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p-adic L-functions via local–global interpolation: the case of $\mathrm{GL}_2 \times \mathrm{GU}(1)$

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Abstract. Let F be a totally real field, and let E/F be a CM quadratic extension. We construct a p-adic L-function attached to Hida families for the group $\mathrm{GL}_{2/F} \times \mathrm{Res}_{E/F}\mathrm{GL}_1$. It is characterized by an exact interpolation property for critical Rankin–Selberg L-values, at classical points corresponding to representations $\pi \boxtimes \chi$ with the weights of χ smaller than the weights of π.

Our p-adic L-function agrees with previous results of Hida when E/F splits above p or $F = \mathbf{Q}$, and it is new otherwise. Exploring a method that should bear further fruits, we build it as a ratio of families of global and local Waldspurger zeta integrals, the latter constructed using the local Langlands correspondence in families.

In an appendix of possibly independent recreational interest, we give a reality-TV-inspired proof of an identity concerning double factorials.

1 Introduction

This paper is a case study in the construction of p-adic L-functions by the "soft" method of glueing ratios of matching families of global and local zeta integrals. The local integrals are constructed and then inserted into the global context by using the local Langlands correspondence in families (see [Dis20] and the references therein). The method, whose deployment seems new for non-abelian families, should be of wide applicability; we give a brief introductory description in Section 1.2.

The specific context and arithmetic interest of our work is the following. Let F be a totally real field, let E/F be a totally imaginary quadratic extension, and let p be a rational prime. We construct a meromorphic function $\mathscr{L}_p(\mathscr{V})$ on Hida families for $\mathrm{GL}_{2/F} \times \mathrm{Res}_{E/F}\mathrm{GL}_1$ that interpolates critical values

$$\frac{L(1/2, \pi_E \otimes \chi)}{L(1, \pi, \mathrm{ad})},$$

for p-ordinary automorphic representations $\pi \boxtimes \chi$ such that χ has lower weights than π. (The precise statement is Theorem A; note that in our normalization, the above numerators are not necessarily central values.) The function $\mathscr{L}_p(\mathscr{V})$ is new (if not surprising) at least when E/F does not split above p; for a discussion of previous related works, see Section 1.1.6.

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The interpolation property of $\mathscr{L}_p(\mathscr{V})$ holds at all classical points satisfying the weight condition and lying outside the polar locus (on which we have partial control), and it provides an entirely explicit and complete characterization of the function, in the spirit of [Hid96]. Its generality and precision are key to some arithmetic applications in [Dis/b], which motivated our choice of case. In that paper, we prove, first, the p-adic Beilinson–Bloch–Kato conjecture in analytic rank 1 for (conjugate-)self-dual motives attached to representations $\pi_E \otimes \chi$ as above; and second, one divisibility in an Iwasawa Main Conjecture for the cyclotomic derivative $\mathrm{d}^\sharp \mathscr{L}_p(\mathscr{V})$ of $\mathscr{L}_p(\mathscr{V})$ along a self-dual locus. Both results, new or partly new even when $F = \mathbf{Q}$ and E/F splits at p, rely on a p-adic Gross–Zagier formula for $\mathrm{d}^\sharp \mathscr{L}_p(\mathscr{V})$. In turn, that formula is proved by analytically continuing formulas from [Dis17, Disa] for the central derivatives of certain cyclotomic p-adic L-functions $\mathscr{L}_p(V_{(\pi,\chi)}, s)$ attached to those representations $\pi \boxtimes \chi$ as above that have minimal weights. The continuation argument thus requires to exactly identify the collection $\{(\mathscr{L}_p(V_{(\pi,\chi)}, s)_{(\pi,\chi)}\}$ of single-variable functions as a set of specializations of a multivariable analytic function, which is indeed our $\mathscr{L}_p(\mathscr{V})$.

It would be interesting to extend our results to the nonordinary case by the method of [AI21, Urb14]. As for further arithmetic directions in the ordinary case,[1] the main remaining goal is perhaps the full Iwasawa Main Conjecture for $\mathscr{L}_p(\mathscr{V})$. This was proved by Skinner and Urban [SU14] and Wan [Wan15] in the split case; in the nonsplit case, results toward it (when $F = \mathbf{Q}$) were recently obtained by Büyükboduk and Lei [BL]. A second goal is the remaining divisibility in the Main Conjecture for the cyclotomic derivative of $\mathscr{L}_p(\mathscr{V})$ (cf. [Dis/b, Theorem E]); in view of the p-adic Gross–Zagier formula of [Dis/b], this is equivalent to a suitable generalization of Perrin-Riou's main conjecture for Heegner points, which in its original form was recently proved by Burungale, Castella, and Kim [BCK21].

1.1 Statement of the main result

We move toward stating our main theorem, leaving a few of the detailed definitions of the objects involved to the body of the paper.

1.1.1 *p*-adic automorphic representations

Consider the algebraic groups over F

$$\mathrm{G} = \mathrm{GL}_{2/F}, \qquad \mathrm{H} := \mathrm{Res}_{E/F}\mathrm{GL}_{1/F}. \tag{1.1.1}$$

If v_0 is a place of $\mathbf{Q}$, we denote

$$\Sigma_{v_0} = \mathrm{Hom}\,(F, \overline{\mathbf{Q}}_{v_0}).$$

A (numerical) v_0-adic weight for G is a tuple $\underline{w} := (w_0, w = (w_\tau)_{\tau\in\Sigma_{v_0}})$ of integers, all of the same parity, such that $w_\tau \geq 0$ for all τ. It is said cohomological if $w_\tau \geq 2$ for all τ.

[1] This discussion has no ambition of being either a comprehensive research program or a comprehensive survey of the growing literature on the subject. Moreover, it entirely leaves out not only the case of nonordinary families, but also the p-adic L-function with complementary (to (1.1.7)) interpolation range introduced in [BDP13].

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A weight for H is a tuple $\underline{l} = (l_0, l = (l_\tau)_{\tau\in\Sigma_{v_0}})$ of integers of the same parity. Finally, if $\underline{w}$ and $\underline{l}$ are weights for G and H, the associated *contracted weight* for G × H is[2]

$$(w_0 + l_0, w, l).$$

If $\underline{w}$ is a p-adic weight (say, for G) and $\iota: \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ is an embedding, we denote $\underline{w}^\iota := (w_0, (w_\tau)_{\iota\circ\tau:F\hookrightarrow\mathbf{C}})$. (In fact, $\underline{w}^\iota$ only depends on $\iota_{|L}$ if $\underline{w}$ is rational over the finite extension L of $\mathbf{Q}_p$ in the sense that $\mathrm{Gal}(\overline{\mathbf{Q}}_p/L)$ fixes $\underline{w}$.) Let $\mathbf{A}$ be the ring of adèles of F. An automorphic representation of archimedean weight $\underline{w}$ is a complex automorphic representation π of $\mathrm{G}(\mathbf{A})$ such that $\pi_\infty = \pi_{\underline{w}} := \otimes_{\tau:F\hookrightarrow\mathbf{R}}\pi_{(w_0,w_\tau)}$, where $\pi_{(w_0,w_\tau)}$ is the discrete series of $\mathrm{GL}_2(F_\tau)$ of weight w_τ and central character $z \mapsto z^{w_0}$. If L is p-adic, we define an *automorphic representation of* $\mathrm{G}(\mathbf{A})$ *of weight* $\underline{w}$ *over* L to be a representation π of $\mathrm{G}(\mathbf{A}^\infty)$ on an L-vector space, such that for every $\iota: L \hookrightarrow \mathbf{C}$, the representation

$$\pi^\iota := \pi \otimes_{L,\iota} \pi_{\infty,\underline{w}^\iota}$$

of $\mathrm{G}(\mathbf{A})$ is automorphic.[3]

To the representation π over L is attached a two-dimensional representation V_π of $G_F := \mathrm{Gal}(\overline{F}/F)$; denoting by $V_{\pi,v}$ its restriction to a decomposition group at a place v of F, the representation V_π is characterized by $L(V_{\pi,v}, s) = L(s + 1/2, \pi_v)$ for all v (this is the "Hecke" normalization of the Langlands correspondence, cf. [Del73, Section 3.2]). We say that π is *ordinary*[4] if, for each place $v|p$ of F, there is a nontrivial G_{F_v}-stable filtration

$$0 \to V^+_{\pi,v} \to V_{\pi,v} \to V^-_{\pi,v} \to 0 \tag{1.1.2}$$

such that the character $\alpha^\circ_{\pi,v}: F_v^\times \to L^\times$ corresponding to $V^+_{\pi,v}(-1)$ has values in $\mathscr{O}_L^\times$.

Let L be a p-adic field splitting E, suppose chosen for each $\tau: F \hookrightarrow L$ an extension τ' to E,[5] and let $\tau'^c = \tau' \circ c$ for the complex conjugation c of E/F. A Hecke character of H of weight $\underline{l}$ over the p-adic field L is a locally algebraic character $\chi: E^\times \backslash \mathbf{A}_E^{\infty,\times} \to L^\times$ such that

$$\chi(t_p) = \prod_{\tau:F\hookrightarrow L} \tau'(t_p)^{(l_\tau+l_0)/2}\tau'^c(t_p)^{(-l_\tau+l_0)/2}$$

for all t_p in some neighborhood of $1 \in E_p^\times$. We let V_χ be the one-dimensional G_E-representation corresponding to χ.

1.1.2 *L*-values

Let π (respectively, χ) be a complex automorphic representation of $\mathrm{G}(\mathbf{A})$ (respectively, $\mathrm{H}(\mathbf{A})$), and let π_E denote the base change of π to E. Let us also introduce the

[2] A contracted weight is the same as a weight for $(\mathrm{G}\times\mathrm{H})^1 := \{(g, h) : \det(g) = N_{E/F}(h)\} \subset \mathrm{G}\times\mathrm{H}$. This is in fact the true group governing our constructions.

[3] This definition is slightly different from, but equivalent to, the one adopted in [Dis/b], whose flexibility will not be needed here.

[4] The literature often adds to the definition the extra restriction that $\alpha^\circ_{\pi,v}$ should be unramified, and calls "nearly ordinary" what we call "ordinary."

[5] This will only intervene in the numerical labeling of the weights.

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convenient notation

$$\text{“}V_{(\pi,\chi),v} := (V_{\pi,v} \otimes \mathrm{Ind}_{F_v}^{E_v}\chi_v) \ominus \mathrm{ad}(V_{\pi,v})(1)\text{''} \tag{1.1.3}$$

(to be thought of as referring to a “virtual motive”).

Let $\eta\colon F^\times\backslash\mathbf{A}^\times \to \{\pm 1\}$ be the character associated with E/F, and let

$$\mathscr{L}(V_{(\pi,\chi),v},0) := \frac{\zeta_{F,v}(2)L(1/2,\pi_{E,v}\otimes\chi_v)}{L(1,\eta_v)L(1,\pi_v,\mathrm{ad})}\cdot\begin{cases}1, & \text{if } v\nmid\infty\\ \pi^{-1}, & \text{if } v|\infty\end{cases} \quad \in \mathbf{C},$$

$$\mathscr{L}(V_{(\pi,\chi)},0) := \prod_v \mathscr{L}(V_{(\pi,\chi),v},0),$$

where the product (in the sense of analytic continuation) is over all places. These are the L-values we will interpolate.

1.1.3 Interpolation factors

Let L be a finite extension of $\mathbf{Q}_p$, let π be an ordinary automorphic representation of $\mathrm{G}(\mathbf{A})$ over L, with a locally algebraic central character $\omega_\pi\colon \mathbf{A}^{\infty,\times}\to L^\times$, let $\chi\colon \mathrm{H}(F)\backslash\mathrm{H}(\mathbf{A})\to L^\times$ be a locally algebraic character, and set $\omega_\chi := \chi_{|\mathbf{A}^{\infty,\times}}$. Let $\iota\colon L\hookrightarrow\mathbf{C}$ be an embedding, and let $\psi = \prod_v \psi_v\colon F\backslash\mathbf{A}\to\mathbf{C}^\times$ be the standard additive character such that $\psi_\infty(x) = e^{2\pi i \mathrm{Tr}_{F_\infty/\mathbf{R}}(x)}$.

If $v|p$, let $\mathrm{ad}(V_{\pi,v})(1)^{++} := \mathrm{Hom}\,(V_{\pi,v}^-, V_{\pi,v}^+)$, and define

$$e_v(V_{(\pi',\chi')}) = \frac{\prod_{w|v}\gamma(\iota\mathrm{WD}[V^+_{\pi,v|G_{E,w}}\otimes V_{\chi,w}],\psi_{E,w})^{-1}}{\gamma(\iota\mathrm{WD}[\mathrm{ad}(V_{\pi,v})(1)^{++}],\psi_v)^{-1}}\cdot\mathscr{L}(V_{(\pi',\chi'),v})^{-1}, \tag{1.1.4}$$

where $\iota\mathrm{WD}$ is the functor from potentially semistable Galois representations to complex Weil–Deligne representations of [Fon94], the inverse Deligne–Langlands γ-factor is $\gamma(W,\psi_v)^{-1} = L(W)/\varepsilon(W,\psi_v)L(W^*(1))$,[6] and $\psi_{E,w} = \psi_v\circ\mathrm{Tr}_{E_w/F_v}$.

Let $k_0\in\mathbf{Z}$ be such that the archimedean component of $\omega=\omega_\pi\omega_\chi$ is $\omega_\infty(x) = x^{k_0}$. We define

$$e_\infty(V_{(\pi',\chi')}) = i^{k_0[F:\mathbf{Q}]}$$

and

$$e_{p\infty}(V_{(\pi',\chi')}) := e_\infty(V_{(\pi',\chi')})\cdot\prod_{v|p} e_v(V_{(\pi',\chi')}). \tag{1.1.5}$$

1.1.4 Hida families

Let $U^p_{\mathrm{G}}\subset\mathrm{G}(\mathbf{A}^{p\infty})$ be any open compact subgroup, and for $R=\mathbf{Z}_p,\mathbf{Q}_p$, let $\mathbf{T}^{\mathrm{sph,\,ord}}_{U^p_{\mathrm{G}},R}$ be the p-(nearly) ordinary spherical Hecke R-algebra acting on ordinary p-adic modular cuspforms for G of tame level U^p_{G}.

[6] The normalizations of L- and ε-factors are as in [Tat79].

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A *cuspidal Hida family* $\mathscr{X}_{\mathrm{G}}$ is an irreducible component of the space $\mathscr{Y}_{\mathrm{G},U^p_{\mathrm{G}}} := \operatorname{Spec} \mathbf{T}^{\mathrm{sph,\,ord}}_{U^p_{\mathrm{G}},\mathbf{Q}_p}$ for some U^p_{G}. It is a scheme finite flat over $\operatorname{Spec} \mathbf{Z}_p[\![T_1,\ldots,T_{[F:\mathbf{Q}]+1+\delta_{F,p}}]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ (where $\delta_{F,p}$ is the p-Leopoldt defect of F), coming with a dense ind-finite subscheme

$$\mathscr{X}^{\mathrm{cl}}_{\mathrm{G}} \subset \mathscr{X}_{\mathrm{G}}$$

of *classical points*, and a locally free sheaf $\mathscr{V}_{\mathrm{G}}$ of rank 2 endowed with an $\mathscr{O}_{\mathscr{X}_{\mathrm{G}}}$-linear action of G_F. To each $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$ is associated an automorphic representation π_x of $\mathrm{G}(\mathbf{A}^\infty)$ over $\mathbf{Q}_p(x)$, and the fiber $\mathscr{V}_{\mathrm{G}|x} \cong V_{\pi_x}$. The (numerical) weight of x is defined to be the weight of π_x.

Let $U^p_{\mathrm{H}} \subset \mathrm{H}(\mathbf{A}^{p\infty})$ be an open compact subgroup. We define

$$\mathscr{Y}_{\mathrm{H}} = \mathscr{Y}_{\mathrm{H},U^p_{\mathrm{H}}} := \operatorname{Spec} \mathbf{Z}_p[\![\mathrm{H}(F)\backslash \mathrm{H}(\mathbf{A}^{p\infty})/U^p_{\mathrm{H}}]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p, \tag{1.1.6}$$

where the topology on $\mathrm{H}(F)\backslash \mathrm{H}(\mathbf{A}^{p\infty})/U^p_{\mathrm{H}}$ is profinite; it comes with a universal character $\chi_{\mathrm{univ}}\colon \mathrm{H}(F)\backslash \mathrm{H}(\mathbf{A}^\infty) \to \mathscr{O}(\mathscr{Y}_{\mathrm{H}})^\times$, identified with a G_E-representation $\mathscr{V}_{\mathrm{H}}$ of rank 1, and a dense ind-finite subscheme $\mathscr{Y}^{\mathrm{cl}}_{\mathrm{H}} \subset \mathscr{Y}_{\mathrm{H}}$, whose points y correspond to U^p_{H}-invariant locally algebraic Hecke characters χ_y of H over $\mathbf{Q}_p(y)$. The weight of y is defined to be the weight of χ_y.

Finally, the ordinary eigenvariety for $\mathrm{G} \times \mathrm{H}$ of level $U^p_{\mathrm{G}} \times U^p_{\mathrm{H}}$ is

$$\mathscr{Y}_{\mathrm{G}\times\mathrm{H}} = \mathscr{Y}_{\mathrm{G}\times\mathrm{H},U^p_{\mathrm{G}}\times U^p_{\mathrm{H}}} := \mathscr{Y}_{\mathrm{G}} \hat{\times} \mathscr{Y}_{\mathrm{H}} := \operatorname{Spec} \mathbf{T}^{\mathrm{sph,\,ord}}_{U^p_{\mathrm{G}},\mathbf{Z}_p} \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[\![\mathrm{H}(F)\backslash \mathrm{H}(\mathbf{A}^{p\infty})/U^p_{\mathrm{H}}]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

Its subset of classical points is $\mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}\times\mathrm{H}} := \mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}} \times \mathscr{Y}^{\mathrm{cl}}_{\mathrm{H}}$. A Hida family for $\mathrm{G}\times\mathrm{H}$ is an irreducible component of $\mathscr{Y}_{\mathrm{G}\times\mathrm{H},U^p_{\mathrm{G}}\times U^p_{\mathrm{H}}}$ for some U^p_{G}, U^p_{H}.

We now isolate an interesting subspace of $\mathscr{Y}_{\mathrm{G}\times\mathrm{H}}$. Let $\omega_{\mathrm{G}}\colon F^\times\backslash \mathbf{A}^{\infty,\times} \to \mathscr{O}(\mathscr{Y}_{\mathrm{G}})^\times$ be the character giving the action of the center of $\mathrm{G}(\mathbf{A})$ on p-adic modular forms, let $\omega_{\mathrm{H}} := \chi_{\mathrm{univ}|\mathbf{A}^{\infty,\times}}$, and let

$$\omega := \omega_{\mathrm{G}}\omega_{\mathrm{H}}\colon F^\times\backslash \mathbf{A}^{\infty,\times} \to \mathscr{O}(\mathscr{Y}_{\mathrm{G}\times\mathrm{H}})^\times.$$

The *self-dual locus*

$$\mathscr{Y}^{\mathrm{sd}}_{\mathrm{G}\times\mathrm{H}} \subset \mathscr{Y}_{\mathrm{G}\times\mathrm{H}}$$

is the closed subspace defined by $\omega = \mathbf{1}$. If $\mathscr{X}$ is a Hida family for $\mathrm{G}\times\mathrm{H}$, we denote $\mathscr{X}^{\mathrm{cl}} := \mathscr{X} \cap \mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}\times\mathrm{H}}$, $\mathscr{X}^{\mathrm{sd}} := \mathscr{X} \cap \mathscr{Y}^{\mathrm{sd}}_{\mathrm{G}\times\mathrm{H}}$, and $\mathscr{X}^{\mathrm{cl,sd}} := \mathscr{X}^{\mathrm{cl}} \cap \mathscr{X}^{\mathrm{sd}}$.

1.1.5 Main theorem

Throughout this paper, if $\mathscr{X}$ is a scheme over a characteristic-zero field L, we identify a geometric point $x \in \mathscr{X}(\mathbf{C})$ with a pair (x_0, ι), where $x_0 \in \mathscr{X}$ is the scheme point image of (as a synonym, *underlying*) x and $\iota\colon L(x_0) \hookrightarrow \mathbf{C}$ is an embedding. If $\mathscr{X}$ is integral, we denote by $\mathscr{K}(\mathscr{X})$ the local ring of the generic point, which we call the field of meromorphic functions on $\mathscr{X}$.

If $\mathscr{X}$ is a Hida family for $\mathrm{G} \times \mathrm{H}$, we define $\mathscr{X}^{\mathrm{cl,wt}} \subset \mathscr{X}^{\mathrm{cl}}$ to be the subset of points (x_0, y_0) whose contracted weight (k_0, w, l) satisfies

$$|l_v| \le w_v - 2, \qquad |k_0| \le w_v - |l_v| - 2 \qquad \text{for all } v \in \Sigma_p. \tag{1.1.7}$$

We denote $\mathscr{X}^{\mathrm{cl,sd,wt}} := \mathscr{X}^{\mathrm{cl,sd}} \cap \mathscr{X}^{\mathrm{cl,wt}}$.

Theorem A *Let $\mathscr{X}$ be a Hida family for $\mathrm{G} \times \mathrm{H}$ whose self-dual locus $\mathscr{X}^{\mathrm{sd}}$ is nonempty. There exists a unique meromorphic function*

$$\mathscr{L}_p(\mathscr{V}) \in \mathscr{K}(\mathscr{X}),$$

whose polar locus $\mathscr{D}$ does not intersect $\mathscr{X}^{\mathrm{cl,sd,wt}}$ such that for each $(x, y) \in \mathscr{X}^{\mathrm{cl,wt}}(\mathbf{C}) - \mathscr{D}(\mathbf{C})$, we have

$$\mathscr{L}_p(\mathscr{V})(x, y) = e_{p\infty}(V_{(\pi_x, \chi_y)}) \cdot \mathscr{L}(V_{(\pi_x, \chi_y)}, 0). \tag{1.1.8}$$

Here, if $(x_0, y_0) \in \mathscr{X}^{\mathrm{cl}}$ is the point underlying (x, y) and $\iota\colon \mathbf{Q}_p(x_0, y_0) \hookrightarrow \mathbf{C}$ is the corresponding embedding, we have denoted $\pi_x = \pi^{\iota}_{x_0}$, $\chi_y = \chi^{\iota}_{y_0}$, and the interpolation factor is as in (1.1.5).

The value of the interpolation factor agrees with the general conjectures of Coates and Perrin-Riou (see [Coa91]). (The notation $\mathscr{V}$ is meant to evoke some "universal virtual Galois representation interpolating (1.1.3).")

1.1.6 Previous related work

When E/F splits above p, Theorem A may be essentially deduced from the main result of [Hid91] (see also [Hid09]). Hida's method uses the Rankin–Selberg integral, whereas ours uses Waldspurger's variant [Wal85] based on the Weil representation (as discussed below).

The numerator of our L-value is a special case of the standard L-function for $\mathrm{GL}_2 \times \mathrm{GL}_1$ over E, and when so considered, our p-adic L-function is a multiple of the restriction to some base-change locus of one constructed by Januszewski [Jan] using the method of modular symbols; however, that function is not uniquely characterized by its interpolation property, which involves unspecified periods.

Finally, when $F = \mathbf{Q}$, variants of $\mathscr{L}_p(\mathscr{V})$ were constructed by Hida [Hid88, Theorem 5.1a] and, more recently, by Loeffler and Büyükboduk and Lei (see [BL, Section B.4]) under various local restrictions.

1.2 Idea of proof, organization of the paper, and discussion of the method

The proof combines the strategy of Hida [Hid91] with an enhanced version of that of [Dis17, Proof of Theorem A], where we had constructed the "slices" $\mathscr{L}_p(\mathscr{V})(x, -)$ for $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$ of weight 2.

We start from Waldspurger's [Wal85] integral representation of Rankin–Selberg type

$$(f, I(\phi, \chi)) = \mathscr{L}(V_{(\pi,\chi)}, 0) \cdot \prod_v R_v(W_v, \phi_v, \chi_v), \tag{1.2.1}$$

where $(\ ,\ )$ is a normalized Petersson product, f is a form in π with Whittaker function $\otimes_v W_v$, the form $I(\phi, \chi)$ is a mixed theta-Eisenstein series depending on a certain Schwartz function ϕ, and the R_v are normalized local integrals.

In Section 2, we discuss the general setup. In Section 3, we make a judicious choice of ϕ_v at the places $v|p\infty$ and interpolate the ordinary projection of $I(\phi, \chi)$ into a $\mathscr{Y}_{\mathrm{G}\times\mathrm{H}}$-adic modular form. In Section 4, we interpolate R_v for $v \nmid p\infty$ using sheaves of local Whittaker functions over $\mathscr{X}$ provided by the local Langlands correspondence in families (Section 4.4); we compute R_v for $v|p\infty$ (Section 4.3), which yield the interpolation factors in (1.1.8); and finally (Section 4.5), we use (1.2.1) to define $\mathscr{L}_p(\mathscr{V})$ as a glued quotient of the global and local (away from $p\infty$) families of zeta integrals.

In Appendix A, we give a TV-inspired bijective proof of a combinatorial lemma occurring in Section 3.3.

The method of constructing p-adic L-functions as ratios of arbitrary matching families of global and local zeta integrals should be applicable whenever an integral representation for the corresponding complex L-function is available, at least if the groups involved are products of general linear groups: for example, for Rankin–Selberg L-functions. It can be compared to the "hard" constructions from much of the existing literature on p-adic L-functions, which rely on fine choices of local data at all places, computation of the associated integrals, and bounds on the ramification of the data (see [Hsi21] for an excellent example of the state of the art). To be sure, the two approaches should be viewed as complementary rather than alternative: while the "soft" construction provides a flexibility useful for some applications (such as in [Dis17]), explicit choices and computations can still be plugged into it, and are likely still indispensable to address finer issues such as integrality.

For another brief general discussion of our method focused on the role of the local Langlands correspondence in families (LLCF), as well as some results on local interpolation, we refer to [Dis20, Sections 1.2 and 5];[7] see also the very recent work of Cai and Fan [CF] for a related study in the context of periods attached to spherical varieties. Abelian antecedents of the construction, for which the LLCF is not needed, can be found in [Dis17, LZZ18].

The local–global approach may in principle introduce poles coming from zeros of the families of local integrals. In our specific setup, the Waldspurger local integrals are not easy to control (at least for this author) away from the self-dual locus. This is why Theorem A, while sufficient for the arithmetic applications in [Dis/b], is not as strong as it could be: one may at least expect that the condition that $\mathscr{X}^{\mathrm{sd}}$ be nonempty is superfluous, and that the polar locus of $\mathscr{L}_p(\mathscr{V})$ should not intersect $\mathscr{X}^{\mathrm{cl,wt}}$. As noted by a referee, approaching $\mathscr{L}_p(\mathscr{V})$ via the well-understood Rankin–Selberg integrals for $\mathrm{GL}_2 \times \mathrm{GL}_2$ would likely yield such a strengthening.

2 *p*-adic modular forms and Hida families

The material of this section is largely due to Hida (see [Hid91, Sections 1–3 and 7] and the references therein).

[7] In Appendix B, we correct a couple of blundered statements from [Dis20].

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2.1 Notation and preliminaries

The notation introduced in the present subsection (or in the introduction) will be used throughout the paper unless otherwise noted, in particular, the groups G and H defined in (1.1.1).

2.1.1 General notation

The following notational choices are largely standard.

- The fields F and E are as fixed in the introduction unless specified otherwise; if $*$ denotes a place of $\mathbf{Q}$ or a finite set thereof, we denote by S_* the set of places of F above $*$.
- We denote by D_F, D_E, and $D_{E/F}$, respectively, the absolute discriminants of F and E and relative discriminant of E/F; for a finite place v of F, we denote by $d_v \in F_v$ a generator of the different ideal of F and by $D_v \in F_v$ a generator of the relative discriminant ideal.
- We denote by $<$ the partial order on F given by $x < y$ if and only if $\tau(x) < \tau(y)$ for all $\tau \in \Sigma_\infty$; we denote $\mathbf{R}^+ := \{x \in \mathbf{R} \,|\, x > 0\}$ and $F^+ := \{x \in F \,|\, x > 0\} \subset F^\times$.
- $\mathbf{A}$ is the ring of adèles of F; if S is a finite set of places of a number field F, we denote $\mathbf{A}^S = \prod'_{v \notin S} F_v$, and $F_S := \prod_{v \in S} F_v$; when S consists of the set of places of F above some finite set of places of $\mathbf{Q}$ (for instance, the place p), we use the same notation with those places of $\mathbf{Q}$ instead of S (for instance, $F_p = F_{S_p}$). We denote $F_\infty^+ = \{x \in F_\infty \mid x_\tau > 0 \text{ for all } \tau \in \Sigma_\infty\}$ and $\mathbf{A}^+ := \mathbf{A}^\infty \times F_\infty^+$.
- We denote by $\psi\colon F\backslash\mathbf{A} \to \mathbf{C}^\times$ the standard additive character as in Section 1.1.3.
- If R/R_0 is a ring extension, A is an R_0-algebra, and X is an R_0-scheme, we denote

$$A_R := A \otimes_{R_0} R, \qquad X_R := X \times_{\operatorname{Spec} R_0} R.$$

- We denote by G_K the absolute Galois group of a field K.
- If K is a finite extension of F, its class number is denoted by

$$h_K := |K^\times \backslash \mathbf{A}_K^{\infty,\times} / \widehat{\mathscr{O}}_K^\times|.$$

- For a place v of F, we denote by ϖ_v a fixed uniformizer at v, and by $q_{F,v}$ the cardinality of the residue field; we denote $q_{F,p} := (q_{F,v})_{v \in S_p}$.
- The class field theory isomorphism is normalized by sending uniformizers to geometric Frobenii; for K a number field (respectively, a local field), we will then identify characters of G_K with characters of $K^\times\backslash\mathbf{A}_K^\times$ (respectively, $K^\times$) without further comment.
- If I is a finite index set and $x = (x_i)_i$, $y = (y_i)$ are real vectors, we define $(xy)_i = x_i y_i$ and $x^y := \prod_i x_i^{y_i}$ whenever that makes sense. Moreover, we often identify an integer w_0 with the constant vector $(w_0)_{i \in I} \in \mathbf{Z}^I$.
- We denote by $\mathbf{1}[\cdot]$ the $\{0,1\}$-valued function on logical propositions such that $\mathbf{1}[\phi] = 1$ if and only if ϕ is true.

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2.1.2 Subgroups of GL_2 and special elements

We denote by Z, A, and N, respectively the center, diagonal torus, and upper unipotent subgroup of $\mathrm{G} = \mathrm{GL}_{2/F}$; we let $P = AN$ and $P^1 := P \cap \mathrm{SL}_{2/F}$. We define a map $\mathbf{a}\colon \mathrm{GL}_{1/F} \to \mathrm{G}$ by

$$\mathbf{a}(y) := \begin{pmatrix} y & \\ & 1 \end{pmatrix}.$$

We denote

$$w := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \in \mathrm{GL}_2(F)$$

or its image in $\mathrm{GL}_2(R)$ for any F-algebra R. (The context will prevent any confusion with the notation for the weights of G.) For $r \in \mathbf{Z}_{\geq 1}^{S_p}$, we define

$$w_{r_v,v} := \begin{pmatrix} & 1 \\ -\varpi_v^{r_v} & \end{pmatrix} \in \mathrm{GL}_2(F_v), \qquad w_{r,p} := \prod_{v|p} w_{r_v,v} \in \mathrm{GL}_2(F_p),$$

as well as a sequence of compact subgroups

$$U_{v,r_v} := U_1^1(\varpi_v^{r_v}) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathscr{O}_{F,v}) \ : \ a-1 \equiv d-1 \equiv c \equiv 0 (\mathrm{mod}\ \varpi_v^{r_v})\} \subset \mathrm{GL}_2(F_v),$$

$$U_{p,r} := \prod_{v \in S_p} U_{v,r_v}.$$

For $\theta \in (\mathbf{R}/2\pi\mathbf{Z})^{S_\infty}$, we denote $r_\theta := \left(\begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix}\right)_v \in \mathrm{SO}(2, F_\infty)$.

2.1.3 Hecke algebras

Let S be a finite set of non-archimedean places of F, and let $U^S = \prod_{v \notin S} U_v \subset \mathrm{GL}_2(\mathbf{A}^{S\infty})$ be an open compact subgroup. For each finite set of finite places S, we define the Hecke algebra

$$\mathscr{H}_{U^S} := C_c(U^S \backslash \mathrm{G}(\mathbf{A}^{S\infty})/U^S, \mathbf{Z}).$$

It carries an involution

$$T \mapsto T^\vee \tag{2.1.1}$$

arising from the map $g \mapsto g^{-1}$ on the group G.

Let $A_p := A(F_p) \subset \mathrm{G}(F_p)$ be the diagonal torus, and let A_p^+ be the set of $t = \left(\begin{smallmatrix} t_1 & \\ & t_2 \end{smallmatrix}\right)$ such that $v(t_1) \geq v(t_2)$ for all $v|p$. The involution

$$t \mapsto t^\vee := \det(t)^{-1} t \tag{2.1.2}$$

preserves A_p^+. For S a finite set of places of F disjoint from $S_p \cup S_\infty$, we define the ordinary Hecke algebra

$$\mathscr{H}_{U^{Sp}}^{\mathrm{ord}} := \mathscr{H}_{U^{Sp},\mathbf{Z}_p} \otimes \mathbf{Z}_p[A_p]$$

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over $\mathbf{Z}_p$, which will act on spaces of ordinary modular forms (here and in the rest of the text, a subscript Sp is shorthand for $S \cup S_p$). It is endowed with the involution $\vee$ deduced from (2.1.1) and (2.1.2).

If $U = \prod_v U_v \subset \mathrm{G}(\mathbf{A}^\infty)$ (respectively, $U^p = \prod_{v \nmid p} U_v \subset \mathrm{G}(\mathbf{A}^{p\infty})$) are open compact subgroups, and S (respectively, S^p) is the set of places such that U_v is not maximal, we define

$$\mathscr{H}_U^{\mathrm{sph}} := \mathscr{H}_{U^S}, \qquad \mathscr{H}_{U^p}^{\mathrm{sph,\, ord}} := \mathscr{H}_{U^{Sp}}^{\mathrm{ord}}.$$

(These depend on S, but their images in endomorphisms rings of spaces of modular forms do not.)

2.1.4 Measures

We use the same notation and conventions for Haar measures and integration as in [YZZ12, Section 1.6] and [Dis17, Section 1.9]. In particular, we have a regularized integration functional

$$\int^*_{E^\times\backslash \mathbf{A}_E^\times/\mathbf{A}^\times} f(t)\, dt,$$

which satisfies the following.

Lemma 2.1 *Let f be a smooth function on $\mathbf{A}_E^\times$ that is invariant under $E_\infty^\times$. Let $\mu \subset \mathscr{O}_E^\times$ be a finite index subgroup fixing f (under the scaling action). Then*

$$\int^*_{E^\times\backslash \mathbf{A}_E^\times/\mathbf{A}^\times} \sum_{x\in E^\times} f(xt)\, dt = \frac{2L(1,\eta)}{h_E}[\mathscr{O}_E^\times : \mu] \int_{\mathbf{A}_E^{\infty,\times}} \sum_{\alpha\in\mu} f(\alpha t)\, d^\bullet t,$$

where $d^\bullet t$ is the Haar measure giving volume 1 to $\widehat{\mathscr{O}}_E^\times$.

Proof Let $U \subset \mathbf{A}_E^{\infty,\times}$ be any compact open subgroup fixing f. Since both sides are independent of μ, we may assume that $\mu = \mathscr{O}_E^\times \cap U$. By [YZZ12, equation (1.6.1) and the following paragraphs], we have

$$\begin{aligned}\int^*_{E^\times\backslash \mathbf{A}_E^\times/\mathbf{A}^\times} f(t)\, dt &= \mathrm{vol}(E^\times\backslash \mathbf{A}_E^\times/\mathbf{A}^\times) \fint_{E^\times\backslash \mathbf{A}_E^\times/\mathbf{A}^\times} f(t)\, dt \\ &= \frac{\mathrm{vol}(E^\times\backslash \mathbf{A}_E^\times/\mathbf{A}^\times)}{|E^\times\backslash \mathbf{A}_E^{\infty,\times}/U|} \sum_{t\in E^\times\backslash \mathbf{A}_E^{\infty,\times}/U} f(t).\end{aligned} \tag{2.1.3}$$

Now, by a coset identity,

$$|E^\times\backslash \mathbf{A}_E^{\infty,\times}/U| = h_E \frac{|\widehat{\mathscr{O}}_E^\times/U|}{[\mathscr{O}_E^\times : \mu]},$$

and by [YZZ12, Section 1.6.3], $\mathrm{vol}(E^\times\backslash \mathbf{A}_E^\times/\mathbf{A}^\times) = 2L(1,\eta)$. Hence, (2.1.3) equals

$$\frac{2L(1,\eta)}{h_E} \frac{[\mathscr{O}_E^\times : \mu]}{|\widehat{\mathscr{O}}_E^\times/U|} \sum_{t\in E^\times\backslash \mathbf{A}_E^{\infty,\times}/U} f(t).$$

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If we compose with the operator $f(\cdot) \mapsto \sum_{x\in E^\times} f(x\cdot) = \sum_{x\in\mu\backslash E^\times} \sum_{\alpha\in\mu} f(\alpha x\cdot)$, we obtain

$$\frac{2L(1,\eta)}{h_E}\frac{[\mathscr{O}_E^\times:\mu]}{|\widehat{\mathscr{O}}_E^\times/U|}\sum_{t\in \mathbf{A}_E^{\infty,\times}/U}\sum_{\alpha\in\mu} f(\alpha t) = \frac{2L(1,\eta)}{h_E}[\mathscr{O}_E^\times:\mu]\int_{\mathbf{A}_E^{\infty,\times}}\sum_{\alpha\in\mu} f(\alpha t)\, d^\bullet t.$$

■

2.2 Modular forms and their q-expansions

Let $\mathfrak{h} \subset \mathbf{C}$ be the upper half-plane. We view $\mathrm{G}(F_\infty)$ as acting on $\mathfrak{h}^{\Sigma_\infty}$ by Möbius transformations, and identify

$$C_\infty^+ := (\mathbf{R}_+\mathrm{SO}(2,\mathbf{R}))^{\Sigma_\infty} \subset \mathrm{G}(F_\infty)$$

with the neutral connected component of the stabilizer of $\mathrm{i} := (\sqrt{-1},\ldots,\sqrt{-1}) \in \mathfrak{h}^{\Sigma_\infty}$.

2.2.1 Nearly holomorphic modular forms

Let $\underline{w}$ be an (∞-adic Section 1.1.1) weight for G, let $U \subset \mathrm{G}(\mathbf{A}^\infty)$ be a compact open subgroup, and let $m = (m_\tau) \in \mathbf{Z}_{\geq 0}^{\Sigma_\infty}$. A complex *nearly holomorphic (Hilbert) modular form* of weight $\underline{w}$, level U, and degree $\leq m$ is a function

$$f\colon \mathrm{G}(\mathbf{A}) \to \mathbf{C}$$

satisfying the following two conditions:

(1) For all $g \in \mathrm{G}(\mathbf{A})$, $\gamma \in \mathrm{G}(F)$, and $k \in UC_\infty^+$,

$$f(\gamma g k) = j_{\underline{w}}(k_\infty, \mathrm{i})^{-1} f(g),$$

where for $z \in \mathfrak{h}^{\Sigma_\infty}$,

$$j_{\underline{w}}\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right), z\right) := (ad-bc)^{(w_0-w)/2}(cz+d)^w.$$

(2) There is a Whittaker–Fourier expansion

$$f\left(\begin{pmatrix} y & x\\ & 1\end{pmatrix}\right) = |y| \sum_{a\in F} W_{f,a}^{\mathbf{C}}(y)(Y)\,\mathbf{q}^a \tag{2.2.1}$$

for all $y \in \mathbf{A}^+$, $x \in \mathbf{A}$, where:

- we have

  $$W_{f,0}^{\mathbf{C}}(y) = y_\infty^{(w_0+w-2)/2}\mathrm{W}_{f,0}(y), \quad W_{f,a}^{\mathbf{C}}(y) = (ay_\infty)^{(w_0+w-2)/2}\mathrm{W}_{f,a}(y) \quad (a \neq 0)$$

  for polynomials

  $$\mathrm{W}_{f,a}(y) \in \mathbf{C}[(T_\tau)_{\tau\in\Sigma_\infty}]$$

  of degree $\leq m_\tau$ in the variables T_τ, evaluated at $Y := (Y_\tau)_{\tau\in\Sigma_\infty}$ with $Y_\tau = (4\pi y_\tau)^{-1}$;
- we denote

  $$\mathbf{q}^a := \psi(ax)\psi_\infty(\mathrm{i}ay_\infty).$$

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The polynomial $\mathrm{W}_{f,a}(y)$ only depends on the class of ay modulo $\mathbf{a}^{-1}(U)$, so that defining

$$\mathrm{W}_f(a) := W_{f,1}(a) \tag{2.2.2}$$

for $a \in \mathbf{A}^+$, for all $a \in F^+$, and for $y \in \mathbf{A}^+$, we have $W_{f,a}(y) = W_f(ay)$. We say that f is *cuspidal* if $W_0(y) = 0$ for all y.

If f is nearly holomorphic of degree 0 (that is, $\leq (0,\dots,0)$), we simply say that f is a (holomorphic) modular form. If $R \subset \mathbf{C}$ is any subring, we denote by

$$S_{\underline{w}}(U,R) \subset M_{\underline{w}}(U,R) \subset N_{\underline{w}}^{\leq m}(U,R),$$

respectively, the spaces of cuspidal forms and holomorphic forms of level U and weight $\underline{w}$, and of nearly holomorphic forms of level U, weight $\underline{w}$, and degree $\leq m = (m_\tau)$, such that for all $a \in \mathbf{A}^+$, the polynomials $W_f(a)$ have coefficients in R. We write $N_{\underline{w}}(U,R) := \varinjlim_m N_{\underline{w}}^{\leq m}(U,R)$, and $\square(R) := \varinjlim_U \square(U,R)$ if $\square$ stands for the notation for any of the spaces of forms defined above (or below).

Finally, we define the space $S_{\underline{w}}^{\mathrm{a}}(U,\mathbf{C})$ of *antiholomorphic cuspforms* to be the $\mathbf{C}$-vector space image of $S_{\underline{w}}(U,\mathbf{C})$ under complex conjugation. The formula

$$f \mapsto f^{\mathrm{a}} := \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} f \tag{2.2.3}$$

(where $\left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right) \in \mathrm{G}(F) \subset \mathrm{G}(\mathbf{A})$ acts, as usual, by right translation) defines $\mathbf{C}$-linear bijections from $S_{\underline{w}}(U,\mathbf{C})$ to $S_{\underline{w}}^{\mathrm{a}}(U,\mathbf{C})$ and vice versa.

2.2.2 Twisted modular forms

A *twisted nearly holomorphic (Hilbert) modular form* of weight $\underline{w}$, level U, and degree $\leq m = (m_\tau)$ is a function

$$f\colon \mathrm{G}(\mathbf{A}) \times \mathbf{A}^\times \to \mathbf{C}$$

satisfying the following two conditions:

(1) For all $g \in \mathrm{G}(\mathbf{A})$, $\gamma \in \mathrm{G}(F)$, and $k \in UC_\infty^+$,

$$f(\gamma g k, \det(\gamma)^{-1} u) = j_{\underline{w}}(k_\infty, \mathrm{i})^{-1} f(g,u).$$

(2) There is a Whittaker–Fourier expansion

$$f\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}, u\right) = |y| \sum_{a \in F} W_{f,a}^{\mathbf{C}}(y,u)(Y)\, \mathbf{q}^a \tag{2.2.4}$$

for all $x \in \mathbf{A}$ and $y, u \in \mathbf{A}^\times$ such that $(uy)_\infty > 0$, where:

$$W_{f,0}^{\mathbf{C}}(y,u) = y_\infty^{(w_0+w-2)/2} \mathrm{W}_{f,0}(y,u),$$
$$W_{f,a}^{\mathbf{C}}(y,u) = (a y_\infty)^{(w_0+w-2)/2} \mathrm{W}_{f,a}(y,u) \quad (a \neq 0)$$

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for polynomials

$$\mathrm{W}_{f,a}(y,u) \in \mathbf{C}[(T_\tau)_{\tau:F\hookrightarrow\mathbf{R}}]$$

of degree $\leq m_\tau$ in the variables T_τ, evaluated at $Y := (Y_\tau)_{\tau:F\hookrightarrow\mathbf{R}}$ with $Y_\tau = (4\pi y_\tau)^{-1}$.

If $R \subset \mathbf{C}$ is any subring, we denote by $M^{\mathrm{tw}}_{\underline{w}}(U,R) \subset N^{\mathrm{tw},\leq m}_{\underline{w}}(U,R)$ the spaces of holomorphic and nearly holomorphic forms of level U, weight $\underline{w}$, and degree $\leq m = (m_\tau)$, such that all the polynomials $\mathrm{W}_{f,a}(y,u)$ have coefficients in R.

2.2.3 Contracted product

For any open compact subgroup $U_F \subset \widehat{\mathscr{O}}^\times_F$, let

(2.2.5)

$$\mu_{U_F} := F^\times \cap U_F, \qquad \nu_{U_F} := |\{\pm 1\} \cap \mu_{U_F}|, \qquad c_{U_F} = \frac{\nu_{U_F} \cdot 2^{[F:\mathbf{Q}]} h_F}{[\mathscr{O}^\times_F : \mu^2_{U_F}]}.$$

Let $\varphi\colon \mathbf{A}^\times \to \mathbf{C}$ be a Schwartz function, invariant under a subgroup of the form $\mu^2_{U'_F} \subset F^\times$ as above. Then the sum

$$\sum^{\star}_{u\in F^\times} \varphi(u) := c_{U_F} \sum_{u\in\mu^2_{U_F}\backslash F^\times} \varphi(u) \tag{2.2.6}$$

is well defined independently of $U_F \subset U'_F$, and for any such choice, the support of the sum is finite.

If f_1, f_2 are twisted nearly holomorphic forms, we may thus define a (plain) nearly holomorphic form $f_1 \star f_2$ by

$$f_1 \star f_2(g) := \sum^{\star}_{u\in F^\times} f_1(g,u) f_2(g,u). \tag{2.2.7}$$

2.2.4 Differential operators

We attach to a nearly holomorphic (genuine or twisted) form f the function

$$\begin{aligned} f^{\mathfrak{h}}\colon \mathrm{G}(\mathbf{A}^\infty) \times \mathfrak{h}^{\Sigma_\infty} &\to \mathbf{C} \\ (g^\infty, z = g_\infty \mathrm{i}) &\mapsto j_{\underline{w}}(g_\infty, \mathrm{i}) f(g_\infty); \end{aligned}$$

the map $f \mapsto f^{\mathfrak{h}}$ is injective.

The Maass–Shimura differential operators on functions on $\mathfrak{h}^{\Sigma_\infty}$ are defined as follows. For $\tau\colon F \hookrightarrow \mathbf{R}$ and $w \in \mathbf{Z}$, let

$$\delta^{\tau,\mathfrak{h}}_w := \frac{1}{2\pi i}\left(\frac{w}{2iy_\tau} + \frac{\partial}{\partial z_\tau}\right), \qquad d^\tau := \frac{1}{2\pi i}\frac{\partial}{\partial z_\tau},$$

a differential operator on the upper half-plane $\mathfrak{h}$. For $w, k \in \mathbf{Z}^{\Sigma_\infty}_{\geq 0}$, let

$$\delta^{k,\mathfrak{h}}_w := \prod_\tau \delta^{\tau,\mathfrak{h}}_{w_\tau+2k_\tau} \circ \cdots \circ \delta^{\tau,\mathfrak{h}}_{w_\tau+2} \circ \delta^{\tau,\mathfrak{h}}_{w_\tau}, \qquad d^k := \prod_\tau (d^\tau)^{k_\tau}.$$

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Then, for any ring $\mathbf{Q} \subset R \subset \mathbf{C}$, this operator defines a map

$$\delta_w^k \colon N_{\underline{w}}^{(\mathrm{tw}),\leq m}(U,R) \to N_{\underline{w}+(0;2k)}^{(\mathrm{tw}),\leq m+k}(U,R)$$

such that $\delta_w^k(f)^{\mathfrak{h}} = \delta_w^{k,\mathfrak{h}}(f^{\mathfrak{h}})$. (For a proof of the intuitive fact that the archimedean operator $\delta_{\underline{w}}^k$ indeed preserves the rationality properties of finite Whittaker–Fourier coefficients, see [Hid91, Proposition 1.2], whose calculations also apply to the twisted case.) The subscript w will be omitted if it is clear from the context.

By [Shi81, equation (1.16)], for all $k \in \mathbf{Z}_{\geq 0}^{\Sigma_\infty}$, we have

$$\delta_w^k = \sum_{0\leq j\leq k} \prod_{\tau\in\Sigma_\infty} \binom{k_\tau}{j_\tau} \frac{\Gamma(w_\tau + k_\tau)}{\Gamma(w_\tau + j_\tau)} (-4\pi y_\tau)^{j_\tau - k_\tau}\, d^j. \tag{2.2.8}$$

If $w \geq 2m+1$, any $f \in N_{\underline{w}}^{(\mathrm{tw}),\leq m}(U,R)$ can be written uniquely as

$$f = \sum_{0\leq r\leq m} \delta_{w-2r}^r f_r$$

with $f_r \in M_{\underline{w}+(0;-2r)}^{(\mathrm{tw})}(U,R)$. (The proof in [Shi76, Lemma 7] carries over to our context.) Thus, the linear map

$$\begin{aligned} e^{\mathrm{hol}} \colon N_{\underline{w}}^{(\mathrm{tw}),\leq \mathrm{m}}(U,R) &\to M_{\underline{w}}^{(\mathrm{tw})}(U,R) \\ f &\mapsto f_0 \end{aligned} \tag{2.2.9}$$

is well defined.

2.3 *p*-adic modular forms

We study the completions of spaces of modular forms for certain p-adic norms.

2.3.1 Arithmetic *q*-expansion

Let $\underline{w}$ be a weight for G, and let $U \subset \mathrm{G}(\mathbf{A}^\infty)$ be a compact open subgroup. The *q-expansion* map

$$f \mapsto (a \mapsto \mathrm{W}_f(a) = (2.2.2))$$

sends $S_{\underline{w}}(U,\mathbf{C})$ to $\mathbf{C}^{\mathbf{A}^+/U_F F_\infty^+}$, where $U_F = \mathbf{a}^{-1}(U)$. By the q-expansion principle (see [Dis17, Proposition 2.1.1] for a version in our setting), the map is injective. We denote its image by $\mathrm{S}_{\underline{w}}(U,\mathbf{C})$ and view the map $S_{\underline{w}}(U,\mathbf{C}) \to \mathrm{S}_{\underline{w}}(U,\mathbf{C})$ as an identification.

If R is any ring admitting embeddings into $\mathbf{C}$, we denote by

$$\mathrm{S}_\bullet(U,R) \subset R^{\mathbf{A}^+/U_F F_\infty^+}$$

the set of those sequences

$$\mathrm{f} = (\mathrm{W}_{\mathrm{f}}(a))_a \tag{2.3.1}$$

such that for any $\iota\colon R \hookrightarrow \mathbf{C}$, the sequence $\mathrm{f}^\iota := (\iota \mathrm{W}_\mathrm{f}(a))_a$ is the q-expansion of a cuspform

$$f^\iota \in S_\bullet(U, \mathbf{C}) = \bigoplus_{\underline{w}} S_{\underline{w}}(U, \mathbf{C}).$$

(In (2.3.1), the notation W_f can be thought of as simply synonymous to f; it is introduced in order to match the identification of the previous paragraph.) By [Hid91, Theorem 2.2(i)] (together with a consideration of Galois actions mixing the weights), for any such ring R, we have $\mathrm{S}_\bullet(U, R) = \mathrm{S}_\bullet(U, \mathbf{Z}) \otimes R$. For more general rings, the previous equality is taken to be the definition of $\mathrm{S}_\bullet(U, R)$.

2.3.2 *p*-adic modular forms

Let L be a finite extension of $\mathbf{Q}_p$ splitting F. A *p-adic L-valued (cohomological) weight* $\underline{w} = (w_0, (w_\tau)_{\tau: F \hookrightarrow L})$ is a tuple of integers, all having the same parity, such that $w_\tau \geq 1$ for all $\tau\colon F \hookrightarrow L$. As in Section 1.1.1, if $\underline{w}$ is an L-valued weight and $\iota\colon L \hookrightarrow \mathbf{C}$ is an embedding, we define the complex weight $\underline{w}^\iota = (w_0, (w_\tau)_{\iota\circ\tau})$.

Let $U \subset \mathrm{G}(\mathbf{A}^\infty)$ be a compact open subgroup, and let $\underline{w}$ be an L-valued weight. We define $\mathrm{S}_{\underline{w}}(U, L)$ to be the set of q-expansions f such that for every $\iota\colon L \hookrightarrow \mathbf{C}$, the expansion f^ι belongs to $\mathrm{S}_{\underline{w}^\iota}(U, \mathbf{C})$. The *p-adic q-expansion* of $\mathrm{f} = (\mathrm{W}_\mathrm{f}(a))_a \in \mathrm{S}_{\underline{w}}(U, L)$ is the sequence

$$f = (W_f(a)) := (W_\mathrm{f}(a)), \qquad W_\mathrm{f}(a) := a_p^{(w_0+w-2)/2}\mathrm{W}_\mathrm{f}(a),$$

so that

$$W^{\mathbf{C}}_{f^\iota, a}(y) := (ay)_\infty^{(w_0+w^\iota-2)/2}\, \iota\left((ay_p)^{(-w_0-w+2)/2} W_f(a)\right) \tag{2.3.2}$$

is the Whittaker–Fourier coefficient of f^ι as in (2.2.1). (In other words, we have two embeddings $\mathrm{S}_{\underline{w}}(U, L) \hookrightarrow L^{\mathbf{A}^+/U_F F_\infty^+}$: the q-expansion $\mathrm{f} \mapsto (\mathrm{W}_\mathrm{f}(a))_a$, and the p-adic q-expansion $\mathrm{f} \mapsto (W_\mathrm{f}(a))_a$.)

Let $U^p \subset \mathrm{G}(\mathbf{A}^{p\infty})$ be a compact open subgroups, let $U_F^p := \mathbf{a}^{-1}(U^p)$, and for any L-valued weight $\underline{w}$, we denote

$$\mathrm{S}_{\underline{w}}(U^p, L) := \varinjlim_n \mathrm{S}_{\underline{w}}(U^p U_{p,n}, L).$$

The space of cuspidal *p-adic modular forms*

$$\mathbf{S}(U^p, L) \subset L^{\mathbf{A}^{\infty,\times}/U_F^p} \subset L^{\mathbf{A}^{\infty,\times}} = L^{\mathbf{A}^+/F_\infty^+}$$

is the completion of $\mathrm{S}_{\underline{w}}(U^p, L)$ for the norm $||\mathrm{f}|| := \sup_a |W_\mathrm{f}(a)|$, for any $\underline{w}$. By a fundamental result of Hida (see [Hid91, paragraph after Theorem 3.1]), the space $\mathbf{S}(U^p, L)$ is independent of the choice of $\underline{w}$. In particular, if L is Galois over $\mathbf{Q}_p$, this space is stable by the action of $\mathrm{Gal}(L/\mathbf{Q}_p)$ and so it is of the form $\mathbf{S}(U^p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} L$ for a space $\mathbf{S}(U^p, \mathbf{Q}_p)$.

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2.3.3 Nearly holomorphic forms as p-adic modular forms

We may attach a p-adic q-expansion to a nearly holomorphic form with coefficients in a p-adic subfield of $\mathbf{C}$.

Let L be a finite extension of $\mathbf{Q}_p$, and let $\underline{w}$ be a p-adic L-valued weight. We say that

$$f = (W_f(a)) \in L^{\mathbf{A}^+/F^+_\infty}$$

is a *p-adic nearly holomorphic cuspform* of weight $\underline{w}$ and level $U^p \subset \mathrm{G}(\mathbf{A}^{p\infty})$ if the following condition holds. For each $\iota\colon L \hookrightarrow \mathbf{C}$, there exists a cuspidal nearly holomorphic form

$$f^\iota \in N^{\leq \lfloor (w+1)/2 \rfloor}_{\underline{w}^\iota}(U^p U_{p,n}, \mathbf{C})$$

for some $n \in \mathbf{Z}^{S_p}_{\geq 1}$, whose Whittaker–Fourier polynomials have constant terms satisfying

$$\mathrm{W}_{f^\iota,1}(a)(0) = \iota\left(a_p^{(-w_0-w+2)/2} W_f(a)\right). \tag{2.3.3}$$

The notion of a p-adic *twisted* nearly holomorphic cuspform is defined similarly by the identity $\mathrm{W}_{f^\iota,a}(y,u)(0) = \iota\left((ay)_p^{(-w_0-w+2)/2} W_f(a)(y,u)\right)$.

Proposition 2.2 *If f is a p-adic nearly holomorphic cuspform over L of level U^p, then it belongs to the space $\mathbf{S}(U^p, L)$ of p-adic modular cuspforms of level U^p.*

Proof This is the first assertion of [Hid91, Proposition 7.3]. ∎

2.3.4 Hecke operators and ordinary projection

The space $N_{\underline{w}}(U, \mathbf{C})$ is endowed with the usual action of $\mathscr{H}_U$. By writing down the effect of this action on Whittaker–Fourier coefficients of cuspforms, we may descend it to a bounded action of $\mathscr{H}_{U^p,L}$ on $\mathrm{S}_{\underline{w}}(U^p, L)$, and hence on $\mathbf{S}(U^p, L)$, for any p-adic field L.

For $t \in A_p^+$ or $y \in \prod_{v|p} \mathscr{O}_{F,v} - \{0\}$, and any $n \in \mathbf{Z}^{S_p}_{\geq 1}$, define the double coset operators

$$\begin{aligned} \mathrm{U}_t &:= [U_{p,n}\, t\, U_{p,n}], & \mathrm{U}_t^{\circ,\underline{w}} &:= t_1^{2-w} \det(t)^{(-w_0+w-2)/2}\mathrm{U}_t, \\ \mathrm{U}_y &:= \mathrm{U}_{\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right)}, & \mathrm{U}_y^{\circ,\underline{w}} &:= y^{(-w_0-w+2)/2}\mathrm{U}_y. \end{aligned} \tag{2.3.4}$$

If L is a finite extension of $\mathbf{Q}_p$, then for all $y \in \prod_{v|p} \mathscr{O}_{F,v} - \{0\}$, we also define the operator

$$\begin{aligned} \mathrm{U}_y^\circ\colon \mathbf{S}(U^p, L) &\to \mathbf{S}(U^p, L) \\ W_{\mathrm{U}_y^\circ f}(c) &:= W_f(cy). \end{aligned}$$

This is compatible with the previous definition in the following sense (see [Hid91, equation (2.2b)], where U_y is denoted by $T(y)$): if f is a p-adic nearly holomorphic form of weight $\underline{w}$ over L, then for all $\iota\colon L \hookrightarrow \mathbf{C}$, we have

$$(\mathrm{U}_y^\circ f)^\iota = \mathrm{U}_y^{\circ, \underline{w}^\iota} f^\iota.$$

The superscript $\underline{w}$ will be omitted when understood from the context. The ordinary projector is

$$e^{\mathrm{ord}} := \lim_{n\to\infty} (\mathrm{U}_p^\circ)^{n!} \quad \in \quad \mathrm{End}_L(\mathbf{S}(U^p, L)) \tag{2.3.5}$$

for any tame level U^p and p-adic field L. Its image is denoted by

$$\mathbf{S}^{\mathrm{ord}}(U^p, L) := e^{\mathrm{ord}}\mathbf{S}(U^p, L).$$

The operator e^{ord} preserves $\mathrm{S}_{\underline{w}}(U^p, L)$, and we denote $\mathrm{S}_{\underline{w}}^{\mathrm{ord}}(U^p, L) := e^{\mathrm{ord}}\mathbf{S}(U^p, L)$ and $\mathrm{S}^{\mathrm{ord}}(U^p, L) := \bigoplus_{\underline{w}} \mathrm{S}_{\underline{w}}^{\mathrm{ord}}(U^p, L)$.

If $f^{\mathbf{C}}$ is a complex modular form arising as $f^{\mathbf{C}} = f^\iota$ for a form $f \in \mathrm{S}(L)$ for some finite extension L of $\mathbf{Q}_p$ and some $\iota\colon L \hookrightarrow \mathbf{C}$, we define

$$e^{\mathrm{ord},\iota}(f^{\mathbf{C}}) := (e^{\mathrm{ord}} f)^\iota.$$

2.3.5 Differential operators after ordinary and holomorphic projections

Let L be a finite extension of $\mathbf{Q}_p$, and let f_1, f_2 be p-adic twisted nearly holomorphic forms over L. For any $\iota\colon L \hookrightarrow \mathbf{C}$ and $k \in \mathbf{Z}_{\geq 0}^{\Sigma_\infty}$, we have

$$[e^{\mathrm{ord}}(f_1 \star d^k f_2)]^\iota = e^{\mathrm{ord},\iota}[e^{\mathrm{hol}}(f_1^\iota \star \delta^k f_2^\iota)]; \tag{2.3.6}$$

the proof of [Hid91, Proposition 7.3] carries over to the twisted case.

2.4 Hida families

We gather the fundamental notions concerning Hida families and the associated sheaves of modular forms.

2.4.1 Weight space

Let $U_{F,p}^\circ = \prod_{v|p} U_{F,v}^\circ \subset \mathscr{O}_{F,p}^\times$ be a compact open subgroup (which will be fixed once and for all in Section 2.4.5). Let $U_F^p \subset \mathbf{A}^{p\infty,\times}$ be a compact open subgroup, and consider the topological groups (with the profinite topology)

$$[Z]_{U_F^p} := Z(F)U_F^p \backslash Z(\mathbf{A}^\infty), \qquad [Z]_{U_F^p} \times U_{F,p}^\circ;$$

the latter is isomorphic to $\Delta \times \mathbf{Z}_p^{1+[F:\mathbf{Q}]+\delta_{F,p}}$, where Δ is a finite group and $\delta_{F,p}$ is the p-Leopoldt defect of F. It is embedded into $A(\mathbf{A}^\infty)$ by

$$(z, y_p) \mapsto \begin{pmatrix} zy_p & \\ & z \end{pmatrix}.$$

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The *weight space* (of tame level U_F^p) is

$$\mathfrak{W} = \mathfrak{W}_{U_F^p} := \operatorname{Spec} \mathbf{Z}_p[\![[Z]_{U_F^p} \times U_{F,p}^\circ]\!]_{\mathbf{Q}_p}. \tag{2.4.1}$$

A point $\underline{\kappa} \in \mathfrak{W}$ is identified with the pair of characters

$$\left(\kappa_0 := \underline{\kappa}_{|[Z]_{U_F^p}}, \qquad \kappa = \underline{\kappa}_{|U_{F,p}^\circ} \right). \tag{2.4.2}$$

We have an involution defined as

$$\underline{\kappa}^\vee(t) := \kappa_0(\det t)^{-1}\underline{\kappa}(t).$$

If $\underline{k}$ is a p-adic weight for G, we say that $\underline{\kappa}$ is *classical of weight* $\underline{k}$ if for all $v|p$,

$$\kappa_0^{\rm sm}(z_p) := \kappa_0(z_p) z_p^{-k_0}, \qquad \kappa_v^{\rm sm}(y) := \kappa_v(y) \prod_{\tau|v} \tau(y)^{(-k_0-k_\tau+2)/2}$$

are smooth characters of $F_p^\times$ (respectively, $U_{F,v}^\circ$); in the second equation, $\kappa_v := \kappa_{|U_{F,v}^\circ}$, and the product runs over the $\tau \in \Sigma_p$ inducing the place $v \in S_p$. For a classical weight $\underline{\kappa}$, we define $\kappa^{\rm sm} := \otimes_{v|p} \kappa_v^{\rm sm}$ and

$$\kappa' := \kappa^{\rm sm} \kappa_0^{\rm sm,-1} = \kappa^{\vee,\rm sm}, \tag{2.4.3}$$

a smooth character of $U_{F,p}^\circ$.

We denote by

$$\mathfrak{W}^{\rm cl} \subset \mathfrak{W}$$

the set of points of classical weight, which has the structure of an ind-étale ind-finite scheme over $\mathbf{Q}_p$. If $\underline{\kappa}$ is classical of weight $\underline{k} = (k_0, k)$, then $\underline{\kappa}^\vee$ is classical of weight $\underline{k}^\vee = (-k_0, k)$. We let $\mathfrak{W}^{\rm cl, \geq 2}$ be the set of classical points satisfying $k \geq 2$.

2.4.2 Hida schemes

In light of the examples of the previous and following paragraphs, it will be convenient to introduce a suitable category of spaces.[8] Define the category of *Hida rings* to consist of finite flat $\mathbf{Z}_p[\![X_1, \ldots, X_n]\!]$-algebras A° (for some n) and $\mathbf{Z}_p$-algebra morphisms, and the category of *Hida algebras* to be the image of Hida rings under the functor $\otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Define the category of affine Hida schemes to be dual to the category of Hida algebras. A *Hida scheme* is an open subset of an affine Hida scheme. If A_i° are Hida rings (for $i = 1, 2$) and $\mathscr{X}_i = \operatorname{Spec}(A_i^\circ \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$ (for $i = 1, 2$), we define

$$\mathscr{X}_1 \hat{\times} \mathscr{X}_2 := \operatorname{Spec}(A_1^\circ \hat{\otimes} A_2^\circ)_{\mathbf{Q}_p},$$

where $\hat{\otimes}$ is the completed tensor product.

[8] The treatment proposed here is minimal and somewhat ad hoc, but it will be sufficient for our purposes. We believe that a more systematic treatment of the geometry of Hida theory should be based on the theory of uniformly rigid spaces developed in [Kap12].

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2.4.3 Hida families

Let

$$\mathbf{T}^{\mathrm{sph,\,ord}}_{U^p,\mathbf{Q}_p} \subset \mathbf{T}^{\mathrm{ord}}_{U^p,\mathbf{Q}_p} \subset \mathrm{End}\,(\mathbf{S}^{\mathrm{ord}}(U_p,\mathbf{Q}_p))$$

be the images of the Hecke algebras $\mathscr{H}^{\mathrm{sph,\,ord}}_{U^p,\mathbf{Q}_p}$, $\mathscr{H}^{\mathrm{ord}}_{U^p,\mathbf{Q}_p}$ from Section 2.1.3. We let

$$\mathscr{Y}_{\mathrm{G}} = \mathscr{Y}_{\mathrm{G},U^p} := \mathrm{Spec}\,\mathbf{T}^{\mathrm{sph,\,ord}}_{U^p,\mathbf{Q}_p}$$

be the *ordinary eigenvariety* for G of tame level U^p. (The subscript U^p will be omitted when unimportant or understood from the context.) The space $\mathscr{Y}_{\mathrm{G},U^p}$ is a union of finitely many irreducible components, called *Hida families* of tame level (dividing) U^p. It carries an involution $\vee$ deduced from the one on $\mathscr{H}^{\mathrm{sph,\,ord}}_{U^p,\mathbf{Q}_p}$.

Letting $U^p_F := U^p \cap Z(\mathbf{A}^{p\infty})$, we have a *weight-character* map

$$\underline{\kappa}_{\mathrm{G}}\colon \mathscr{Y}_{\mathrm{G},U^p} \to \mathfrak{W}_{U^p_F}$$

that, when identified with a pair $(\kappa_{\mathrm{G},0},\kappa_{\mathrm{G}})$ of $\mathscr{O}(\mathscr{Y}_{\mathrm{G}})^\times$-valued characters as in (2.4.2), is $\kappa_{\mathrm{G},0}(z)$ = the Hecke operator acting by right translation by z on modular forms, $\kappa_G(y_p) = \mathrm{U}^\circ_{y_p}$. The weight map is finite and flat, and it intertwines the involutions $\vee$.

The set of classical points of $\mathscr{Y}_{\mathrm{G}}$ is

$$\mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}} := \mathscr{Y}_{\mathrm{G}} \times_{\mathfrak{W}} \mathfrak{W}^{\mathrm{cl},\ge 2} \subset \mathscr{Y}_{\mathrm{G}}.$$

If $x_0 \in \mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}}$, we denote by π_{x_0} the automorphic representation of $\mathrm{G}(\mathbf{A})$ over $\mathbf{Q}_p(x_0)$ on which $\mathscr{H}^{\mathrm{sph}}_{U^p}$ acts by the $\mathbf{Q}_p(x_0)$-character corresponding to x_0. If $x \in \mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}}(\mathbf{C})$ corresponds to $(x_0 \in \mathscr{Y}^{\mathrm{cl}}_{\mathrm{G}}, \iota\colon \mathbf{Q}_p(x_0) \hookrightarrow \mathbf{C})$, we denote $\pi_x := \pi^{\iota}_{x_0}$.

2.4.4 Families of ordinary forms

By construction, for each $U^{p\prime} \subset U^p$, the ordinary eigenvariety $\mathscr{Y}_{\mathrm{G},U^p}$ (respectively, the weight space $\mathfrak{W}_{U^p_F}$) carries a (coherent) sheaf

$$\mathscr{S}^{U^{p\prime}}$$

(respectively, $\mathscr{S}^{U^{p\prime}}_{\mathfrak{W}} := \kappa_{\mathrm{G},*}\mathscr{S}^{U^{p\prime}}$), whose modules of global sections are $\mathbf{S}^{\mathrm{ord}}(U^{p\prime},\mathbf{Q}_p)$. We set $\mathscr{S}_{(\mathfrak{W})} := \varinjlim_{U^{p\prime}} \mathscr{S}^{U^p}_{(\mathfrak{W})}$. By Hida's Control Theorem (see [Hid91, Corollary 3.3]), the restriction of $\mathscr{S}^{U^p}_{\mathfrak{W}}$ to $\mathfrak{W}^{\mathrm{cl}}$ is the sheaf attached to $\mathrm{S}^{\mathrm{ord}}(U^p,\mathbf{Q}_p)$.

For each $x \in \mathscr{Y}^{\mathrm{cl}}_{\mathrm{G},U^p}$ of weight $\underline{w}$, there exists a unique (up to isomorphism) ordinary automorphic representation π_x of $\mathrm{G}(A)$ over $L := \mathbf{Q}_p(x)$ of weight $\underline{w}$ such that there is an $\mathscr{H}^{\mathrm{ord}}_{U^p,L}$-isomorphism

$$\mathscr{S}_{|x} \cong \pi^{\mathrm{ord}}_x := e^{\mathrm{ord}}\pi_x := [\lim_n (\mathrm{U}^{\circ,\underline{w}}_p)^{n!}]\,\pi_x;$$

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the isomorphism is unique up to scalars. This defines a bijection between $\mathscr{Y}^{\mathrm{cl}}_{\mathrm{G},U^p}(\overline{\mathbf{Q}}_p)$ and the set of isomorphism classes of ordinary automorphic representation π of $\mathrm{G}(\mathbf{A})$ over $\overline{\mathbf{Q}}_p$ with $\pi^{U^p}\neq 0$.

Lemma 2.3 *Let $U^p\subset \mathrm{G}(\mathbf{A}^{p\infty})$ be a compact open subgroup, and let $U^p_F := U^p\cap Z(\mathbf{A}^{p\infty})$. Let $\mathscr{X}$ be a Hida scheme endowed with a map $\varphi\colon \mathscr{X}\to\mathfrak{W}=\mathfrak{W}_{U^p_F}$. Then we have an $\mathscr{O}_{\mathscr{X}}$-linear injective q-expansion map*

$$\mathscr{S}^{U^p}_{\mathscr{X}} := \mathscr{S}^{U^p}_{\mathfrak{W}}\otimes_{\mathscr{O}_{\mathfrak{W}}}\mathscr{O}_{\mathscr{X}}\longrightarrow \mathscr{O}_{\mathscr{X}}^{\mathbf{A}^{\infty,\times}/U^p_F}\subset \mathscr{O}_{\mathscr{X}}^{\mathbf{A}^{\infty,\times}} \tag{2.4.4}$$

characterized by the property that for every $\kappa\in\mathfrak{W}^{\mathrm{cl}}$, every closed point $z\in\varphi^{-1}(\kappa)$, and every $a\in\mathbf{A}^{\infty,\times}$, we have

$$W_{\mathbf{f}}(a)(z)=W_{\mathbf{f}(z)}(a),$$

where the right-hand side is the p-adic q-expansion coefficient of the classical modular form $\mathbf{f}(z)\in(\varphi^*\mathscr{S}_{\mathfrak{W}})_{|z}\subset \mathrm{S}^{\mathrm{ord}}(U^p,\mathbf{Q}_p(z))$.

Moreover, the image of (2.4.4) equals the space of those sequences $(W(a))_a$ for which there exists a set of closed points $\Sigma\subset\varphi^{-1}(\mathfrak{W}^{\mathrm{cl}})$ that is dense in $\mathscr{X}$ such that for all $z\in\Sigma$, the sequence $(W(a)(z))_a$ is the p-adic q-expansion of a modular form $\mathbf{f}_z\in(\varphi^*\mathscr{S}_{\mathfrak{W}})_{|z}$.

Note that the sheaf $\mathscr{S}^{U^p}$ on $\mathscr{Y}_{\mathrm{G}}$ is identified with $(\mathscr{S}^{U^p}_{\mathfrak{W}}\otimes_{\mathscr{O}_{\mathfrak{W}}}\mathscr{Y}_{\mathrm{G}})^{\mathscr{O}_{\mathscr{Y}_{\mathrm{G}}}}$, the subsheaf of invariants for the diagonal $\mathscr{O}_{\mathfrak{W}}$-linear action of $\mathscr{O}_{\mathscr{Y}_{\mathrm{G}}}$. In particular, we deduce from (2.4.4) a q-expansion map

$$\mathscr{S}^{U^p}\to\mathscr{O}_{\mathscr{Y}_{\mathrm{G}}}^{\mathbf{A}^{\infty,\times}/U^p_F}. \tag{2.4.5}$$

Proof It suffices to construct (2.4.4) for $\mathscr{X}=\mathfrak{W}$ as the general case follows by base change. Let $A^\circ := \mathbf{Z}_p[\![[Z]_{U^p_F}\times U^\circ_{F,p}]\!]$, and let $\mathbf{S}^{\mathrm{ord}}(U^p,\mathbf{Z}_p)$ be the space of ordinary forms with $\mathbf{Z}_p$-coefficients; this is an A°-module and a $\mathbf{Z}_p$-lattice in $\mathscr{S}^{U^p}_{\mathfrak{W}}(\mathfrak{W})=\mathbf{S}^{\mathrm{ord}}(U^p,\mathbf{Q}_p)$. For $\kappa\in\mathfrak{W}^{\mathrm{cl}}$, let $\mathfrak{p}_\kappa\subset A^\circ$ be the corresponding prime ideal. Let $n\in\mathbf{N}$, and let M range among finite subsets of $\mathfrak{W}$; the filtered system of ideals

$$I_{n,M} := (p^n)+\bigcap_{\kappa\in M}\mathfrak{p}_\kappa$$

forms a fundamental system of neighborhoods of $0\in A^\circ$, i.e., $A^\circ=\varprojlim_{n,M}A^\circ/I_{n,M}$. The p-adic q-expansion maps $\mathbf{S}^{\mathrm{ord}}(U^p,\mathbf{Z}_p)\otimes_{A^\circ}A^\circ/\mathfrak{p}_\kappa\to\mathbf{Z}_p(\kappa)^{\mathbf{A}^{\infty,\times}}$ yield a compatible family of maps

$$\mathbf{S}^{\mathrm{ord}}(U^p,\mathbf{Z}_p)\otimes_{A^\circ}A^\circ/I_{n,M}\to(A^\circ/I_{n,M})^{\mathbf{A}^{\infty,\times}}$$

and after taking projective limits, the desired map $\mathbf{S}^{\mathrm{ord}}(U^p,\mathbf{Z}_p)\to(A^\circ)^{\mathbf{A}^{\infty,\times}}$. It is injective by the q-expansion principle and the preservation of injectivity under inverse limits.

We now consider the second statement. It is clear that, for any fixed Σ as in the lemma, the space $\tilde{\mathscr{S}}^{U^p,\Sigma}_{\mathscr{X}}\subset\mathscr{O}_{\mathscr{X}}^{\mathbf{A}^{\infty,\times}}$ described contains the image of (2.4.4); we show the opposite containment. We may assume that $\mathscr{X}=\operatorname{Spec}B^\circ_{\mathbf{Q}_p}$ for a Hida ring B°, and

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consider (2.4.4) as a map

(2.4.6) $\quad \mathbf{S}^{\mathrm{ord}}(U^p, \mathbf{Z}_p) \otimes_{A^\circ} B^\circ \to \tilde{\mathbf{S}}^{\mathrm{ord}}_{B^\circ}(U^p)^{\Sigma} := (B^\circ)^{\mathbf{A}^{\infty,\times}} \cap \tilde{\mathscr{S}}^{U^p}_{\mathscr{X}}(\mathscr{X}).$

For $z \in \Sigma$, let $\mathfrak{p}_z \subset B^\circ$ be the corresponding prime ideal. Let $n \in \mathbf{N}$, and let N range among finite subsets of Σ; then the filtered system of ideals $J_{n,N} := (p^n) + \bigcap_{z \in N} \mathfrak{p}_z$ forms a fundamental system of neighborhoods of $0 \in B^\circ$. By assumption, for each $z \in \Sigma$, the map (2.4.6) is an isomorphism modulo $\mathfrak{p}_z$; hence, it is an isomorphism modulo $J_{n,N}$ for all (n, N), hence an isomorphism. ∎

We call elements of $\mathscr{S}^{U^p}_{\mathscr{X}}$ (respectively, $\mathscr{S}^{U^p}_{\mathscr{X}} \otimes_{\mathscr{O}_{\mathscr{X}}} \mathscr{K}(\mathscr{X})$) *$\mathscr{X}$-adic ordinary modular cuspforms* (respectively, *meromorphic* $\mathscr{X}$-adic ordinary modular cuspforms) of weight $\varphi \colon \mathscr{X} \to \mathfrak{W}$.

2.4.5 Weight-character map for H

Let $U^p_{\mathrm{H}} \subset \mathrm{H}(\mathbf{A}^{p\infty})$ be an open compact subgroup, and let

$$\mathscr{Y}_{\mathrm{H}} = \mathscr{Y}_{\mathrm{H},U^p_{\mathrm{H}}} := \operatorname{Spec} \mathbf{Z}_p[\![\mathrm{H}(F)\backslash \mathrm{H}(\mathbf{A}^{p\infty})/U^p_{\mathrm{H}}]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

as in (1.1.6). A (Hida) *family* for H is a connected component of $\mathscr{Y}_{\mathrm{H}}$.

Fix a sufficiently small open compact subgroup $U^{\circ,\surd}_{F,p} = \prod_{v|p} U^{\circ,\surd}_{F,v} \subset \mathscr{O}^\times_{F,p}$ and an injective group homomorphism

$$j'' \colon (U^{\circ,\surd}_{F,p}) \to \mathscr{O}^{\times,1}_{E,p} := \{t \in \mathscr{O}^\times_{E,p} \mid N_{E_p/F_p}(t) = 1\},$$

and let

$$U^\circ_{F,p} := (U^{\circ,\surd}_{F,p})^2 \subset \mathscr{O}^\times_{F,p},$$

which is now fixed as promised in Section 2.4.1. Let $\sqrt{\ } \colon U^\circ_{F,p} \to U^{\circ,\surd}_{F,p}$ be the (uniquely determined, up to shrinking $U^{\circ,\surd}_{F,p}$) square root, and let $j, j' \colon U^\circ_{F,p} \to \mathscr{O}^\times_{E,p}$ be the maps[9]

(2.4.7) $$j'(a) := j''(\sqrt{a})/\sqrt{a}, \qquad j(a) = j'(a)a.$$

For any open compact $U^p_E \subset \mathbf{A}^{p\infty,\times}_E$ and $U^p_F := U^p_E \cap \mathbf{A}^{p\infty,\times}$, define a map

(2.4.8) $$\begin{aligned} \underline{\kappa}_{\mathrm{H}} \colon \mathscr{Y}_{\mathrm{H},U^p_E} &\to \mathfrak{W}_{U^p_F} \\ y \mapsto \underline{\kappa}_{\mathrm{H}}(y) &= \left(\kappa_0 = \chi_{y|[Z]_{U^p_F}}, \kappa := \chi_y \circ j\right). \end{aligned}$$

The set of classical points is

$$\mathscr{Y}^{\mathrm{cl}}_{\mathrm{H}} := \mathscr{Y}_{\mathrm{H}} \times_{\mathfrak{W}} \mathfrak{W}^{\mathrm{cl}}.$$

Note that if $y \in \mathscr{Y}_{\mathrm{H}}$ is a classical point such that $\underline{\kappa}_{\mathrm{H}}(y)$ has weight (l_0, l), then χ_y has weight (l_0, l) as defined in the introduction.

[9] If $v|p$ splits in E, then for $a \in U^\circ_{F,v}$, we have $j(a) = (a, 1)$ under some isomorphism $E^\times_v \cong F^\times_v \times F^\times_v$.

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2.4.6 Hida families for $\mathrm{G} \times \mathrm{H}$

These are defined as in Section 1.1.4.

2.4.7 Universal automorphic sheaf on a Hida family

Let $\mathscr{X}_{\mathrm{G}}$ be a Hida family for G, and let $\mathscr{X}_{\mathrm{G}}^{\mathrm{cl}} := \mathscr{X}_{\mathrm{G}} \cap \mathscr{Y}_{\mathrm{G}}^{\mathrm{cl}}$. For each sufficiently small U^p, we may view $\mathscr{X} \subset \mathscr{Y}_{\mathrm{G},U^p}$ and we define

$$\Pi^{U^p} = \Pi^{U^p}_{\mathscr{X}_{\mathrm{G}}} := \mathscr{S}^{U^p}_{|\mathscr{X}_{\mathrm{G}}}.$$

For each $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$, by Hida's Control Theorem (see, for instance, [Hid91, Corollary 3.3]) and the theory of newforms, we have an isomorphism of $\mathscr{H}_{U^p}$-modules,

$$\Pi^{U^p}_{|x} \cong \pi_x^{U^p,\mathrm{ord}} := e^{\mathrm{ord}}\pi_x^{U^p}. \tag{2.4.9}$$

Let $U^p_{\mathscr{X}_{\mathrm{G}}}$ be minimal such that $\mathscr{X}_{\mathrm{G}}$ is a component of $\mathscr{Y}_{\mathrm{G},U^p_{\mathscr{X}_{\mathrm{G}}}}$. By [Hid91, Section 3], there is a unique

$$\mathbf{f}_0 = \mathbf{f}_{0,\mathscr{X}_{\mathrm{G}}} \in \Pi^{U^p_{\mathscr{X}_{\mathrm{G}}}}(\mathscr{X}_{\mathrm{G}}) \tag{2.4.10}$$

(the *normalised primitive form* over $\mathscr{X}_{\mathrm{G}}$) such that $W_{\mathbf{f}_0}(1) = 1 \in \mathscr{O}(\mathscr{X}_{\mathrm{G}})$ for the q-expansion map deduced from (2.4.5). Any $\mathbf{f} \in \Pi^{U^p}$ can be written as $\mathbf{f} = T\mathbf{f}_0$ for some Hecke operator T supported at the places $v \nmid p\infty$ such that U^p is not maximal.

2.4.8 Universal Galois sheaf on a Hida family and local–global compatibility

Let $\mathscr{X}_{\mathrm{G}}$ be a Hida family for G. By results of Hida and Wiles (see [Dis/b, Proposition 3.2.4]), there exist an open subset $\mathscr{X}_{\mathrm{G}}' \subset \mathscr{X}_{\mathrm{G}}$ containing $\mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$ and a locally free sheaf $\mathscr{V}_{\mathrm{G}}$ of rank 2, endowed with a Galois action

$$G_F \to \mathrm{End}_{\mathscr{O}_{\mathscr{X}_{\mathrm{G}}'}}(\mathscr{V}_{\mathrm{G}})$$

such that for all $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$, the fiber $\mathscr{V}_{\mathrm{G}|x}$ is the Galois representation attached to π_x by the global Langlands correspondence.

Let S be a finite set of finite places of F, disjoint from S_p, such that for all $v \notin S$, the tame level $U^p = U^{Sp}U_S$ of $\mathscr{X}_{\mathrm{G}}$ is maximal at v. We define

$$\Pi^{U^{Sp}}_{\mathscr{X}_{\mathrm{G}}} := \varinjlim_{U_S'} \Pi^{U^{Sp}U_S'}_{\mathscr{X}_{\mathrm{G}}}, \tag{2.4.11}$$

which is a finitely generated $\mathscr{O}_{\mathscr{X}_{\mathrm{G}}}[\mathrm{G}(F_S)]$-module. On the other hand, [Dis20, Theorem 4.4.1] attaches to the restriction $\mathscr{V}_{\mathrm{G},v} := \mathscr{V}_{\mathrm{G}|G_{F_v}}$ an $\mathscr{O}_{\mathscr{X}_{\mathrm{G}}'}[\mathrm{G}(F_S)]$-module

$$\Pi(\mathscr{V}_{\mathrm{G},v}),$$

which is torsion-free and *co-Whittaker* in the sense of [Dis20, Definition 4.2.2].

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Proposition 2.4 *After possibly replacing $\mathscr{X}'_{\mathrm{G}} \subset \mathscr{X}_{\mathrm{G}}$ with a smaller open subset still containing $\mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$, there exists a line bundle $\Pi^{\circ}_{\mathscr{X}'_{\mathrm{G}}}$ over $\mathscr{X}'_{\mathrm{G}}$ with trivial $\mathrm{G}(F_S)$-action, such that*

$$\Pi^{U^{Sp}}_{|\mathscr{X}'_{\mathrm{G}}} \cong \Pi^{\circ}_{\mathscr{X}'_{\mathrm{G}}} \otimes \bigotimes_{v \in S} \Pi(\mathscr{V}_{\mathrm{G},v})$$

as $\mathscr{O}_{\mathscr{X}'_{\mathrm{G}}}[\mathrm{G}(F_S)]$-modules.

Proof By the local–global compatibility of the Langlands correspondence for Hilbert modular forms (see [Car86] or [Dis/b, Theorem 2.5.1]), for all $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$ and all places v, the $\mathrm{G}(F_v)$-representation $\pi_{x,v}$ corresponds, under local Langlands, to the Weil–Deligne representation $V_{x,v}$ attached to $\mathscr{V}_{\mathrm{G}|x|G_{F_v}}$. Then the result follows from [Dis20, Theorem 4.4.3]. ∎

3 Theta-Eisenstein family

In this section, we define the kernel of the Rankin–Selberg convolution giving the p-adic L-function.

3.1 Weil representation

We recall the definition of the Weil representation for groups of similitudes; this subsection is largely identical to [Dis17, Section 3.1].

3.1.1 Local case

Let $V = (V, q)$ be a quadratic space of even dimension over a local field F of characteristic not 2. Fix a nontrivial additive character ψ of F. For $u \in F^{\times}$, we denote by V_u the quadratic space (V, uq). We let $\mathrm{GL}_2(F) \times \mathrm{GO}(V)$ act on the usual space of Schwartz functions $\mathcal{S}'(V \times F^{\times})$ as follows (here, $\nu\colon \mathrm{GO}(V) \to \mathbf{G}_m$ denotes the similitude character):

- $r(h)\phi(x,u) = \phi(h^{-1}x, \nu(h)u)$ for $h \in \mathrm{GO}(V)$;
- $r(n(b))\phi(x,u) = \psi(buq(x))\phi(x,u)$ for $n(b) \in N(F) \subset \mathrm{GL}_2(F)$;
- $r\left(\begin{pmatrix} a & \\ & d \end{pmatrix}\right)\phi(x,u) = \chi_{V_u}(a)\left|\frac{a}{d}\right|^{\frac{\dim V}{4}}\phi(at, d^{-1}a^{-1}u)$;
- $r(w)\phi(x,u) = \gamma(V_u)\hat{\phi}(x,u)$ for $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Here, $\chi_V = \chi_{(V,q)}$ is the quadratic character attached to V, $\gamma(V,q)$ is a fourth root of unity, and $\hat{\phi}$ denotes Fourier transform in the first variable with respect to the self-dual measure for the character $\psi_u(x) = \psi(ux)$. We will need to note the following facts (see, for instance, [JL70]): χ_V is trivial if V is a quaternion algebra over F or $V = F \oplus F$, and $\chi_V = \eta$ if V is a separable quadratic extension E of F with associated character η.

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3.1.2 Fock model and reduced Fock model

Assume that $F = \mathbf{R}$ and V is positive definite. Then we will prefer to consider a modified version of the previous setting. Let the *Fock model* $\mathcal{S}(V \times \mathbf{R}^\times, \mathbf{C})$ be the space of functions spanned by those of the form

$$H(u)P(x)e^{-2\pi|u|q(x)},$$

where H is a compactly supported smooth function on $\mathbf{R}^\times$ and P is a complex polynomial function on V. This space is not stable under the action of $\mathrm{GL}_2(\mathbf{R})$, but it is so under the restriction of the induced $(\mathfrak{gl}_{2,\mathbf{R}}, \mathbf{O}_2(\mathbf{R}))$-action on the usual Schwartz space (see [YZZ12, Section 2.1.2]).

We will also need to consider the *reduced Fock space* $\overline{\mathcal{S}}(V \times \mathbf{R}^\times)$ spanned by functions of the form

$$\phi(x,u) = (P_1(uq(x)) + \mathrm{sgn}(u)P_2(uq(x)))e^{-2\pi|u|q(x)},$$

where P_1, P_2 are polynomial functions with rational coefficients.

By [YZZ12, Sections 3.4.1 and 4.4.1], there is a surjective quotient map

$$\begin{aligned} &\mathcal{S}(V \times \mathbf{R}^\times, \mathbf{C}) \to \overline{\mathcal{S}}(V \times \mathbf{R}^\times) \otimes_{\mathbf{Q}} \mathbf{C} \\ &\Phi \mapsto \phi(x,u) = \overline{\Phi}(x,u) = \int_{\mathbf{R}^\times} \int_{\mathbf{O}(V)} r(ch)\Phi(x,u)\, dh\, dc. \end{aligned} \tag{3.1.1}$$

We let $\mathcal{S}(V \times \mathbf{R}^\times) \subset \mathcal{S}(V \times \mathbf{R}^\times, \mathbf{C})$ be the preimage of $\overline{\mathcal{S}}(V \times \mathbf{R}^\times)$. For the sake of uniformity, when F is non-archimedean, we set $\overline{\mathcal{S}}(V \times F^\times) = \mathcal{S}(V \times F^\times) := \mathcal{S}'(V \times F^\times)$.

3.1.3 Global case

Let $(\mathbf{V}, q)$ be an even-dimensional quadratic space over the adèles $\mathbf{A}$ of a totally real number field F, and suppose that $\mathbf{V}_\infty$ is positive definite; we say that $\mathbf{V}$ is *coherent* if it has a model over F and *incoherent* otherwise. Given an $\widehat{\mathscr{O}}_F$-lattice $\mathbf{V}^\circ \subset \mathbf{V}$, we define the space $\mathcal{S}(\mathbf{V} \times \mathbf{A}^\times)$ as the restricted tensor product of the corresponding local spaces, with respect to the spherical elements

$$\phi_v(x,u) = \mathbf{1}_{\mathbf{V}_v^\circ}(x)\mathbf{1}_{\varpi_v^{n_v}}(u),$$

if ψ_v has level n_v. We call such ϕ_v the *standard Schwartz function* at a non-archimedean place v. We define similarly the reduced space $\overline{\mathcal{S}}(\mathbf{V} \times \mathbf{A}^\times)$, which admits a quotient map

$$\mathcal{S}(\mathbf{V} \times \mathbf{A}^\times) \to \overline{\mathcal{S}}(\mathbf{V} \times \mathbf{A}^\times) \tag{3.1.2}$$

defined by the product of the maps (3.1.1) at the infinite places and of the identity at the finite places. The Weil representation of $\mathrm{GO}(\mathbf{V}) \times \mathrm{G}(\mathbf{A}^\infty) \times (\mathfrak{gl}_{2,F_\infty}, \mathbf{O}(\mathbf{V}_\infty))$ is the restricted tensor product of the local representations.

For a quadratic space $\mathbf{V} = (\mathbf{V}, q)$ over $\mathbf{A}$, we define $\varepsilon(\mathbf{V}) = +1$ (respectively, -1) if and only if there exists (respectively, does not exist) a quadratic space V over F such that $V \otimes_F \mathbf{A} = \mathbf{V}$.

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3.1.4 The quadratic spaces of interest

Let us go back to our usual notation: thus, F is our chosen totally real field and E its chosen CM quadratic extension. In this paper, we will consider the quadratic spaces $\mathbf{V} = (\mathbf{B}, q)$, where $\mathbf{B}$ is a quaternion algebra over $\mathbf{A}$, definite at all the archimedean places and split at p, and endowed with an $\mathbf{A}$-embedding $\mathbf{A}_E \hookrightarrow \mathbf{B}$, and $q\colon \mathbf{B} = \mathbf{V} \to \mathbf{A}$ is its reduced norm. It has a decomposition

$$\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2,$$

where $\mathbf{V}_1 = \mathbf{A}_E$ (on which the restriction of q coincides with $N_{E/F}$) and $\mathbf{V}_2$ is the q-orthogonal complement. Thus, $\varepsilon(\mathbf{V}) = \varepsilon(\mathbf{V}_2)$. We denote by r_1 the restriction of r to a representation of $\mathbf{A}_E^\times = \mathrm{GO}(\mathbf{V}_1)$ on $\overline{\mathcal{S}}(\mathbf{V}_1 \times \mathbf{A}^\times)$.

For each place v, we have

$$\varepsilon(\mathbf{B}_v) = \varepsilon(\mathbf{V}_v) = \begin{cases} +1, & \text{if } \mathbf{B}_v \cong M_2(F_v), \\ -1, & \text{if } \mathbf{B}_v \text{ is a division algebra.} \end{cases} \tag{3.1.3}$$

We have $\varepsilon(\mathbf{V}) := \prod_v \varepsilon(\mathbf{V}_v) = (-1)^{[F:\mathbf{Q}]} \prod_{v \nmid p} \varepsilon(\mathbf{V}_v)$.

3.2 Theta series

Let $\phi_1 \in \overline{\mathcal{S}}(\mathbf{V}_1 \times \mathbf{A}^\times)$. We define a function on $\mathrm{G}(\mathbf{A}) \times \mathbf{A}^\times$ by

$$\theta(g, u, \phi_1) := \sum_{x \in E} r(g)\phi_1(x, u). \tag{3.2.1}$$

It satisfies

$$\theta(zg, u, \phi_1) = \eta(z)\theta(g, u, r(z^{-1}, 1)\phi_1) \tag{3.2.2}$$

for all $z \in \mathbf{A}^\times$ (here, we view $(z, 1) \in \mathrm{G}(\mathbf{A}) \times \mathbf{A}_E^\times$).

For a complex weight $\underline{l}$ for H, let

$$\begin{aligned} \phi_{1,\underline{l},\infty} &:= \otimes_{v|\infty} \phi_{1,\underline{l},v}, \\ \phi_{1,\underline{l},v}(t, u) &:= \mathbf{1}_{\mathbf{R}^+}(u) \begin{cases} t^{l_v} |u|^{(-l_0+l_v)/2} e^{-2\pi u q(t)}, & \text{if } l_v \geq 0, \\ (\overline{t})^{-l_v} |u|^{(-l_0-l_v)/2} e^{-2\pi u q(t)}, & \text{if } l_v \leq 0, \end{cases} \end{aligned} \tag{3.2.3}$$

and let

$$\theta(g, u, \phi_1^\infty; \underline{l}) := \theta(g, u, \phi_1^\infty \phi_{1,\underline{l},\infty}).$$

Define $|\underline{l}| := (l_0, (|l_v|)_v)$.

Lemma 3.1 *The series $\theta(g, u, \phi_1^\infty; \underline{l})$ is a twisted modular form of weight $(0, \underline{1}) + |\underline{l}|$.*

Proof The usual proof that classical theta series are automorphic shows that our θ is twisted automorphic. The archimedean component of the central character is easy to determine by (3.2.2). The weight is computed as in [Xue07, Section A1 on p. 350]. ∎

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The Whittaker–Fourier expansion of $\theta(\underline{l})$ is standard: for all $g = \left(\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right) \in \mathrm{G}(\mathbf{A})$ with $y \in \mathbf{A}^+$,

$$\begin{aligned}\theta(g, u, \phi_1^\infty; \underline{l}) &= \sum_{a \in F^\times} \sum_{x \in E^\times : uq(x) = a} r(g)\phi_1(x, u) \\ &= \eta(y)|y|y_\infty^{\frac{l_0+|l|}{2}} \sum_{a \in F^\times} \sum_{\substack{x \in E^\times \\ uq(x)=a}} \phi_1^\infty(yx, y^{-1}u)\, \mathbf{q}^a.\end{aligned} \tag{3.2.4}$$

The following expansion result will be used in Section 3.4.

Lemma 3.2 *Let $\chi: E^\times \backslash \mathbf{A}_E^\times \to \mathbf{C}^\times$ be a locally algebraic character of weight $\underline{l}$, and let $E(g, u)$ be any twisted modular form such that $E(g, u_\infty u) = E(g, u)$ for all $u_\infty \in F_\infty^+$. Suppose that $\phi_1^\infty(0, u) = 0$ for all u. Then, for all $g = \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \in \mathrm{G}(\mathbf{A})$, we have*

$$\begin{aligned}&\int^*_{E^\times \backslash \mathbf{A}_E^\times / \mathbf{A}^\times} \chi(t)\theta(g, u, r(t)\phi_1^\infty; \underline{l}) \star E(g, q(t)u)\, dt \\ &= 4|D_{E/F}|^{1/2} \sum_{a \in F^\times} \mathbf{1}_{F_\infty^+}(y_\infty^{-1}a)\eta(y)|y|^{1/2} y_\infty^{\frac{|l|+l_0}{2}} \int_{\mathbf{A}_E^{\infty,\times}} \chi^\infty(t) r_1(t)\phi_1^\infty(y, y^{-1}a) \\ &\quad \times E(g, q(t)a)\, d^\bullet t\, \mathbf{q}^a.\end{aligned}$$

Proof We may assume that U_F is so small that $E(u)$ is invariant under $u \in U_F$ and $v_{U_F} = 1$. Taking fundamental domains for $\mu_{U_F}^2 \backslash F^\times$, the expression of interest is

$$c_{U_F}\eta(y)|y|^{1/2} \int^*_{E^\times \backslash \mathbf{A}_E^\times / \mathbf{A}^\times} \chi(t) \sum_{u \in \mu_{U_F}^2 \backslash F^\times} \sum_{a \in \mu_{U_F}^2 \backslash F^\times} \sum_{x \in E^\times} \phi_1(t^{-1}xy, y^{-1}q(t)u)\mathbf{1}[uq(x) = a]$$
$$E(q(t)u, g)\, dt.$$

Since the integrand is invariant under $E_\infty^\times$, by Lemma 2.1 with $\mu = \mu_{U_F}$ and a change of variables $a = uq(x)$, this equals

$$\frac{2L(1, \eta)c_{U_F}}{h_E[\mathcal{O}_E^\times : \mu_{U_F}]}\eta(y)|y|^{1/2} \int_{\mathbf{A}_E^{\infty,\times}} \sum_{a \in \mu_{U_F}^2 \backslash F^\times} \sum_{\alpha \in \mu_{U_F}}$$
$$\chi^\infty(t\alpha)\phi_1^\infty(t^{-1}\alpha^{-1}y, y^{-1}aq(t)\alpha^2))\mathbf{1}_{F_\infty^+}(y_\infty^{-1}a) y_\infty^{\frac{|l|+l_0}{2}} E(g, q(t\alpha)a)\, \mathbf{q}^{\alpha^2 a}\, d^\bullet t.$$

By the invariance properties under U_F, this can be brought into the desired expression by a change of variables $a' = \alpha^2 a$ and the calculation

$$\frac{2L(1, \eta)c_{U_F}}{h_E[\mathcal{O}_E^\times : \mu_{U_F}]} = \frac{L(1, \eta)[\mathcal{O}_E^\times : \mathcal{O}_F^\times]}{h_E/h_F} = 4|D_{E/F}|^{1/2},$$

which follows from the definition of $c_{U_F} = (2.2.5)$ and the class number formula. ∎

3.3 Eisenstein series

Let $\mathbf{V}_2$ be a two-dimensional quadratic space over $\mathbf{A}$, totally definite at the archimedean places. Let $\phi_2 \in \overline{\mathcal{S}}(\mathbf{V}_2 \times \mathbf{A}^\times)$ be a Schwartz function, and let $\xi: F^\times \backslash \mathbf{A}^\times \to$

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$\mathbf{C}^\times$ be a locally algebraic character such that $\xi_\infty(x) = x^{k_0}$ for some integer k_0 and for all $x \in F_\infty^+$. Define the automorphic Eisenstein series[10]

$$E_r(g, u, \phi_2; \xi) = \frac{L^{(p\infty)}(1, \eta\xi)}{L^{(p\infty)}(1, \eta)} \sum_{\gamma \in P^1(F)\backslash SL_2(F)} \delta_{\xi,r}(\gamma g w_{r,p}) r(\gamma g)\phi_2(0, u),$$

where (with $s \in \mathbf{C}$)

(3.3.1)

$$\delta_{\xi,r}(g) := \delta_{\xi,r,0}(g),$$

$$\delta_{\xi,r,s}(g) := \begin{cases} \xi(d)^{-1}|a/d|^{s/2}\psi(k_0\theta), & \text{if } g = \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) h \text{ with } h = h^\infty r_\theta \in U_{p,r}SO(2, F_\infty), \\ 0, & \text{if } g \notin P(\mathbf{A})U_{p,r}SO(2, F_\infty). \end{cases}$$

(The defining sum is absolutely convergent for $\mathfrak{R}(s)$ sufficiently large, and otherwise it is interpreted by analytic continuation.) It satisfies

$$E_r(zg, u, \phi_2; \xi) = \eta\xi^{-1}(z)E_r(g, u, r(x, 1)\phi_2, \xi).$$

3.3.1 Schwartz function at ∞

Let $P_{k_0,k} \in \mathbf{R}[X]$ be the (rescaled) Laguerre polynomial

$$P_{k_0,k}(X) := (2\pi i)^{-k_0}(4\pi)^{-k}(k + k_0)! \sum_{j=0}^{k} \binom{k}{j} \frac{(-X)^j}{j!}. \tag{3.3.2}$$

For $k = (k_0, (k_v)) \in \mathbf{Z} \times \mathbf{Z}_{\geq 0}^{\Sigma_\infty}$ such that $k_v + k_0 \geq 0$ for all v, define

$$E_r(g, u, \phi_2^\infty; \xi, k) = E_r(g, u, \phi_2^\infty \phi_{2,\infty,k}; \xi),$$

where $\phi_{2,\infty,k} = \otimes_{v|\infty}\phi_{2,v,k_v}$ with

$$\phi_{2,v,k_v}(x, u) = \mathbf{1}_{\mathbf{R}^+}(u)P_{k_0,k_v}(4\pi u q(x))e^{-2\pi u q(x)}. \tag{3.3.3}$$

The series $E_r(\phi_2^\infty; \xi, k)$ belongs to $N^{\leq k}_{\mathrm{tw},(-k_0,k+k_0)}(\mathbf{C})$.

3.3.2 Whittaker–Fourier expansion

The following standard result is essentially [Dis17, Proposition 3.2.1].

Proposition 3.3 *We have*

$$E_r\left(\left(\begin{smallmatrix} y x 1 & \\ u & \end{smallmatrix}\right)\phi_2; \xi\right) = \sum_{a \in F} W_{a,r}\left(\left(\begin{smallmatrix} y 1 & \\ & u \end{smallmatrix}\right), \phi_2; \xi\right)\psi(ax),$$

where

$$W_{a,r}(g, u, \phi_2; \xi) = \prod_v W_{a,r,v}(g, u, \phi_{2,v}; \xi_v)$$

[10] For $k_0 = 0$, this is $L^{(p\infty)}(1, \eta\xi)/L^{(p\infty)}(1, \eta)$ times the series defined in [Dis17].

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with, for each v and $a \in F_v$,

$$W_{a,r,v}(g,u,\phi_{2,v};\xi_v) = \frac{L^{(p\infty)}(1,\eta_v\xi_v)}{L^{(p\infty)}(1,\eta_v)} \int_{F_v} \delta_{\xi,r,v}(wn(b)gw_{r,v})r(wn(b)g)\phi_{2,v}(0,u)\psi_v(-ab)\,db.$$

Here, $L^{(p\infty)}(s,\xi'_v) := L(s,\xi'_v)$ if $v \nmid p\infty$ and $L^{(p\infty)}(s,\xi'_v) := 1$ if $v|p\infty$, and we use the convention that $r_v = 0$ if $v \nmid p$.

(Note that the functions $W_{a,r}(\phi_2,\xi)$ correspond to the $W^{\mathbf{C}}_{E_r(\phi_2,\xi),a}$ of Section 2.2.2. We prefer to use lighter notation in this section.)

We choose convenient normalizations for the local Whittaker functions: let $\gamma_{u,v} = \gamma(\mathbf{V}_{2,v}, uq)$ be the Weil index, and for $a \in F_v^\times$, set

$$W^\circ_{a,r,v}(g,u,\phi_{2,v};\xi_v) := \gamma_{u,v}^{-1} L^{(p)}(1,\eta_v) W_{a,r,v}(g,u,\phi_{2,v};\xi_v).$$

Then, for the global Whittaker functions, we have

$$W_{a,r}(g,u,\phi_2;\xi) = \frac{-\varepsilon(\mathbf{V}_2)}{L^{(p)}(1,\eta)} \prod_v W^\circ_{a,r,v}(g,u,\phi_{2,v};\xi_v) \tag{3.3.4}$$

if $a \in F^\times$, where $\varepsilon(\mathbf{V}_2) = \prod_v \gamma_{u,v}$ equals -1 if $\mathbf{V}_2$ is coherent or $+1$ if $\mathbf{V}_2$ is incoherent. We similarly define $W^\circ_{0,r}(g,u,\phi_2,\xi)$ by the identity

$$W_{0,r}(g,u,\phi_2;\xi) = \frac{-\varepsilon(\mathbf{V}_2)}{L^{(p)}(1,\eta)} W^\circ_{0,r}(g,u,\phi_2;\xi). \tag{3.3.5}$$

A simple calculation shows that for all v and $a \neq 0$,

$$W^\circ_{a,r,}\left(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right), u, \phi_{2,v};\xi_v\right) = \eta\xi^{-1}(y)|y|^{1/2} W^\circ_{ay,r,}(1, y^{-1}u, \phi_{2,v};\xi_v). \tag{3.3.6}$$

We will sometimes drop $\phi_{2,v}$ from the notation.

The following sufficient condition for cuspidality will simplify matters a little later on.

Lemma 3.4 *Assume that there is a place $v \nmid p\infty$, at which ξ_v is unramified, such that*

$$\phi_{2,v}(0,u) = 0 \tag{3.3.7}$$

for all u. Then, for all $g = \left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right)$ with $y \in \mathbf{A}^+$, $x \in \mathbf{A}$, we have

$$W_{0,r}(g,u,\phi_2;\xi) = 0.$$

Proof This is a special case of [YZZ12, Proposition 6.10]. ∎

3.3.3 Archimedean Whittaker functions

We compute them explicitly based on our explicit choice of Schwartz function.

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Lemma 3.5 *Let $v|\infty$, let $\xi_v(x) = x^{k_0}$ for some $k_0 \in \mathbf{Z}$, and let $\phi_{2,v}(x,u) := \mathbf{1}_{\mathbf{R}^+}(u)(uq(x))^k e^{-2\pi uq(x)}$ for some $k \in \mathbf{Z}_{\geq 0}$ with $k \geq -k_0$. Let $a \in \mathbf{R}^\times$. Then*

$$W^\circ_{a,v}(1,u,\phi_{2,v,k_v};\xi_v) = \begin{cases} 2(2\pi i)^{k_0} \dfrac{k!}{(k+k_0)!} a^{k+k_0} e^{-2\pi a}, & \text{if } a, u > 0, \\ 0, & \text{if } a < 0 \text{ or } u < 0. \end{cases}$$

Proof We drop the subscript v and write $\delta_{\xi,s}$ for $\delta_{\xi,r,s}$. We have

$$wn(b) = \begin{pmatrix} (1+b^2)^{-1/2} & -b(1+b^2)^{-1/2} \\ & (1+b^2)^{1/2} r_{\theta_b} \end{pmatrix}$$

with $e^{i\theta_b} = (i-b)/(1+b^2)^{1/2}$. Then $\delta_{\xi,s}(wn(b)) = i^{k_0}(1+b^2)^{-s/2}(1-ib)^{-k_0}$. Since $L(1,\eta) = \pi^{-1}$ and $\gamma_v = i$, we have

(3.3.8)

$$\begin{aligned} & i^{-k_0} W^\circ_a(s,1,u,\phi_{2,k}) \\ & := i^{-1}\pi^{-1} \int_{\mathbf{R}} \delta_{\xi,s}(wn(b)) r(wn(b)) \phi_{2,k}(0,u) \psi(-ab)\, db \\ & = \pi^{-1} \int_{\mathbf{R}} (1+b^2)^{-s/2}(1-ib)^{-k_0} \int_{\mathbf{C}} \phi_{2,k}(x,u) \psi(ubq(x)) d_u x\, \psi(-ab) db \\ & = \pi^{-1} \int_{\mathbf{R}} (1+b^2)^{-s/2}(1-ib)^{-k_0} \int_{\mathbf{C}} u^{k+1} q(x)^k e^{-2\pi uq(x)} \psi(ubq(x)) d_1 x\, \psi(-ab) db, \end{aligned}$$

where we recall that $d_u x = |u| d_1 x$ and $d_1 x$ is twice the usual Lebesgue measure. The integral over $\mathbf{C}$ is

$$2u^{k+1} \sum_{j=0}^{k} \int_{\mathbf{R}} \int_{\mathbf{R}} \binom{k}{j} x_1^{2j} x_2^{k-j} e^{-2\pi u(1-ib)(x_1^2+x_2^2)} dx_1 dx_2.$$

Since $\int_{\mathbf{R}} x^{2j} e^{-Ax^2}\, dx = A^{-j-1/2}\Gamma(j+1/2)$, this equals

$$2u^k \sum_{j=0}^{k} \binom{k}{j} \Gamma(j+1/2)\Gamma(k-j+1/2) \cdot (2\pi u)^{-k-1}(1-ib)^{-k-1} = 2^{-k}\pi^{-k} k!\, (1-ib)^{-k-1}$$

by the combinatorial identity (see Appendix A)

$$\sum_{j=0}^{k} \binom{k}{j} \Gamma(j+1/2)\Gamma(k-j+1/2) = \pi k!. \tag{3.3.9}$$

Therefore, when $u > 0$,

$$W^\circ_a(s,1,u,\phi_{2,k}) = i 2^{-k}\pi^{-k} k! \int_{\mathbf{R}} (1+ib)^{-s/2}(1-ib)^{-(s+2k+2k_0+2)/2} e^{-2\pi iab} db.$$

The integral is the same one appearing in [YZZ12, bottom of p. 55] with $d = 2 + 2k + 2k_0$. By [YZZ12, Proposition 2.11] (whose normalization differs from ours by

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$L(1, \eta_v \xi_v) = \pi i$), we find

$$W_a^\circ(0, 1, u, \phi_{2,k}) = i^{k_0} 2^{-k} \pi^{-k-1} k! \frac{(2\pi)^{1+k+k_0}}{\Gamma(1+k+k_0)} a^{k+k_0} e^{-2\pi a}$$
$$= 2(2\pi i)^{k_0} \frac{k!}{(k+k_0)!} a^{k+k_0} e^{-2\pi a}$$

if $a, u > 0$, as well as simpler formulas implying the desired ones in the other cases. ∎

We deduce the following. Let

$$Q_{k_0,k}(X) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+k_0)!}{(j+k_0)!} (-X)^{k-j}, \tag{3.3.10}$$

which satisfies $Q_{k_0,k}(0) = 1$.

Proposition 3.6 *Let $v|\infty$, let $a \in \mathbf{R}$, and let $k_0 \in \mathbf{Z}$ and $k \in \mathbf{Z}_{\geq 0}$ with $k \geq -k_0$. Then, for $a \neq 0$, we have*

$$W_{a,v}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, u, \phi_{2,v,k_v}; \xi_v\right) = (-1)^k \eta_v \xi_v^{-1}(y)|y|^{1/2} \cdot 2(ay)^{k+k_0} Q_{k_0,k}((4\pi a y)^{-1}) e^{-2\pi a y}$$

if $ay > 0$, $uy > 0$, and $W_{a,v}(0, \begin{pmatrix} y & \\ & 1 \end{pmatrix}, u, \phi_{2,v,k_v}; \xi_v) = 0$ otherwise.

Proof After recalling the definition of $P_{k_0,k}$ in (3.3.2), by Lemma 3.5 and (3.3.6), we find the asserted vanishing and that for $ay, uy > 0$ we have (dropping subscripts v):

$$W_{a,v}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, u, \phi_{2,k}; \xi\right)$$
$$= \eta \xi^{-1}(y)|y|^{1/2} \cdot 2(4\pi)^{-k}(-4\pi)^{-k_0}(k+k_0)! \sum_{j=0}^{k} \binom{k}{j} \frac{(k+k_0)!}{(j+k_0)!} (-4\pi a y)^{j+k_0} e^{-2\pi a y},$$

which is equal to the asserted formula. ∎

Corollary 3.7 *Let $\xi \colon F^\times \backslash \mathbf{A}^\times \to \mathbf{C}^\times$ with $\xi(x_\infty) = x_\infty^{k_0}$ for some $k_0 \in \mathbf{Z}$. For each $k \in \mathbf{Z}_{\geq 0}^{\Sigma_\infty}$ with $k \geq -k_0$, we have*

$$E_r(g, u, \phi_2^\infty; \xi, k) = (-1)_\infty^k \delta^k E_r(g, u, \phi_2^\infty; \xi, 0).$$

Proof This follows from Proposition 3.6 and (2.2.8). ∎

3.3.4 Schwartz function at *p*

Let $U_{F,p}^\circ \subset \mathscr{O}_{F,p}^\times$ be as fixed in Section 2.4.5, and let $\kappa_2' \colon U_{F,p}^\circ \to \mathbf{C}^\times$ be a smooth character. We define

$$\phi_{2,p,\kappa_2'}(x, u) := \mathbf{1}_{\mathbf{V}_{2,p}^\circ}(x) \cdot \delta_{U_{F,p}^\circ}(u) \kappa_2'(u), \tag{3.3.11}$$

where $\mathbf{V}_{2,p}^\circ \subset \mathbf{V}_{2,p}$ is a fixed $\mathscr{O}_{F,p}$-lattice and $\delta_{U_{F,p}^\circ}(u) := \dfrac{\mathrm{vol}(\mathscr{O}_{F,p}^\times)}{\mathrm{vol}(U_{F,p}^\circ)} \cdot \mathbf{1}_{U_{F,p}^\circ}(u)$. It is invariant under $N(\mathscr{O}_{F,p})$.

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Let

$$E_r(\phi_2^{p\infty}; \xi, \kappa_2', k) := E_r(\phi_2^{p\infty}\phi_{2,p,\kappa_2'}\phi_{2,\infty,k}; \xi), \tag{3.3.12}$$

and denote its normalized Whittaker functions by

$$W^\circ_{a,r,v}(g, u, \phi_2^{p\infty}; \xi, \kappa_2', k);$$

depending on the place v, we will drop the unnecessary elements from the notation.

3.3.5 Non-archimedean Whittaker functions

We study the functions $W^\circ_{a,v}$ and the q-expansion of E_r.

Proposition 3.8 *Let v be a non-archimedean place of F.*

(1) *If $v \nmid p$, then $W^\circ_{a,v,r} = W^\circ_{a,v}$ does not depend on r, and for all $a \in F_v$,*

$$W^\circ_{a,v}(1, u, \xi) = |d_v|^{1/2} L(1, \eta_v\xi_v)(1 - \xi_v(\varpi_v)) \sum_{n=0}^{\infty} \xi_{F,v}(\varpi_v)^n q_{F,v}^n \int_{D_n(a)} \phi_{2,v}(x_2, u)\, d_u x_2,$$

where $d_u x_2$ is the self-dual measure on $(\mathbf{V}_{2,v}, uq)$ and

$$D_n(a) = \{x_2 \in \mathbf{V}_{2,v} \,|\, uq(x_2) \in a + p_v^n d_v^{-1}\}.$$

(When the sum is infinite, it is to be understood in the sense of analytic continuation from characters $\xi|\cdot|^s$ with $s > 0$.)

(2) *For all finite places v, $|d|_v^{-3/2}|D_v|^{-1/2} W^\circ_{a,v}(1, u, \xi) \in \mathbf{Q}[\xi, \phi_{2,v}]$, and for almost all v, we have*

$$|d|_v^{-3/2}|D_v|^{-1/2} W^\circ_{a,v}(1, u, \xi) = \begin{cases} 1, & \text{if } v(a) \geq -v(d_v) \text{ and } v(u) = -v(d_v), \\ 0, & \text{otherwise.} \end{cases}$$

(3) *If $v|p$, then*

$$W^\circ_{a,r,v}(1, u; \xi, \kappa_2') = \begin{cases} |d_v|^{3/2}|D_v|^{1/2}\xi_v(-1)\kappa'_{2,v}(u), & \text{if } v(a) \geq -v(d) \text{ and } u \in U^\circ_{F,v}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof See [Dis17, Proposition 3.2.3 and Lemma 3.2.4] for parts 1 and 2. For part 3, we drop subscripts v and compute

$$\delta_{\xi,r}(wn(b)w_r) = \xi(-1)\mathbf{1}_{\mathscr{O}_F}(b), \quad \gamma_u^{-1} r(wn(b))\phi_{2,\kappa_2'}(0, u) = |d_v||D_v|^{1/2}\mathbf{1}_{\mathscr{O}_F^\times}(u)\kappa_2'(u)$$

(the latter if $b \in \mathscr{O}_F$), so that

$$W^\circ_a(1, u; \xi, \kappa_2') = |d_v||D_v|^{1/2}\xi(-1)\mathbf{1}_{\mathscr{O}_F^\times}(u)\kappa_2'(u) \int_{\mathscr{O}_F} \psi(-ab)\, db,$$

which gives the asserted value. ∎

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Corollary 3.9 *For $g = \left(\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right)$ with $y \in \mathbf{A}^+$ and $x \in \mathbf{A}$, we have*

$$E_r(g, u, \phi_2^\infty; \xi, k) = \frac{-\varepsilon(\mathbf{V}_2)}{L^{(p)}(1,\eta)} \eta\xi^{-1}(y)|y|^{1/2}$$
$$\cdot \left(W_0^\circ(y,u) + \sum_{a \in F^+} 2^{[F:\mathbf{Q}]}(ay_\infty)^{k+k_0} W_{a,r}^{\circ,\infty}(0,1,y^{-1}u,\phi_2,\xi) Q_{k_0,k}((4\pi a y_\infty)^{-1})\, \mathbf{q}^a \right),$$

where $W_{a,r}^{\circ,\infty}(0,1,u,\phi_2;\xi) = \prod_{v\nmid\infty} W_{a,r,v}^\circ(0,1,y^{-1}u,\phi_{2,v};\xi)$.

Let $\phi_2^{p\infty}$ satisfy (3.3.7), so that by Lemma 3.4, the corresponding Eisenstein series is cuspidal. For ξ a locally algebraic p-adic character of $\mathbf{A}^\times$ of weight k_0, consider the (bounded) sequence of coefficients in $\mathbf{Q}_p(\xi)$

$$\mathrm{E}_r(u,\phi_2^\infty;\xi,k) = \left(\lambda \cdot \eta\xi^{-1}(y) 2^{[F:\mathbf{Q}]}(by_p)^{k+k_0}|D_F|^{1/2}|D_E|^{1/2} W_{by}^{\circ,\infty}(1,y^{-1}u,\phi_2^\infty;\xi)\right)_{b\in F^+},$$

where $\lambda = -\varepsilon(\mathbf{V}_2)/|D_{E/F}|^{1/2}L^{(p)}(1,\eta)$. This is the p-adic q-expansion attached to E_r. Analogously to Corollary 3.7, we have

$$\mathrm{E}_r(u,\phi_2;\xi,k) = (-1)_\infty^k d^k \mathrm{E}_r(u,\phi_2;\xi,0). \tag{3.3.13}$$

3.3.6 *p*-adic interpolation of Whittaker functions

For a place $v \nmid p\infty$ of F, denote $\mathscr{Y}_v := \operatorname{Spec} \mathbf{Q}_p[F_v^\times]$ and (for a later use) $\mathscr{Y}_{\mathrm{H},v} := \operatorname{Spec} \mathbf{Q}_p[E_v^\times]$, the spaces of characters of $F_v^\times$ (respectively, $E_v^\times$). We say that a meromorphic function Φ on an integral scheme *has poles controlled by* the (nonzero) meromorphic function Φ' if Φ/Φ' is regular.

Proposition 3.10 *Let $v \nmid p\infty$. For each $a \in F_v^\times$, $y \in F_v^\times$, and $\phi_{2,v} \in \mathcal{S}(\mathbf{V}_2^{p\infty} \times \mathbf{A}^{p\infty,\times}, \mathbf{Q}_p)$, there is a meromorphic function*

$$\mathscr{W}_{a,v}^\circ(y,u,\phi_{2,v}) \in \mathscr{K}(\mathscr{Y}_v),$$

regular if $\phi_{2,v}$ is standard and otherwise with poles controlled by $L(1,\eta_v\xi_v)$, satisfying

$$\mathscr{W}_{a,v}^\circ(y,u,\phi_{2,v};\xi_v) = |d_v|^{-3/2}|D_v|^{-1/2} W_{a,r,v}^\circ\left(\left(\begin{smallmatrix} y_v & \\ & 1 \end{smallmatrix}\right), u, \phi_{2,v}(\xi_v);\xi_v\right)$$

for all $\xi_v \in \mathscr{Y}_v(\mathbf{C})$ whose underlying scheme point is not a pole.

Proof Part 1 is proved as in [Dis17, Lemma 3.3.1], except that we write the arbitrary $\phi_{2,v} = c\phi_{2,v}^\circ + \phi_{2,v}'$ without the extra factor of equation (3.3.2) *ibid.* Then the argument shows that (only) when $\phi_{2,v}' \neq 0$, there may be a pole controlled by $L(1,\eta_v\xi_v)$. ■

3.4 Theta-Eisenstein family

Fix a compact open subgroup $U^p \subset \mathrm{G}(\mathbf{A}^{p\infty})$ (which will be usually omitted from all the notation), and let $U_F^p := U^p \cap \mathbf{A}^{p\infty,\times}$. Let $\phi^{p\infty} \in \mathcal{S}(\mathbf{V}^{p\infty} \times \mathbf{A}^{p\infty,\times})$ be a Schwartz function fixed by U^p. Let $\xi\colon F^\times\backslash\mathbf{A}^\times \to \mathbf{C}^\times$ be a locally algebraic character fixed by U_F^p such that $\xi(x_\infty) = x_\infty^{k_0}$ for some $k_0 \in \mathbf{Z}$, and let $k \in \mathbf{Z}_{\geq 0}^{\Sigma_\infty}$ satisfy $k_v + k_0 \geq 0$ for all v.

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We fix a choice of a Schwartz function in $\mathcal{S}(\mathbf{V}_{2,p} \times F_p^\times)$ as follows. Let $U_{F,p}^\circ \subset \mathscr{O}_{F,p}^\times$ be as fixed in Section 2.4.5. For $r \in \mathbf{Z}_{\geq 1}^{S_p}$ and $\kappa_1' \colon U_{F,p}^\circ \to \mathbf{C}^\times$ a smooth character, we define

$$\phi_{1,r,\kappa_1',p}(x_1,u) = \delta_{1,r,p}(x)\mathbf{1}_{U_{F,p}^\circ}\kappa_1'(u) := \prod_{v|p}\phi_{1,r_v,\kappa_1',v}(x_{1,v},u_v),$$

(3.4.1)

$$\phi_{1,r_v,\kappa_1',v}(x_{1,v},u_v) := \frac{\mathrm{vol}(\mathscr{O}_{E,v},dt)}{\mathrm{vol}(1+\varpi_v^{r_v}\mathscr{O}_{E,v},d^\times t)}\mathbf{1}_{1+\varpi_v^{r_v}\mathscr{O}_{E,v}}(x_{1,v})\mathbf{1}_{U_{F,p}^\circ}(u)\kappa_1'(u),$$

which is invariant under $N(\mathscr{O}_{F,p}) = \prod_{v|p} N(\mathscr{O}_{F,v})$.

Let $\kappa_2' \colon U_{F,p}^\circ \to \mathbf{C}^\times$ be a smooth character. For $t \in \mathbf{A}_E^\times$, $r \geq \underline{1}$, and $\phi^{p\infty} = \phi_1^{p\infty} \otimes \phi_2^{p\infty} \in \mathcal{S}(\mathbf{V}^{p\infty} \times \mathbf{A}^{p\infty,\times})$, define a form in $N_{(l_0-k_0,2+l+k_0+2k)}^{\leq k}(\mathbf{C})$ by

$$I_r(t,\phi^{p\infty};\kappa_1',\underline{l},\xi,\kappa_2',k) := |D_F|^{-1/2}\cdot\theta(u,r_1(t,1)\phi_1^{p\infty}\phi_{1,\kappa_1',r,p};\underline{l}) \star E_r(q(t)u,\phi_2^{p\infty};\xi,\kappa_2',k).$$

where the product $\star$ is (2.2.7), and $E_r(\cdot) = (3.3.12)$.

Fix a compact open subgroup $U_{\mathrm{H}}^p \subset \mathbf{A}_E^{p\infty,\times}$ (which will be omitted from all the notation). Let $\underline{l}$ be a complex weight for H, let $\chi \colon E^\times\backslash\mathbf{A}_E^\times \to \mathbf{C}^\times$ be a locally algebraic character of weight $\underline{l}$ fixed by U_{H}^p, and assume that for all $w|v|p$, the integer $r_v \geq 1$ is greater than the conductors of χ_w, ξ_v, $\kappa_{2,v}'$. Then we define

$$\text{(3.4.2)}\quad I(\phi^{p\infty};\chi,\xi,\kappa_2',k) := \int^*_{E^\times\backslash\mathbf{A}_E^\times/\mathbf{A}^\times}\chi(t)I_r(t,\phi^{p\infty};\kappa_{1,\chi,p}',\underline{l},\xi,\kappa_2',k)\,dt,$$

which does not depend on the choice of r; here, $\kappa_{1,\chi,p}'$ is as in (2.4.3), namely

$$\text{(3.4.3)}\qquad \kappa_{1,\chi,p}' := \chi_p \circ j'$$

for $j' \colon U_{F,p}^\circ \to \mathscr{O}_{E,p}^\times$ as in (2.4.7).

Lemma 3.11 *For each $c \in \mathbf{A}^+$ satisfying $v(c) \geq 1$ for some $v|p$, the c^{th} Whittaker–Fourier coefficient of $I(\phi^{p\infty};\chi,\xi,\kappa_2',k)$ is*

$$W_I^{\mathbf{C}}(c) = \frac{-\varepsilon(\mathbf{V}_2)\cdot 2^{[F:\mathbf{Q}]}}{L^{(p)}(1,\eta)}\frac{|D_{E/F}|^{1/2}}{|D_F|^{1/2}}\cdot\sum_{a/c\in F,\ 0<a/c<1}\chi^\infty(a)a_\infty^{(|l|+l_0)/2}\xi^{\infty,-1}(c-a)(a-c)_\infty^k$$
$$\prod_{v\nmid p\infty}J_v(a,c,\phi_v;\chi_v,\xi_v)\prod_{v|p}J_v(a,c;\chi_v,\xi_v,\kappa_{2,v}').$$

Here, taking $\phi_{1,v} = \phi_{1,r_v,\kappa_{1,\chi,v}'}$ with $\kappa_{1,\chi,v}' := \chi_v \circ j_v$ and $\phi_{2,v} = \phi_{2,\kappa_{2,v}'}$ if $v|p$, we define, for all $v \nmid p\infty$,

(3.4.4)

$$J_v(a,c,(\phi_v);\chi_v,\xi_v,(\kappa_{2,v}'))$$
$$:= \int_{E_v^\times}\chi_v(t)r_1(t)\phi_{1,v}(1,a)W^\circ_{c-q(t)a,v}(1,q(t)a,(\phi_{2,v});\xi,(\kappa_{2,v}'))\,d^\bullet t_v,$$

for the measure $d^\bullet t_v$ giving volume 1 to $\mathscr{O}_{E,v}^\times$.

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Proof We lighten some of the notation. The assumptions of Lemma 3.2 are satisfied; therefore, for $c \in F^+$ and $g = \left(\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right)$ with $y \in \mathbf{A}^+$ and $x \in \mathbf{A}$,

$$\begin{aligned} &I(g;\chi,\xi,\kappa_2',k) \\ &= \frac{|D_{E/F}|^{1/2}}{|D_F|^{1/2}}\eta(y)|y|^{1/2}y_\infty^{\frac{|l|+l_0}{2}}\sum_{a\in F^+}\int_{\mathbf{A}_E^{\infty,\times}}\chi^\infty(t)r_1(t)\phi_1^\infty(y,y^{-1}a)\,E(g,q(t)a)\,d^\bullet t\,\mathbf{q}^a \\ &= \frac{|D_{E/F}|^{1/2}}{|D_F|^{1/2}}|y|^{1/2}\eta\chi(y)y_\infty^{(|l|-l_0)/2}\sum_{a\in F^+}\int_{\mathbf{A}_E^{\infty,\times}}\chi^\infty(t)r_1(t)\phi_1^\infty(1,ay)\,E(g,q(t)ay^2)\,d^\bullet t\,\mathbf{q}^a. \end{aligned}$$

Now, for $\lambda = -\varepsilon(\mathbf{V}_2)/L^{(p)}(1,\eta)$, by Corollary 3.9, we have

$$\begin{aligned} &E(g,q(t)ay^2) = \lambda\cdot|y|^{1/2}\eta\xi^{-1}(y) \\ &\quad\times\left(W_0(y,u) + (-1)_\infty^k\sum_{b\in F^+}2^{[F:\mathbf{Q}]}(by_\infty)^{k+k_0}Q_{k_0,\underline{k}}((4\pi b y_\infty)^{-1})W_{by}^{\circ,\infty}(1,q(t)ay,\xi)\,\mathbf{q}^b\right) \end{aligned}$$

so that

$$I(g;\chi,\xi,k) = |y|\sum_{c\in F^\times}W_{I,c}^{\mathbf{C}}(y)\mathbf{q}^c$$

for some coefficients $W_{I,c}^{\mathbf{C}}(y)$, which we now explicitly calculate if c satisfies $v(c)\geq 1$ for some $v|p$. Under this condition, we have

$$\begin{aligned} W_{I,c}^{\mathbf{C}}(y) &= \frac{-\varepsilon(\mathbf{V}_2)\cdot 2^{[F:\mathbf{Q}]}}{L^{(p)}(1,\eta)}\frac{|D_{E/F}|^{1/2}}{|D_F|^{1/2}}\sum_{\substack{a\in F\\0<a<c}}(ay)_\infty^{(|l|-l_0)/2}\int_{\mathbf{A}_E^{\infty,\times}}\chi(ay)\chi^\infty(t)r_1(t)\phi_1^\infty(1,ay) \\ &\cdot(-1)_\infty^k\xi^{-1}((c-a)y)((c-a)y_\infty)^{k+k_0}W_{(c-q(t)a)y}^{\circ,\infty}(1,q(t)ay,\xi)Q_{k_0,\underline{k}}((4\pi(c-a)y_\infty)^{-1})\,dt, \end{aligned}$$

where we have noted that, since we have assumed $v(c)\geq 1$ and the choice of $\phi_{1,p}$ implies $v(a)=0$, the constant term (corresponding to $a=c$) of the Eisenstein series does not contribute. Finally, we rewrite the resulting formula with c in place of cy. ■

Lemma 3.12 *Let $a, c\in\mathbf{A}^{\infty,\times}$, let $\phi_2^{p\infty}\in\mathcal{S}(\mathbf{V}_2^{p\infty}\times\mathbf{A}^{p\infty,\times})$, and let $\xi\colon F^\times\backslash\mathbf{A}^\times\to\mathbf{C}^\times$, $\chi\colon E^\times\backslash\mathbf{A}_E^\times\to\mathbf{C}^\times$ be locally algebraic characters.*

(1) *For $v|p$, if $v(c)\geq 1$, then*

$$|d_v|^{-3/2}|D_v|^{-1/2}\cdot J_v(a,c;\chi_v,\xi_v,\kappa_{2,v}') = \xi_v(-1)\mathbf{1}_{U_{F,v}^\circ}(a_v)\kappa_{1,\chi,v}'\kappa_{2,v}'(a_v).$$

(2) *For all but finitely many $v\nmid p$,*

$$|d_v|^{-3/2}|D_v|^{-1/2}J_v(a,c,\phi_v;\chi_v,\xi_v) = 1.$$

(3) *For all $v\nmid p\infty$, there is a function $\mathbf{J}_v(a,c,,\phi_v)\in\mathscr{O}(\mathscr{Y}_{\mathrm{H},v}\times\mathscr{Y}_v)$ such that for all $(\chi_v,\xi_v)\in\mathscr{Y}_{\mathrm{H},v}\times\mathscr{Y}_v(\mathbf{C})$,*

$$\mathbf{J}_v(a,c,\phi_v)(\chi_v,\xi_v) = |d_v|^{-3/2}|D_v|^{-1/2}J_v(a,c,\phi_v;\chi_v,\xi_v).$$

Proof Part 1 follows from the definitions and Proposition 3.8.3. Part 2 follows from Proposition 3.8.2 and a simple calculation. Finally, since the integrand in (3.4.4) is compactly supported, part 3 follows from Proposition 3.10. ■

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Let

$$\lambda := \frac{-\varepsilon(\mathbf{V}_2)\cdot 2^{[F:\mathbf{Q}]}}{|D_{E/F}|^{1/2}L^{(p)}(1,\eta)} \in \mathbf{Q}^\times. \tag{3.4.5}$$

For the sake of simplicity, we momentarily introduce the assumption that the weight $\underline{l}$ of χ satisfies $l \geq 0$. We will see in Corollary 3.15 that this does not affect our main construction.

Proposition 3.13 *Let $\chi \in \mathscr{Y}_{\mathrm{H}}^{\mathrm{cl}}$ have weight $\underline{\kappa}_1 = \underline{\kappa}_{\mathrm{H}}(\chi)$, and let $\underline{\kappa}_2 = (\kappa_{2,0}, \kappa_2) \in \mathfrak{W}^{\mathrm{cl}}$. Let $U_F^\circ := U_F^p U_{F,p}^\circ$. Write $\xi = \kappa_{2,0}^{-1}$ and*

$$\begin{aligned} \kappa_{1,0,p} &= \kappa_{1,0,p}^{\mathrm{sm}}(\cdot)^{l_0}, & \kappa_{2,0,p} &= \kappa_{2,0,p}^{\mathrm{sm}}(\cdot)^{-k_0},\\ \kappa_1 &= \kappa_1^{\mathrm{sm}}(\cdot)^{(l+l_0)/2}, & \kappa_2 &= \kappa_2^{\mathrm{sm}}(\cdot)^{k}. \end{aligned} \tag{3.4.6}$$

Assume that $l_\tau, k_\tau, k_\tau + k_0 \geq 0$ for all $\tau \in \Sigma_p$. Let

$$\kappa_1', \kappa_2' \colon U_{F,p}^\circ \to \mathbf{C}^\times$$

be as in (2.4.3). Let $n \in \mathbf{N}$ be sufficiently large (depending on ξ, χ). The coefficients

$$\begin{aligned} W_{(\mathrm{U}^\circ)_p^{n!}\,\mathrm{I}(\phi^{p\infty})}(c)(\chi, \underline{\kappa}_2) := \lambda \cdot \sum_{\substack{a/c\in F\\ 0<a/c<p^{n!}}} &\mathbf{1}_{U_{F,p}^\circ}(a_p)\kappa_1(a_p)\kappa_2((a-p^{n!}c)_p)\\ &\cdot \kappa_{1,0}^{p\infty}(a)\kappa_{2,0}^{p\infty}(-a)\prod_{v\nmid p}\mathbf{J}_v(a,c,\phi_v;\xi_v,\chi_v), \end{aligned} \tag{3.4.7}$$

for $c \in \mathbf{A}^+$ with $v(c) \geq 0$ for all $v|p$, define a p-adic modular form

$$(\mathrm{U}_p^\circ)^{n!}\,\mathrm{I}(\phi^{p\infty};\chi,\underline{\kappa}_2) \in \mathbf{S}(\mathbf{Q}_p(\chi,\underline{\kappa}_2))$$

such that for every $\iota\colon \mathbf{Q}_p(\chi,\underline{\kappa}_2) \hookrightarrow \mathbf{C}$, we have

$$(\mathrm{U}_p^\circ)^{n!}\mathrm{I}(\phi^{p\infty};\chi,\underline{\kappa}_2)^\iota = |D_E|^{1/2}|D_F|(\mathrm{U}_p^\circ)^{n!}I(\phi^{p\infty};\chi^\iota,\xi^\iota,\iota\kappa_2',k).$$

Moreover, if $\phi_2^{p\infty}$ satisfies (3.3.7), then

$$e^{\mathrm{ord}}\mathrm{I}(\phi^{p\infty};\chi,\underline{\kappa}_2) = e^{\mathrm{ord},\iota}[e^{\mathrm{hol}}(|D_E|^{1/2}|D_F|(\mathrm{U}_p^\circ)^{n!}I(\phi^{p\infty};\chi^\iota,\xi^\iota,\iota\kappa_2',k)]. \tag{3.4.8}$$

Proof From Lemmas 3.11 and 3.12, we find an expression which can be brought into the above form, after replacing $p^{n!}c$ with c whenever it occurs as the argument of a smooth function.

The second assertion follows from (2.3.6), Corollary 3.7, and (3.3.13). ∎

With notation as in Proposition 3.13, let

$$\mathbf{J}^p(a,c,\phi^{p\infty};\chi,\kappa_{2,0}^{-1}) := \prod_{v\nmid p}\kappa_{2,0,v}^{-1}(a)\chi_v(-a)\mathbf{J}_v(a,c,\phi_v;\chi_v,\kappa_{2,0,v}^{-1}).$$

It is easy to verify, using only that $\mathbf{J}^p(\cdot)$ is a Schwartz function of $a \in \mathbf{A}^{p\infty,\times}$, that the Riemann sums

$$\mu_n(\mathbf{J}^p, c, \phi^{p\infty})(\varphi) := \sum_{a/c\in F^\times}\varphi(a)\cdot\mathbf{1}[0<a/c<p^{n!}]\mathbf{1}[a\in U_{F,p}^\circ]\cdot\mathbf{J}^p(a,c,\phi^{p\infty};\chi,\kappa_{2,0}^{-1}),$$

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for $\varphi\colon U^{\circ}_{F,p} \to \mathbf{Q}_p(\kappa_2, \xi)$ locally constant, converge to a measure (bounded distribution)

$$\mu(\mathbf{J}^p, c, \phi^{p\infty}; \chi^{p\infty}, \underline{\kappa}_2^{p\infty})$$

on $U^{\circ}_{F,p}$ valued in $\mathbf{Q}_p(\chi, \underline{\kappa}_2)$. Then, by the same argument of the standard result in [Kob77, Theorem 6 on p. 39], any continuous $\mathbf{Q}_p(\chi, \underline{\kappa}_2)$-valued function is integrable for this measure. The same holds with $\mathbf{Q}_p(\chi, \underline{\kappa}_2)$ replaced by $\mathscr{O}(\mathscr{Y}_{\mathrm{H}} \hat{\times} \mathfrak{W})$ and $\chi, \underline{\kappa}_2$ by the universal characters, $\mathbf{J}^p$ by the universal function; for this universal situation, we will use the same notation without $\chi, \underline{\kappa}_2$.

Corollary 3.14 *Let S^{bad} be the set of places of F at which $\phi^{p\infty}$ is not the standard Schwartz function. Denote $\underline{\kappa}_1 = \underline{\kappa}_{\mathrm{H}}(\chi_{\mathrm{univ}})$. Assume that $\phi_2^{p\infty}$ satisfies (3.3.7). The sequence of coefficients*

$$\begin{aligned} &W_{\mathbf{I}^{\mathrm{ord}}(\phi^{p\infty})}(c)(\chi, \underline{\kappa}_2) \\ &\quad := \lambda \cdot \mathbf{1}_{\mathscr{O}_{F,p}}[c_p] \cdot \int_{U^{\circ}_{F,p}} \kappa_1\kappa_2(a)\, d\mu(\mathbf{J}^p, c, \phi^{p\infty})(a) \in \mathscr{K}(\mathscr{Y}_{\mathrm{H}} \hat{\times} \mathfrak{W}), \quad c \in \mathbf{A}^{\infty,+}, \end{aligned}$$

has poles controlled by $\prod_{v\in S^{\mathrm{bad}}} L(1, \eta_v \kappa_{2,0,v}^{-1})$. It defines an ordinary meromorphic $\mathscr{Y}_{\mathrm{H}} \hat{\times} \mathfrak{W}$-adic modular form (Section 2.4.4)

$$\mathbf{I}^{\mathrm{ord}}(\phi^{p\infty}; \chi, \underline{\kappa}_2)$$

of weight $\underline{\kappa}_1\underline{\kappa}_2$, which satisfies the following property.

For all $\chi \in \mathscr{Y}_{\mathrm{H}}^{\mathrm{cl}}(\mathbf{C})$, $\underline{\kappa}_2 \in \mathfrak{W}^{\mathrm{cl}}(\mathbf{C})$ with underlying numerical weights $\underline{l}$, $\underline{k}$ such that $l_\tau, k_\tau, k_\tau + k_0 \geq 0$ for all τ, we have

$$\mathbf{I}^{\mathrm{ord}}(\phi^{p\infty}; \chi, \underline{\kappa}_2) = e^{\mathrm{ord},\iota} |D_E|^{1/2} |D_F| e^{\mathrm{hol}} I(\phi^{p\infty}; \chi^{\iota}, \kappa_{2,0}^{-1,\iota}, \iota\kappa_2', k), \tag{3.4.9}$$

where $\iota\colon \mathbf{Q}_p(\chi, \underline{\kappa}_2) \hookrightarrow \mathbf{C}$ is the embedding attached to the complex geometric point $(\chi, \underline{\kappa}_2)$.

Proof The interpolation property (3.4.9) at the level of q-expansions follows from Proposition 3.13 and the previous discussion. The simplification in the argument of κ_2 in the interpolated coefficient (3.4.7) is justified by the fact that $\kappa_2(a - p^{n!}c) - \kappa_2(a) \to 0$ uniformly in a, and that the expression of interest is a bounded function of $\kappa_2(\cdot)$. Lemma 2.4.4 then shows the existence of the $\mathscr{Y}_{\mathrm{H}} \hat{\times} \mathfrak{W}$-adic modular form $\mathbf{I}^{\mathrm{ord}}(\phi^{p\infty}; \chi, \underline{\kappa}_2)$. ∎

Consider the weight map

$$\begin{aligned} \varphi\colon (\mathscr{Y}_{\mathrm{G}} \hat{\times} \mathscr{Y}_{\mathrm{H}}) &\to \mathfrak{W} \\ (x, y) &\mapsto \underline{\kappa}_x. \end{aligned}$$

Recycling notation (in a way that should cause no confusion), define an ordinary meromorphic $(\mathscr{Y}_{\mathrm{G}} \hat{\times} \mathscr{Y}_{\mathrm{H}})$-adic modular cuspform of weight φ by

$$\mathbf{I}^{\mathrm{ord}}(\phi^{p\infty}; x, y) := \mathbf{I}^{\mathrm{ord}}(\phi^{p\infty}, \chi_y, \underline{\kappa}_x \underline{\kappa}_y^{-1}), \tag{3.4.10}$$

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where the right-hand side is the form of Corollary 3.14. We denote by

$$\mathbf{I}^{\mathrm{ord},\vee}(\phi^{p\infty}, x, y)$$

the pullback of $\mathbf{I}^{\mathrm{ord}}(\phi^{p\infty}, x, y)$ under the involution $\mathscr{Y}_{\mathrm{G}} \times \mathscr{Y}_{\mathrm{H}} \xrightarrow{\vee\times\mathrm{id}} \mathscr{Y}_{\mathrm{G}} \times \mathscr{Y}_{\mathrm{H}}$.

Corollary 3.15 *Assume that $\phi_2^{p\infty}$ satisfies (3.3.7). For all $x \in \mathscr{Y}_{\mathrm{G}}^{\mathrm{cl}}(\mathbf{C})$ and $y \in \mathscr{Y}_{\mathrm{H}}^{\mathrm{cl}}(\mathbf{C})$, of weights $\underline{w}$, $\underline{l}$ such that for all $\tau \in \Sigma_\infty$,*

$$|l_\tau| \leq w_\tau - 2, \qquad |w_0 + l_0| \leq w_\tau - 2 - |l_\tau|,$$

we have

(3.4.11)
$$\mathbf{I}^{\mathrm{ord},\vee}(\phi^{p\infty}; x, y) = e^{\mathrm{ord},\iota}|D_E|^{1/2}|D_F|e^{\mathrm{hol}}I(\phi^{p\infty}; \chi^\iota, \xi^\iota_{x,y}, \iota\kappa'_{2,x,y}, k_{x,y}),$$

where:

- *$\xi = \xi_{x,y} = \omega_x\omega_y$, whose weight we denote by k_0;*
- *$k_{x,y} = (w - 2 - |l| - k_0)/2$;*
- *$\kappa'_{2,x,y} = \kappa_x^{\vee\prime} \cdot \kappa_y^{\prime -1}$ (with notation as in (2.4.3));*
- *$\iota\colon \mathbf{Q}_p(x, y) \hookrightarrow \mathbf{C}$ is the embedding attached to the complex geometric point (x, y).*

Proof The interpolation property at characters satisfying $l \geq 0$ follows from Proposition 3.13 via Corollary 3.14; the same argument also goes through without the assumption $l \geq 0$, since the weight of the chosen theta-Eisenstein series does not depend on $\underline{l}$ (or y) but only on x. The inequalities on the weights come from the conditions $k, k + k_0 \geq 0$. ∎

4 Zeta integrals

In this final section, we interpolate global and local (away from $p\infty$) zeta integrals, compute the archimedean and p-adic integrals, and construct the p-adic L-function.

As preliminary, we recall gamma factors introduced in the introduction. Let F_v and L be p-adic fields. The (inverse) Deligne–Langlands gamma factor of a potentially semistable representation ρ of $\mathrm{Gal}(\overline{F}_v/F_v)$ over L, with respect to a nontrivial character $\psi_v\colon F_v \to \mathbf{C}^\times$ and an embedding $\iota\colon L \hookrightarrow \mathbf{C}$, is defined as

$$\gamma(\iota\rho, \psi_v)^{-1} := \frac{L(\iota\mathrm{WD}(\rho))}{\varepsilon(\iota\mathrm{WD}(\rho), \psi_v)L(\iota\mathrm{WD}(\rho^*(1)))},$$

where $\iota\mathrm{WD}$ is Fontaine's functor [Fon94] to complex Weil–Deligne representations. We also define $\gamma(s, W, \psi_v) := \gamma(W \otimes |\cdot|^s, \psi_v)$.

Let $\mathbf{Q}_p^{\mathrm{ab}}$ be the abelian closure of $\mathbf{Q}_p$. If W_χ is the Weil–Deligne representation corresponding to a smooth character $\chi\colon F_v \to \mathbf{Q}_p^{\mathrm{ab},\times}$, then for any $\sigma \in G^{\mathrm{ab}}_{\mathbf{Q}_p}$ corresponding to $a \in \mathbf{Q}_p^\times$ under the reciprocity map, we have

(4.0.1)
$$\gamma(W_\chi, \psi_v)^\sigma = \chi(a)\gamma(W_{\chi^\sigma}, \psi_v).$$

For now until the final Section 4.5.3, we fix an embedding

(4.0.2)
$$\iota^{\mathrm{ab}}\colon \mathbf{Q}_p^{\mathrm{ab}} \hookrightarrow \mathbf{C},$$

by which we identify the fixed standard character $\psi^\infty\colon \mathbf{A}^\infty \to \mathbf{C}$ with one valued in $\mathbf{Q}_p^{\mathrm{ab}}$ (still denoted by ψ^∞).

4.1 Petersson product

Let π be an ordinary automorphic representation of $\mathrm{G}(\mathbf{A})$ over a finite extension L of $\mathbf{Q}_p^{\mathrm{ab}}$. For $v|p$, let $\omega_{\pi,v}, \alpha_{\pi,v}\colon F_v^\times \to L^\times$ be the central character and, respectively, U_v-eigencharacter of π_v.

Define $\gamma(\mathrm{ad}(V_{\pi,p})(1)^{++}, \psi_p) := \prod_{v|p} \gamma(\mathrm{ad}(V_{\pi,v})(1)^{++}, \psi_v)$, where $\mathrm{ad}(V_{\pi,p})(1)^{++}$ is the character $\omega_{\pi,v}^{-1}\alpha_{\pi,v}^2|\cdot|^2$ of $F_v^\times$. For $\iota\colon L \hookrightarrow \mathbf{C}$, let

$$e_p(\mathrm{ad}(V_{\pi^\iota})(1)) := \frac{\gamma(\iota\mathrm{ad}(V_{\pi,p})(1)^{++}, \psi_p)^{-1}\zeta_{F,p}(2)}{L(1, \pi_p^\iota, \mathrm{ad})}.$$

Let $(\,,\,)$ denote the Petersson pairing

$$(f, f') := \int_{\mathbf{A}^\times\backslash \mathrm{G}(\mathbf{A})} f(g) f'(g)\, dg$$

of automorphic forms on $\mathrm{G}(\mathbf{A})$.

Lemma 4.1 *Let π be an ordinary cuspidal automorphic representation of $\mathrm{G}(\mathbf{A})$ of weight $\underline{w}$ over a finite extension L of $\mathbf{Q}_p^{\mathrm{ab}}$. There exists a bilinear pairing*

$$\langle\,,\,\rangle\colon \pi^{\mathrm{ord}} \otimes_L N_{\underline{w}^\vee}(L) \to L$$

such that for all $\iota\colon L \hookrightarrow \mathbf{C}$ extending ι^{ab} = (4.0.2) and all sufficiently large $r \in \mathbf{N}^{S_p}$,

(4.1.1)
$$\iota\langle f, g\rangle = \frac{|D_F|^{1/2}\zeta_F(2)}{\omega_{\pi^\iota,p}(-1)e_p(\mathrm{ad}(V_{\pi^\iota})(1)) \cdot 2^{\sum_\sigma -1-w_\sigma} \cdot L(1, \pi^\iota, \mathrm{ad})} q_{F,p}^r (w_{r,p}\mathrm{U}_p^{-r} f^{\iota,\mathrm{a}}, g^\iota).$$

The pairing $\langle\,,\,\rangle$ satisfies the following properties.

1. *For all $f \in \pi^{\mathrm{ord}}$ and $g \in N_{\underline{w}^\vee}(L)$, we have $\langle f, g\rangle = \langle f, e^{\mathrm{ord}}e^{\mathrm{hol}}g\rangle$.*
2. *If $f_0 \in \pi^\infty$ and $f_0^\vee \in \pi^{\vee,\infty}$ are ordinary forms, new at places away from p, holomorphic at the infinite places, and with first Fourier coefficients equal to 1, then*

$$\langle f_0, f_0^\vee\rangle = c_{\pi^\infty} \qquad (4.1.2)$$

for some constant $c_{\pi^\infty} \in L^\times$ depending only on the Bernstein components and the monodromy of π_v for all $v \nmid p\infty$.

Proof The existence and (4.1.2) follow from the factorization of the Petersson inner product into parings in the Whittaker models [CST14, Proposition 2.1], together with the local calculations of [CST14, Proposition 3.11] away from p and [Dis/b, Lemma A.3.3] at p. Since the elements $f^{\iota,\mathrm{a}}$ in the right-hand side of (4.1.1) are antiholomorphic, it is clear that the pairing factors through e^{hol}. ∎

Proposition 4.2 *Let $\mathscr{X}_\mathrm{G} \subset \mathscr{Y}_\mathrm{G}$ be a Hida family of tame level U^p, let S be a finite set of places such that U^{Sp} is maximal, and let $\Pi = \Pi_{\mathscr{X}_\mathrm{G}}^{U^{Sp}}$. There is a unique*

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$\mathscr{O}(\mathscr{Y}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}})$*-bilinear pairing*

$$\langle\langle\ ,\ \rangle\rangle\colon \Pi \otimes_{\mathscr{O}(\mathscr{Y}_{\mathrm{G}})} \mathscr{S}^{\vee}(\mathscr{Y}_{\mathrm{G}})_{\mathbf{Q}_p^{\mathrm{ab}}} \to \mathscr{K}(\mathscr{X}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}})$$

such that for all $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}}$*, corresponding to an ordinary representation* $\pi = \pi_x$ *over* $\mathbf{Q}_p^{\mathrm{ab}}(x)$*, and for all* $\mathbf{f} \in \Pi_{\mathbf{Q}_p^{\mathrm{ab}}}$*,* $\mathbf{g} \in \mathscr{S}^{\vee}(\mathscr{Y}_{\mathrm{G}})_{\mathbf{Q}_p^{\mathrm{ab}}}$*, we have*

$$\langle\langle \mathbf{f}, \mathbf{g}\rangle\rangle(x) = \langle \mathbf{f}_x, \mathbf{g}_x\rangle.$$

Proof The construction is very similar to that of the pairing denoted by $H^{-1}l_\lambda$ in [Hid91, p. 380]. In this case, let $\mathbf{f}_0^{\vee} = (2.4.10)$ be the normalized primitive form in $\Pi^{\vee}$, let $U_0^p \subset \mathrm{G}(\mathbf{A}^{p\infty})$ be a maximal open compact subgroup fixing $\mathbf{f}_0^{\vee}$, and let

$$e_{\mathbf{f}_0^{\vee}}\colon \mathscr{S}^{\vee} \to \mathscr{K}(\mathscr{X}_{\mathrm{G}})\,\mathbf{f}_0^{\vee}$$

be the unique $\mathscr{H}^{\mathrm{sph}}_{U_0^p}$-equivariant idempotent that factors through the idempotent projection $\mathscr{S}^{\vee} \to \mathscr{S}^{U_0^p,\vee}$. Then we define $\langle\langle \mathbf{f}_0, -\rangle\rangle$ by

$$\langle\langle \mathbf{f}_0, \mathbf{g}\rangle\rangle \mathbf{f}_0^{\vee} = c_{\mathscr{X}_{\mathrm{G}}}^{-1} \cdot e_{\mathbf{f}_0^{\vee}}(\mathbf{g}),$$

where $c_{\mathscr{X}_{\mathrm{G}}} := c_{\pi^\infty}$ for any automorphic representation π such that $\pi^{U^{Sp},\mathrm{ord}} \cong \Pi_{|x}$, for some $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}}$. Let us explain why this is well defined independently of x. As noted before, c_{π^∞} only depends on the Bernstein component and the (rank of the) monodromy of $\pi_{x,v}$ for $v \nmid p\infty$. (In plain terms, the rank of the monodromy is 1 if $\pi_{x,v}$ is a special representation and it is 0 otherwise.) The Bernstein component is an invariant of connected families. As for the rank of the monodromy, by the local–global compatibility result of Proposition 2.4, it is the rank of the monodromy of the Weil–Deligne representation attached to $\mathscr{V}_{\mathrm{G}|x}$. Since the latter is pure, the desired constancy along $\mathscr{X}_{\mathrm{G}}$ follows from [Dis20, Proposition 3.3.1].

In general, we may write $\mathbf{f} = T\mathbf{f}_0$ for some Hecke operator T supported away from p. We then define $\langle\langle \mathbf{f}, \mathbf{g}\rangle\rangle := \langle\langle \mathbf{f}_0, T^{\vee}\mathbf{g}\rangle\rangle$. The interpolation property follows from the definitions, the interpolation property proved in [Hid91, Lemma 9.3], and (4.1.2). ■

4.2 Waldspurger's Rankin–Selberg integral

We recall the local and global theory of Waldspurger's [Wal85] integral representation of our L-function.

4.2.1 Setup

Let $y \in \mathscr{Y}^{\mathrm{cl}}_{\mathrm{H}}(\mathbf{C})$, let $\chi = \chi_y$ be the corresponding character of $E^\times\backslash\mathbf{A}_E^\times$, let $\underline{\kappa}_\chi \in \mathfrak{W}^{\mathrm{cl}}(\mathbf{C})$ be its weight, let $\underline{l}$ be its numerical weight, and let $\omega_\chi := \chi_{|\mathbf{A}^\times}$.

Let $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}(\mathbf{C})$, corresponding to a point $x_0 \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$ and an embedding $\iota\colon \mathbf{Q}_p(x) \hookrightarrow \mathbf{C}$. Let π_0 be the ordinary automorphic representation of $\mathrm{G}(\mathbf{A})$ over $\mathbf{Q}_p(x_0)$ attached to x_0, and let $\pi = \pi_0^\iota$. We denote by

$$\underline{\kappa}_\pi \in \mathfrak{W}^{\mathrm{cl}}(\mathbf{C}), \quad \underline{w}, \quad \omega_\pi,$$

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respectively, the weight, numerical weight, and central character of π. We let $\alpha = \otimes_v \alpha_v \colon F_p^\times \to \mathbf{C}^\times$ be the character such that $\mathrm{U}_y f^\iota = \alpha(t) f^\iota$ for any $f \in \pi_0^{\mathrm{ord}}$ and $t \in F_p^\times$. Then

$$\kappa_{\pi,0}(z) = \omega_\pi(z) z^{w_0}, \qquad \kappa_\pi(t) = \alpha_{|U_{F,p}^\circ}(t) t^{(w+w_0)/2}$$

are the decompositions of $\kappa_{\pi,0}$ and κ_π into a product of a smooth and an algebraic character.

Define, as in Corollary 3.15, a numerical weight $\underline{k}$ and a smooth character κ_2' of $U_{F,p}^\circ$ by

$$\begin{aligned} \kappa_\chi' \kappa_2' &= \kappa_{\pi^\vee}' = \alpha_{|U_{F,p}^\circ}, & \xi &= \omega_\pi \omega_\chi, \\ k &= (w - 2 - |l| - k_0)/2, & k_0 &= w_0 + l_0, \end{aligned} \tag{4.2.1}$$

and let $\underline{\kappa}_2 \in \mathfrak{W}(\mathbf{C})$ be the associated weight as in (3.4.6).

For $v|p$, we choose a Schwartz function $\Phi_v = \phi_v \in \mathcal{S}(\mathbf{V}_v \times F_v^\times)$ as in (3.4.1) and (3.4.3) (for ϕ_1), and (3.3.11) and (4.2.1) (for ϕ_2); then

$$\phi_v(x,u) = \delta_{r,v}(x_1) \mathbf{1}_{\mathbf{V}_{2,v}^\circ}(x_2) \delta_{U_{F,v}^\circ}(u) \alpha_v(u). \tag{4.2.2}$$

For $v|\infty$, let $\Phi_v = \Phi_{l_0,l,k_0,k,v}$ be a preimage, under the map (3.1.1), of

$$\phi_{l_0,l,k_0,k,v}(x_1,x_2,u) = \phi_{1,l_0,l,v}(x_1,u)\phi_{2,k_0,k,v}(x_2,u),$$

where the factors are defined in (3.2.3) and (3.3.3).

4.2.2 Waldspurger's integral

The next proposition gives an integral representation for the L-function we are interested in. We first define the local terms. Let $f_0 \in \pi_0^{\mathrm{ord}}$, let $f := f_0^\iota$, and let

$$\overline{W}(g) := \int_{F\backslash \mathbf{A}} f^{\mathrm{a}} \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \right) \psi(x)\, dx$$

be the Whittaker function of f^{a} with respect to ψ^{-1}. It is related to the q-expansion (2.2.1) of f by

$$\overline{W}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} \right) = W_f^{\mathbf{C}}(y).$$

We assume that $W \colon \mathrm{G}(\mathbf{A}) \to \mathbf{C}$ is factorizable as $\overline{W} = \otimes_v \overline{W}_v$.

For $\Phi = \otimes_v \Phi_v \in \mathcal{S}(\mathbf{V} \times \mathbf{A}^\times)$, let

$$\begin{aligned} &R_{r,v}(\overline{W}_v, \Phi_v, \chi_v) \\ &\quad := \int_{Z(F_v)N(F_v)\backslash \mathrm{G}(F_v)} \overline{W}_v(g) \delta_{\xi_v,r}(g) \int_{T(F_v)} \chi_v(t) r(g w_{r,v}^{-1}) \Phi_v(t^{-1}, q(t))\, dt\, dg, \end{aligned}$$

where $\delta_{\xi,r}$ is as in (3.3.1). Note that the integral $R_{r,v}$ does not depend on $r \geq \underline{1}$ unless $v|p$; we will accordingly simplify the notation in these cases. We also define normalized versions. For $v|p\infty$, let Φ_v be as fixed in Section 4.2.1. Then we put

$$R_v^\natural(\overline{W}_v, \Phi_v, \chi_v) := |d_v|^{-2}|D_v|^{-1/2}\frac{\zeta_{F,v}(2)L(1,\eta_v\xi_v)}{L(1/2,\pi_{E,v}\otimes\chi_v)}R_v(\overline{W}_v,\Phi_v,\chi_v), \qquad \text{if } v\nmid p\infty,$$

$$R_{r,v}^\dagger(\overline{W}_v, \chi_v, \alpha_v) := |d_v|^{-2}|D_v|^{-1/2}\frac{\zeta_{F,v}(2)L(1,\eta_v)}{L(1/2,\pi_{E,v}\otimes\chi_v)}q_{F,v}^r\alpha_v^{-r}R_{r,v}(\overline{W}_v,\Phi_v,\chi_v), \qquad \text{if } v|p,$$

$$R_v^\dagger(\overline{W}_v, \chi_v, k) := \frac{\zeta_{F,v}(2)L(1,\eta_v)}{L(1/2,\pi_{E,v}\otimes\chi_v)}R_v(\overline{W}_v,\Phi_v,\chi_v), \qquad \text{if } v|\infty.$$

By a result of Waldspurger (see [Dis17, Lemma 5.3.2]), for a place v such that π_v and χ_v are unramified, ϕ_v is standard, and $\overline{W}_v$ is unramified, we have

$$R_v^\natural(\overline{W}_v, \Phi_v, \chi_v) = \overline{W}_v(1). \tag{4.2.3}$$

Proposition 4.3 *Let $f_0 \in \pi_0^{\mathrm{ord}}$ and assume that $f := f_0^{\iota,\mathrm{a}}$ has a factorizable ψ^{-1}-Whittaker function $\overline{W} = \otimes_v \overline{W}_v$. Let $\phi^{p\infty} \in \mathcal{S}(\mathbf{V}^{p\infty} \times \mathbf{A}^{p\infty,\times})$. For sufficiently large $r = (r_v)_{v|p}$, we have*

$$\iota q_{F,p}^r\alpha(\varpi_p)^{-r}(f, w_{r,p}^{-1}I(\phi^{p\infty};\chi,\xi,\kappa_2',k)) = |D_F|^{-1}|D_E|^{-1/2}\frac{L(1/2,\pi_E\otimes\chi)}{\zeta_F(2)L(1,\eta)}$$
$$\cdot\prod_{v\nmid p\infty}R_{r,v}^\natural(\overline{W}_v,\phi_v,\chi_v)\prod_{v|p}R_{r,v}^\dagger(\overline{W}_v,\chi_v,\alpha_v)\prod_{v|\infty}R_v^\dagger(\overline{W}_v,\chi_v,k),$$

where all but finitely many of the factors in the infinite product are equal to 1.

Proof As in [Dis17, Proof of Proposition 3.5.1], corrected in [Dis/a, Appendix B, under "*Proposition 2.4.4.1*"] to include the factor $q_{F,p}^r$. ■

4.2.3 Nonvanishing of the local integrals

We recall a fundamental nonvanishing result for our zeta integrals for self-dual $\pi \boxtimes \chi$, as well as a useful refinement.

Lemma 4.4 *Let $v \nmid p\infty$ be a place of F, and let L be a field of characteristic zero. Let π_v be a smooth irreducible representation of $\mathrm{G}(F_v)$ over L, with central character $\omega_{\pi,v}$, and let $\chi_v: E_v^\times \to L^\times$ be a smooth character. Assume the self-duality condition $\omega_{\pi,v}\chi_{|F_v^\times} = \mathbf{1}$.*

There exist:

- *a four-dimensional quadratic space $\mathbf{V}_v = \mathbf{B}_v$ over F_v of the type described in Section 3.1.4, uniquely determined by*

$$\varepsilon(\mathbf{B}_v) = \eta_v\chi_v(-1)\varepsilon(\pi_{E,v}\otimes\chi_v),$$

- *a function $\overline{W}_v$ in the Whittaker model of π_v,*
- *a Schwartz function $\phi_v \in \mathcal{S}(\mathbf{V}_v \times F_v^\times, L)$,*

such that

$$R_v(\overline{W}_v, \phi_v, \chi_v) \neq 0.$$

If moreover all the data are unramified at a place v inert in E, it is possible to choose $\overline{W}_v$ and $\phi_v = \phi_{1,v}\phi_{2,v}$ such that $\phi_{2,v}(0,u) = 0$ for all u (condition (3.3.7)).

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Proof The argument in [Dis17, Proof of Proposition 3.7.1, second paragraph] applies verbatim to prove the first statement. Let us prove the second one. We drop all subscripts v. Fix an isomorphism $\mathbf{V}_2 \cong E$, and let us choose $\overline{W}$ to be a new vector, $\phi_{1,v}$ to be the standard Schwartz function, and

$$\phi_2(x_2, u) = \mathbf{1}_{\mathcal{O}_E^\times}(x_2)\mathbf{1}_{\mathcal{O}_F^\times}(u).$$

Writing $\doteq$ for an equality up to nonzero scalars, by the Iwasawa decomposition

$$R(\overline{W}, \phi, \chi) \doteq \int_{F^\times} \overline{W}\left(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)\right) \int_{E^\times} \chi(t) \int_{\mathrm{GL}_2(\mathcal{O}_F)} r(g)\phi(t^{-1}y, y^{-1}q(t))\, dg\, d^\times y\, dt.$$

Let $U_0(\varpi^r) \subset U_0 := \mathrm{GL}_2(\mathcal{O}_F)$ be the set of matrices which are upper-triangular modulo ϖ^r. It is easy to verify that ϕ_2 is invariant under $U_0(\varpi^r)$ for some r, and that $U_0 = U_0(\varpi^r) \sqcup \bigsqcup_{b \in \mathcal{O}_{F,v}/\varpi^t} \left(\begin{smallmatrix} 1 & \\ b & 1 \end{smallmatrix}\right) U_0(\varpi)$. Thus, the integral in dg is a constant multiple of

$$\begin{aligned}
&\int_{\mathcal{O}_F} r(w)[\psi(y^{-1}bq(tx'))\hat{\phi}(x', y^{-1}q(t))]_{|x=t^{-1}y}\, db\\
&= \int_{\mathcal{O}_F} \int_{E\times E} \psi(\mathrm{Tr}_{E/F}tx_1)\psi(y^{-1}q(t)\cdot bq(x))\mathbf{1}_{\mathcal{O}_E}(x_1)\widehat{\mathbf{1}_{\mathcal{O}_E^\times}}(x_2)\mathbf{1}^\times_{\mathcal{O}_F}(y^{-1}q(t))]\, dxdb\\
&= \mathbf{1}^\times_{\mathcal{O}_F}(y^{-1}q(t)) \int_{\mathcal{O}_F} \widehat{\mathbf{1}_{\mathcal{O}_E}}(t) \int_E \psi(y^{-1}q(t)bq(x_2)\widehat{\mathbf{1}_{\mathcal{O}_E^\times}}(x_2)\, dx_2\, db \doteq \mathbf{1}_{\mathcal{O}_E}(t)\mathbf{1}^\times_{\mathcal{O}_F}(y^{-1}q(t)),
\end{aligned}$$

where the last equality follows from interchanging the order of integration and observing that $\widehat{\mathbf{1}_{\mathcal{O}_E^\times}}(x_2) = \mathrm{vol}(\mathcal{O}_E^\times)$ for $x_2 \in \mathcal{O}_E$.

The last quantity equals $\phi^\circ(t^{-1}y, y^{-1}q(t))$ for the standard Schwartz function ϕ°; therefore, the integral R is a constant multiple of the unramified integral, in particular, it is nonzero by (4.2.3). ∎

4.3 Evaluation of the integrals at p and ∞

We explicitly compute the local integrals at the places $v|p\infty$.

4.3.1 p-adic integrals

Define, for $v|p$,

$$\begin{aligned}
e_v(V_{\pi_E\otimes\chi}) &:= \frac{L(1/2, \pi_{E,v} \otimes \chi_v)}{\zeta_{F,v}(2)L(1, \eta_v)} \prod_{w|v} \gamma(\chi_w \alpha_{\pi,v}|\cdot| \circ N_{E_w/F_v}, \psi_v)^{-1},\\
e_p(V_{\pi_E\otimes\chi}) &:= \prod_{v|p} e_v(V_{\pi_E\otimes\chi}).
\end{aligned} \tag{4.3.1}$$

Lemma 4.5 *Let $v|p$, and assume that $\overline{W}_v$ is normaliz,ed by $\overline{W}_v(1) = 1$. Then for any sufficiently large r (depending on χ_v, π_v), we have*

$$R^\dagger_{r,v}(\overline{W}_v, \chi_v, \alpha_v) = \frac{\chi_v(-1)}{L(1, \eta_v)} e_v(V_{\pi_E\otimes\chi}).$$

Proof By [Dis17, Proposition A.2.2] (with the discriminant factors corrected as in [Dis/a, Appendix B]), we have

$$R^{\dagger}_{r,v}(\overline{W}_v, \phi_v, \chi_v) = \frac{L(1/2, \pi_E \otimes \chi_v)}{\zeta_{F,v}(2) L(1, \eta_v)^2} Z_v,$$

where Z_v are integrals defined in [Dis17, Lemma A.1.1]. By [Dis/a, Lemma A.1.1], we have

$$Z_v = \chi_v(-1) \prod_{w|v} \gamma(\chi_w \alpha_v |\cdot| \circ N_{E_w/F_v}, \psi_v)^{-1}.$$

The asserted formula follows. ∎

4.3.2 Archimedean integrals

We compute the local integrals $R^{\dagger}_v$ when $v|\infty$. The *standard antiholomorphic Whittaker function* for ψ^{-1} of weight (w_0, w) is

$$\overline{W}^{(w_0,w),\mathrm{a}}\left(\begin{pmatrix} z & \\ & z\end{pmatrix}\begin{pmatrix} y & x \\ & 1\end{pmatrix} r_\theta\right) = z^{w_0} \mathbf{1}_{\mathbf{R}^+}(y)|y|^{(w+w_0)/2} \psi(-x + iy)\psi(-w\theta). \tag{4.3.2}$$

Lemma 4.6 *Let $v|\infty$, and let $\overline{W}_v$ be the standard antiholomorphic Whittaker function of weight (w_0, w) for ψ^{-1}. Then*

$$R^{\dagger}_v(\overline{W}_v, \chi_v, k) = i^{-k_0} 2^{-1-w}.$$

Proof By the Iwasawa decomposition, we can uniquely write any $g \in \mathrm{GL}_2(\mathbf{R})$ as

$$g = \begin{pmatrix} 1 & \\ x & \end{pmatrix} 1 \begin{pmatrix} z & \\ & z \end{pmatrix}\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with $x \in \mathbf{R}$, $z \in \mathbf{R}^\times$, $y \in \mathbf{R}^\times$ and $\theta \in \mathbf{R}/2\pi\mathbf{Z}$; the local Tamagawa measure is then $dg = dx d^\times z \frac{d^\times y}{|y|} \frac{d\theta}{2}$. Let $\Phi_v = \Phi_{l_0,l,k_0,k,v}$. We drop all subscripts v. Since the weights match, the integration over $\mathrm{SO}(2,\mathbf{R})$ yields 1, and we have

$$\begin{aligned} R = R(\overline{W}, \Phi, \chi) = &\int_{\mathbf{R}^\times \times (\mathbf{R}^\times \backslash \mathbf{C}^\times) \times \mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}^\times} \\ &\chi(tz)\omega_\pi(z)|y|^{(w+w_0)/2} e^{-2\pi y} \xi^{-1}(z)|y|\Phi(yzt^{-1}, y^{-1}z^{-2}q(t))\, d^\times z \frac{d\theta}{2} \frac{d^\times y}{|y|}\, dt. \end{aligned}$$

By definition, $\omega_\pi \chi \xi^{-1}(z) = 1$, so that the integration in $d^\times z$ simply realizes the map $\Phi \mapsto \phi$. Then

(4.3.3)

$$\begin{aligned} R &= \pi \int_{\mathbf{R}^\times} \int_{\mathbf{R}^\times \backslash \mathbf{C}^\times} \chi(t)|y|^{(w+w_0)/2} e^{-2\pi y} |y| \mathbf{1}_{\mathbf{R}^+}(y) P_k(0) y^{(|l|+l_0)/2} \chi(t)^{-1} e^{-2\pi y} \frac{d^\times y}{|y|}\, dt \\ &= 2\pi P_{k_0,k}(0) \int_{\mathbf{R}^+} y^{(w+|l|+w_0+l_0)/2} e^{-4\pi y} d^\times y, \end{aligned}$$

where $2 = \mathrm{vol}(\mathbf{R}^\times \backslash \mathbf{C}^\times)$.

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Recall from (4.2.1) and (3.3.2) that $k_0 = w_0 + l_0$ and $k = (w - 2 - |l| - k_0)/2$ and that $P_{k_0,k}(0) = (2\pi i)^{-k_0}(4\pi)^{-k}(k + k_0)!$. Then, after a change of variables, we have

$$R = (2\pi)^{1-k_0} i^{-k_0}(4\pi)^{-(w-|l|-k_0-2)/2}\Gamma(\frac{w-|l|+k_0}{2})(4\pi)^{-(w+|l|+k_0)/2}\Gamma(\frac{w+|l|+k_0}{2})$$
$$= i^{-k_0}2^{-1-w}\pi^2\Gamma_{\mathbf{C}}(\frac{w-l+k_0}{2})\Gamma_{\mathbf{C}}(\frac{w+l+k_0}{2}).$$

Now, the result follows from identifying

$$\pi^2\Gamma_{\mathbf{C}}(\frac{w-l+k_0}{2})\Gamma_{\mathbf{C}}(\frac{w+l+k_0}{2}) = \frac{L(1,\pi_{E,v}\otimes\chi)}{\zeta_{F,v}(2)L(1,\eta_v)}. \qquad \blacksquare$$

4.4 Interpolation of the local zeta integral

Let $\mathscr{X} = \mathscr{X}_{\mathrm{G}}\hat{\times}\mathscr{X}_{\mathrm{H}}$ be a Hida family for $\mathrm{G}\times\mathrm{H}$, let $v \nmid p\infty$ be a place of F, and let $\Pi_v := \Pi(\mathscr{V}_{\mathrm{G},v})$ be as in Section 2.4.8. Let $\mathscr{X}_{\mathrm{G}}^{(v)} \subset \mathscr{X}$ be the open subset containing $\mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$ over which Π_v is defined, and let $\mathscr{X}^{(v)} = \mathscr{X}_{\mathrm{G}}^{(v)}\hat{\times}\mathscr{X}_{\mathrm{H}}$. Let $\mathscr{W}_v$ be the ψ_v-Whittaker model of $\Pi_{v,\mathbf{Q}_p^{\mathrm{ab}}}$, which exists since Π_v is co-Whittaker (see [Dis20, Section 4.2]); it is $\mathscr{O}_{\mathscr{X}^{(v)}}[\mathrm{G}(F_v)]$-isomorphic to the tensor product of Π_v and an invertible sheaf with trivial $\mathrm{G}(F_v)$-action. The space $\mathscr{W}_v$ is, as usual, a space of functions on $\mathrm{G}(F_v)$, ψ_v-invariant under the action of the unipotent subgroup $N(F_v)$. For any $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$ and any $\mathbf{W}_v \in \mathscr{W}_v$, the twisted specialization

$$\overline{\mathbf{W}}_{v|x}(g) := \mathbf{W}_{v|x}\left(\begin{pmatrix}-1 & \\ & 1\end{pmatrix} g \begin{pmatrix}-1 & \\ & 1\end{pmatrix}\right)$$

belongs to the ψ^{-1}-Whittaker model of π_x.

Proposition 4.7 *Let $v \nmid p\infty$. There exists an $\mathscr{O}(\mathscr{X}^{(v)}_{\mathbf{Q}_p^{\mathrm{ab}}})$-linear map*

$$\mathbf{R}_v \colon \mathscr{W}_v \otimes_{\mathscr{O}(\mathscr{X}^{(v)}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}})} \mathscr{O}(\mathscr{X}^{(v)}_{\mathbf{Q}_p^{\mathrm{ab}}}) \otimes_{\mathbf{Q}_p} \mathcal{S}(\mathbf{V}_v \times F_v^\times) \to \mathscr{O}(\mathscr{X}^{(v)}_{\mathbf{Q}_p^{\mathrm{ab}}})$$

such that for all $\mathbf{W}_v \in \mathscr{W}_v$, all $\phi_v \in \mathcal{S}(\mathbf{V}_v \times F_v^\times)$, and all $(x,y) \in \mathscr{X}^{\mathrm{cl}}_{\mathbf{Q}_p^{\mathrm{ab}}}(\mathbf{C})$, with underlying embedding $\iota\colon \mathbf{Q}_p^{\mathrm{ab}}(x,y) \hookrightarrow \mathbf{C}$, we have

$$\mathbf{R}_v(\mathbf{W}_v, \phi_v)(x,y) = R_v^\natural(\iota\overline{\mathbf{W}}_{v|x}, \iota\phi_v, \chi^\iota_{y,v}).$$

Proof In fact, we may prove a stronger statement by replacing $\mathscr{X}_{\mathrm{H}}$ by (its image in) $\mathscr{Y}_{\mathrm{H},v}$, or equivalently any connected component $\mathscr{Y}^\circ_{\mathrm{H},v}$ thereof (which is an étale torsor for $\mathbf{G}^{\{w|v\}}_{m,\mathbf{Q}_p}$, the action being induced by multiplication by the uniformizers in $E_v^\times = \prod_{w|v} E_w^\times$).

The proof is largely similar to that of [Dis20, Proposition 5.2.3] (whose statement is corrected in Appendix B); we refer to *loc. cit.* and the sections preceding it for more details on the notions we use. Since $\mathscr{W}_v \cong \Pi(\mathscr{V}_{\mathrm{G},v})$ is in the image of the local Langlands correspondence, there exists an irreducible component $\mathfrak{X}^\circ$ of the extended Bernstein variety of [Dis20, Section 3.3] and a map $\mathscr{X}_{\mathrm{G}}^{(v)} \to \mathfrak{X}^\circ$, such that $\mathscr{W}_v$ is a quotient of the universal co-Whittaker module over $\mathfrak{X}^\circ$. We may further extend scalars to $\mathbf{C}$ and replace $\mathfrak{X}^\circ$ by a cover of the form

$$\widetilde{\mathfrak{X}}^\circ = \mathbf{G}_m^d$$

for $d = 1$ or 2; then the pull-back $\widetilde{\mathscr{W}_v}$ of the universal co-Whittaker module has one of the following shapes:

(a) $\widetilde{\mathscr{W}_v} = \mathrm{Ind}_{P_v}^{\mathrm{G}(F_v)}(\beta_1 \boxtimes \beta_2)$, where $d = 2$ and $\beta_i \colon F_v^\times \to \mathscr{O}(\widetilde{\mathfrak{X}}^\circ)^\times$ are the universal characters;
(b) $\widetilde{\mathscr{W}_v} = \mathrm{St} \otimes \beta_1$, where $d = 1$ and $\beta_1 \colon F_v^\times \to \mathscr{O}(\widetilde{\mathfrak{X}}^\circ)^\times$ is the universal character;
(c) $\widetilde{\mathscr{W}_v} = \pi_0 \otimes \beta_1$, where π_0 is a complex supercuspidal representation of $\mathrm{G}(F_v)$, $d = 1$, and $\beta_1 \colon F_v^\times \to \mathscr{O}(\widetilde{\mathfrak{X}}^\circ)^\times$ is the universal character.

In all cases, we need to show that for every $\mathbf{W}_v \in \widetilde{\mathscr{W}_v}$, there is an element $\mathbf{R}_v(\mathbf{W}_v, \phi_v) \in \mathscr{O}(\widetilde{\mathfrak{X}}^\circ \times \mathscr{Y}_{\mathrm{H},v})$ such that

$$\mathbf{R}_v(\mathbf{W}_v, \phi_v)(x, y) = L(1/2, \pi_{x,E,v} \otimes \chi_{y,v})^{-1} R_v(\phi_v, \overline{\mathbf{W}}_{v|x}, \chi_{y,v})$$

for all x, y; in other words, that the power series in $X_i^{\pm 1} := \beta_i(\varpi_v)^{\pm 1}$ and $Y_w^{\pm 1} := \chi_{\mathrm{univ}}(\varpi_w)^{\pm 1}$ obtained from the integral defining R_v is a Laurent-polynomial multiple of the inverse of the Laurent polynomial $L(1/2, \pi_{x,E,v} \otimes \chi_{y,v})$. This is proved by the same argument as in [Dis17, Proof of Proposition 3.6.1]: since $\widetilde{\mathscr{W}_v}$ is torsion-free, it embeds in the representation $\widetilde{\mathscr{W}_v} \otimes \mathscr{K}(\widetilde{\mathfrak{X}}^\circ)$ over the field $\mathscr{K}(\widetilde{\mathfrak{X}}^\circ)$, so that the usual explicit description of the Kirillov model used in *loc. cit.* applies. ∎

4.5 The *p*-adic *L*-function

Let

$$\mathscr{X} \subset \mathscr{Y}_{\mathrm{G}} \hat{\times} \mathscr{Y}_{\mathrm{H}}$$

be a Hida family with $\mathscr{X}^{\mathrm{sd}} \neq \varnothing$, of tame level $U^p = U_{\mathrm{G}}^p \times U_{\mathrm{H}}^p$. Let S be a finite set of places of F, disjoint from $S_{p\infty}$ and containing all those at which the tame level of $\mathscr{X}$ is not maximal, and let $\Pi := \Pi_{\mathscr{X}_{\mathrm{G}}}^{U_{\mathrm{G}}^{Sp}}$. If $\mathscr{X}'$ is an (ind-)scheme over $\mathbf{Q}_p^{\mathrm{ab}}$, we define

$$\mathscr{X}'_{/\mathbf{Q}_p^{\mathrm{ab}}}(\mathbf{C}) \subset \mathscr{X}'(\mathbf{C})$$

to be the subset of geometric points over ι^{ab} (that is, those such that the composition $\operatorname{Spec} \mathbf{C} \to \mathscr{X}'_{\mathbf{Q}_p^{\mathrm{ab}}} \to \operatorname{Spec} \mathbf{Q}_p^{\mathrm{ab}}$ is $\iota^{\mathrm{ab},\sharp}$).

4.5.1 Whittaker models and *q*-expansions in families

For $v \in S$, let $\mathscr{X}_{(\mathrm{G})}^{(v)} \subset \mathscr{X}_{(\mathrm{G})}$ and $\mathscr{W}_v$ be as in Section 4.4, and let $\mathscr{X}'_{(\mathrm{G})} := \bigcap_{v \in S} \mathscr{X}_{(\mathrm{G})}^{(v)}$; it contains $\mathscr{X}_{(\mathrm{G})}^{\mathrm{cl}}$.

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Lemma 4.8 *There is an isomorphism of $\mathscr{O}_{\mathscr{X}'_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}}}[\mathrm{G}(F_S)]$-modules*

$$\mathbf{W}_{-,S}\colon \Pi_{\mathbf{Q}_p^{\mathrm{ab}}} \xrightarrow{\cong} \bigotimes_{v\in S} \mathscr{W}_v, \qquad \mathbf{f} \mapsto \mathbf{W}_{\mathbf{f},S} = \otimes_v \mathbf{W}_{\mathbf{f},v} \tag{4.5.1}$$

such that for all classical points $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}}$ and all $a_S \in F_S^\times$, we have

$$\mathbf{W}_{\mathbf{f},S}\left(\begin{pmatrix} a_S & \\ & 1\end{pmatrix}\right)(x) = W_{\mathbf{f}(x)}(a_S 1^{S\infty}), \tag{4.5.2}$$

where the right-hand side is the p-adic q-expansion of $\mathbf{f}(x)$ defined in Section 2.3.2.

Proof By Proposition 2.4 and [Dis20, Theorem 4.4.3], after possibly shrinking $\mathscr{X}'_{\mathrm{G}}$, there exist an invertible sheaf $\mathscr{W}^{S,U^S}$ over $\mathscr{X}'_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}}$ with trivial $\mathrm{G}(F_S)$-action and an $\mathscr{O}_{\mathscr{X}'_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}}}[\mathrm{G}(F_S)]$-isomorphism

$$\mathbf{W}\colon \Pi_{\mathbf{Q}_p^{\mathrm{ab}}} \xrightarrow{\cong} \mathscr{W}^{S,U^S} \otimes \bigotimes_{v\in S} \mathscr{W}_v, \tag{4.5.3}$$

unique up to $\mathscr{O}^\times_{\mathscr{X}'_{\mathrm{G},\mathbf{Q}_p^{\mathrm{ab}}}}$, that we may write locally as

$$\mathbf{f} \longmapsto \mathbf{W}^S(1^S) \otimes \mathbf{W}_{\mathbf{f},S} = \mathbf{W}^S_{\mathbf{f}}(1^S) \otimes \otimes_{v\in S}\mathbf{W}_{\mathbf{f},v},$$

where $\mathbf{W}^S(1^S)$ is a section trivializing $\mathscr{W}^{S,U^S}$.

For $v \notin S \cup S_{p\infty}$ and $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$, let $\lambda_{x,v}\colon F_v^\times \to \mathbf{Q}_p(x)$ be the smooth function such that $W_v(a) = \lambda_{x,v}(a) W_v(1)$ for any spherical element W_v in the Kirillov model of $\pi_{x,v}$; by the standard formulas (see, for instance, [Wal85, p. 190]), there are functions $\lambda_v\colon F_v^\times \to \mathscr{O}(\mathscr{X}_{\mathrm{G}})$ such that $\lambda_v(x) = \lambda_{x,v}$ for all $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$. Let $\lambda^{Sp} := \otimes_{v\notin S\cup S_{p\infty}} \lambda_v$, and let $\alpha_p^\circ\colon F_p^\times \to \mathscr{O}(\mathscr{X}_{\mathrm{G}})^\times$ be the U_p°-eigencharacter. Then we may define a pair of injective maps in $\mathrm{Hom}_{\mathscr{H}^{\mathrm{ord}}_{\mathrm{G}}}(\Pi, \mathscr{O}^{\mathbf{A}^{\infty,\times}}_{\mathscr{X}'_{\mathrm{G}}})$ by

$$\mathbf{f} \mapsto (W_{\mathbf{f}}(a)), \qquad \mathbf{f} \mapsto (\alpha^\circ(a_p)\lambda^{Sp}(a^{Sp})\mathbf{W}_{\mathbf{f},S}(a_S)),$$

where the former arises from (2.4.5) and interpolates the q-expansions $(W_{\mathbf{f}_x}(a))$ for $x \in \mathscr{X}^{\mathrm{cl}}_{\mathrm{G}}$. By [Dis20, Lemma 4.2.5], the maps differ by a scalar in $\mathscr{O}_{\mathscr{X}'_{\mathrm{G}}}$. It follows that the invertible sheaf $\mathscr{W}^{S,U^S}$ is trivial, and that, from (4.5.3), we may deduce an isomorphism (4.5.1) normalized so as to satisfy (4.5.2) ∎

4.5.2 Definition of the *p*-adic *L*-function and interpolation property

For each classical point $(x, y) \in \mathscr{X}^{\mathrm{cl,sd}}$ and each place $v \nmid p\infty$, let $\mathbf{V}_{(x,y),v}$ be the quadratic space given by the application of Lemma 4.4 to π_x and χ_y.

Lemma 4.9 *The quadratic space $\mathbf{V}_v = \mathbf{V}_{(x,y),v}$ is independent of $(x, y) \in \mathscr{X}^{\mathrm{cl,sd}}$.*

Proof This follows from the characterization in (3.1.3) and the constancy results for epsilon factors of [Dis20, Corollary 5.3.3]. ∎

Let $\mathbf{V}^{p\infty} := \otimes_{v\nmid p\infty} \mathbf{V}_v$, and assume that S is not disjoint from the set S' of inert places v where U^p is maximal. Let

$$\mathcal{A} \subset (\Pi_{\mathscr{K}(\mathscr{X}_{\mathrm{G}})} - \{0\}) \times \mathcal{S}(\mathbf{V}^{p\infty} \times \mathbf{A}^{p\infty,\times})$$

be the set of those pairs $(\mathbf{f}, \phi^{p\infty})$ such that $\phi^{Sp\infty}$ is standard, (3.3.7) holds at an inert place $v \in S \cap S'$, and the meromorphic function $\mathbf{R}_v(\mathbf{W}_{\mathbf{f},v}, \phi_v)$ on $\mathscr{X}$ is nonzero for all $v \in S$.

For $(\mathbf{f}, \phi^{p\infty}) \in \mathcal{A}$, we define a meromorphic function

$$\mathscr{L}_p(\mathscr{V}, \mathbf{f}, \phi^{p\infty}) \in \mathscr{K}(\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}}),$$

$$\mathscr{L}_p(\mathscr{V}, \mathbf{f}, \phi^{p\infty})(x,y) := C \frac{\langle\langle \mathbf{f}_x, \mathbf{I}^{\mathrm{ord},\vee}(\phi^{p\infty}; x, y)\rangle\rangle}{\prod_{v\in S} \mathbf{R}_v(\mathbf{W}_{\mathbf{f}|x,v}, \phi_v, \chi_{y,v})}, \tag{4.5.4}$$

where we still denote by $\mathbf{I}^{\mathrm{ord},\vee}(\phi^{p\infty})$ the restriction to $\mathscr{X}$ of the $(\mathscr{Y}_{\mathrm{G}} \hat{\times} \mathscr{Y}_{\mathrm{H}})$-adic form of (3.4.10), and

$$C = C(x,y) := \omega_x^{p\infty} \omega_y^{p\infty}(-1) L(1, \eta_p) \frac{|D_F|^{-1/2} \zeta_F(2)}{\pi^{[F:\mathbf{Q}]}}$$

is a constant in $\mathbf{Q}^\times$; here, $\omega_x = \omega_{\pi_x}$ and $\omega_y = \omega_{\chi_y}$. Note that the (base change of the) functional $\langle\langle \mathbf{f}, -\rangle\rangle$ may be applied to $\mathbf{I}^{\mathrm{ord},\vee}(\phi^{p\infty})$, thanks to Lemma 2.3.

Proposition 4.10 *The collection*

$$(\mathscr{L}_p(\mathscr{V}, \mathbf{f}, \phi^{p\infty}))_{(\mathbf{f}, \phi^{p\infty}) \in \mathcal{A}}$$

of meromorphic functions on $\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}}$ has the following properties.

(1) *Let $(x,y) \in \mathscr{X}^{\mathrm{cl}}_{\mathbf{Q}_p^{\mathrm{ab}}/\mathbf{Q}_p^{\mathrm{ab}}}(\mathbf{C})$ have contracted weight (k_0, w, l) satisfying*

$$|l_\tau| \le w_\tau - 2, \qquad |k_0| \le w_\tau - 2 - |l_\tau|. \tag{4.5.5}$$

If (x,y) is outside the polar locus of $\mathscr{L}_p(\mathscr{V}, \mathbf{f}, \phi^{p\infty})$, we have

$$\mathscr{L}_p(\mathscr{V}, \mathbf{f}, \phi^{p\infty})(x,y) = e_{p\infty}(V_{(\pi,\chi)}) \cdot \mathscr{L}(V_{(\pi,\chi)}, 0), \tag{4.5.6}$$

where $\pi = \pi_x$, $\chi = \chi_y$.

(2) *For each $(x,y) \in \mathscr{X}^{\mathrm{cl,sd}}$, there is a pair $(\mathbf{f}, \phi^{p\infty}) \in \mathcal{A}$ such that $\mathscr{L}_p(\mathscr{V}, \mathbf{f}, \phi^{p\infty})$ does not have a pole at (x,y).*

Note that the right-hand side of (4.5.6) is the same as in (1.1.8) and independent of $(\mathbf{f}, \phi^{p\infty})$. This will enable us to glue the various $\mathscr{L}(\mathscr{V}, \mathbf{f}, \phi^{p\infty})$ into the sought-for p-adic L-function.

Proof The second statement follows from Lemma 4.4.

It remains to prove the interpolation property. Abbreviate $\mathscr{L}_p = \mathscr{L}_p(\mathscr{V}, \mathbf{f}, \phi^{p\infty})$, and let $\overline{W}_v := \overline{\mathbf{W}}_{\mathbf{f}|x}$, $\alpha = \alpha_\pi$. Denote by $(x_0, y_0) \in \mathscr{X}^{\mathrm{cl}}$ and $\iota \colon \mathbf{Q}_p^{\mathrm{ab}}(x_0, y_0) \hookrightarrow \mathbf{C}$ the data corresponding to (x,y). Let $k_{x,y}$ and $\xi_{x,y}$ (respectively, k_{x_0,y_0}, ξ_{x_0,y_0}, $\underline{\kappa}_2 = \underline{\kappa}_{2,x_0,y_0}$) be defined by (4.2.1) (respectively, by the analogous formulas for the objects attached to (x_0, y_0) instead of (x,y)).

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By the definitions and the defining property of $\langle\langle\ ,\ \rangle\rangle$ in Proposition 4.2, and of $\mathbf{R}_v$ in Proposition 4.7, we have

$$\begin{aligned}\mathscr{L}_p(x,y) &= C\cdot\frac{\iota\langle \mathbf{f}_{x_0}, e^{\mathrm{ord}}\mathrm{I}(\phi^{p\infty};\chi_{y_0},\xi_{x_0,y_0},k_{x_0,y_0}\rangle}{\prod_{v\in S}R_v^{\natural}(\iota\overline{W}_v,\phi_v,\chi_v)}\\ &= C\cdot\frac{|D_F|^{1/2}\zeta_F(2)\cdot|D_F||D_E|^{1/2}}{\omega_{x,p}(-1)\cdot e_p(\mathrm{ad}(V_\pi)(1))\cdot 2^{\sum_{v|\infty}-1-w_v}}\cdot\frac{q_{F,p}^r\alpha_\pi^{-r}(\mathbf{f}_x^{\mathrm{a}},w_{r,p}^{-1}I_r(\phi^{p\infty};\chi,\xi_{x,y},\kappa_2',k_{x,y})}{L(1,\pi,\mathrm{ad})\cdot\prod_{v\in S}R_v^{\natural}(\iota\overline{W}_v,\phi_v,\chi_v)},\end{aligned}$$

where $r\in(\mathbf{Z}_{\geq 1})^{S_p}$ is sufficiently large, and the second equality follows from the interpolation properties of $\mathbf{I}^{\mathrm{ord},\vee}$ in (3.4.11), and of $\langle\ ,\ \rangle$ in Lemma 4.1.

Using first Waldspurger's integral representation as in Proposition 4.3, and then the calculations of local integrals in Lemmas 4.5 and 4.6, we find

$$\begin{aligned}\mathscr{L}_p(x,y) &= C\cdot\frac{\prod_{v|p}R_{r,v}^{\dagger}(\overline{W}_v,\chi_v,\alpha_v)}{2^{\sum_\tau -1-w_\tau}\omega_{x,p}(-1)\cdot e_p(\mathrm{ad}(V_\pi)(1))}\cdot\frac{|D_F|^{1/2}L(1/2,\pi_E\otimes\chi)}{L(1,\eta)L(1,\pi,\mathrm{ad})}\prod_{v|\infty}R_v^{\dagger}(\overline{W}_v,\chi_v,k_v)\\ &= C\cdot i^{-k_0[F:\mathbf{Q}]}\omega_{x,p}\omega_{y,p}^{\mathrm{sm}}(-1)\frac{e_p(V_{\pi_E\otimes\chi})}{L(1,\eta_p)\cdot e_p(\mathrm{ad}(V_\pi)(1))}\cdot\frac{|D_F|^{1/2}L(1/2,\pi_E\otimes\chi)}{L(1,\eta)L(1,\pi,\mathrm{ad})}.\\ &= i^{k_0[F:\mathbf{Q}]}\cdot e_p(V_{(\pi,\chi)})\cdot\mathscr{L}(V_{(\pi,\chi)},0),\end{aligned}$$

as desired. ∎

Remark 4.11 The interpolation factors $e_{p\infty}(V_{(\pi,\chi)})$ are easily seen to agree with the predictions of Coates and Perrin-Riou (see [Coa91]) for a (cyclotomic) p-adic L-function attached to the "virtual motive" (1.1.3), up to a subtlety that we now explain. With the notation used in (1.1.4), for $v|p$, consider the G_{F_v}-representations

$$\begin{aligned}\mathrm{ad}(V_{\pi,v})(1) &:= \mathrm{End}^0(V_\pi)(1)\\ &\qquad\supset \mathrm{ad}(V_{\pi,v})(1)^+ := \mathrm{Ker}\left[\mathrm{ad}(V_\pi)(1)\to\mathrm{Hom}\left(V_{\pi,v}^+,V_{\pi,v}^-\right)(1)\right]\\ &\qquad\supset \mathrm{ad}(V_{\pi,v})(1)^{++} = \mathrm{Hom}\left(V_{\pi,v}^-,V_{\pi,v}^+\right)(1)),\end{aligned}$$

where "0" denotes trace-0 elements, and the cokernel of the second containment is isomorphic to the cyclotomic character. Then (1.1.4) differs from the ratio of the v-adic Coates–Perrin-Riou factors for the hypothetical L-functions of $V_\pi\otimes\mathrm{Ind}_{G_F}^{G_E}V_\chi$ and of $\mathrm{ad}(V_\pi)(1)$ by the appearance of $\gamma(\iota\mathrm{ad}(V_{\pi,v})(1)^{++},\psi_v)^{-1}$ in place of $\gamma(\iota\mathrm{ad}(V_{\pi,v})(1)^{+},\psi_v)^{-1}$. This discrepancy removes the trivial zero $\gamma(\mathbf{C}(1),\psi_v)^{-1}$ from the latter inverse gamma factor.

4.5.3 Rationality and completion of the proof of Theorem A

By Proposition 4.10 and the density of classical points, the functions $\mathscr{L}_p(\mathscr{V},\mathbf{f},\phi^{p\infty}) =$ (4.5.4) $\in\mathscr{K}(\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}})$ glue to a function

$$\mathscr{L}_p(\mathscr{V})\in\mathscr{K}(\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}}),$$

which satisfies the required interpolation property, and whose polar locus does not meet the set $\mathscr{X}^{\mathrm{cl,sd}}$. All that is left to show is that $\mathscr{L}_p(\mathscr{V})$ descends to $\mathscr{K}(\mathscr{X})$. It will be a consequence of the following.

Proposition 4.12 *Let $\mathscr{X}^{\mathrm{cl},\|} \subset \mathscr{X}^{\mathrm{cl}}$ be the sub-ind-scheme of those (x,y) corresponding to a representation $\pi \boxtimes \chi$ whose contracted weight (k_0, w, l) satisfies (4.5.5) and is parallel.*[11] *There is a function*

$$\mathrm{L} \in \mathscr{O}(\mathscr{X}^{\mathrm{cl},\|})$$

such that for any $z = (x,y) \in \mathscr{X}^{\mathrm{cl},\|}(\mathbf{C})$ corresponding to a point $z_0 \in \mathscr{X}^{\mathrm{cl},\|}$ and an embedding $\iota\colon \mathbf{Q}_p(z_0) \hookrightarrow \mathbf{C}$, with attached representation $\pi \boxtimes \chi$, we have

(4.5.7)
$$\mathrm{L}(z) = \iota\mathrm{L}(z_0) = i^{-(1+k_0)[F:\mathbf{Q}]}\gamma(1, \eta^\infty\omega^\infty, \psi^\infty)^{-1}\frac{\zeta_F(2)L(1/2, \pi_E \otimes \chi)}{\pi^{[F:\mathbf{Q}]}L(1, \pi, \mathrm{ad})}.$$

Here, we denote by ω_π the central character of π, set $\omega_\chi := \chi_{|\mathbf{A}^\times}$ $\omega = \omega_\pi\omega_\chi$, and define $\gamma(s, \omega'^\infty, \psi^\infty) := \prod_{v \nmid \infty}\gamma(s, \omega'_v, \psi_v)$.

Remark 4.13 The construction of this paper gives an alternative proof of this result. However, due to the occurrence of the additive character ψ in the definition of the form I (via the Weil representation), keeping track of rationality requires some burdensome bookkeeping.

Proof This is a consequence of a well-known algebraicity theorem of Shimura [Shi78, Theorem 4.2], applied to the newform in the representation π and the CM form attached to χ, whose central character is $\eta\omega_\chi$. (For the comparison of Shimura's periods and adjoint L-values, see [CST14, Proposition 1.11].) ■

Corollary 4.14 *The function $\mathscr{L}_p(\mathscr{V})$ belongs to $\mathscr{K}(\mathscr{X}) \subset \mathscr{K}(\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}})$.*

Proof We need to show that

(4.5.8)
$$\mathscr{L}(\mathscr{V})^\sigma = \mathscr{L}(\mathscr{V})$$

for all $\sigma \in \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{ab}}/\mathbf{Q}_p)$. Let $\mathscr{X}^{\mathrm{cl},\|,\mathrm{reg}}_{\mathbf{Q}_p^{\mathrm{ab}}}$ be the intersection of $\mathscr{X}^{\mathrm{cl},\|}_{\mathbf{Q}_p^{\mathrm{ab}}}$ with the complement of the polar locus of $\mathscr{L}_p(\mathscr{V})$. Since this set is dense in $\mathscr{X}_{\mathbf{Q}_p^{\mathrm{ab}}}$, it suffices to show that (4.5.8) holds for the restriction $\mathrm{L}_p(\mathscr{V})$ of $\mathscr{L}_p(\mathscr{V})$ to $\mathscr{X}^{\mathrm{cl},\|,\mathrm{reg}}_{\mathbf{Q}_p^{\mathrm{ab}}}$; in other words, that $\mathrm{L}_p(\mathscr{V})$ belongs to $\mathscr{O}(\mathscr{X}^{\mathrm{cl},\|,\mathrm{reg}})$.

By (4.5.6) and (4.5.7),

$$\mathrm{L}_p(\mathscr{V})(z) = \frac{i^{[F:\mathbf{Q}]}\gamma(1,\eta^\infty,\psi^\infty)}{L(1,\eta)} \cdot \frac{\gamma(1,\eta^\infty\omega^\infty,\psi^\infty)}{\gamma(1,\eta^\infty,\psi^\infty)\gamma(1,\omega^\infty,\psi^\infty)} \cdot \frac{1}{\gamma(1,\omega^{p\infty},\psi^{p\infty})^{-1}} \\ \cdot \frac{e_p(V_{(x,y)})}{\gamma(1,\omega_p,\psi_p)^{-1}} \cdot \mathrm{L}(z).$$

We show that all factors belong to $\mathscr{O}(\mathscr{X}^{\mathrm{cl},\|,\mathrm{reg}})$:

- By the class number formula and standard results on Gauß sums, the ratio $L(1,\eta)/i^{[F:\mathbf{Q}]}\gamma(1,\eta^\infty)$ is rational, as both numerator and denominator are rational multiples of $|D_{E/F}|^{-1/2}$.

[11] That is, w_τ is independent of $\tau \in \Sigma_\infty$ and so is l_τ. Without this condition, we may have a slightly weaker result.

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- By (4.0.1), the second and fourth ratios are values of functions on $\mathscr{O}(\mathscr{X}^{\mathrm{cl},\|})$, as $e_p(V_{(x,y)})$ is a ratio of inverse gamma factors of characters whose ratio is ω_p.
- As $\mathscr{X}$ is connected and it contains points z with $\omega_z = \mathbf{1}$, the character $\omega_{z,v}$ is unramified for all $z \in \mathscr{X}$ and all $v \nmid p$; thus, for those v, the quantity $\gamma(\omega_v)$ is a ratio of L-values, and hence the third factor is also the value of a function in $\mathscr{O}(\mathscr{X}^{\mathrm{cl},\|})$.
- Finally, $\mathrm{L} \in \mathscr{O}(\mathscr{X}^{\mathrm{cl},\|})$ by Proposition 4.12.

This completes the proof of the corollary and of Theorem A. ∎

A Reality shows and double-factorial identities

Consider the identity

$$(*) \qquad \sum_{k=0}^{n} \binom{n}{k} (2k-1)!!(2n-2k-1)!! = 2^n n!,$$

where we recall that $(2m-1)!! = 1 \cdot 3 \cdot 5 \cdots (2m-1)$ is the number of perfect matchings (into pairs) of a $2m$-element set. Since $\Gamma(j+1/2) = \frac{(2j-1)!!}{2^j}\sqrt{\pi}$, the identity $(*)$ is equivalent to (3.3.9).

Quick analytic proofs of $(*)$ have appeared in [AA10] and [GQ12, Theorem 3]. As we were not able to find a bijective proof in the literature, we give one here. Another bijective proof was communicated to the author by David Callan.

A *reality TV show format* is an algorithm whose inputs are called *players' choices* and whose outputs are called *outcomes* (the set of players is partitioned into two disjoint sets, the *producers* and the *participants*). A format is said to be *bijective* if its set of players' choices is in bijection with its set of outcomes.

We will describe two bijective formats for reality TV shows, with different sets of players' choices but the same set of outcomes. In each case, there are $2n$ participants forming an ordered set of n heterosexual couples[12]; there are two tropical islands, Q and H, and in each case, the outcome is:

- a new matching of the participants into n disjoint couples (which may be homosexual or heterosexual), and
- an assignment of each participant to either island Q or island H, such that
- each person lives in the same island as both their old and their new partners.

Show 1. The producers choose a set of couples, send all their members to island Q, and send all the other participants to island H. Within each island, people mingle until they form new disjoint couples (heterosexual or homosexual) as they wish.

Show 2. The producers pick a permutation $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$, then they do the following.

- Initialize: $i = 1$ and the set variable $C = \varnothing$ (where C is for "cycle"; to be thought of as the set of couples embarked in the show's boat at a given time).

12 These TV shows, for simplicity or close mindedness, assume the gender binary.

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- Process:
  (a) Consider couple i, set $C_{\mathrm{new}} = C_{\mathrm{old}} \cup \{i\}$, and interview couple member p_i where: if $C = \{i\}$, then p_i is the woman; if $C \supsetneq \{i\}$, then p_i is the one that does not yet have a new partner.
  The possible answers to the interview question are "H" and "Q".
  (b) If $j = \sigma(i) \notin C$ and p_i responds H (resp. Q):
  – rematch p_i with the person of opposite (respectively, the same) sex of couple $j = \sigma(i)$. Set $i_{\mathrm{new}} = j$. Return to (a).
  (c) If $j = \sigma(i) \in C$ and p_i responds H (resp. Q):
  – rematch p_i with the unique nonrematched person of couple j, and send all members of the "original couples" in C to island H (respectively, Q); set $C_{\mathrm{new}} = \varnothing$;
  – if everyone has been rematched, STOP. Else: set $i_{\mathrm{new}} \in \{1, \ldots, n\}$ to be the smallest such that neither member of couple i_{new} has been rematched. Return to (a).

Proof of $(*)$ The number of possible players' choices in Show 1 is the left-hand side of $(*)$. The number of possible players' choices in Show 2 is the right-hand side of $(*)$. However, the shows are bijective with the same set of outcomes. ■

B Errata to [Dis20]

The conclusions of the statements of Lemma 5.2.2 and Propositions 5.2.3 and 5.2.4 should, respectively, have $A[T^{\pm 1}]$, $\mathscr{O}_X[T^{\pm 1}]$, and $\mathscr{O}_X[T^{\pm 1}]$ instead of $A[T]$, $\mathscr{O}_X[T]$, and $\mathscr{O}_X[T]$.

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