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SOME NEW LOWER BOUNDS FOR THE KIRCHHOFF INDEX OF A GRAPH

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Abstract

Let G be a simple connected graph with n vertices and m edges and $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ its sequence of vertex degrees. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ are the Laplacian eigenvalues of G, then the Kirchhoff index of G is $Kf(G) = n \sum_{i=1}^{n-1} \mu_i^{-1}$. We prove some new lower bounds for $Kf(G)$ in terms of some of the parameters $\Delta = d_1$, $\Delta_2 = d_2$, $\Delta_3 = d_3$, $\delta = d_n$, $\delta_2 = d_{n-1}$ and the topological index $NK = \prod_{i=1}^{n} d_i$.

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1. Introduction

Let G be a simple connected graph with n vertices, $V = \{1, 2, \ldots, n\}$, and m edges. Let $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ be its sequence of vertex degrees and $\Delta = d_1$, $\Delta_2 = d_2$, $\Delta_3 = d_3$, $\delta = d_n$, $\delta_2 = d_{n-1}$. Let $\mathbf{A}$ be the adjacency matrix of G and $\mathbf{D} = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ the diagonal matrix of its vertex degrees. The Laplacian matrix of G is defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$. The eigenvalues of $\mathbf{L}$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, form the Laplacian spectrum of G.

The Wiener index, $W(G)$, originally termed the 'path number', is a topological graph index defined by

$$W(G) = \sum_{i<j} d_{ij},$$

where d_{ij} is the shortest path between vertices i and j in G. The first investigations of the Wiener index were made by Harold Wiener in 1947 (see [20]) to explain correlations between the boiling points of paraffin and the structure of the molecules. Since then it has become one of the most frequently used topological indices in chemistry, as molecules are usually modelled as undirected graphs. Based on its success, many other topological indices of chemical graphs, based on information in the distance matrix of the graph, have been developed.

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In [8], Klein and Randić introduced the notion of resistance distance, r_{ij}, as the second distance function on the vertex set of a graph. It is defined as the resistance between the nodes i and j in an electrical network corresponding to the graph G in which all edges are replaced by unit resistors. The sum of resistance distances of all pairs of vertices of a graph G is the Kirchhoff index, that is,

$$Kf(G) = \sum_{i<j} r_{ij}. \tag{1.1}$$

Klein and Randić proved that $Kf(G) \leq W(G)$ with equality if and only if G is a tree. There are several equivalent ways to define the resistance distance. Gutman and Mohar [6] (see also [4]) proved that the Kirchhoff index can also be represented as

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}. \tag{1.2}$$

From a computational point of view, (1.2) is more appropriate then (1.1) and this has aroused interest in study of this invariant and its applications in various areas, such as spectral graph theory, molecular chemistry [23], computer science [11] etc.

Narumi and Katayama [15] defined a new topological index, NK, later called the Narumi–Katayama index, by

$$NK = NK(G) = \det \mathbf{D} = \prod_{i=1}^{n} d_i.$$

This is the oldest multiplicative vertex-degree-based topological index. Details of this index can be found in [5, 18].

Before we proceed, let us define one special class of d-regular graphs Γ_d (see [16]). Let $N(i)$ be the set of all neighbours of the vertex i, that is, $N(i) = \{k \mid k \in V, k \sim i\}$, and let $d(i, j)$ be the distance between vertices i and j. Denote by Γ_d the set of all d-regular graphs, $1 \leq d \leq n-1$, with diameter 2 and $|N(i) \cap N(j)| = d$ for $i \nsim j$.

In this paper we derive lower bounds for the Kirchhoff index of a connected (molecular) graph in terms of its structural parameters such as the number of vertices (atoms), the number of edges (bonds), maximum vertex degree (valency), second maximum vertex degree, minimum vertex degree and the topological index $NK(G)$.

2. Preliminaries

In this section we recall some lower bounds for $Kf(G)$ needed in our work.

In [12] (see also [13]), lower bounds for $Kf(G)$ in terms of the pair of parameters n and Δ, and also the pair of parameters n and m, are obtained, namely

$$Kf(G) \geq \frac{n(n-1) - \Delta}{\Delta} \tag{2.1}$$

and

$$Kf(G) \geq \frac{n^2(n-1) - 2m}{2m}, \tag{2.2}$$

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with equality if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$. Lower bounds for $Kf(G)$ that depend on the same parameters as in (2.1) and (2.2) were also considered in [16, 17, 23].

In [3] (see also [2]), a lower bound for $Kf(G)$ in terms of n, m, Δ and δ is derived:

$$Kf(G) \geq n\Big(\frac{1}{1+\Delta} + \frac{1}{\delta} + \frac{(n-3)^2}{2m-\Delta-\delta-1}\Big), \tag{2.3}$$

with equality if and only if $G \cong K_{1,n-1}$, or $G \cong 2K_1 \vee K_{n-2}$, or $G \cong (K_1 \cup K_{n-2}) \vee K_1$. In [4], a lower bound for $Kf(G)$ depending on n, m, Δ, δ and Δ_2 is established:

$$Kf(G) \geq \frac{n}{1+\Delta} + \frac{n}{2m-\Delta-1}\Big((n-2)^2 + \frac{(\Delta_2-\delta)^2}{\Delta_2\delta}\Big), \tag{2.4}$$

with equality if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$. In [1], a lower bound for $Kf(G)$ in terms of the parameters n, m, Δ and Δ_2 is obtained:

$$Kf(G) \geq n\Big(\frac{1}{1+\Delta} + \frac{1}{\Delta_2} + \frac{(n-3)^2}{2m-\Delta-\Delta_2-1}\Big). \tag{2.5}$$

In [12] (see also [13]), the following inequalities are proved:

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{(n-1)^3}{2m-\Delta}, \tag{2.6}$$

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{n-1}{\delta} + \frac{(n-1)(n-2)^2}{2m-\Delta-\delta}, \tag{2.7}$$

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{n-1}{\Delta_2} + \frac{(n-1)(n-2)^2}{2m-\Delta-\Delta_2}, \tag{2.8}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

In [22], a number of lower bounds for $Kf(G)$ are reported. Particularly interesting for our work is the inequality between the Kirchhoff index and the inverse degree index of G, given by $ID = \sum_{i=1}^{n} 1/d_i$:

$$Kf(G) \geq -1 + (n-1)\sum_{i=1}^{n}\frac{1}{d_i}. \tag{2.9}$$

Equality holds if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$. At first glance, the inequality (2.9) may not seem helpful since it requires knowing degrees of all vertices of a graph to compute ID. However, the invariant ID is well studied and we will use available lower bounds for ID to determine new lower bounds for $Kf(G)$ in terms of various other graph invariants and parameters.

3. Main result

The following theorem gives a lower bound for $Kf(G)$ in terms of the number of vertices, n, number of edges, m, maximum vertex degree, Δ, and minimum vertex degree, δ.

THEOREM 3.1. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n^2(n-1)-2m}{2m} + \frac{(n-1)(\sqrt{\Delta}-\sqrt{\delta})^2}{2\Delta\delta}, \tag{3.1}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

PROOF. Let $a = (a_i)$, $i = 1, 2, \ldots, n$, be real numbers such that $0 < r \leq a_i \leq R < +\infty$. From [19],

$$n\sum_{i=1}^{n} a_i^2 - \Big(\sum_{i=1}^{n} a_i\Big)^2 \geq \frac{n}{2}(R-r)^2. \tag{3.2}$$

For $a_i = 1/\sqrt{d_i}$, $i = 1, 2, \ldots, n$, $r = 1/\sqrt{\Delta}$ and $R = 1/\sqrt{\delta}$, this inequality becomes

$$n\sum_{i=1}^{n} \frac{1}{d_i} - \Big(\sum_{i=1}^{n} \frac{1}{\sqrt{d_i}}\Big)^2 \geq \frac{n(\sqrt{\Delta}-\sqrt{\delta})^2}{2\Delta\delta}. \tag{3.3}$$

Let $p = (p_i)$, $i = 1, 2, \ldots, n$, be a sequence of positive real numbers and let $a = (a_i)$, $b = (b_i), \ldots, c = (c_i)$, $i = 1, 2, \ldots, n$, be r sequences of nonnegative real numbers, all monotonic in the same sense. From [7] (see also [14]),

$$\Big(\sum_{i=1}^{n} p_i\Big)^{r-1} \sum_{i=1}^{n} p_i a_i b_i \cdots c_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \cdots \sum_{i=1}^{n} p_i c_i. \tag{3.4}$$

For $r = 3$, $p_i = 1/\sqrt{d_i}$, $a_i = b_i = c_i = \sqrt{d_i}$, $i = 1, 2, \ldots, n$, this inequality transforms into

$$\Big(\sum_{i=1}^{n} \frac{1}{\sqrt{d_i}}\Big)^2 \sum_{i=1}^{n} d_i \geq n^3,$$

that is,

$$\Big(\sum_{i=1}^{n} \frac{1}{\sqrt{d_i}}\Big)^2 \geq \frac{n^3}{2m}. \tag{3.5}$$

According to (3.3) and (3.5),

$$\sum_{i=1}^{n} \frac{1}{d_i} \geq \frac{n^2}{2m} + \frac{(\sqrt{\Delta}-\sqrt{\delta})^2}{2\Delta\delta}.$$

Finally, from this and inequality (2.9), we obtain (3.1).

Equality in (3.5) holds if and only if $d_1 = d_2 = \cdots = d_n$. Equality in (2.9) holds if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$. Therefore, equality in (3.1) holds if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$. □

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REMARK 3.2. Since $(\sqrt{\Delta}-\sqrt{\delta})^2 \geq 0$, the inequality (3.1) is stronger than (2.2).

Since $2m \leq n\Delta$, we have the following corollary of Theorem 3.1.

COROLLARY 3.3. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n(n-1)-\Delta}{\Delta} + \frac{(n-1)(\sqrt{\Delta}-\sqrt{\delta})^2}{2\Delta\delta}, \tag{3.6}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

REMARK 3.4. The inequality (3.6) is stronger than (2.1). If G is a d-regular graph with $1 \leq d \leq n-1$, then (3.6) implies the following inequality from [16]:

$$Kf(G) \geq \frac{n(n-1)-d}{d}.$$

REMARK 3.5. The lower bounds for $Kf(G)$ in (2.3) and (3.1) depend on the same parameters n, m, Δ and δ. However, they are not comparable. Thus, for example, the lower bound (3.1) is better than (2.3) for $G \cong K_{n/2,n/2}$, but the opposite holds for $G \cong 2K_1 \vee K_{n-2}$.

The following theorem gives a lower bound for $Kf(G)$ in terms of the parameters n, m, Δ, Δ_2 and δ.

THEOREM 3.6. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + (n-1)\Big(\frac{(n-1)^2}{2m-\Delta} + \frac{(\sqrt{\Delta_2}-\sqrt{\delta})^2}{2\Delta_2\delta}\Big), \tag{3.7}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.

PROOF. The inequality (3.2) can be written in the form

$$(n-1)\sum_{i=2}^{n} a_i^2 - \Big(\sum_{i=2}^{n} a_i\Big)^2 \geq \frac{n-1}{2}(R-r)^2.$$

For $a_i = 1/\sqrt{d_i}$, $i = 2, \ldots, n$, $R = 1/\sqrt{\delta}$ and $r = 1/\sqrt{\Delta_2}$,

$$(n-1)\sum_{i=2}^{n} \frac{1}{d_i} - \Big(\sum_{i=2}^{n} \frac{1}{\sqrt{d_i}}\Big)^2 \geq \frac{n-1}{2}\Big(\frac{1}{\sqrt{\delta}} - \frac{1}{\sqrt{\Delta_2}}\Big)^2,$$

that is,

$$(n-1)\sum_{i=2}^{n} \frac{1}{d_i} \geq \Big(\sum_{i=2}^{n} \frac{1}{\sqrt{d_i}}\Big)^2 + \frac{(n-1)(\sqrt{\Delta_2}-\sqrt{\delta})^2}{2\Delta_2\delta}. \tag{3.8}$$

Similarly, the inequality (3.4) can be written as

$$\Big(\sum_{i=2}^{n} p_i\Big)^{r-1} \sum_{i=2}^{n} p_i a_i b_i \cdots c_i \geq \sum_{i=2}^{n} p_i a_i \sum_{i=2}^{n} p_i b_i \cdots \sum_{i=2}^{n} p_i c_i.$$

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For $r = 3$, $p_i = 1/\sqrt{d_i}$, $a_i = b_i = c_i = \sqrt{d_i}$, $i = 2, \ldots, n$, this inequality becomes

$$\Big(\sum_{i=2}^{n} \frac{1}{\sqrt{d_i}}\Big)^2 \sum_{i=2}^{n} d_i \geq (n-1)^3,$$

that is,

$$\Big(\sum_{i=2}^{n} \frac{1}{\sqrt{d_i}}\Big)^2 \geq \frac{(n-1)^3}{2m-\Delta}. \tag{3.9}$$

According to (3.8) and (3.9),

$$(n-1)\sum_{i=2}^{n} \frac{1}{d_i} \geq \frac{(n-1)^3}{2m-\Delta} + \frac{(n-1)(\sqrt{\Delta_2}-\sqrt{\delta})^2}{2\Delta_2\delta},$$

that is,

$$(n-1)\sum_{i=1}^{n} \frac{1}{d_i} \geq \frac{n-1}{\Delta} + (n-1)\Big(\frac{(n-1)^2}{2m-\Delta} + \frac{(\sqrt{\Delta_2}-\sqrt{\delta})^2}{2\Delta_2\delta}\Big). \tag{3.10}$$

From (3.10) and (2.9), we arrive at (3.7).

Equality in (3.9) holds if and only if $d_2 = d_3 = \cdots = d_n$. Equality in (2.9) holds if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n,2}$, or $G \in \Gamma_d$. So, equality in (3.7) holds if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n,2}$, or $G \in \Gamma_d$. □

Remark 3.7. The inequality (3.7) is stronger than (2.6). It is also stronger than (2.4) when $G \cong K_{n/2,n,2}$ or $G \in \Gamma_d$. However, it remains an open question whether the lower bound for $Kf(G)$ given by (3.7) is always better than (2.4).

The following results can be proved by similar arguments to those used in the proofs of Theorems 3.1 and 3.6.

Theorem 3.8. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n-1-\delta}{\delta} + (n-1)\Big(\frac{(n-1)^2}{2m-\delta} + \frac{(\sqrt{\Delta}-\sqrt{\delta_2})^2}{2\Delta\delta_2}\Big),$$

with equality if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

Theorem 3.9. *Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then*

$$Kf(G) \geq \frac{(n-1)(\Delta+\delta)-\Delta\delta}{\Delta\delta} + (n-1)\Big(\frac{(n-2)^2}{2m-\Delta-\delta} + \frac{(\sqrt{\Delta_2}-\sqrt{\delta_2})^2}{2\Delta_2\delta_2}\Big), \tag{3.11}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

Theorem 3.10. *Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then*

$$Kf(G) \geq \frac{(n-1)(\Delta+\Delta_2)-\Delta\Delta_2}{\Delta\Delta_2} + (n-1)\Big(\frac{(n-2)^2}{2m-\Delta-\delta} + \frac{(\sqrt{\Delta_3}-\sqrt{\delta})^2}{2\Delta_3\delta}\Big), \tag{3.12}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

REMARK 3.11. The inequality (3.11) is stronger than (2.7), but it is not comparable with (2.3). The inequality (3.12) is stronger than (2.8). However, it is questionable whether inequality (3.12) is always stronger than (2.5).

The following theorem gives a lower bound for $Kf(G)$ depending on the parameters n, m and the topological index $NK(G)$.

THEOREM 3.12. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{(n^3 - 2m)(NK(G))^{1/n} - 2mn}{2m(NK(G))^{1/n}}, \tag{3.13}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

PROOF. Let $a = (a_i)$, $i = 1, 2, \ldots, n$, be a sequence of positive real numbers. From [21] (see also [10]),

$$(n-1)\sum_{i=1}^{n} a_i \geq \Big(\sum_{i=1}^{n} \sqrt{a_i}\Big)^2 - n\Big(\prod_{i=1}^{n} a_i\Big)^{1/n}. \tag{3.14}$$

For $a_i = 1/d_i$, $i = 1, 2, \ldots, n$, it follows that

$$(n-1)\sum_{i=1}^{n} \frac{1}{d_i} \geq \Big(\sum_{i=1}^{n} \frac{1}{\sqrt{d_i}}\Big)^2 - n\Big(\prod_{i=1}^{n} \frac{1}{d_i}\Big)^{1/n}.$$

From this inequality and (3.5),

$$(n-1)\sum_{i=1}^{n} \frac{1}{d_i} \geq \frac{n^3}{2m} - \frac{n}{(NK(G))^{1/n}}. \tag{3.15}$$

The inequality (3.13) is a direct consequence of (3.15) and (2.9).

Equality holds in (3.15) if and only if $d_1 = d_2 = \cdots = d_n$. Equality holds in (2.9) if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$. Therefore, equality holds in (3.13) if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$. □

Since $2m \leq n\Delta$, we have the following corollary of Theorem 3.12.

COROLLARY 3.13. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{(n^2 - \Delta)(NK(G))^{1/n} - n\Delta}{\Delta(NK(G))^{1/n}},$$

with equality if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

The next theorem gives a lower bound for $Kf(G)$ in terms of n, m, Δ and $NK(G)$.

THEOREM 3.14. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{(n-1)^2}{n-2}\Big(\frac{(n-1)^2}{2m-\Delta} - \Big(\frac{\Delta}{NK(G)}\Big)^{1/(n-1)}\Big), \tag{3.16}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

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Proof. The inequality (3.14) can be written as

$$(n-2)\sum_{i=2}^{n} a_i \geq \Big(\sum_{i=2}^{n} \sqrt{a_i}\Big)^2 - (n-1)\Big(\prod_{i=2}^{n} a_i\Big)^{1/(n-1)}.$$

For $a_i = 1/d_i$, $i = 2, 3, \ldots, n$, this inequality becomes

$$(n-2)\sum_{i=2}^{n} \frac{1}{d_i} \geq \Big(\sum_{i=2}^{n} \frac{1}{\sqrt{d_i}}\Big)^2 - (n-1)\Big(\prod_{i=2}^{n} \frac{1}{d_i}\Big)^{1/(n-1)}.$$

From this inequality and (3.9),

$$(n-2)\sum_{i=2}^{n} \frac{1}{d_i} \geq \frac{(n-1)^3}{2m-\Delta} - (n-1)\frac{\Delta^{1/(n-1)}}{(NK(G))^{1/(n-1)}},$$

that is,

$$\sum_{i=1}^{n} \frac{1}{d_i} \geq \frac{1}{\Delta} + \frac{n-1}{n-2}\Big(\frac{(n-1)^2}{2m-\Delta} - \frac{\Delta^{1/(n-1)}}{(NK(G))^{1/(n-1)}}\Big). \tag{3.17}$$

Inequality (3.16) is a direct consequence of (3.17) and (2.9).

Equality in (3.17) holds if and only if $d_2 = d_3 = \cdots = d_n$. Equality in (2.9) holds if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$. Therefore, equality in (3.16) holds if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$. □

Corollary 3.15. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq \frac{(n-1)(n^2-n-1)-(n-2)\Delta}{(n-2)\Delta} - \frac{(n-1)^2}{n-2}\Big(\frac{\Delta}{NK(G)}\Big)^{1/(n-1)},$$

with equality if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.

Since $NK(G) \geq n-1$ (see [5]), Theorem 3.14 has the following corollary.

Corollary 3.16. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{(n-1)^2}{n-2}\Big(\frac{(n-1)^2}{2m-\Delta} - \Big(\frac{\Delta}{n-1}\Big)^{1/(n-1)}\Big),$$

with equality if and only if $G \cong K_{1,n-1}$.

Let $t = t(G)$ be the total number of spanning trees of G. In [9], it was proved that

$$NK(G) \geq (n-1)t$$

and this leads to a further corollary of Theorem 3.14.

Corollary 3.17. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{(n-1)^2}{n-2}\Big(\frac{(n-1)^2}{2m-\Delta} - \Big(\frac{\Delta}{t(n-1)}\Big)^{1/(n-1)}\Big),$$

with equality if and only if $G \cong K_{1,n-1}$.

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The following theorems can be proved in a similar way to Theorems 3.12 and 3.14.

THEOREM 3.18. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n-1-\delta}{\delta} + \frac{(n-1)^2}{n-2}\left(\frac{(n-1)^2}{2m-\delta} - \left(\frac{\delta}{NK(G)}\right)^{1/(n-1)}\right),$$

with equality if and only if $G \cong K_n$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

THEOREM 3.19. *Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then*

$$Kf(G) \geq \frac{(n-1)(\Delta+\delta)-\Delta\delta}{\Delta\delta} + \frac{(n-1)(n-2)}{n-3}\left(\frac{(n-2)^2}{2m-\Delta-\delta} - \left(\frac{\Delta\delta}{NK(G)}\right)^{1/(n-2)}\right),$$

with equality if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$, or $G \in \Gamma_d$.

COROLLARY 3.20. *Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then*

$$Kf(G) \geq \frac{(n-1)(\Delta+\delta)-\Delta\delta}{\Delta\delta} + \frac{(n-1)(n-2)}{n-3}\left(\frac{(n-2)^2}{2m-\Delta-\delta} - \left(\frac{\Delta\delta}{(n-1)t}\right)^{1/(n-2)}\right),$$

with equality if and only if $G \cong K_{1,n-1}$.

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