame, we add the points s_w to $W_h^{\forall V}$ where $\widehat{s_w} = z_w$. We then add (w, s_w) to $R_h^{\forall V}$ and take the reflexive and transitive closure. We also add (s_w, s) to $E_h^{\forall V}$ and generate the least equivalence relation.

$\to$-*step* (*Vertical*): Let $W_h^{\to} = W_h^{\forall V}$, $R_h^{\to} = R_h^{\forall V}$, and $E_h^{\to} = E_h^{\forall V}$. For each $t \in W_h^{\forall H} \backslash W_{h-1}^{\forall H}$ (hence including any points added in the horizontal steps above, but not in the previous vertical step), consider all $\alpha \to \sigma \in \Sigma^{\to}(t)$ such that there is no $s \in W_h^{\to}$ already such that $tR_h^{\to}s$, $\widehat{s} \vDash \alpha$, and $\widehat{s} \nvDash \sigma$. Consider

$$A = [W \setminus v(\alpha \to \sigma)] \cap \bigcap_{\psi \in \mathsf{Sub}(\varphi)} \{v(\psi) : \widehat{t} \vDash \psi\}.$$

Then A is clopen and $\widehat{t} \in A$, so by Lemma 5.2(3) there is $z \in A$ with $z \in \max E(A)$, $\widehat{t}Qz$, and $E(z)$ clean. We add the point s to $W_h^{\to}$ where $\widehat{s} = z$ (s is a distinct new copy of z) and (t, s) to $R_h^{\to}$.

REMARK 5.3. It is at this step that we have altered the construction given in [25, Section 10.3], in which witnesses for implications are added in the same manner

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as in the $\forall V$-step. In our version, we took an original Q-relation and turned it into an R-relation. The reason for this is that we cannot guarantee the existence of an R-successor of t that is maximal with respect to $\alpha \to \sigma$ and at the same time belongs to a clean cluster.

Before wrapping up the step, we show two properties of the chosen points.

LEMMA 5.4. *The point $\widehat{s} = z$, as chosen above, is maximal with respect to $\alpha \to \sigma$.*

PROOF. Let zRu for some $u \not\vDash \alpha \to \sigma$. Since zRu and each $v(\psi)$ in $\{v(\psi) : \widehat{t} \vDash \psi\}$ is an upset, we have $u \in A$. Because $z \in \max A$, we obtain $z = u$. Thus, z is maximal with respect to $\alpha \to \sigma$. ⊣

LEMMA 5.5. $E(\widehat{t}\,) \neq E(\widehat{s}\,)$.

PROOF. If $E(\widehat{t}\,) = E(\widehat{s}\,)$, then $\widehat{t} \in \max E(A)$ by Lemma 5.1. Since $\widehat{t} \in A$, we have $\widehat{t} \in \max A$. Therefore, the same argument as in the proof of the previous lemma yields that $\widehat{t}$ is maximal with respect to $\alpha \to \sigma$. This contradicts the fact that $\alpha \to \sigma \in \Sigma^{\to}(t)$. ⊣

We wrap up the $\to$-step the same way as the $\forall V$-step. Since W satisfies commutativity, for each $w \in E_h^{\forall V}(t)$ there is $z_w \in W$ with $\widehat{w}Rz_w$ and $z_w E\widehat{s}$. We add the points s_w to $W_h^{\forall V}$ where $\widehat{s_w} = z_w$. We then add (w, s_w) to $R_h^{\to}$ and take the reflexive and transitive closure. We also add (s_w, s) to $E_h^{\to}$ and generate the least equivalence relation.

To end this stage of the construction, we let $\mathfrak{F}_h = (W_h, R_h, E_h)$ where

$$W_h = W_h^{\to}, \;\; R_h = R_h^{\to}, \;\; E_h = E_h^{\to}.$$

LEMMA 5.6. *$\mathfrak{F}_h$ is a finite* $\mathsf{M^+IPC}$*-frame.*

PROOF. First we show that R_h is a partial order. Since in the $\exists$- and $\forall H$-steps we only added reflexive arrows to R_{h-1}, the relation $R_h^{\forall H}$ is a partial order. By moving from $R_h^{\forall H}$ to R_h we finished by taking the reflexive and transitive closure, hence R_h is clearly reflexive and transitive. Antisymmetry of R_h follows from the fact that every R-arrow added in the $\forall V$-step and $\to$-step is either reflexive or an arrow from a previously existing point into a freshly added point.

That E_h is an equivalence relation is clear from the construction. Moreover, the extra points added in the $\forall V$-step and $\to$-step make sure that commutativity is satisfied. In fact, the added points assure commutativity for immediate successors and by transitivity this implies commutativity for the whole frame. Therefore, $\mathfrak{F}_h$ is an MIPC-frame.

It follows from the construction that $\mathfrak{F}_h$ is finite. Thus, by Lemma 4.4, it is left to show that $\mathfrak{F}_h$ has clean clusters. Note that in the $\exists$-step and $\forall H$-step all freshly introduced E_h-relations are of the shape (s, t) where either $s \in W_h$ and $t \in W_h^{\forall H} \setminus W_{h-1}$ or $s, t \in W_h^{\forall H}$. Since no non-reflexive R_h-arrows are introduced in these steps, no dirty cluster could have been built. We have already discussed the shape of the R_h arrows introduced in the $\forall V$-step and $\to$-step. This guarantees that no cluster in $W_h^{\forall H}$ is made dirty. The freshly introduced E_h-relations in these steps are of the shape (s, t) where $s, t \in W_h \setminus W_h^{\forall H}$. Since no non-reflexive R_h relations exist between these points, we infer that all clusters are clean. ⊣

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5.2. Auxiliary lemmas. To prove that our construction terminates after finitely many steps, we require several auxiliary lemmas.

LEMMA 5.7. *Let $x, y \in W_h$.*

1. *If $xR_h y$ and $x \neq y$, then $\widehat{x}Q\widehat{y}$ and $E(\widehat{x}) \neq E(\widehat{y})$.*
2. *If $xE_h y$, then $\widehat{x}E\widehat{y}$.*
3. *If $xQ_h y$, then $\widehat{x}Q\widehat{y}$.*

PROOF. (1) Observe that in the construction each non-trivial R_h-relation between immediate successors comes from either a non-trivial R-relation (as in the case of points added for commutativity or in the $\forall V$-step) or a non-trivial Q-relation (as in the case of points added in the $\to$-step). In the former case, it is obvious that $\widehat{x}Q\widehat{y}$ and $E(\widehat{x}) \neq E(\widehat{y})$. In the latter case, there is $w \in W$ with $\widehat{x} \neq w$, $\widehat{x}Rw$, and $wE\widehat{y}$. In that case we obviously have $\widehat{x}Q\widehat{y}$ and by Lemma 5.5, $E(\widehat{x}) \neq E(\widehat{y})$. If the relation $xR_h y$ was added by transitivity, there is a chain $x = x_0 R_h x_1 R_h \cdots R_h x_n = y$ of immediate R_h-successors to which the previous applies, hence $\widehat{x}Q\widehat{y}$ and $E(\widehat{x}) \neq E(\widehat{y})$ by induction.

(2) It is obvious that each E_h-relation in W_h comes from a pre-existing E-relation in W.

(3) If $xQ_h y$, then there is z with $xR_h z$ and $zE_h y$. If $x = z$, then $xE_h y$, so $\widehat{x}E\widehat{y}$ by (2), and hence $\widehat{x}Q\widehat{y}$. If $x \neq z$, then $\widehat{x}Q\widehat{z}$ by (1). Also, $zE_h y$ implies $\widehat{z}E\widehat{y}$ by (2). Thus, $\widehat{x}Q\widehat{y}$. ⊣

LEMMA 5.8 (Persistence). *If $uR_h w$, then $\widehat{u} \vDash \psi$ implies $\widehat{w} \vDash \psi$ for all $\psi \in \mathsf{Sub}(\varphi)$.*

PROOF. Suppose $uR_h w$, $\psi \in \mathsf{Sub}(\varphi)$, and $\widehat{u} \vDash \psi$. It suffices to show the result for an immediate R_h-successor w of u, the general result then follows by induction. We consider how the R_h-arrow from u to w was added. By construction, either $\widehat{u}R\widehat{w}$ or w was added to witness some implication in $\Sigma^{\to}(u)$. If $\widehat{u}R\widehat{w}$, then clearly $\widehat{u} \vDash \psi$ implies $\widehat{w} \vDash \psi$. If w was added in a $\to$-step, then w is specifically chosen so that $\widehat{w} \in v(\gamma)$ for all $\gamma \in \mathsf{Sub}(\varphi)$ such that $\widehat{u} \vDash \gamma$. Thus, $\widehat{u} \vDash \psi$ implies $\widehat{w} \vDash \psi$. ⊣

LEMMA 5.9.

1. *If $tE_h u$, then $\Sigma^{\exists}(t) = \Sigma^{\exists}(u)$, $\Sigma^{\forall H}(t) = \Sigma^{\forall H}(u)$, and $\Sigma^{\forall V}(t) = \Sigma^{\forall V}(u)$.*
2. *If $tR_h v$ and $\exists\gamma \in \Sigma^{\exists}(t) \cap \Sigma^{\exists}(v)$, then there are u, w such that $tE_h u$, $uR_h w$, $wE_h v$, $\widehat{u} \models \gamma$, and $\widehat{w} \models \gamma$.*
3. *If $tR_h v$ and $t \neq v$, then $\Sigma^{\forall H}(t) \cap \Sigma^{\forall H}(v) = \varnothing$.*
4. *Along an R_h-chain, each formula in*

$$\{\forall\psi \mid \forall\psi \in \mathsf{Sub}(\varphi)\} \cup \{\exists\psi \mid \exists\psi \in \mathsf{Sub}(\varphi)\}$$

   can serve at most once as a reason to enlarge a cluster in a horizontal step.
5. *If $tR_h u$, then $\Sigma^{\forall V}(u) \subseteq \Sigma^{\forall V}(t)$ and if u was added as an immediate R_h-successor to t because of $\forall\alpha \in \Sigma^{\forall V}(t)$, then $\Sigma^{\forall V}(u) \subset \Sigma^{\forall V}(t)$.*
6. *If $tR_h u$, then $\Sigma^{\to}(u) \subseteq \Sigma^{\to}(t)$ and if u was added as an immediate R_h-successor to t because of $\alpha \to \beta \in \Sigma^{\to}(t)$, then $\Sigma^{\to}(u) \subset \Sigma^{\to}(t)$.*

PROOF. (1) Suppose $tE_h u$. Then $\widehat{t}E\widehat{u}$ by Lemma 5.7(2). Therefore, $E(\widehat{t}) = E(\widehat{u})$ and $Q(\widehat{t}) = Q(\widehat{u})$. Thus, $\widehat{t} \vDash \exists\gamma$ iff $\widehat{u} \vDash \exists\gamma$, and $\widehat{t} \vDash \forall\gamma$ iff $\widehat{u} \vDash \forall\gamma$. Moreover, since $E(\widehat{t})$ is a clean cluster, $\widehat{t}$ is not maximal wrt $\forall\gamma$ iff $\widehat{u}$ is not maximal wrt $\forall\gamma$. Consequently, $\Sigma^{\exists}(t) = \Sigma^{\exists}(u)$, $\Sigma^{\forall H}(t) = \Sigma^{\forall H}(u)$, and $\Sigma^{\forall V}(t) = \Sigma^{\forall V}(u)$.

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(2) Suppose tR_hv and $\exists\gamma \in \Sigma^{\exists}(t) \cap \Sigma^{\exists}(v)$. By the construction, there is u with tE_hu and $\widehat{u} \vDash \gamma$. Since $\mathfrak{F}_h$ satisfies commutativity, there is w with uR_hw and wE_hv. By Lemma 5.8, $\widehat{w} \vDash \gamma$.

(3) Suppose tR_hv and $t \neq v$. Then $\widehat{t}Q\widehat{v}$ and $E(\widehat{t}\,) \neq E(\widehat{v}\,)$ by Lemma 5.7(1), so $\widehat{t} \neq \widehat{v}$. Thus, if $\forall\psi \in \Sigma^{\forall H}(t)$, then $\widehat{v} \vDash \forall\psi$ by maximality of $\widehat{t}$, so $\forall\psi \notin \Sigma^{\forall H}(v)$. Conversely, if $\forall\psi \in \Sigma^{\forall H}(v)$, then $\widehat{t}$ cannot be maximal with respect to $\forall\psi$, so $\forall\psi \notin \Sigma^{\forall H}(t)$.

(4) Let $\{v_i \mid i \in \mathbb{N}\}$ be an R_h-chain in W_h, i.e., $v_iR_hv_{i+1}$ for all $i \in \mathbb{N}$. Suppose $\exists\psi \in \mathsf{Sub}(\varphi)$. Let k be the least stage at which the formula $\exists\psi$ has been used to enlarge the cluster $E_h(v_k)$ in a horizontal step. By (2), all $E_h(v_l)$ for $l > k$ already contain a witness for ψ, so no cluster above will need to be enlarged in a horizontal step to witness the formula $\exists\psi$. Now suppose $\forall\psi \in \mathsf{Sub}(\varphi)$. Let l be a stage at which the formula $\forall\psi$ has been used to enlarge the cluster $E_h(v_l)$ in a horizontal step. Then $\forall\psi \in \Sigma^{\forall H}(v_l)$. By (3), $\forall\psi \notin \Sigma^{\forall H}(v_k)$ for $k \neq l$. Thus, $\forall\psi$ is responsible for enlarging a cluster at most once in a horizontal step.

(5) We show the statement for immediate R_h-successors only, the general case follows by induction. Suppose tR_hu and $\forall\psi \in \Sigma^{\forall V}(u)$. If $t = u$, then the result is clear. Suppose $t \neq u$. Since tR_hu, either $\widehat{t}R\widehat{u}$ in W or u was added as a successor of t in some $\to$-step. If $\widehat{t}R\widehat{u}$, then $\forall\psi \in \Sigma^{\forall V}(t)$ by persistence (see Lemma 5.8). Suppose u was added as an R_h-successor to t as a witness to some implication. By the choice of u, we have $\widehat{u} \vDash \chi$ for all $\chi \in \mathsf{Sub}(\varphi)$ with $\widehat{t} \vDash \chi$. Therefore, if $\widehat{t} \vDash \forall\psi$, then we would have $\widehat{u} \vDash \forall\psi$, contradicting $\forall\psi \in \Sigma^{\forall V}(u)$. Thus, we must have $\widehat{t} \not\vDash \forall\psi$. Moreover, since $\widehat{t}R\widehat{u}$, $\widehat{t} \neq \widehat{u}$, and $\widehat{u} \not\vDash \forall\psi$, we have that $\widehat{t}$ is not maximal with respect to $\forall\psi$, so $\forall\psi \in \Sigma^{\forall V}(t)$. Consequently, we have $\Sigma^{\forall V}(u) \subseteq \Sigma^{\forall V}(t)$.

Suppose that u was added as an immediate R_h-successor to t because of $\forall\alpha \in \Sigma^{\forall V}(t)$. Since $\forall\alpha \in \Sigma^{\forall V}(t)$, we have $\widehat{t} \not\vDash \forall\alpha$ but $\widehat{t}$ is not maximal with respect to $\forall\alpha$. Since u was added as an immediate R_h-successor of t because of $\forall\alpha$, we specifically chose u so that $\widehat{u} \not\vDash \forall\alpha$ maximally, hence $\forall\alpha \notin \Sigma^{\forall V}(u)$.

(6) We show the statement for immediate R_h-successors only, the general case follows by induction. Suppose tR_hu and $\alpha \to \beta \in \Sigma^{\to}(u)$. Then $\widehat{u} \not\vDash \alpha \to \beta$ and $\widehat{u} \not\vDash \alpha$. If $t = u$, then the result is clear. Suppose $t \neq u$. Since tR_hu, either $\widehat{t}R\widehat{u}$ in W or u was added as a successor of t in some $\to$-step. If $\widehat{t}R\widehat{u}$, then $\alpha \to \beta \in \Sigma^{\to}(t)$ by persistence (see Lemma 5.8). Suppose u was added as an R_h-successor to t as a witness to some implication. By the choice of u, we have $\widehat{u} \vDash \psi$ for all $\psi \in \mathsf{Sub}(\varphi)$ with $\widehat{t} \vDash \psi$. Therefore, we must have $\widehat{t} \not\vDash \alpha \to \beta$ and $\widehat{t} \not\vDash \alpha$, so $\Sigma^{\to}(u) \subseteq \Sigma^{\to}(t)$. Moreover, if u was added as an immediate R_h successor to t because of $\alpha \to \beta$, then $\widehat{u}$ refutes $\alpha \to \beta$ maximally (Lemma 5.4), and hence $\widehat{u} \vDash \alpha$. Thus, $\alpha \to \beta \notin \Sigma^{\to}(u)$. ⊣

5.3. Termination of the construction. With the aid of the auxiliary lemmas of the previous section, we will now prove that the end result of our construction is a finite frame. We will do this by looking at three important parameters of our frame: cluster size, R-branching, and R-depth.

DEFINITION 5.10.

1. A frame $\mathfrak{F}$ has *bounded cluster size* if there exists $k \in \mathbb{N}$ such that $|E(t)| \leq k$ for all $t \in W$.

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2. A frame $\mathfrak{F}$ has *bounded R-branching* if there exists $m \in \mathbb{N}$ such that t has at most m distinct immediate R-successors for all $t \in W$.
3. A frame $\mathfrak{F}$ has *bounded R-depth* if there exists $n \in \mathbb{N}$ such that there is no R-chain in $\mathfrak{F}$ with more than n distinct elements.

We call $\mathfrak{F} = (W, R, E)$ *rooted* if there exists $w \in W$, called a *root* of $\mathfrak{F}$, such that $W = Q(w)$.

LEMMA 5.11. *Let $\mathfrak{F} = (W, R, E)$ be a partially ordered rooted augmented Kripke frame. If $\mathfrak{F}$ has bounded cluster size, bounded R-branching, and bounded R-depth, then $\mathfrak{F}$ is finite.*

PROOF. Suppose $\mathfrak{F} = (W, R, E)$ is a partially ordered rooted augmented Kripke frame with bounded cluster size, R-branching, and R-depth. Consider the quotient $(W/E, R_E)$ whose worlds are the clusters $E(x)$ where $x \in W$ and $E(x) R_E E(y)$ iff xQy. To see that R_E is well defined, suppose xQy, $x' \in E(x)$, and $y' \in E(y)$. Then $x'ExQyEy'$, so $x'Qy'$, and hence R_E is well defined.

Because Q is reflexive and transitive, so is R_E. Since R is a partial order and $\mathfrak{F}$ has bounded R-depth, from xQy and yQx it follows that xEy by [4, Lemma 3(b)]. This shows that R_E is anti-symmetric, and hence a partial order. Clearly $(W/E, R_E)$ is rooted since so is $\mathfrak{F}$. Using commutativity in $\mathfrak{F}$ it is easy to verify that $(W/E, R_E)$ inherits bounded depth and bounded branching from $\mathfrak{F}$. Since every rooted partial order with these properties is finite, we have that W/E is finite. Because W has bounded cluster size, we conclude that W is finite too. ⊣

Let m_1, m_2, m_3 be the non-negative integers

$$m_1 = |\{\exists\psi \mid \exists\psi \in \mathsf{Sub}(\varphi)\}|,$$
$$m_2 = |\{\forall\psi \mid \forall\psi \in \mathsf{Sub}(\varphi)\}|,$$
$$m_3 = |\{\psi \to \chi \mid \psi \to \chi \in \mathsf{Sub}(\varphi)\}|.$$

LEMMA 5.12. *For all $h < \omega$, the cluster size of $\mathfrak{F}_h = (W_h, R_h, E_h)$ is bounded by $1 + m_1 + m_2$.*

PROOF. Recall how the clusters of our frame are built. The 'bottom cluster' of the starting point t_0 contains points added via the horizontal $\exists$- and $\forall H$-steps. After this, no more points are added to this cluster.

All other clusters are constructed as follows. First points of a new cluster are added via the vertical $\forall V$- or $\to$-steps, and then the cluster is enlarged by the points added for commutativity. We refer to this stage as the 'building phase' of the cluster. In the next round of the construction, the cluster is (possibly) enlarged via the two horizontal steps. After this, no more points are added to the cluster. In the horizontal steps, we enlarge the cluster for only two different reasons:

$$\exists\gamma \in \Sigma^{\exists}(t) \text{ or } \forall\gamma \in \Sigma^{\forall H}(t).$$

Thus, each enlargement of a cluster after its building phase is due to a formula in $\{\forall\psi \mid \forall\psi \in \mathsf{Sub}(\varphi)\} \cup \{\exists\psi \mid \exists\psi \in \mathsf{Sub}(\varphi)\}$. At the end of its building phase, the bottom cluster contains just one point. Observe that every cluster can be reached

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from the bottom cluster by an R_h-chain. It follows from Lemma 5.9(4) that every formula in $\{\forall\psi \mid \forall\psi \in \mathsf{Sub}(\varphi)\} \cup \{\exists\psi \mid \exists\psi \in \mathsf{Sub}(\varphi)\}$ can serve at most once as a reason to enlarge a cluster after its building phase along an R_h-chain. This entails that every cluster has size at most $1 + m_1 + m_2$. ⊣

LEMMA 5.13. *For all $h < \omega$, the R_h-branching of $\mathfrak{F}_h = (W_h, R_h, E_h)$ is bounded by $m_2 + (1 + m_1 + m_2) \cdot m_3$.*

PROOF. Immediate R_h-successors are added in the $\forall V$-step and $\to$-step. First observe that since we are adding points to witness commutativity, every point in a cluster has the same number of immediate R_h-successors by the end of a stage. Thus, it is enough to count the immediate successors of a point t that we picked in the $\forall V$-step.

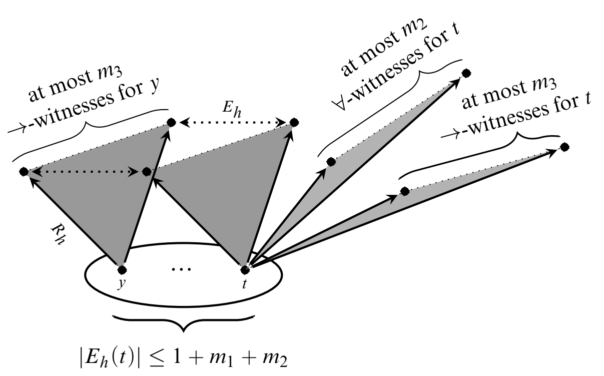
To such a point t we add immediate R_h-successors for three different reasons:

1. $\forall\gamma \in \Sigma^{\forall V}(t)$,
2. $\alpha \to \sigma \in \Sigma^{\to}(t)$, or
3. $\alpha \to \sigma \in \Sigma^{\to}(y)$ for some $y \in E_h(t)$ with $y \neq t$.

The last reason covers the case where we add an R_h-successor to t to witness commutativity. Note that all reasons occur at most once for each formula in the respective sets. Therefore, reason (1) occurs at most m_2-times and reason (2) at most m_3-times. Finally, reason (3) occurs at most $(m_1 + m_2) \cdot m_3$ times since by Lemma 5.12 there are at most $m_1 + m_2$ points apart from t in the cluster of t. Thus, the R_h-branching of $\mathfrak{F}$ is bounded by

$$m_2 + m_3 + (m_1 + m_2) \cdot m_3 = m_2 + (1 + m_1 + m_2) \cdot m_3$$

(see the diagram below):



⊣

LEMMA 5.14. *For all $h < \omega$, the R_h-depth of $\mathfrak{F}_h = (W_h, R_h, E_h)$ is bounded by $(1 + m_1 + m_2) \cdot (m_2 + m_3)$.*

PROOF. The reason for adding an immediate successor to $t \in W_h$ via an R_h-relation is due to either a formula in $\Sigma^{\forall V}(t)$ or a formula in $\Sigma^{\to}(y)$ for some $y \in E_h(t)$ (as discussed in the proof of Lemma 5.13). Let s be a (not necessarily immediate) R_h-successor of t. Then s could have been added via direct formula witnessing, i.e., there is an immediate predecessor t' of s with $tR_ht'R_hs$ and s was added due to a formula in $\Sigma^{\forall V}(t')$ or $\Sigma^{\to}(t')$, or else s was added to satisfy commutativity.

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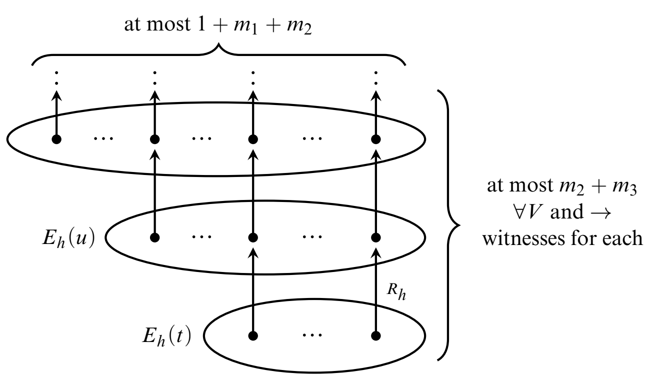
As we saw in Lemma 5.9, moving up along an R_h-chain, the cardinality of the sets $\Sigma^{\forall V}(t)$ and $\Sigma^{\to}(t)$ does not increase, and it in fact decreases whenever an R_h-successor is added by direct formula witnessing. In particular, each point can have at most $m_2 + m_3$ R_h-successors that have been added via direct formula witnessing and since in each cluster there are at most $1 + m_1 + m_2$ points (Lemma 5.12), we have that the total R_h-depth cannot exceed

$$(1 + m_1 + m_2) \cdot (m_2 + m_3)$$

(see the diagram below):



Lemma 5.15. *There is $h \in \mathbb{N}$ such that $\mathfrak{F}_{h'} = \mathfrak{F}_h$ for all $h' \geq h$.*

Proof. All points in the bottom cluster are added in round 1 and in each round we enlarge the R_h-length of a path by at most one. Thus, in stage k of the construction, all R_h-chains are bounded by k. The construction continues only until vertical witnesses are required. Since, by Lemma 5.14, the R_h-depth of $\mathfrak{F}_k$ is bounded by $m = (1 + m_1 + m_2) \cdot (m_2 + m_3)$, we have $\mathfrak{F}_{h'} = \mathfrak{F}_{m+1}$ for all $h' \geq m + 1$. ⊣

Set $\mathfrak{F}' = (W', R', E')$ where

$$W' = W_h, \quad R' = R_h, \quad E' = E_h,$$

and h is as in Lemma 5.15. Then $\mathfrak{F}'$ is a finite $\mathsf{M^+IPC}$-frame by Lemma 5.6.

5.4. Truth lemma. Define a valuation v' on W' by

$$v'(p) = \{t \in W' \mid \widehat{t} \in v(p)\}$$

for $p \in \mathsf{Sub}(\varphi)$ and $v'(q) = \varnothing$ for variables q not occurring in φ. That v' is well defined follows from Lemma 5.8, which ensures that the sets $v'(\psi)$ are in $\mathsf{Up}(\mathfrak{F}')$ for each $\psi \in \mathsf{Sub}(\varphi)$.

Lemma 5.16 (Truth Lemma). *For all $t \in W'$ and $\psi \in \mathsf{Sub}(\varphi)$, we have $t \vDash' \psi$ iff $\widehat{t} \vDash \psi$.*

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Proof. The proof is by induction on the complexity of ψ. The base cases $\psi = \bot$ and $\psi = p$ (p a propositional variable) follow from the definition, and the cases $\psi = \psi_1 \wedge \psi_2$ and $\psi = \psi_1 \vee \psi_2$ are easily verified. So we focus on the cases $\psi = \psi_1 \to \psi_2$ (and hence $\psi = \neg\psi_1 = \psi_1 \to \bot$), $\psi = \exists\psi_1$, and $\psi = \forall\psi_1$.

$\to$ *case*: Let $\psi = \psi_1 \to \psi_2$ and $t \in W'$. Suppose $t \not\vDash' \psi_1 \to \psi_2$. Then $tR's$ for some $s \in W'$ with $s \vDash' \psi_1$ and $s \not\vDash' \psi_2$. By the inductive hypothesis, $\widehat{s} \vDash \psi_1$ and $\widehat{s} \not\vDash \psi_2$. Thus, $\widehat{s} \not\vDash \psi_1 \to \psi_2$. Since $tR's$, we have $\widehat{t} \not\vDash \psi_1 \to \psi_2$ by persistence (Lemma 5.8).

Conversely, suppose $\widehat{t} \not\vDash \psi_1 \to \psi_2$. If $\widehat{t} \vDash \psi_1$, then we have $\widehat{t} \vDash \psi_1$ but $\widehat{t} \not\vDash \psi_2$. By the inductive hypothesis, $t \vDash' \psi_1$ but $t \not\vDash' \psi_2$. By construction, $tR't$. Therefore, $t \not\vDash' \psi_1 \to \psi_2$. If $\widehat{t} \not\vDash \psi_1$, then in the $\to$-step of the stage immediately after t is added to W', we add s to W' and $tR's$ where $\widehat{s} \not\vDash \psi_1 \to \psi_2$ maximally (Lemma 5.4). Thus, $\widehat{s} \vDash \psi_1$ and $\widehat{s} \not\vDash \psi_2$, so by the inductive hypothesis, $s \vDash' \psi_1$ and $s \not\vDash' \psi_2$. Since $tR's$, we conclude that $t \not\vDash' \psi_1 \to \psi_2$.

$\exists$ *case*: Let $\psi = \exists\psi_1$ and $t \in W'$. Suppose $t \vDash' \exists\psi_1$. Then $tE's$ for some $s \in W'$ with $s \vDash' \psi_1$. By the inductive hypothesis, $\widehat{s} \vDash \psi_1$, and $tE's$ implies $\widehat{t}E\widehat{s}$ by Lemma 5.7(2). Thus, $\widehat{t} \vDash \exists\psi_1$.

Conversely, suppose $\widehat{t} \vDash \exists\psi_1$. Then $\exists\psi_1 \in \Sigma^{\exists}(t)$, so in the $\exists$-step of the next stage of the construction after t is added, we add s to W' and (t, s) to E' where s is a copy of some $\widehat{s} \in W$ with $\widehat{t}E\widehat{s}$ and $\widehat{s} \vDash \psi_1$. By the inductive hypothesis, $s \vDash' \psi_1$. Since $tE's$, we conclude that $t \vDash' \exists\psi_1$.

$\forall$ *case*: Let $\psi = \forall\psi_1$ and $t \in W'$. Suppose $t \not\vDash' \forall\psi_1$. Then $tQ'w$ for some $w \in W'$ with $w \not\vDash' \psi_1$. By the inductive hypothesis, $\widehat{w} \not\vDash \psi_1$, and $tQ'w$ implies $\widehat{t}Q\widehat{w}$ by Lemma 5.7(3). Thus, $\widehat{t} \not\vDash \forall\psi_1$.

Conversely, suppose $\widehat{t} \not\vDash \forall\psi_1$. If $\widehat{t}$ is maximal with respect to $\forall\psi_1$, then $\forall\psi_1 \in \Sigma^{\forall H}(t)$, so at some point in the construction of the next stage after t is added, we add s to W' and (t, s) to E' where s is a copy of some $\widehat{s} \in W$ with $\widehat{t}E\widehat{s}$ and $\widehat{s} \not\vDash \psi_1$. By the inductive hypothesis, $s \not\vDash' \psi_1$, so $t \not\vDash' \forall\psi_1$. If $\widehat{t}$ is not maximal, then we add s to W' and (t, s) to R' where s is a copy of some $\widehat{s} \in W$ and $\widehat{s}$ is maximal with respect to $\forall\psi_1$. Therefore, $\forall\psi_1 \in \Sigma^{\forall H}(s)$, and in the next stage we add w to W' and (s, w) to E' where $\widehat{w} \in W$ and $\widehat{w} \not\vDash \psi_1$. But then $tQ'w$, and by the inductive hypothesis, $w \not\vDash' \psi_1$. Thus, $t \not\vDash' \forall\psi_1$. ⊣

The FMP of $\mathsf{M^+IPC}$ is now an immediate consequence of the above.

Theorem 5.17. $\mathsf{M^+IPC}$ *has the finite model property.*

Proof. Suppose $\mathsf{M^+IPC} \nvdash \varphi$. By completeness of $\mathsf{M^+IPC}$ with respect to descriptive frames, there are a descriptive $\mathsf{M^+IPC}$-frame $\mathfrak{F}$ and a valuation v on $\mathfrak{F}$ such that $(\mathfrak{F}, v) \not\vDash \varphi$. Let $\mathfrak{F}'$ be the finite $\mathsf{M^+IPC}$-frame constructed above. Since t_0 was chosen so that $\widehat{t_0}$ refutes φ in $\mathfrak{F}$, by Lemma 5.16, t_0 refutes φ in $\mathfrak{F}'$. We thus found a finite $\mathsf{M^+IPC}$-frame refuting φ. ⊣

Since $\mathsf{M^+IPC}$ is finitely axiomatizable and has the finite model property, it is decidable, meaning that there is an effective method for determining whether an arbitrary formula is a theorem of $\mathsf{M^+IPC}$.

Corollary 5.18. $\mathsf{M^+IPC}$ *is decidable.*

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REMARK 5.19. Another consequence of Theorem 5.17 is that $\mathsf{M^+IPC}$ is the monadic fragment of $\mathsf{IQ^+C}$. This can be seen by utilizing the Translation Theorem of Ono and Suzuki (see [37, Theorem 3.5]).

§6. The finite model property of $\mathsf{M^+Grz}$. In this section we prove that $\mathsf{M^+Grz}$ has the finite model property. Our proof, which consists of three steps, is a mixture of selective and standard filtration techniques. The main reasons why the same technique as for $\mathsf{M^+IPC}$ does not work are the lack of persistence in $\mathsf{M^+Grz}$-models and the fact that witnesses for $\forall$-formulas cannot be chosen maximally wrt Q-relations. A rough structure of the proof is as follows.

Suppose that $\mathsf{M^+Grz} \nvdash \varphi$. Then there is a descriptive $\mathsf{M^+Grz}$-frame $\mathfrak{F}_0 = (W_0, R_0, E_0)$ and a valuation v_0 on W_0 such that $\mathfrak{F}_0 \not\models_0 \varphi$. We build a finite $\mathsf{M^+Grz}$-frame from $\mathfrak{F}_0$ in three steps:

1. First we select a (possibly infinite) partially ordered $\mathsf{MS4}$-frame $\mathfrak{F}_1 = (W_1, R_1, E_1)$ from $\mathfrak{F}_0$, in which all clusters are clean and φ is refuted. An important feature of this step is that R_1 is not simply the restriction of R_0 to W_1, but rather its strengthening. Its construction resembles the construction of R-relations from Q-relations in the $\to$-step of the $\mathsf{M^+IPC}$-construction.
2. Next we construct a (possibly infinite) partially ordered $\mathsf{MS4}$-frame $\mathfrak{F}_2$ from $\mathfrak{F}_1$, in which all clusters are both clean and finite and φ is refuted. In this step we use standard filtration to collapse E_1-clusters of $\mathfrak{F}_1$ so that each cluster contains only one point representing all points that satisfy the same formulas of $\mathsf{Sub}(\varphi)$.
3. Finally, as in Step 1, we use selective filtration to construct a finite partially ordered $\mathsf{MS4}$-frame $\mathfrak{F}_3$ from $\mathfrak{F}_2$, in which all clusters are clean (hence $\mathfrak{F}_3$ is an $\mathsf{M^+Grz}$-frame) and φ is refuted. This step resembles the $\mathsf{M^+IPC}$-construction, but in order for $\mathfrak{F}_3$ to inherit the bounded cluster size from $\mathfrak{F}_2$, we need to add only a single copy of an original point in $\mathfrak{F}_2$ to a cluster.

6.1. Step 1: Ensuring all clusters are clean. Let $\mathfrak{F}_0 = (W_0, R_0, E_0)$ be as above. For $x, y \in W_0$ let

$$x \overrightarrow{Q_0}\, y \quad \text{iff} \quad \text{there is } w \in W_0 \text{ such that } w \neq x,\ xR_0w, \text{ and } wE_0y.$$

We construct $\mathfrak{F}_1 = (W_1, R_1, E_1)$ as follows:

- $W_1 = \{x \in W_0 \mid x \in \max_{R_0} E_0(A)$ for some clopen A of $\mathfrak{F}_0\}$.
- $xR_1y \Leftrightarrow x = y$ or $x \overrightarrow{Q_0}\, y$ and $x \models_0 \Box\psi \Rightarrow y \models_0 \Box\psi$ for all $\Box\psi \in \mathsf{Sub}(\varphi)$.
- $xE_1y \Leftrightarrow xE_0y$.
- We define a valuation v_1 on $\mathfrak{F}_1$ by $v_1(p) = \{x \mid x \in v_0(p)\}$ for all $p \in \mathsf{Sub}(\varphi)$, and $v_1(q) = \varnothing$ for all other propositional variables q.

We first show that there is a point in W_1 which refutes φ (in $\mathfrak{F}_0$).

LEMMA 6.1. *There is $v \in W_0$ such that $v \not\models_0 \varphi$ and $v \in \max_{R_0} E_0(v(\neg\varphi))$(hence $E_0(v)$ is clean and $v \in W_1$).*

PROOF. Since $\mathfrak{F}_0 \not\models_0 \varphi$, there is $t \in W_0$ such that $t \not\models_0 \varphi$. Then $t \in v_0(\neg\varphi)$, so $t \in E_0(v_0(\neg\varphi))$. Because $v_0(\neg\varphi)$ is clopen, $E_0(v_0(\neg\varphi))$ is clopen. Therefore, Lemma 2.26(2) yields $u \in \max_{R_0} E_0(v_0(\neg\varphi))$ with tR_0u. Since $u \in E_0(v_0(\neg\varphi))$, there is $v \in W_0$ with uE_0v and $v \not\models_0 \varphi$. We now show that v is our desired point. Because

$u \in \max_{R_0} E_0(v_0(\neg\varphi))$, the cluster $E_0(u) = E_0(v)$ is clean (Lemma 4.8). Therefore, $v \in \max_{R_0} E_0(v_0(\neg\varphi))$ by Lemma 4.7(1). Now, since $v \in \max_{R_0} E_0(v_0(\neg\varphi))$ and $v \in v_0(\neg\varphi)$, it is easy to see that v is R_0-maximal with respect to φ, hence is our desired point. ⊣

We next highlight some fundamental properties of $\mathfrak{F}_1$.

LEMMA 6.2.

1. *$E_0(x) \subseteq W_0$ is a clean cluster in $\mathfrak{F}_0$ for all $x \in W_1$.*
2. *If $x \in W_1$, then $E_0(x) \subseteq W_1$.*
3. *$x \overrightarrow{Q_0} y$ iff xQ_0y but $x\not{E}_0y$ for all $x, y \in W_1$.*
4. *The restriction of $\overrightarrow{Q_0}$ to W_1 is a strict partial order.*
5. *R_1 is a partial order.*
6. *E_1 is an equivalence relation.*
7. *R_1 and E_1 satisfy commutativity.*
8. *$\mathfrak{F}_1$ has clean clusters.*
9. *For $x \in W_1$ and $\Box\gamma \in \mathsf{Sub}(\varphi)$, if $x \not\vDash_0 \Box\gamma$, then there is $y \in W_1$ such that xR_1y, $y \in A \cap \max_{R_0} E_0(A)$, where*

$$A = v_0(\neg\Box\gamma) \cap \bigcap\{v_0(\Box\psi) \mid \Box\psi \in \mathsf{Sub}(\varphi) \text{ and } x \vDash_0 \Box\psi\},$$

and $y \not\vDash_0 \Box\gamma$ R_0-maximally.

PROOF. (1) This is an immediate consequence of Lemma 4.8.

(2) Let $x \in W_1$ and $y \in E_0(x)$. Then $x \in \max_{R_0} E(A)$ for some clopen $A \subseteq W_0$. Therefore, $E_0(x)$ is clean by (1), and so $y \in \max_{R_0} E(A)$ by Lemma 4.7(1). Thus, $y \in W_1$.

(3) The implication from right to left is obvious. For the converse, suppose that $x, y \in W_1$ and there is $w \in W_0$ such that $w \neq x$, xR_0w, and wE_0y. Then clearly xQ_0y. Also, since x is from a clean cluster, $x\not{E}_0w$. Thus, $x\not{E}_0y$.

(4) Irreflexivity of $\overrightarrow{Q_0}$ on W_1 follows from the reflexivity of E_0 and (3). We show that $\overrightarrow{Q_0}$ is transitive on W_1. Suppose $x \overrightarrow{Q_0} y \overrightarrow{Q_0} z$ for $x, y, z \in W_1$. Then there are $y' \neq x$ and $z' \neq y$ with xR_0y', $y'E_0y$ and yR_0z' and $z'E_0z$. By commutativity, there is z'' with $y'R_0z''$ and $z''E_0z$. Therefore, xR_0z'' and $z''E_0z$. If we had $x = z''$, then we would obtain $xR_0y'R_0x$, and so $x = y'$ by Lemma 4.7(2). The latter contradicts the choice of y'. Thus, $z'' \neq x$ and so $x \overrightarrow{Q_0} z$.

(5) R_1 is reflexive by definition. To see that R_1 is transitive, suppose $x, y, z \in W_1$ with xR_1yR_1z. Without loss of generality we may assume that x, y, z are pairwise distinct. Then $x \overrightarrow{Q_0} y$ and $y \overrightarrow{Q_0} z$, so $x \overrightarrow{Q_0} z$ by (4). Moreover, if $x \vDash \Box\psi$ for $\Box\psi \in \mathsf{Sub}(\varphi)$, then since xR_1yR_1z, we have $y \vDash \Box\psi$ and so $z \vDash \Box\psi$. Therefore, R_1 is transitive. Finally, if xR_1yR_1x and $x \neq y$, then $x \overrightarrow{Q_0} y \overrightarrow{Q_0} x$. The latter implies $x \overrightarrow{Q_0} x$ by transitivity of $\overrightarrow{Q_0}$, which contradicts irreflexivity of $\overrightarrow{Q_0}$. Thus, R_1 is antisymmetric.

(6) This is immediate since E_0 is an equivalence relation.

(7) Suppose that xR_1y and xE_1z. Without loss of generality we may assume that $x \neq y$ and $x \neq z$. Then $x \overrightarrow{Q_0} y$, so there is $u \in W_0$ such that $x \neq u$, xR_0u, and uE_0y. By commutativity in W_0, there is v such that zR_0v and vE_0u. We show that v is the required witness for commutativity in W_1. From vE_0u and uE_0y we have

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vE_0y, so $v \in W_1$ by (2). Because $x \neq u$, xR_0u, and x is from a clean cluster, we have $x\not{E}_0u$. Thus, $z\not{E}_0v$. In particular, $z \neq v$, and so $z \overrightarrow{Q_0} v$. Moreover, zR_0v gives that if $z \vDash_0 \Box\gamma$, then $v \vDash_0 \Box\gamma$, so zR_1v. From vE_0y we have vE_1y, yielding commutativity in W_1.

(8) Suppose there are $x, y \in W_1$ with $x \neq y$, xE_1y, and xR_1y. Since xE_1y, we have xE_0y, and because xR_1y and $x \neq y$, we have $x \overrightarrow{Q_0} y$. Thus, there is $w \in W_0$ with $x \neq w$, xR_0w, and wE_0y. From xE_0y and yE_0w we have xE_0w. By (1), x is chosen from a clean cluster in W_0, so xR_0w and xE_0w imply $x = w$, a contradiction.

(9) Suppose $x \nvDash_0 \Box\gamma$. Consider

$$A = v_0(\neg\Box\gamma) \cap \bigcap\{v_0(\Box\psi) \mid \Box\psi \in \mathsf{Sub}(\varphi) \text{ and } x \vDash_0 \Box\psi\}.$$

Clearly $x \in A$, so $x \in E_0(A)$. We have $x \in \max_{R_0} E_0(A)$ or $x \notin \max_{R_0} E_0(A)$.

Case 1: $x \in \max_{R_0} E_0(A)$

If $x \in \max_{R_0} E_0(A)$, then from xR_0w and $x \neq w$ it follows that $w \notin E_0(A)$, so $w \notin A$. But xR_0w implies $w \in \bigcap\{v_0(\Box\psi) \mid \Box\psi \in \mathsf{Sub}(\varphi) \text{ and } x \vDash_0 \Box\psi\}$, so we must have $w \notin v_0(\neg\Box\gamma)$. Therefore, $w \vDash_0 \Box\gamma$. Since $x \nvDash_0 \Box\gamma$ but $w \vDash_0 \Box\gamma$ for all $w \neq x$ with xR_0w, we must have $x \nvDash_0 \Box\gamma$ R_0-maximally.

Case 2: $x \notin \max_{R_0} E_0(A)$

If $x \notin \max_{R_0} E_0(A)$, then Lemma 2.26(2) yields $t \in \max_{R_0} E_0(A)$ such that $x \neq t$ and xR_0t. But then tE_0y for some $y \in A$. Since $t \in \max_{R_0} E_0(A)$, we have $t \in W_1$, so $y \in W_1$ by (2). From $x \neq t$ and xR_0t it follows that $x \overrightarrow{Q_0} y$. Since $y \in A$, if $x \vDash_0 \Box\psi$ then $y \vDash_0 \Box\psi$ for all $\Box\psi \in \mathsf{Sub}(\varphi)$, so xR_1y. From $y \in A$ it follows that $y \nvDash_0 \Box\gamma$. We show that $y \nvDash_0 \Box\gamma$ R_0-maximally. Suppose yR_0z and $z \nvDash_0 \Box\gamma$. If $x \vDash_0 \Box\psi$, then $y \vDash_0 \Box\psi$ (as $y \in A$), so yR_0z implies $z \vDash_0 \Box\psi$. Thus, $z \in A$, hence $z \in E_0(A)$, and maximality of y in $E_0(A)$ yields $y = z$. Consequently, y is R_0-maximal with respect to $\Box\gamma$. ⊣

We conclude Step 1 by proving the truth lemma for $\mathfrak{F}_1$.

LEMMA 6.3 (Truth Lemma). *For all $x \in W_1$ and $\psi \in \mathsf{Sub}(\varphi)$,*

$$(\mathfrak{F}_0, x) \vDash_0 \psi \Leftrightarrow (\mathfrak{F}_1, x) \vDash_1 \psi.$$

PROOF. The proof is by induction on the complexity of ψ. The base case $\psi = p$ is clear from the definition of v_1. The cases of $\psi = \psi_1 \vee \psi_2$ and $\psi = \neg\psi_1$ are straightforward, so we focus on the cases $\psi = \forall\psi_1$ and $\psi = \Box\psi_1$.

Suppose $\psi = \forall\psi_1$. If $x \nvDash_0 \forall\psi_1$, then xE_0y for some $y \nvDash_0 \psi_1$. By Lemma 6.2(2), $y \in W_1$, so $y \nvDash_1 \psi_1$ by the inductive hypothesis. From xE_0y we have xE_1y by the definition of E_1. Thus, $x \nvDash_1 \forall\psi_1$. The proof of the converse implication is immediate.

Suppose $\psi = \Box\psi_1$. If $x \nvDash_0 \Box\psi_1$, then by Lemma 6.2(9), there is $y \in W_1$ such that xR_1y and $y \nvDash_0 \psi_1$. By the inductive hypothesis, $y \nvDash_1 \psi_1$, hence $x \nvDash_1 \Box\psi_1$. Conversely, if $x \nvDash_1 \Box\psi_1$, then there is $y \in W_1$ such that xR_1y and $y \nvDash_1 \psi_1$. By the inductive hypothesis, $y \nvDash_0 \psi_1$. If $x = y$, then $x \nvDash_0 \psi_1$, hence $x \nvDash_0 \Box\psi_1$. If $x \neq y$, then as xR_1y, we have $x \overrightarrow{Q_0} y$ and $x \vDash_0 \Box\gamma$ implies $y \vDash_0 \Box\gamma$ for all $\Box\gamma \in \mathsf{Sub}(\varphi)$. Since $y \nvDash_0 \psi_1$, we have $y \nvDash_0 \Box\psi_1$. Thus, $x \nvDash_0 \Box\psi_1$. ⊣

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6.2. Step 2: Ensuring all clusters are finite. In this step we use the standard filtration technique to construct $\mathfrak{F}_2$ from $\mathfrak{F}_1$ by 'collapsing' E_1-clusters into finitely many classes. Thus, each cluster in $\mathfrak{F}_2$ will be finite.

Define an equivalence relation $\sim$ on W_1 by

$$x \sim y \Leftrightarrow \left(xE_1y \text{ and } x \vDash_1 \gamma \Leftrightarrow y \vDash_1 \gamma \text{ for all } \gamma \in \mathsf{Sub}(\varphi)\right).$$

We construct $\mathfrak{F}_2 = (W_2, R_2, E_2)$ as follows:

- $W_2 = W_1/{\sim} = \{[x] \mid x \in W_1\}$ where $[x]$ is the $\sim$-equivalence class of x.
- For $[x], [y] \in W_2$, $[x]R_2[y] \Leftrightarrow [x] = [y]$ or xR_1y.
- For $[x], [y] \in W_2$, $[x]E_2[y] \Leftrightarrow xE_1y$.
- $v_2(p) = \{[x] \mid x \in v_1(p)\}$ for all $p \in \mathsf{Sub}(\varphi)$, and $v_2(q) = \varnothing$ for all other propositional variables q.

LEMMA 6.4. *The relations E_2 and R_2 are well defined, and so is the valuation v_2.*

PROOF. It is easy to see that E_2 and v_2 are well defined. We show that R_2 is well defined. Let $x, y, x', y' \in W_1$ with $x \sim x'$, $y \sim y'$, and $[x]R_2[y]$. Then $[x] = [y]$ or xR_1y. If $[x] = [y]$, we have $[x'] = [x] = [y] = [y']$, and so $[x']R_2[y']$. If xR_1y, then $x = y$ or $x \overrightarrow{Q_0} y$ and $x \vDash_0 \Box\gamma$ implies $y \vDash_0 \Box\gamma$ for all $\Box\gamma \in \mathsf{Sub}(\varphi)$. The former case implies $[x] = [y]$ which we have already considered. In the latter case, from $x \overrightarrow{Q_0} y$ it follows that xQ_0y and $x\not{E}_0y$ by Lemma 6.2(3). Note that $x' \sim x$ implies $x'E_1x$ and so $x'E_0x$. Similarly, $y'E_0y$. By transitivity of Q_0 we thus have $x'Q_0y'$. Moreover, $x'E_0x$, $y'E_0y$, and $x\not{E}_0y$ imply that $x'\not{E}_0y'$. Thus, $x' \overrightarrow{Q_0} y'$ by Lemma 6.2(3). If $\Box\gamma \in \mathsf{Sub}(\varphi)$ and $x' \vDash_0 \Box\gamma$, then $x \vDash_0 \Box\gamma$ since $x' \sim x$. So $y \vDash_0 \Box\gamma$ by assumption. But then $y' \vDash_0 \Box\gamma$ since $y' \sim y$. This shows that $x'R_1y'$, so $[x']R_2[y']$. ⊣

In the following lemma we highlight some properties of $\mathfrak{F}_2$.

LEMMA 6.5.

1. *R_2 is a partial order.*
2. *E_2 is an equivalence relation.*
3. *R_2 and E_2 satisfy commutativity.*
4. *$\mathfrak{F}_2$ has clean clusters.*
5. *For $[x] \in W_2$, $|E_2([x])| \leq 2^n$, where $n = |\mathsf{Sub}(\varphi)|$.*
6. *For $[x] \in W_2$ and $\Box\gamma \in \mathsf{Sub}(\varphi)$, if $x \nvDash_1 \Box\gamma$, then there is $[y] \in W_2$ such that $[x]R_2[y]$ and $y \nvDash_1 \Box\gamma$ R_1-maximally.*

PROOF. (1) Reflexivity of R_2 is immediate from the definition, and transitivity and antisymmetry follow from transitivity and antisymmetry of R_1.

(2) This follows from E_1 being an equivalence relation.

(3) This follows from R_1 and E_1 satisfying commutativity.

(4) Suppose there are $[x] \neq [y]$ in W_2 with $[x]R_2[y]$ and $[x]E_2[y]$. Then $x \neq y$, so by the definition of R_2 and E_2, we have xR_1y and xE_1y which yields a dirty cluster in $\mathfrak{F}_1$, contradicting Lemma 6.2(8).

(5) This follows from the fact that there are at most 2^n $\sim$-equivalence classes in each cluster (see, e.g., [12, Proposition 5.24]).

(6) Suppose $x \nvDash_1 \Box\gamma$. By Lemma 6.3, $x \nvDash_0 \Box\gamma$, so by Lemma 6.2(9), there is $y \in W_1$ such that xR_1y, $y \in A \cap \max_{R_0} E_0(A)$, and $y \nvDash_0 \Box\gamma$ R_0-maximally, where

$$A = v_0(\neg\Box\gamma) \cap \bigcap\{v_0(\Box\psi) \mid \Box\psi \in \mathsf{Sub}(\varphi) \text{ and } x \vDash_0 \Box\psi\}.$$

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Then $[x]R_2[y]$ and by Lemma 6.3, $y \not\models_1 \Box\gamma$. We show that y is R_1-maximal with respect to $\Box\gamma$. Suppose yR_1z and $z \not\models_1 \Box\gamma$. By Lemma 6.3, $z \not\models_0 \Box\gamma$, and from yR_1z it follows that $y = z$ or $y\,\overrightarrow{Q_0}\,z$ and $y \models_0 \Box\psi$ implies $z \models_0 \Box\psi$ for all $\Box\psi \in \mathsf{Sub}(\varphi)$. Suppose the latter. Since $z \not\models_0 \Box\gamma$, we have $z \in v_0(\neg\Box\gamma)$. If $x \models_0 \Box\psi$ for $\Box\psi \in \mathsf{Sub}(\varphi)$, then $y \in A$ implies $y \models_0 \Box\psi$. So yR_1z then gives $z \models_0 \Box\psi$. Therefore, $z \in \bigcap\{v_0(\Box\psi) \mid \Box\psi \in \mathsf{Sub}(\varphi)$ and $x \models_0 \Box\psi\}$, and hence $z \in A$. As $y\,\overrightarrow{Q_0}\,z$, there is $w \in W_0$ such that $y \neq w$, yR_0w, and wE_0z. Then $w \in E_0(A)$, and maximality of y in $E_0(A)$ yields $y = w$, contradicting $y \neq w$. Thus, $y = z$, and so y is R_1-maximal with respect to $\Box\gamma$. ⊣

We conclude Step 2 by showing the truth lemma for $\mathfrak{F}_2$.

LEMMA 6.6 (Truth Lemma). *For all $x \in W_1$ and $\psi \in \mathsf{Sub}(\varphi)$,*

$$(\mathfrak{F}_1, x) \models_1 \psi \Leftrightarrow (\mathfrak{F}_2, [x]) \models_2 \psi.$$

PROOF. The proof is by induction on the complexity of ψ. The base case $\psi = p$ follows from the definition of v_2. The cases of $\psi = \psi_1 \vee \psi_2$ and $\psi = \neg\psi_1$ are straightforward, and the $\forall$-case follows from the definition of E_2. Suppose that $\psi = \Box\psi_1$. If $x \not\models_1 \Box\psi_1$, then there is $y \in W_1$ with xR_1y and $y \not\models_1 \psi_1$. Therefore, $[x]R_2[y]$ and $[y] \not\models_2 \psi_1$ by the inductive hypothesis. Thus, $[x] \not\models_2 \Box\psi_1$. Conversely, if $[x] \not\models_2 \Box\psi_1$, then there is $y \in W_1$ with $[x]R_2[y]$ and $[y] \not\models_2 \psi_1$. By the inductive hypothesis, $y \not\models_1 \psi_1$. If $[x] = [y]$, then $x \not\models_1 \Box\psi_1$ by definition of $\sim$. If $[x] \neq [y]$, then xR_1y and again $x \not\models_1 \Box\psi_1$. ⊣

6.3. Step 3: Ensuring a finite frame. We are ready for our final step, in which we construct $\mathfrak{F}_3 = (W_3, R_3, E_3)$ by selective filtration from $\mathfrak{F}_2$. This is done by constructing a sequence of finite partially ordered MS4-frames with clean clusters $\mathfrak{F}_{3.h} = (W_{3.h}, R_{3.h}, E_{3.h})$ such that $\mathfrak{F}_{3.h} \subseteq \mathfrak{F}_{3.h+1}$ for all $h < \omega$. We then show that this construction eventually terminates.

Similar to the construction for $\mathsf{M^+IPC}$, for each point $[x] \in W_2$ that we select, we create a copy of the point, give it a new name, say t, and let $\widehat{t} = [x]$ denote the original point in W_2 that t represents and will behave similar to. However, we take a bit more care with the copies in this construction than in the construction for $\mathsf{M^+IPC}$. In particular, we will never create two copies of the same original point within one cluster. This will ensure that the cluster size in $\mathfrak{F}_3$ has the same bound as the cluster size in $\mathfrak{F}_2$.

Before we begin the construction, we highlight an important property we will need for selecting our points.

LEMMA 6.7. *For $[x] \in W_2$ and $\Box\gamma \in \mathsf{Sub}(\varphi)$, if $[x] \not\models_2 \Box\gamma$, then there is $[y] \in W_2$ such that $[x]R_2[y]$ and $[y] \not\models_2 \Box\gamma$ R_2-maximally.*

PROOF. Suppose $[x] \not\models_2 \Box\gamma$. By Lemma 6.6, $x \not\models_1 \Box\gamma$, and by Lemma 6.5(6), there is $[y] \in W_2$ such that $[x]R_2[y]$ and $y \not\models_1 \Box\gamma$ R_1-maximally. Applying Lemma 6.6 again yields $[y] \not\models_2 \Box\gamma$. To see that $[y]$ is R_2-maximal with respect to $\Box\gamma$, suppose $[y]R_2[z]$ and $[z] \not\models_2 \Box\gamma$. By definition of R_2, either $[y] = [z]$ or yR_1z. If yR_1z, then by R_1-maximality of y, we have $y = z$, so $[y] = [z]$, and hence $[y]$ must be R_2-maximal with respect to $\Box\gamma$. ⊣

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Throughout the construction, for each $t \in W_{3.h}$, we associate the following sets of formulas:

$$\Sigma^{\forall}(t) = \{\forall\delta \in \mathsf{Sub}(\varphi) \mid \widehat{t} \not\vDash_2 \forall\delta\},$$
$$\Sigma^{\Box}(t) = \{\Box\gamma \in \mathsf{Sub}(\varphi) \mid \widehat{t} \not\vDash_2 \Box\gamma, \widehat{t} \vDash_2 \gamma\}.$$

We start with $\mathfrak{F}_{3.0} = (W_{3.0}, R_{3.0}, E_{3.0})$ where

$$W_{3.0} = \{t_0\}, \quad R_{3.0} = W_{3.0}^2, \quad E_{3.0} = W_{3.0}^2$$

and $\widehat{t_0} = [x_0] \in W_2$ is a point with $[x_0] \not\vDash_2 \varphi$. This will be a root of our frame and has Q_3-depth 1. Let $W_{3.-1} = R_{3.-1} = E_{3.-1} = \varnothing$. Suppose $\mathfrak{F}_{3.h-1} = (W_{3.h-1}, R_{3.h-1}, E_{3.h-1})$ has already been constructed and is a partially ordered MS4-frame with clean clusters. We construct $\mathfrak{F}_{3.h}$ by the following steps.

$\forall$-*step* (*Horizontal*): Let $W_{3.h}^{\forall} = W_{3.h-1}$, $R_{3.h}^{\forall} = R_{3.h-1}$, and $E_{3.h}^{\forall} = E_{3.h-1}$. For each cluster $E_{3.h}(t) \subseteq W_{3.h-1} \setminus W_{3.h-1}^{\forall}$, consider $\forall\delta \in \Sigma^{\forall}(t)$. If there is no $s \in W_{3.h}^{\forall}$ already such that $tE_{3.h}^{\forall}s$ and $\widehat{s} \not\vDash_2 \delta$, we add a witness to our new frame as follows. Since $\widehat{t} \not\vDash_2 \forall\delta$, there exists $[x] \in W_2$ such that $\widehat{t}E_2[x]$ and $[x] \not\vDash_2 \delta$. We add the point s to $W_{3.h}^{\forall}$ where $\widehat{s} = [x]$ (s is a distinct new copy of $[x]$), the relations (s, s) to $R_{3.h}^{\forall}$, the relations (t, s) to $E_{3.h}^{\forall}$ and generate the least equivalence relation.

$\Box$-*step* (*Vertical*): Let $W_{3.h}^{\Box} = W_{3.h}^{\forall}$, $R_{3.h}^{\Box} = R_{3.h}^{\forall}$, and $E_{3.h}^{\Box} = E_{3.h}^{\forall}$. For $t \in W_{3.h}^{\forall} \setminus W_{3.h-1}^{\forall}$ (hence including any points added in the horizontal step), consider $\Box\gamma \in \Sigma^{\Box}(t)$ where $\widehat{t} \not\vDash_2 \Box\gamma$, but $\widehat{t} \vDash_2 \gamma$ (thus, t isn't witnessing the formula $\Box\gamma$ itself), and there is no $s \in W_{3.h}^{\Box}$ already such that $tR_{3.h}^{\Box}s$ and $\widehat{s} \not\vDash_2 \Box\gamma$ R_2-maximally. For each such $\Box\gamma$, since $\widehat{t} \not\vDash_2 \Box\gamma$ and $\widehat{t} = [w]$ for some $[w] \in W_2$, we have $[w] \not\vDash_2 \Box\gamma$. By Lemma 6.7, there is $[x] \in W_2$ such that $[w]R_2[x]$ and $[x]$ is R_2-maximal with respect to $\Box\gamma$. We add the point s to $W_{3.h}^{\Box}$ where $\widehat{s} = [x]$, (t, s) and (s, s) to $R_{3.h}^{\Box}$ and close under transitivity, and add (s, s) to $E_{3.h}^{\Box}$. To make sure commutativity is satisfied, for each $w \in E_{3.h}^{\Box}(t)$, if there is already $s_w \in E_{3.h}^{\Box}(s)$ such that $\widehat{w}R_2\widehat{s_w}$, we simply add the relation (w, s_w) to $R_{3.h}^{\Box}$. If there is no such s_w, then by commutativity in W_2, there is $[x_w] \in W_2$ such that $\widehat{w}R_2[x_w]$ and $[x_w]E_2[x]$, so we add s_w to $W_{3.h}^{\Box}$, where $\widehat{s_w} = [x_w]$. We then add (w, s_w) to $R_{3.h}^{\Box}$ and close it under reflexivity and transitivity, and add (s_w, s) to $E_{3.h}^{\Box}$ and generate the smallest equivalence relation.

To end this stage of the construction, we let $\mathfrak{F}_{3.h} = (W_{3.h}, R_{3.h}, E_{3.h})$ where

$$W_{3.h} = W_{3.h}^{\Box}, \quad R_{3.h} = R_{3.h}^{\Box}, \quad E_{3.h} = E_{3.h}^{\Box}.$$

LEMMA 6.8. *$\mathfrak{F}_{3.h}$ is a finite partially ordered* MS4-*frame with clean clusters.*

PROOF. That $\mathfrak{F}_{3.h}$ is finite follows from the construction. Since in the $\forall$-step we only added reflexive arrows to $R_{3.h}^{\forall}$, we have that $R_{3.h}^{\forall}$ is a partial order. In the $\Box$-step we close $R_{3.h}^{\Box}$ under reflexivity and transitivity each time we add a new arrow, so $R_{3.h}^{\Box}$ is reflexive and transitive. Moreover, we only add $R_{3.h}^{\Box}$ arrows from points that were already present in $W_{3.h}^{\forall}$ into points that are freshly added in the $\Box$-step of round h. Thus, $R_{3.h}^{\Box}$ is antisymmetric. That $E_{3.h}$ is an equivalence relation and that $\mathfrak{F}_{3.h}$ satisfies commutativity follow from the construction. Finally, to see that $\mathfrak{F}_{3.h}$ has only clean clusters, note that in the $\forall$-step all freshly introduced E_h-relations are of the shape (s, t) where s or $t \in W^{\forall}_{3.h} \setminus W_{3.h-1}$. Since no non-reflexive R_h-arrows are introduced in this step, no dirty cluster could have been built. We have already discussed the shape of the R_h arrows introduced in the $\Box$-step. This guarantees that no cluster in $W^{\forall}_{3.h}$ is made dirty. The freshly introduced E_h-relations in these steps are of the shape (s, t) where $s, t \in W^{\Box}_{3.h} \setminus W^{\forall}_{3.h}$. Since no non-reflexive R_h-relations exist between these points, we infer that all clusters are clean. ⊣

The following lemma summarizes some useful properties of $\mathfrak{F}_3$. In the following let

$$n = |\mathsf{Sub}(\varphi)| \text{ and } m = |\{\Box\psi \mid \Box\psi \in \mathsf{Sub}(\varphi)\}|.$$

LEMMA 6.9. *Let $t, u \in W_{3.h}$.*

1. *If $tE_{3.h}u$, then $\widehat{t}E_2\widehat{u}$.*
2. *If $tE_{3.h}u$, then $\Sigma^{\forall}(t) = \Sigma^{\forall}(u)$. (This ensures that we only need to perform the $\forall$-step once per cluster).*
3. *If $tE_{3.h}u$ and $t \neq u$, then $\widehat{t} \neq \widehat{u}$. (This ensures that one cluster does not contain two different copies of the same point, so our cluster size remains bounded).*
4. *If $[t]\,\overrightarrow{Q_2}[u]$, then $t\,\overrightarrow{Q_0}\,u$.*
5. *If $tR_{3.h}u$, then $\widehat{t}R_2\widehat{u}$.*
6. *If $t\,\overrightarrow{Q_{3.h}}\,u$, then $\widehat{t}\,\overrightarrow{Q_2}\,\widehat{u}$. Thus, if $t\,\overrightarrow{Q_{3.h}}\,u$, then $\widehat{t} \neq \widehat{u}$.*
7. *A formula $\Box\gamma \in \mathsf{Sub}(\varphi)$ can be witnessed at most 2^n times in clusters along an $R_{3.h}$-chain. (This shows that $\Box\gamma$ can be witnessed at most 2^n times per $Q_{3.h}$-chain.)*

PROOF. (1) This follows from the construction.

(2) By (1), $tE_{3.h}u$ implies $\widehat{t}E_2\widehat{u}$, so $\widehat{t} \vDash_2 \forall\gamma$ iff $\widehat{u} \vDash_2 \forall\gamma$.

(3) Suppose $tE_{3.h}u$, $t \neq u$, and $\widehat{t} = \widehat{u}$. Without loss of generality assume that t was added to the cluster before u, so either u is added to witness some formula $\forall\delta_i$ where $\widehat{u} \nvDash_2 \delta_i$, or u is added as a commutativity witness for some point from the cluster immediately below. However, by construction, u would not have been added to witness a formula $\forall\delta_i$, because if $\widehat{u} \nvDash_2 \delta_i$, then $\widehat{t} = \widehat{u}$ implies that $\widehat{t} \nvDash_2 \delta_i$, so t is already a viable witness in the cluster for any such formula, contradicting the $\forall$-step of the construction. Furthermore, u would not be added as a commutativity witness for some point w in the cluster immediately below, because then in W_2 we would have $\widehat{w}R_2\widehat{u}$, so $\widehat{w}R_2\widehat{t}$, and a new $R_{3.h}$-relation would have been added from w to t instead, contradicting the $\Box$-step of the construction. Thus, we must have $\widehat{t} \neq \widehat{u}$.

(4) Suppose $[t]\,\overrightarrow{Q_2}[u]$. Then there is $[w] \in W_2$ with $[t] \neq [w]$, $[t]R_2[w]$, and $[w]E_2[u]$. From the definitions of E_1 and E_2, $[w]E_2[u]$ implies wE_0u. By definition of R_2, $[t]R_2[w]$ and $[t] \neq [w]$ imply tR_1w. Since $[t] \neq [w]$, we have $t \neq w$, so $t\,\overrightarrow{Q_0}\,w$ by the definition of R_1. Then there is $v \in W_1$ with $t \neq v$, tR_0v, and vE_0w. Since vE_0w, we have vE_0u. Thus, $t \neq v$, tR_0v, and vE_0u, and hence $t\,\overrightarrow{Q_0}\,u$.

(5) This follows from the construction.

(6) If $t\,\overrightarrow{Q_{3.h}}\,u$, then there is w such that $t \neq w$, $tR_{3.h}w$, and $wE_{3.h}u$. By (5), $\widehat{t}R_2\widehat{w}$ and $\widehat{w}$ must come from a different cluster in W_2 than $\widehat{t}$, so $\widehat{t} \neq \widehat{w}$. We also have $\widehat{w}E_2\widehat{u}$ by (1), so $\widehat{t}\,\overrightarrow{Q_2}\,\widehat{u}$. Because $\mathfrak{F}_2$ has clean clusters, we must have $\widehat{t} \neq \widehat{u}$.

(7) Suppose that $x_1, \dots, x_{2^n+1}$ are all in different $E_{3.h}$-clusters along an $R_{3.h}$-chain (where $\widehat{x}_1 = [w_1], \dots, \widehat{x}_{2^n+1} = [w_{2^n+1}]$), so $x_1\,\overrightarrow{Q_{3.h}} \dots \overrightarrow{Q_{3.h}}\,x_{2^n+1}$, and all have

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been added to witness a formula $\Box\gamma \in \mathsf{Sub}(\varphi)$. Thus, $\widehat{x_i} \not\models_2 \Box\gamma$ R_2-maximally for $i = 1, \dots, 2^n + 1$. Because there are only 2^n subsets of $\mathsf{Sub}(\varphi)$ (where $n = |\mathsf{Sub}(\varphi)|$), the pigeonhole principle implies that there are some i and j with $i \neq j$ (assume $i < j$) where $\widehat{x_i}$ and $\widehat{x_j}$ satisfy the same subformulas of φ. By (6), $\widehat{x_i}\, \overrightarrow{Q_2}\, \widehat{x_j}$ and $\widehat{x_i} \neq \widehat{x_j}$. If $\widehat{x_i} R_2 \widehat{x_j}$, then R_2-maximality of $\widehat{x_i}$ with respect to $\Box\gamma$ implies $\widehat{x_i} = \widehat{x_j}$, contradicting $\widehat{x_i} \neq \widehat{x_j}$, so we must have $\widehat{x_i} \not R_2 \widehat{x_j}$ and hence $[w_i] \not R_2 [w_j]$. Since $\widehat{x_i}\, \overrightarrow{Q_2}\, \widehat{x_j}$ we have $[w_i]\, \overrightarrow{Q_2} [w_j]$. By (4), we then have $w_i\, \overrightarrow{Q_0}\, w_j$. Since $[w_i]$ and $[w_j]$ satisfy the same formulas in $\mathsf{Sub}(\varphi)$, we have $[w_i] \models_2 \Box\beta \Leftrightarrow [w_j] \models_2 \Box\beta$ for $\Box\beta \in \mathsf{Sub}(\varphi)$. By Lemmas 6.6 and 6.3, $w_i \models_0 \Box\beta \Leftrightarrow w_j \models_0 \Box\beta$. Thus, $w_i R_1 w_j$ and hence $[w_i] R_2 [w_j]$, a contradiction. ⊣

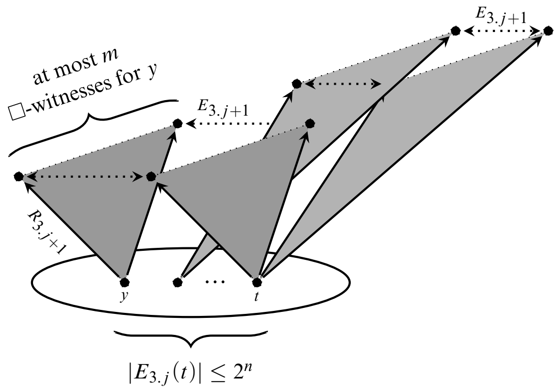
We now prove that the end result of our construction is a finite frame, using the definitions of bounded cluster size, bounded R-branching, and bounded R-depth given in Definition 5.10.

LEMMA 6.10. *For all $h < \omega$, the cluster size of $\mathfrak{F}_{3.h} = (W_{3.h}, R_{3.h}, E_{3.h})$ is bounded by 2^n.*

PROOF. By Lemma 6.5(5), the cluster size in $\mathfrak{F}_2$ is bounded by 2^n, and by Lemma 6.9(3), we do not add copies of the same points to a cluster in $\mathfrak{F}_{3.h}$. Thus, cluster size in $\mathfrak{F}_{3.h}$ is bounded by 2^n. ⊣

LEMMA 6.11. *For all $h < \omega$, the $R_{3.h}$-branching of $\mathfrak{F}_{3.h} = (W_{3.h}, R_{3.h}, E_{3.h})$ is bounded by $2^n \cdot m$.*

PROOF. It is sufficient to show that each $t \in W_{3.j-1}$, for $j \leq h$, has at most $2^n \cdot m$ immediate $R_{3.j}$-successors. By construction, we add at most m-many immediate $R_{3.j}$-successors to t for formulas of the form $\Box\psi \in \mathsf{Sub}(\varphi)$. Each $y \in E_{3.j}(t)$ also needs at most m-many immediate $R_{3.j}$-successors to witness $\Box$-formulas. Since there are at most 2^n-many such y (including t itself), we must add at most $2^n \cdot m$ immediate $R_{3.j}$ successors to t (see the diagram below):



⊣

LEMMA 6.12. *For all $h < \omega$, the $R_{3.h}$-depth of $\mathfrak{F}_{3.h} = (W_{3.h}, R_{3.h}, E_{3.h})$ is bounded by $2^n \cdot m + 1$.*

PROOF. By construction, to make an immediate vertical move from some cluster $E_{3.h}(t)$ to another cluster $E_{3.h}(u)$ (with $t \neq u$), there must be some point $x \in E_{3.h}(t)$ and formula $\Box\psi \in \Sigma^{\Box}(x)$ requiring a witness y, where $y \in E_{3.h}(u)$, $x R_{3.h} y$, and y is

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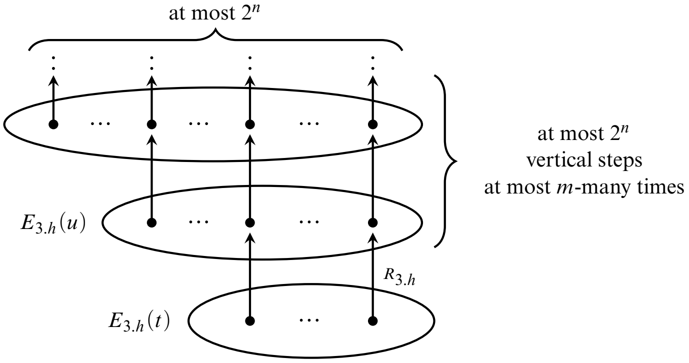
added in the $\Box$-step of the construction. Starting from the bottom cluster $E_{3.h}(t_0)$, by Lemma 6.9(7), each of our m-many $\Box$-formulas can be witnessed at most 2^n times in clusters along an $R_{3.h}$-chain. Thus, we add at most $2^n \cdot m$ elements to an $R_{3.h}$ chain originating from this cluster, with the total length of the chain (including the starting point) being at most $2^n \cdot m + 1$ (see the diagram below):



LEMMA 6.13. *There is $h \in \omega$ such that $\mathfrak{F}_{3.h'} = \mathfrak{F}_{3.h}$ for all $h' \geq h$.*

PROOF. As in the proof of Lemma 5.15, we observe that in stage k of the construction, all $R_{3.h}$-chains are bounded by k. Since, by Lemma 6.12, the $R_{3.h}$-depth of $\mathfrak{F}_{3.h}$ is bounded by $2^n \cdot m + 1$, we have $\mathfrak{F}_{3.h'} = \mathfrak{F}_{2^n \cdot m+1}$ for all $h' \geq 2^n \cdot m + 1$. ⊣

Set $\mathfrak{F}_3 = (W_3, R_3, E_3)$ where

$$W_3 = W_{3.h}, \quad R_3 = R_{3.h}, \quad E_3 = E_{3.h},$$

and h is as in Lemma 6.13. As an immediate consequence of Lemma 6.8, we obtain:

LEMMA 6.14. *$\mathfrak{F}_3 = (W_3, R_3, E_3)$ is a finite* $\mathsf{M^+Grz}$-*frame.*

Finally, we verify that our frame validates precisely the formulas we want it to validate. Define a valuation v_3 on W_3 by $v_3(p) = \{t \in W_3 \mid \widehat{t} \in v_2(p)\}$ for $p \in \mathsf{Sub}(\varphi)$ and $v_3(q) = \varnothing$ for variables q not occurring in φ.

LEMMA 6.15 (Truth Lemma). *For all $x \in W_3$ and $\psi \in \mathsf{Sub}(\varphi)$,*

$$(\mathfrak{F}_2, \widehat{x}) \vDash_2 \psi \Leftrightarrow (\mathfrak{F}_3, x) \vDash_3 \psi.$$

PROOF. The proof is by induction on the complexity of ψ and again we only consider the cases where $\psi = \forall\psi_1$ or $\psi = \Box\psi_1$.

Suppose $\psi = \forall\psi_1$. If $\widehat{x} \nvDash_2 \forall\psi_1$, then $\forall\psi_1 \in \Sigma^{\forall}(x)$, so at some point in the construction of $\mathfrak{F}_3$ we add s to W_3 and (x, s) to E_3 where $\widehat{s} \nvDash_2 \psi_1$. By the inductive hypothesis, $s \nvDash_3 \psi_1$, hence $x \nvDash_3 \forall\psi_1$. Conversely, if $x \nvDash_3 \forall\psi_1$, then there is $w \in W_3$ with xE_3w and $w \nvDash_3 \psi_1$. By the inductive hypothesis, $\widehat{w} \nvDash_2 \psi_1$, and by Lemma 6.9(1), xE_3w implies $\widehat{x}E_2\widehat{w}$, so $\widehat{x} \nvDash_2 \forall\psi_1$.

Suppose $\psi = \Box\psi_1$. If $\widehat{x} \nvDash_2 \Box\psi_1$, then either $\widehat{x} \nvDash_2 \psi_1$ or $\widehat{x} \vDash_2 \psi_1$. If $\widehat{x} \nvDash_2 \psi_1$, then by the inductive hypothesis we have $x \nvDash_3 \psi_1$, hence $x \nvDash_3 \Box\psi_1$. If $\widehat{x} \vDash_2 \psi_1$, then $\Box\psi_1 \in \Sigma^{\Box}(x)$, so at some point in the construction of $\mathfrak{F}_3$ we add s to W_3 and

(x, s) to R_3 where $\widehat{s} \not\models_2 \psi_1$. By the inductive hypothesis, $s \not\models_3 \psi_1$, hence $x \not\models_3 \Box\psi_1$. Conversely, if $x \not\models_3 \Box\psi_1$, then there is $w \in W_3$ with xR_3w and $w \not\models_3 \psi_1$. By the inductive hypothesis, $\widehat{w} \not\models_2 \psi_1$, and by Lemma 6.9(5), xR_3w implies $\widehat{x}R_2\widehat{w}$, so $\widehat{x} \not\models_2 \Box\psi_1$. ⊣

We thus arrive at our desired result:

Theorem 6.16. $\mathsf{M}^+\mathsf{Grz}$ *has the finite model property.*

As an immediate corollary to Theorem 6.16, we have:

Corollary 6.17. $\mathsf{M}^+\mathsf{Grz}$ *is decidable.*

Remark 6.18. Another consequence of Theorem 6.16 is that $\mathsf{M}^+\mathsf{Grz}$ is the monadic fragment of the predicate modal logic obtained by adding to QGrz the Gödel translation of Casari's formula Cas (cf. Remark 5.19).

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