SOME REMARKS ON A PAPER OF D. W. KAHN

C. S. Hoo[1]

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1. Suppose X is a simply-connected CW-complex whose homotopy groups are finite. For each n, let $\mathcal{C}(X, n)$ be the class of torsion groups whose p-primary components are zero for all primes p which do not figure in the homotopy groups $\pi_i(X)$ for $i \leq n$. D. W. Kahn [3] showed that for every unitary bundle ξ over X, the nth Chern class $c_n(\xi)$ is contained in a subgroup of $H^{2n}(X, Z)$ which belongs to $\mathcal{C}(X, 2n-1)$. If $\lambda(n)$ denotes the order of the group $\pi_n(X)$, we shall show further that $\lambda(2n-1)c_n(\xi)$ is contained in a subgroup of $H^{2n}(X, Z)$ which belongs to $\mathcal{C}(X, 2n-2)$. In fact, these results are true for any integral cohomology class of such a space X and are not peculiar to Chern classes, and hold for the odd dimensional classes too. Kahn's result and our result are immediate corollaries of

THEOREM. If X is a simply-connected CW-complex whose homotopy groups are finite, then $H^n(X, Z) \in \mathcal{C}(X, n-1)$ and $\lambda(n-1) H^n(X, Z) \in \mathcal{C}(X, n-2)$ where $\lambda(n-1)$ is the order of $\pi_{n-1}(X)$.

2. In this section all homology and cohomology will be taken with integer coefficients. We first make some preliminary remarks. We observe that the class $\mathcal{C}(X, n)$ is strongly complete, in the terminology of Hu [2]. Suppose $\{X_n, p_n, \pi_n\}$ is a Postnikov system for X. This means that $p_n: X \to X_n$ is an n-equivalence, $\pi_n: X_n \to X_{n-1}$ is a principal $K(\pi_n(X), n)$ fibre space, and

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$\pi_n p_n \simeq p_{n-1}$. Since $\mathcal{C}(X, n)$ is strongly complete, it follows that $H_m(\pi_j(X), j) \in \mathcal{C}(X, n)$ for all $m > 0$ and for all $j \leq n$. It is then easy to show, by induction, that $H_m(X_j) \in \mathcal{C}(X, n)$ for all $m > 0$ and all $j \leq n$. (For example, see Chapter 10 of [2]).

Proof of Theorem. Suppose X_{n-1} is the term in a Postnikov decomposition of X in which we have added all the homotopy groups of X in dimensions less than n. Then we have a map $p: X \to X_{n-1}$ which induces isomorphisms in homotopy in dimensions less than n. If we convert this map into a fibre map, then we have a fibration $F \xrightarrow{i} X \xrightarrow{p} X_{n-1}$ with F being $(n-1)$ connected, and i inducing isomorphisms in homotopy in dimensions greater than $(n-1)$. Since X, and consequently X_{n-1}, is simply connected, this fibration provides an exact sequence, part of which is the following:

$$0 \to H^n(X_{n-1}) \xrightarrow{p^*} H^n(X) \xrightarrow{i^*} H^n(F) \to \dots .$$

Since F is $(n-1)$ connected and $\pi_n(F) \cong \pi_n(X)$, the universal coefficient theorem gives

$$H^n(F) \cong \text{Hom}(H_n(F), Z) \cong \text{Hom}(\pi_n(X), Z) .$$

Since $\pi_n(X)$ is finite, it follows that $H^n(F) = 0$. Hence $p^*: H^n(X_{n-1}) \cong H^n(X)$. Now

$$H^n(X_{n-1}) \cong \text{Hom}(H_n(X_{n-1}), Z) + \text{Ext}(H_{n-1}(X_{n-1}), Z)$$

by the universal coefficient theorem. Since $H_n(X_{n-1})$ and $H_{n-1}(X_{n-1})$ are elements of $\mathcal{C}(X, n-1)$ and hence are finite, we have $H^n(X_{n-1}) \cong H_{n-1}(X_{n-1}) \in \mathcal{C}(X, n-1)$. This proves the first part of the theorem.

The proof of the second part is along the same lines. Let

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X_{n-2} be the term in a Postnikov decomposition of X in which we have added all the homotopy groups in dimensions less than $(n-1)$. Then we have a fibration $F \xrightarrow{i} X \xrightarrow{p} X_{n-2}$ where F is obtained from X by killing all homotopy groups in dimensions less than $(n-1)$. Again we have an exact sequence which ends as follows:

$$\to H^{n-1}(F) \xrightarrow{\tau} H^n(X_{n-2}) \xrightarrow{p^*} H^n(X) \xrightarrow{i^*} H^n(F),$$

where τ is the transgression. Let us consider $H^n(F)$. A Postnikov decomposition of F begins as follows:

$$\begin{array}{ccc} & & \downarrow \\ K(\pi_n(X), n) & \to & G \\ & & \downarrow \\ & & K(\pi_{n-1}(X), n-1) \end{array}$$

with $H^n(F) \cong H^n(G)$. The fibration

$$K(\pi_n(X), n) \to G \to K(\pi_{n-1}(X), n-1)$$

gives an exact sequence:

$$0 \to H^n(\pi_{n-1}(X), n-1) \to H^n(G) \to$$

$$H^n(\pi_n(X), n) \to H^{n+1}(\pi_{n-1}(X), n-1) \to \dots .$$

But $H^n(\pi_n(X), n) \cong \mathrm{Hom}(\pi_n(X), Z) = 0$ since $\pi_n(X)$ is finite. Hence $H^n(F) \cong H^n(G) \cong H^n(\pi_{n-1}(X), n-1)$. But

$$H^n(\pi_{n-1}(X), n-1) \cong \mathrm{Hom}(H_n(\pi_{n-1}(X), n-1), Z) + \mathrm{Ext}(H_{n-1}(\pi_{n-1}(X), n-1), Z)$$

$$\cong \pi_{n-1}(X)$$

since $H_n(\pi_{n-1}(X), n-1) = 0$ by [1], and $\pi_{n-1}(X)$ is finite. Thus

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$H^n(F) \cong \pi_{n-1}(X)$.

Hence $\lambda(n-1)\, H^n(X) \subset \ker i^*$ where $\lambda(n-1)$ is the order of $\pi_{n-1}(X)$. Hence $\lambda(n-1)\, H^n(X) \subset p^* H^n(X_{n-2})$. Since

$$H^n(X_{n-2}) \cong \mathrm{Hom}(H_n(X_{n-2}), Z) + \mathrm{Ext}(H_{n-1}(X_{n-2}), Z)$$
$$\cong H_{n-1}(X_{n-2})$$

$\in \mathcal{C}(X, n-2)$, this completes the proof of the Theorem.

Remarks. The Theorem gives some information regarding the order of some Chern classes. For example, suppose X is 2k-connected and has finite homotopy. Let ξ be a unitary bundle over X . We observe that $\mathcal{C}(X, 2k)$ contains only the group with one element. Hence, according to the Theorem, we have $\lambda(2k+1)\, c_{k+1}(\xi) = 0$ where $\lambda(2k+1)$ is the order of $\pi_{2k+1}(X)$. Thus the order of $c_{k+1}(\xi)$ divides the order of $\pi_{2k+1}(X)$.

Similarly, if X is $(2k-1)$ connected, then clearly $c_k(\xi) \in H^{2k}(X, Z) \cong \mathrm{Hom}(\pi_{2k}(X), Z) = 0$. The order of $c_{k+1}(\xi)$ is a little bit more complicated. The Theorem tells us that $\lambda(2k+1)\, c_{k+1}(\xi)$ is an element of a subgroup of $H^{2k+2}(X, Z)$ which belongs to $\mathcal{C}(X, 2k)$. Let P be the family of all primes p such that the p-primary component of $\pi_{2k}(X)$ is non-zero. For each $p \in P$, let $r(p)$ be the exponent of the highest power of p which divides the order of $H^{2k+2}(X, Z)$. Let $m(k) = \prod_{p \in P} p^{r(p)}$. Then $m(k)\, \lambda(2k+1)\, c_{k+1}(\xi) = 0$. Thus the order of $c_{k+1}(\xi)$ divides $m(k)\, \lambda(2k+1)$.

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University of Alberta, Edmonton

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