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VALUE DISTRIBUTION OF BIAXIALLY SYMMETRIC HARMONIC POLYNOMIALS

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1. Introduction. Consider the biaxially symmetric potential equation

$$(1.1)\qquad L_{\alpha\beta}(\Phi) = \left(\frac{\partial^2}{\partial u^2} + \frac{2\beta+1}{u}\frac{\partial}{\partial u} + \frac{\partial^2}{\partial v^2} + \frac{2\alpha+1}{v}\frac{\partial}{\partial v}\right)\Phi(u, v) = 0$$

where α, $\beta > -1/2$. If $2\alpha + 1$ and $2\beta + 1$ are non-negative integers and if χ corresponds to the hypercircle

$$(1.2)\qquad u = (x_1^2 + \ldots + x_{2\beta+2}^2)^{1/2}, \quad v = (y_1^2 + \ldots + y_{2\alpha+2}^2)^{1/2},$$

then the biaxisymmetric Laplace equation in $\mathbf{E}^{2(\alpha+\beta+2)}$,

$$(1.3)\qquad \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{2\beta+2}^2} + \frac{\partial^2}{\partial y_1^2} + \ldots + \frac{\partial^2}{\partial y_{2\alpha+2}^2}\right)\Phi(\chi) = 0$$

and (1.1) are equivalent. A complete set of solutions for (1.1) which are regular about the origin is given by (cf. [**1**, **2**])

$$(1.4)\qquad \Phi_k(\chi) = \underset{\sim}{\Phi}_k(u, v) = \underset{\sim}{\Phi}_k(r, \theta) = r^{2k}R_k^{(\alpha,\beta)}(\cos 2\theta),$$

where

$$(1.5)\qquad R_k^{(\alpha,\beta)}(\cos 2\theta) = P_k^{(\alpha,\beta)}(\cos 2\theta)/P_k^{(\alpha,\beta)}(1),$$

the $P_k^{(\alpha,\beta)}(x)$ are the Jacobi polynomials, and $u = r\cos\theta$, $v = r\sin\theta$ are the polar coordinates.

It is known that any biaxisymmetric harmonic polynomial (BAHP) of degree $2n$ can be represented in the form

$$(1.6)\qquad H(\chi) = \underset{\sim}{H}(u, v) = \sum_{k=0}^{n} a_k r^{2k} R_k^{(\alpha,\beta)}(\cos 2\theta),$$

where α, $\beta > -1/2$. Until now, the lack of suitable representations for $R_k^{(\alpha,\beta)}(\cos 2\theta)$ had made it difficult to determine a value distribution for BAHP's analogous to the value distribution for axisymmetric harmonic polynomials determined by Morris Marden in [**4**] using the Whittaker formula. However, Tom Koornwinder's Laplace type integral for Jacobi polynomials now allows us to determine information about the value distribution for BAHP's using a convexity argument drawn from the analytic theory of polynomials of one complex variable.

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According to Koornwinder's integral representation, cf. [**3**], if $\alpha > \beta > -1/2$, then

$$\Phi_k(u, v) = r^{2k}R_k^{(\alpha,\beta)}(\cos 2\theta)$$

(1.7) $$= \int_{t=0}^{1} \int_{\phi=0}^{\pi} (u^2 - v^2t^2 + 2iuvt(\cos \phi))^k dm_{\alpha,\beta}(\phi, t),$$

where the non-negative measure

(1.8) $$dm_{\alpha,\beta}(\phi, t) = \frac{2\Gamma(\alpha + 1)(1 - t^2)^{\alpha-\beta-1}t^{2\beta+1}(\sin \phi)^{2\beta}d\phi dt}{\pi^{1/2}\Gamma(\alpha - \beta)\Gamma(\beta + 1/2)}$$

is normalized so that

(1.9) $$\int_0^1 \int_0^\pi dm_{\alpha,\beta}(\phi, t) = 1.$$

2. Value distribution for BAHP's. Let $\underset{\sim}{H}(\chi) = H(u, v)$ be a BAHP as in (1.6), and assume that $\alpha > \beta > -1/2$. Define the associate polynomial of H to be

(2.1) $$h(\xi) = \sum_{k=0}^{n} a_k\xi^k, \quad \xi \in \mathbf{C}, a_n \neq 0$$

so that

(2.2) $$H(\chi) = \underset{\sim}{H}(u, v) = \int_0^1 \int_0^\pi h(z_{u,v}(\phi, t))dm_{\alpha,\beta}(\phi, t)$$

where

(2.3) $$z_{u,v}(\phi, t) = u^2 - v^2t^2 + 2\, iuvt\, (\cos \phi).$$

THEOREM 2.1. *Let H be a BAHP of degree $2n$ as in (2.2) with h as its associate. If h omits the complex value γ in the sector*

(2.4) $$S = \{\xi \in \mathbf{C} : |\arg (\xi - c)| < \pi - \pi/2n\},$$

with vertex at $c \geqq 0$, then on each hypercircle $\chi \in \Omega \subset \mathbf{E}^{2(\alpha+\beta+2)}$ where Ω is the region common to the set

$$x_1^2 + \ldots + x_{2\beta+2}^2 - x_{2\beta+3}^2 - \ldots - x_{2(\alpha+\beta+2)}^2 \geqq c$$

and the hyperbolic cylinder

$$(x_1^2 \ldots + x_{2\beta+2}^2 - y_1^2 - \ldots - y_{2\alpha+2}^2 - c)^2 \tan^2 \pi/2n \geqq 4(x_1^2 + \ldots + x_{2\beta+2}^2)(y_1^2 + \ldots + y_{2\alpha+2}^2),$$

then $H(\chi) \neq \gamma + \eta$ for $\eta = 0$ or for all $|\arg (\eta/a_n)| < \pi/2 \bmod (n + 1)$.

Proof. Suppose $H(\chi_0) = \gamma$ or $\underset{\sim}{H}(u_0, v_0) - \gamma = 0$ for some u_0, v_0 correspond-

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ing to χ_0. Then if

$$(2.5)\qquad h(\xi) - \gamma = a_n \prod_{k=1}^{n} (\xi - \alpha_k),$$

by (2.2)

$$(2.6)\qquad \underset{\sim}{H}(u_0, v_0) - \gamma = \int_0^1 \int_0^\pi w(\phi, t) dm_{\alpha,\beta}(\phi, t) = 0$$

where

$$(2.7)\qquad w(\phi, t) = a_n \prod_{k=1}^{n} (\alpha_k - z_{u_0,v_0}(\phi, t)).$$

Notice that for a fixed u_0, v_0, the region F in the complex plane defined by $z_{u_0,v_0}(\phi, t)$ as ϕ goes from 0 to π and t goes from 0 to 1 is the region bounded by the parabola $y^2 = -4u_0^2(x - u_0)$ and the line $x = u_0^2 - v_0^2$. F is contained in the sector where $|\arg (\xi - c)| \leqq \pi/2n$ by our assumption that $(u_0^2 - v_0^2 - c)$ $\tan \pi/2n \geqq 2\, uv$. Therefore

$$(2.8)\qquad \pi - \pi/2n < \arg \{\alpha_k - z_{u_0,v_0}(\phi, t)\} < \pi + \pi/2n$$

which implies by (2.7) that (2.6) cannot possibly hold since

$$w(\phi, t) \in \{\xi \in \mathbf{C} : |\arg (\xi/a_n) - n\pi| < \pi/2\}, \quad 0 < t < 1, 0 < \phi < \pi$$

and $dm_{\alpha,\beta} \geqq 0$. Consequently, $\underset{\sim}{H}(u_0, v_0) - \gamma \in \{\xi \in \mathbf{C} : |\arg (\xi/a_n) - n\pi| < \pi/2\}$ so that $\underset{\sim}{H}(u_0, v_0) \neq \gamma + \eta$ if $\eta = 0$ or $|\arg (\eta/a_n) - (n + 1)\pi| < \pi/2$.

THEOREM 2.2. *Let H be the BAHP*

$$(2.9)\qquad H(\chi) = \underset{\sim}{H}(u, v) = \sum_{k=0}^{n} a_k r^{2k} R_k^{(\alpha,\beta)}(\cos 2\theta), \quad \alpha > \beta > -1/2,$$

and let γ be an arbitrary constant. If.

$$(2.10)\qquad \nu = 1 + \max \{|a_0 - \gamma|/|a_n|, |a_1/a_n|, \ldots, |a_{n-1}/a_n|\}$$

and χ is a hypercircle in the region Ω defined in Theorem 2.1 *with* $c = \nu \operatorname{cosec} (2\pi/n)$, *then*

$$H(\chi) \neq \gamma + \eta$$

for $\eta = 0$ and for all $|\arg (\eta/a_n)| < \pi/2 \mod (n + 1)$.

Proof. If we denote by $h(\xi)$ the associate of H, then by Cauchy's inequality, (cf. [**5**, p. 123]) the zeros of

$$h(\xi) - \gamma = (a_0 - \gamma) + a_1\xi + a_2\xi^2 + \ldots + a_n\xi^n$$

satisfy the inequality $|\xi| < \nu$, with ν given by (2.10). Therefore, $h(\xi) \neq \gamma$ in the sector S of (2.4) where

$$c = \nu \operatorname{cosec} (2\pi/n) > 0,$$

and the conclusion follows from the previous theorem.

3. Remarks. (*On the set* Ω). It is clear from the proof of Theorem 2.1, that the projection of Ω on the complex plane according to the transformations of (1.2) results in the set

$$\underset{\sim}{\Omega} = \{u + iv : u \geqq 0, v \geqq 0, \text{ and } |\arg ((u + iv)^2 - c)| \leqq \pi/2n\}.$$

Using this description, $\underset{\sim}{\Omega}$ is the intersection, with the first quadrant, of the interior of the hyperbola $u^2/d^2c - v^2/d^2c = 1$ rotated $-\alpha$ $(= -\pi(n-1)/4n)$ radians from the $u-$ axis where $d^2 = \cos 2\alpha$. If $c = 0$, $\underset{\sim}{\Omega} = \{\xi \in \mathbf{C} : 0 \leqq \arg \xi \leqq \pi/4n\}$.

If (x, y, z) are the Cartesian coordinates in $\mathbf{E}^3$, and if we view u as the distance of a point from the x-axis and v as its distance from the y-axis, then

$$(3.1) \qquad u^2 = y^2 + z^2 \text{ and } v^2 = x^2 + z^2.$$

Geometrically, Ω is the set of points in $\mathbf{E}^3$ generated by the intersection of cylinders about the x-axis of radius u and about the y-axis of radius v, where once u is chosen so that $u \geqq c^{1/2}$, then $v \leqq -\cot (\pi/2n)u + (u^2 \operatorname{cosec}^2 (\pi/2n) - c)^{1/2}$.

For example, if in Theorem 2.1, $H(\chi)$ is a *BAHP* of degree 2 and $c > 0$, then $\underset{\sim}{\Omega}$ is the region defined by the interior of the hyperbola $u^2/c - v^2/c = 1$, without rotation, intersected with the first quadrant. Using (3.1), we get that in $\mathbf{E}^3$, $\Omega = \{(x, y, z) : x^2/c - z^2/c \geqq 1)$, the interior of hyperbolic cylinders.

(*On* $\alpha > \beta > -1/2$). First note that the set Ω in Theorem 2.1 depends only on $\alpha > \beta > -1/2$ and not specifically on the values of α and β. If in the expression for H in (1.6), $\beta > \alpha > -1/2$, then one must use the identity

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$

in (1.5) thereby changing (1.6) to

$$\underset{\sim}{H}(u, v) = \sum_{k=1}^{n} c_k r^{2k} R_k^{(\beta,\alpha)}(-\cos 2\theta),$$

and a similar argument to that used in Theorem 2.1, (where u and v are switched in (1.7) due to the $-\cos 2\theta$) using the associated polynomial $h(\xi) = \sum_{k=0}^{n} c_k \xi^k$, will give information about the value distribution for H.

4. The converse problem. The methods found in [**5**] also apply to the converse problem of relating the values of the associate to those (known) values of the *BAHP*. This class of relationships was not considered in [**4**]. However, the reasoning which follows adopts itself to similar considerations for axisymmetric harmonic polynomials.

THEOREM 4.1. *If the BAHP H of degree 2n assumes the value γ on the sphere of radius $R_0 = (u_0^2 + v_0^2)^{1/2}$, then the associate h assumes the value γ at least once in the disc* $|\xi| \leqq R_0^2 \operatorname{cosec} (\pi/2n)$.

Proof. Following [**5**, p. 111], consider the point ξ_0 for which $H(u_0, v_0) = h(\xi_0) = \gamma$ so that

$$\int_0^1 \int_0^\pi [h(z_{u_0,v_0}(\phi, t)) - \gamma] dm_{\alpha,\beta}(\phi, t) = 0. \tag{4.1}$$

By factoring $h(\xi_0) - \gamma$ as in (2.5), it is clear that if $(\alpha_k) \geqq R_0^2 \operatorname{cosec} (\pi/2n)$ for $1 \leqq k \leqq n$, then $\alpha_k - z_{u_0,v_0}(\phi, t)$ satisfies an inequality of the type (2.8) since

$$|z_{u_0,v_0}(\phi, t)| \leqq |z_{u_0,v_0}(0, 1)| = R_0^2$$

Consequently, the integrand of (4.1) is non-vanishing, a contradiction to the fact that $H(u_0, v_0) = \gamma$.

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