VOLUME III JULY, 1903 No. 3

THE GRAPHIC METHOD OF CONSTRUCTING A LIFE TABLE ILLUSTRATED BY THE BRIGHTON LIFE TABLE, 1891—1900.

BY ARTHUR NEWSHOLME, M.D. (Lond.), F.R.C.P., AND
T. H. C. STEVENSON, M.D. (Lond.), D.P.H. (Camb.).

TEN years ago one of us constructed a Life Table for Brighton(1), based on the experience of the ten years 1881—90. This Life Table was constructed by the graphic method originally employed by Milne in the construction of his famous Carlisle Table(2); and the same method of construction is here adopted and described. Dr Hayward has given in this Journal(3) a description of the analytical methods of constructing a Life Table. In the present paper it is proposed to describe and discuss the graphic method of constructing a Life Table. Medical Officers of Health can then decide which method of construction they will adopt. Remarks on the comparative value of the two methods will be made at a later part of this paper. So far as practicable, an example will be given of the working at each stage, in order that the calculations may be followed with a minimum of trouble.

Data.

The data on which a Life Table is formed are the number and ages of the living and the number and ages of the dying. The ideal Life Table would represent "a generation of individuals passing through time" and measure the probabilities of life and death of this generation at birth, and of the survivors at each successive age until the whole generation became extinct. The experience thus watched would be obsolete before it was available. In practice therefore it is preferable to investigate the mortality experience of a population at various ages, from birth to the most advanced age during a recent period.

The present Life Table deals with the experience of Brighton in 1891—1900. The practice of the Registrar-General with regard to the inclusion of deaths occurring in institutions within and exclusion of deaths without Brighton has been strictly followed. No deductions have been made for deaths of visitors occurring in private houses, boarding-houses, and hotels in the town.

The first step is to ascertain the populations in 1891 and 1901, and the deaths in the ten years 1891—1900. These are given in the following table :—

Age	Population of Brighton: Census 1891 Males	Census 1891 Females	Census 1901 Males	Census 1901 Females	Deaths in Brighton 1891—1900 Males	Deaths in Brighton 1891—1900 Females
0—	7189	7185	7226	7425	4368	3804
5—	7260	7290	7088	7185	237	257
10—	6940	7419	7031	7479	144	194
15—	5960	8729	6694	9305	232	229
20—	5035	9179	5891	9992	285	264
25—	9280	14,095	10,339	16,582	688	665
35—	7413	10,569	8888	12,454	966	928
45—	5412	7867	6630	9495	1197	1176
55—	3632	5528	4341	6537	1319	1415
65—	2293	3511	2533	3889	1509	1942
75—	787	1314	976	1650	1140	1790
85 & up.	92	229	122	261	293	591
	61,293	82,915	67,759	92,254		

Number of Years of Life in Age-Groups.

It is then necessary to ascertain the total years of life at risk in the ten years.

The following example illustrates the method of obtaining these for each age-period.

If P_1 is the male census population at ages 10—15 in 1891 and P_2 the same in 1901 and r the population resulting per unit in the ten years, then the total years of life at risk at age 10—15 $= \dfrac{P_1 \times r^{\frac{1}{10}} \times (r-1)}{r^{\frac{1}{10}} - 1}$.

$$r = \frac{P_2}{P_1}, \; P_1 = 6940, \text{ and } P_2 = 7031,$$

therefore $$r = \tfrac{7031}{6940}.$$

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

Therefore $\log r = \log 7031 - \log 6940$

$$\log 7031 = 3{\cdot}8470171$$
$$\log 6940 = 3{\cdot}8413595$$
$$\log r = {\cdot}0056576$$
$$\log r^{\frac{1}{10}} = \tfrac{1}{10}\log r = {\cdot}00056576$$
$$\log r^{\frac{1}{40}} = \tfrac{1}{40}\log r = {\cdot}00014144$$
$$r = 1{\cdot}01311233$$
$$r^{\frac{1}{10}} = 1{\cdot}001303553.$$

From the above formula log. of total years of life at risk required

$$= \log P_1 + \log r^{\frac{1}{40}} + \log(r-1) - \log(r^{\frac{1}{10}} - 1).$$
$$\log P_1 = 3{\cdot}8413595$$
$$\log r^{\frac{1}{40}} = {\cdot}0001414$$
$$\log(r-1) = \bar{2}{\cdot}1176796$$
$$1{\cdot}9591805$$
$$\log(r^{\frac{1}{10}} - 1) = \bar{3}{\cdot}1151286$$
$$4{\cdot}8440519 = \text{log of years of life required.}$$

Therefore total years of life at risk $= 69{,}831$.

N.B. To ensure accuracy the value of $r^{\frac{1}{10}}$ must be calculated by the use of eight figure logarithms (as given in the ordinary table-books for the numbers 10,000 to 10,800). Using seven figure logarithms in the above calculation the result obtained is 69,829.

By the use of the following method the same result is obtained in three stages:

1. The ratio of increase in the ten years is first found as before by the formula

$$P_2 = P_1 \times r,$$
$$7031 = 6940 \times r,$$

therefore $r = \frac{7031}{6940}$.

Therefore $\log r = \log 7031 - \log 6940$

$$\log 7031 = 3{\cdot}8470171$$
$$\log 6940 = 3{\cdot}8413595$$
$$\log r = {\cdot}0056576$$

and as before $\log r^{\frac{1}{10}} = {\cdot}00056576$

and $r^{\frac{1}{10}} = 1{\cdot}001303553 =$ population resulting per unit per annum.

2. The central population of each census year (that is, the population at the middle of the year, three months after the assumed census date, Apr. 1) is then found by the formula $C = P \times r^{\frac{1}{40}}$.

Therefore $\log C_1 = \log P_1 + \frac{1}{40}\log r$,

and $\log C_2 = \log P_2 + \frac{1}{40}\log r$.

Therefore $\log C_1 = 3{\cdot}8413595 + {\cdot}0001414$ or $3{\cdot}8415009$,

and $\log C_2 = 3{\cdot}8470171 + {\cdot}0001414$ or $3{\cdot}8471585$.

Hence $C_1 = 6942{\cdot}260$ and $C_2 = 7033{\cdot}288$.

3. The total years of life at risk in the decennium are now found from the formula

$$\text{years of life at risk} = \frac{C_2 - C_1}{\text{annual increase per unit}} = \frac{91{\cdot}028}{r^{\frac{1}{10}} - 1},$$

therefore log of total years of life at risk required

$$= \log 91{\cdot}028 - \log (r^{\frac{1}{10}} - 1).$$

$$\log 91{\cdot}028 = 1{\cdot}9591750$$

$$\log (r^{\frac{1}{10}} - 1) = \log {\cdot}001303553 = \bar{3}{\cdot}1151286$$

$$4{\cdot}8440464$$

therefore years of life $= 69{,}831$.

Mr A. C. Waters has devised (4) a method of estimating the mean population of an intercensal period which reduces the actual calculation of estimated populations and therefore of years of life at risk into a very simple form. This method is an application of algebra to the method of uniformly changing proportions. He assumes that the whole population increases in geometrical progression at a constant rate, while the proportion of any selected part to the whole changes in arithmetical progression at a constant rate. These assumptions enable consistent estimates to be obtained, in which the summation of the parts is equal to the independent estimate of the population as a whole. Certain factors m and n are obtained in this method, based upon the rate of increase of the population of England and Wales, and these on application to the whole population of Brighton and to each age-group of that population give the mean population. This when multiplied by ten is the total number of years of life at risk.

The value of $m = {\cdot}5445944$, and $\log m = \bar{1}{\cdot}7360732$.

" " $n = {\cdot}4564973$, and $\log n = \bar{1}{\cdot}6594383$.

Thus for males aged 0—5

Total years of life at risk

$$= 10\,(mP_1 + nP_2)$$

$$= 10 \times (m \times 7189 + n \times 7226).$$

$$\log m = \bar{1}{\cdot}7360732$$

$$\log 7189 = 3{\cdot}8566685$$

$$3{\cdot}5927417,$$

therefore $7189 \times m = 3915{\cdot}09$.

$$\log n = \bar{1}{\cdot}6594383$$

$$\log 7226 = 3{\cdot}8588980$$

$$3{\cdot}5183363,$$

therefore $7226 \times n = 3298{\cdot}65$.

Hence number of years of life at risk in 1891–1901

$$= (3915{\cdot}09 + 3298{\cdot}65) \times 10 = 72{,}137.$$

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

The following figures show the years of life of males at each age-group when calculated by the two methods:

Age	Years of life of Males calculated by		Difference
	Old method	Mr Waters's method	
0—	72,067	72,137	70
5—	71,781	71,894	113
10—	69,831	69,891	60
15—	63,016	63,016	0
20—	54,303	54,312	9
25—	97,725	97,735	10
35—	80,913	80,944	31
45—	59,699	59,739	40
55—	39,582	39,596	14
65—	24,050	24,051	1
75—	8,734	8,741	7
85 & upwards	1,055	1,058	3
	642,756	643,114	358

When calculated by means of the factors m and n the number of male years of life is 358 in excess, while the corresponding number of female lives is 428 in excess of the number calculated by the older method.

In the present Life Table the figures obtained by the older method have been used in order that the results for 1891—1900 may be comparable with those for 1881—90. The difference between the results obtained by the two methods is too small materially to affect the Life Table. For those compiling a Life Table for the first time, the use of the factors m and n is recommended, as it saves a considerable number of logarithmic calculations.

Number of Years of Life at Risk and Deaths in Single Years of Life.

Having now obtained a statement of the total number of years of life at risk in quinquennial and decennial groups of ages, the process by which the corresponding numbers for individual years of life have been obtained, must be examined. This, except for the first five years of life, has been done by an adaptation of the graphic method. The reasons for adopting this method, and for regarding it as preferable to the analytical methods are for the sake of convenience given separately on page 314 *et seq.*

The method may be briefly described as follows: Along the abscissa line AZ (Plate XII.) mark off five equal portions, each to represent five years, for the first five quinquennial intervals of age; and let seven other equal portions, each of double length to represent ten years, succeed them for the subsequent decennial intervals of age.

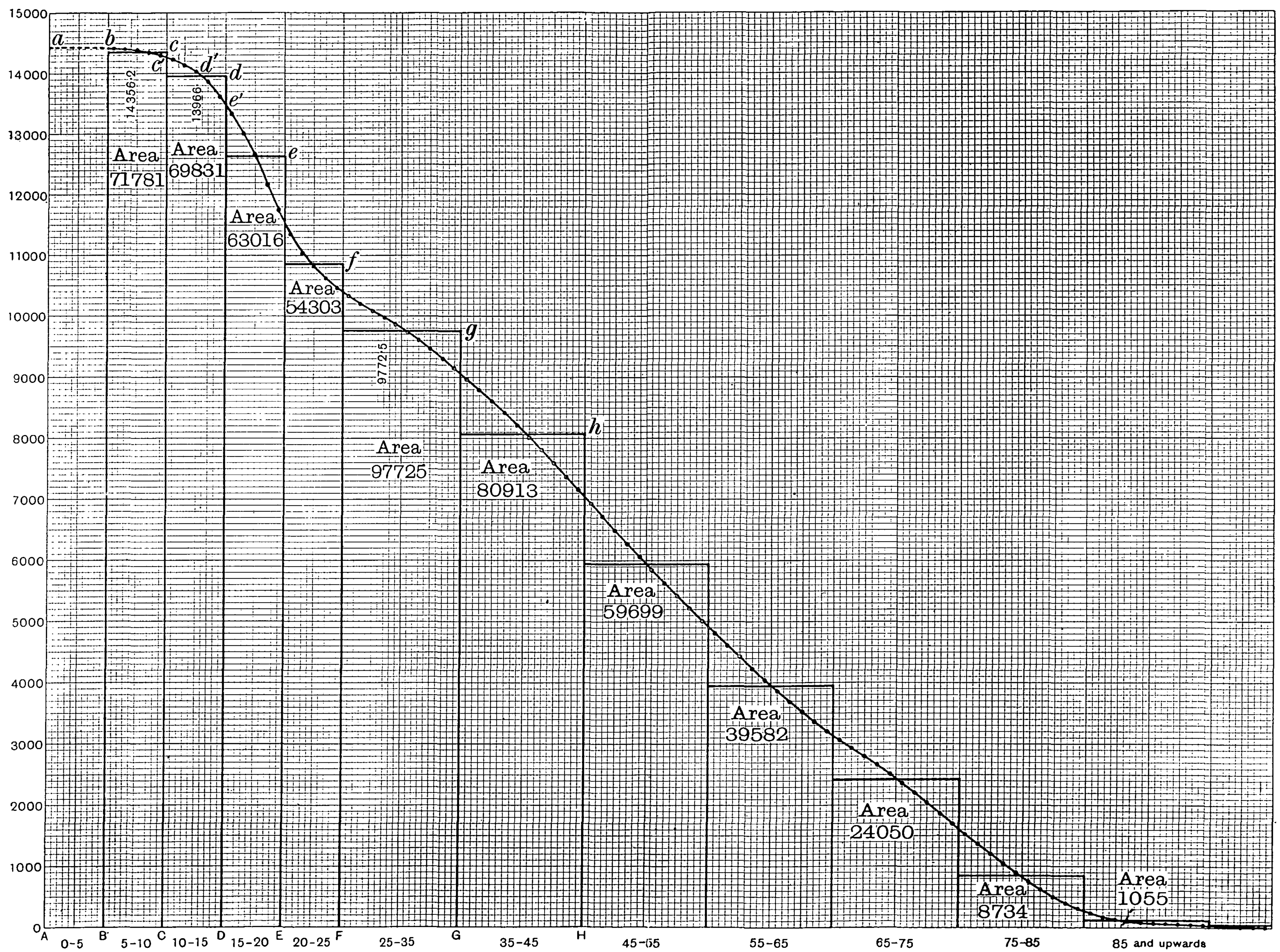
At each of the points, A and B, erect perpendiculars to AZ, and make the perpendicular lines of such a height in accordance with the marginal scale previously decided upon that the parallelogram Ab shall equal in dimensions the population living aged 0—5. Thus in the diagram of the male years of life at risk, $Bb = 14413{\cdot}4$, and this when multiplied by 5, the number of years (0–5) included between A and B $= 72067$. Similarly $Cc = 14356{\cdot}2$, and this when multiplied by 5, gives 71781 as the area of $BbcC$. In the later groups 10 years are taken. Thus $Gg = 9772{\cdot}5$, the area of Fg being 97725. Having thus plotted out the populations living at various groups of ages, the number living at each single year of life is obtained in the following manner.

A curved line is described through the parallelograms already drawn, sweeping as easily as possible through the upper part of these parallelograms from A to Z. This curved line (1) must be as little curved as other conditions will admit of. (2) It must never change its direction abruptly so as to form an angle in its path. (3) The curved line thus described must so cut each of the parallelograms that the area included between the base line below, the corresponding portion of the two ordinates laterally, and the portion of the curved line above, shall equal the area of the parallelogram erected on the same base. Thus the area of the parallelogram $Cd =$ the area of $Cc'd'e'D$. In other words the area cut off must exactly equal the area added.

If now the distances AB, BC, CD, DE, &c., along the abscissa line be divided into equal portions representing one year each, vertical lines drawn from the centre of each of these spaces will give the central population for each year of age.

The accuracy of the curve is confirmed by ascertaining that the sum of the ordinates drawn from the base line within each space to the curved line bounding the space above is equal to the area of the parallelogram drawn on the same base. Thus in Plate XII., $De = 63016$ $=$ the sum of the five ordinates, $13350 + 13030 + 12670 + 12210 + 11750$.

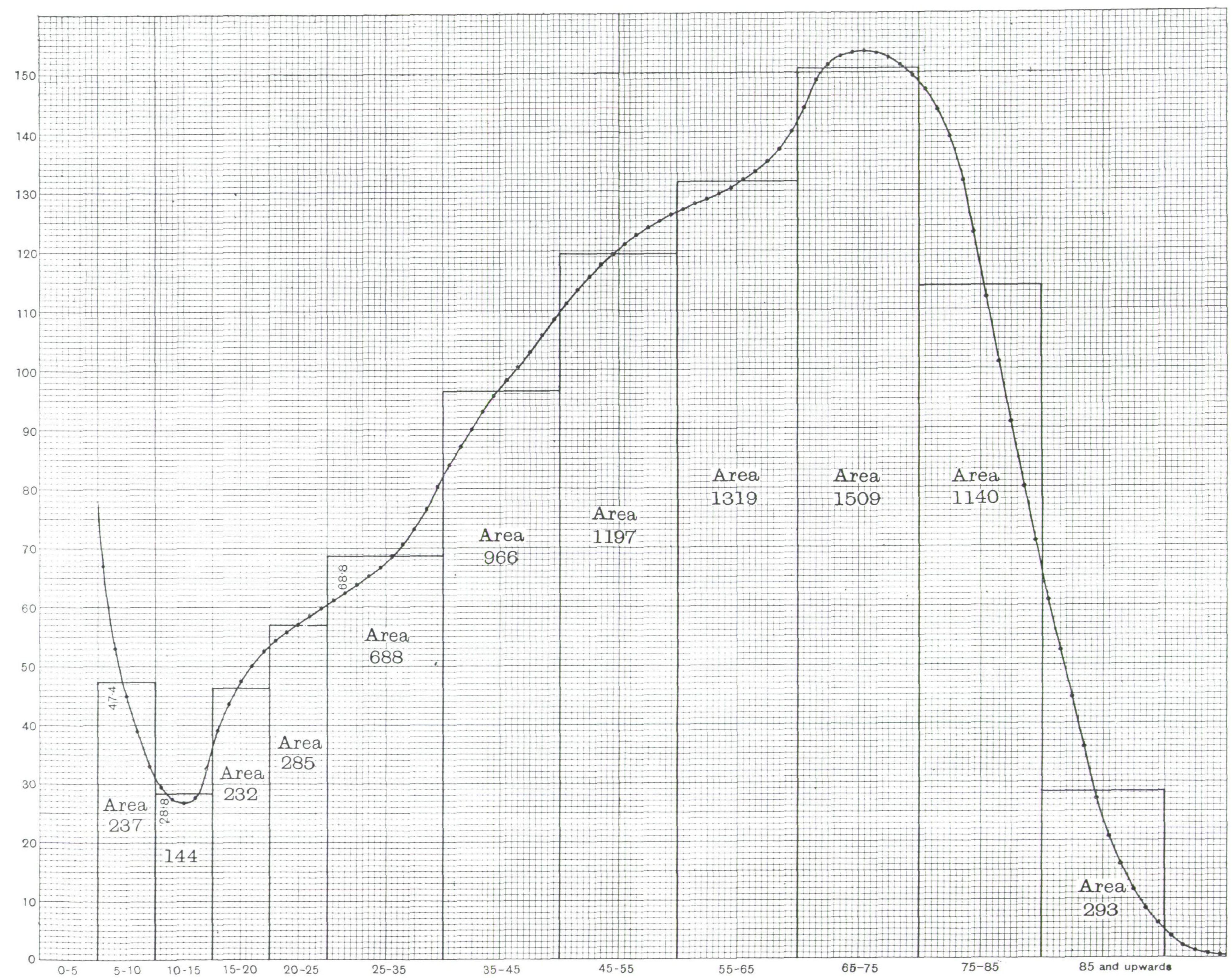
POPULATION.—MALES.



Note. The horizontal dotted line *ab* shewn above is not part of the curve, but represents the upper extremity of the parallelogram Ab.

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

DEATHS.—MALES.



https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

The scale adopted for the curve of lives at risk (Plate XII.) is $\frac{1}{8}'' = 100$, for the deaths (Plate XIII.) $1'' = 8$. The plates as published are reduced to two-fifths of the size of the originals. Experience shows that the errors due to defective measuring of ordinates are insignificant in their effect. A constant check upon such possible errors is maintained by the fact that the sum of the five or ten ordinates must equal the area of the parallelogram for the same age quinquennium or decennium. Hence the error, if any occurs, is merely one of distribution among these 5 or 10 years of life, and affects only the p_x values for these ages. Any possible slight gain or loss at one age is compensated for by a corresponding loss or gain at the immediately previous or successive ages.

If these rules are followed, it is improbable that any material discrepancy will arise when different draughtsmen deal with the same curve.

Having described in full the method by which the central population for each year of life is ascertained, it is unnecessary to describe the same process for the deaths occurring at groups of ages. Plate XIII. dealing with male deaths shows the details of the process. Thus for the deaths $5-10$, the total area is composed of $67{\cdot}0 + 53{\cdot}0 + 45{\cdot}0 + 39{\cdot}0 + 33{\cdot}0 = 5 \times 47{\cdot}4 = 237$.

It is unnecessary to reproduce here the table in which the results obtained as above are set forth. The following extracts sufficiently explain the data.

Total Lives at Risk and Deaths for each Year of Age. Males.

	Population		Deaths	
Age	In original groups	Distributed	In original groups	Distributed
5		14,400		67·0
6		14,380		53·0
7	71,781	14,370	237	45·0
8		14,340		39·0
9		14,291		33·0
10		14,205		30·5
11		14,115		27·5
12	69,831	14,000	144	27·0
13		13,870		27·0
14		13,661		32·0
75		1552		147·3
76		1388		144·0
77		1224		139·0
78		1050		131·7
79		900		123·0
80	8734	750	1140	112·0
81		620		101·0
82		520		91·0
83		410		80·0
84		320		71·0

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

Instead of separately obtaining the years of life at risk and the deaths for each age as shown above, it has been suggested that average p_x values should be calculated for the different age-groups by the formula (see p. 311)

$$p_x = \frac{2P_x - d_x}{2P_x + d_x},$$

and that these average p_x values should then be plotted out as parallelograms, a curve constructed as above described and values of p for each individual year of life read from the curve. But when p_x values for each year of life have been calculated a valuable check on the accuracy of the two curves upon which they are based is constituted by plotting out the p_x values in a curve. If this curve does not run smoothly, the facts as to the lives at risk or deaths in the corresponding years of life can be reconsidered; whereas if the p_x curve be drawn directly no such test can be applied[1]. Furthermore, p_x represents a factorial value, while P_x and d_x are entities. Hence the effect of diminishing or increasing the value of p varies according to the quantity to which it is applied, *i.e.* in the construction of the Life Table, the l_x column. It follows that as the value of l_x is constantly diminishing, a given amount added at ages 25—30 to the average value of p for ages 25—35 must be compensated for by a greater amount subtracted from the values of p at ages 30—35. This is shewn in Plate XIV., p. 310. Thus the rule for checking the accuracy of curves given on p. 302 is not applicable to a p_x curve.

Years of life at risk at ages 0—5.

The graphic method just described gives for a large community accurate results for the years of life from 5 to 85. The first five years of life present special difficulty whatever method of calculating the central population for each of these years of life is adopted. This arises from the fact that the extremely rapid changes in the rate of decrease of mortality at this age-period require very complete data for their exact statement; and the additional data furnished to meet this want by the census and death returns for each age 0—5 are found to be inaccurate. The ages of young children are often erroneously

[1] In our experience the p_x curve has only been found to be irregular to an extent suggesting the desirability of such reconsideration at one place, viz. at 65—75 in the female curve. On reference to the curves of population and deaths there was no difficulty in concluding that the death curve as originally constructed was too much truncated at its apex; and on readjustment a smooth p_x curve was obtained.

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

stated in the census returns. Hence, although the total number aged 0—5 may be accepted as accurate, the distribution of this total at each age under 5 must be found by an independent method.

Two methods have been adopted for this purpose. The first used by Dr Farr [5] was adopted in the Brighton Life Table 1881–90. The second was used in the Life Table for England and Wales for 1881–90. The first method is based on the births during the decennium. The number of children in any one calendar year reaching the exact age of one year may be taken as equal to the births from July 1st to December 31st of the preceding year *plus* the births from January 1st to June 30th in the same year, and *minus* the deaths under one year of age during the same year. Similarly the number of children reaching the age of one year for the ten years 1891–1900 may be taken to be equal to the total births 1890½ to 1899½, *i.e.*, from July 1st 1890 to June 30th 1900 *minus* the number of deaths under one year of age in the ten years, 1891–1900.

Thus having ascertained the total male births from July 1st 1890 to June 30th, 1900, and subtracting from the result the total number of deaths of males under one year of age in the ten years 1891–1900, we obtain the population out of which the deaths at the age 1 —2 occur during the same period. Subtracting the deaths at the age 1—2 we obtain the number out of which the deaths at the age 2—3 occur; subtracting these we obtain the population out of which the deaths at the age 3—4 occur; and subtracting these we obtain the population out of which the deaths 4—5 occur.

The following example will make the above and subsequent steps clear:

First Method:

Male Births July 1, 1890 to June 30, 1900	=18,490·5
Male Deaths under one year, 1891–1900	= 3,036·0
Population at beginning of second year of life	=15,454·5
Male Deaths at age 1—2, 1891–1900	= 718·0
Population at beginning of third year of life	=14,736·5
Male Deaths at age 2—3, 1891–1900	= 307·0
Population at beginning of fourth year of life	=14,429·5
Male Deaths at age 3—4, 1891–1900	= 185·0
Population at beginning of fifth year of life	=14,244·5
Male Deaths at age 4—5, 1891–1900	= 122
Population at beginning of sixth year of life	=14,122·5

Thus the population at birth is	18,490·5
at age 1 is	15,454·5
at age 2 is	14,736·5
at age 3 is	14,429·5
at age 4 is	14,244·5
	77,355·5

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

The mean population for each of these years of life except the first is taken as the geometrical mean of the population at the beginning and at the end of the year, *i.e.* their logs are in arithmetical progression. Thus

$$\begin{aligned} \log 14{,}244{\cdot}5 &= 4{\cdot}1536473 \\ \log 14{,}122{\cdot}5 &= \underline{4{\cdot}1499116} \\ \text{Diff.} &= \ {\cdot}0037357 \\ \tfrac{1}{2}\,\text{Diff.} &= \ {\cdot}00186785 \end{aligned}$$

$$4{\cdot}1499116 + {\cdot}00186785 = 4{\cdot}15177945,$$

therefore mean population during fifth year of life = 14,183·4.

The mean population for the first year of life is obtained by subtracting the deaths under 6 months from the number of births. The results thus obtained are:

Mean population in		1st year of life		= 16,348·5
,,	,,	2nd	,,	= 15,091·2
,,	,,	3rd	,,	= 14,582·2
,,	,,	4th	,,	= 14,336·7
,,	,,	5th	,,	= 14,183·4
Total years of life at risk at ages 0—5				= 74,542·0 = N.

But the same total derived from the census returns is 72,067. Hence each of these five mid-year totals must be reduced in the proportion of $\frac{72067}{74558}$ to make them tally with the census returns. The mean populations thus obtained are:

P_0	15,806
P_1	14,590
P_2	14,098
P_3	13,861
P_4	13,712
	72,067

Second Method:

The second and more accurate method takes into account the births in the $4\frac{1}{2}$ years preceding the beginning of the decennium. An exact description of this method has been given by Dr Hayward [3], and if the steps of the process are exactly followed no difficulty need arise.

Thus the total population at birth in 1891–1900 = 18,490·5.

Population at age 1 equals $\frac{1}{2}$ births in 1889 *plus* births in 9 years 1890–98 *plus* $\frac{1}{2}$ births in 1899 *minus* deaths under 1 year in 10 years 1890—99

$$= 18{,}430{\cdot}5 - 3039 = 15{,}391{\cdot}5.$$

Population at age 2 equals $\frac{1}{2}$ births in 1888 *plus* births in 9 years 1889–97 *plus* $\frac{1}{2}$ births in 1898 *minus* the sum of deaths under 1 year in 1889–98 and at age 1—2 in 1890—99

$$= 18{,}320{\cdot}5 - (2944 + 766) = 14{,}610{\cdot}5.$$

Population at age 3 equals $\frac{1}{2}$ births in 1887 *plus* births in 1888–96 *plus* $\frac{1}{2}$ births in 1898 *minus* the sum of deaths under 1 year in 1888–97, at age 1—2 in 1889–98, and at age 2—3 in 1890–99

$$= 18{,}248{\cdot}5 - (2852 + 801 + 321) = 14{,}274{\cdot}5.$$

Population at age 4 equals ½ births in 1886 *plus* births in 1887–95 *plus* ½ births in 1896 *minus* the sum of deaths under 1 year in 1887–96, of deaths at age 1—2 in 1888–97, at age 2—3 in 1889–98, and at age 3—4 in 1890–99

$$=18{,}216-(2850+819+326+187)=14{,}034.$$

Thus population at birth				=18,490·5
"	"	end of 1st year of life		=15,391·5
"	"	"	2nd "	=14,610·5
"	"	"	3rd "	=14,274·5
"	"	"	4th "	=14,034

The mean population for each of these years of life is obtained for the first year of life by subtracting the deaths under 6 months of age from the population at birth and for each of the next four years by subtracting half the deaths in the same year of life during the ten years 1891–1900. The mean population 0—1 = 16,348·5, at age 1—2 = 15,032·5, at age 2—3 = 14,457, at age 3—4 = 14,182, at age 4—5 = 13,973. These mean populations are reduced in the manner shewn under the first method, so that their sum may coincide with the total population for the age-group 0—5 ascertained from the census enumerations.

The mean populations of the first five years of life by the two methods are as follows:

Old method	New method
15,802	15,923
14,587	14,641
14,095	14,081
13,858	13,813
13,725	13,609
72,067	72,067

In the present Life Table the figures given by the second method have been adopted. The differences in the expectation of life caused by change from the one method to the other are only changes of distribution between these five years of life, no other years of life being affected.

A reference to Plate XIV. (p. 310) and Fig. 1 (p. 316) shows that although the values of p_0 to p_4 inclusive have been obtained by a different process from that adopted for subsequent ages, the two series are not discontinuous to any material extent. The only point at which "smoothing" would have produced a better curve is p_{5-6} in the male curve. No "smoothing" has been attempted, as it had not been done in the preceding Brighton Life Table.

Years of life at risk at age 85 *and upwards.*

From age 85 upwards the application of the graphic method becomes difficult and the results unsatisfactory. On reference to the tables on pp. 298, 301 it will be seen that the male years of life at risk at 85 and upwards number only 1055, while the deaths among these are 293. These numbers give curves, the ordinates of which on the scale adopted for the rest of the Life Table cannot be read with correctness. A difference of 5 when dealing with ordinates of the height of 100 assumes an importance which it does not possess when dealing with an ordinate of the height of 10,000. Apart from this mechanical difficulty, which can be met by constructing the curve of this part of the Life Table on a specially large scale, experience shows that the data as to the number of lives at risk and ages at death at these extreme ages are too scanty to suffice for the calculation of life probabilities, and are less in accordance with facts than those at less advanced ages. Consequently whether the graphic or the analytical method of construction is adopted, the data for this part of the Life Table must be abandoned and the foundation figures must be based on theoretical considerations, involving the application of the laws of probabilities. For instance it may be safely assumed that with very few exceptions the whole of the 100,000 (p. 311) started with at birth have died before reaching the 100th year of life. It may also be taken as certain that the probability of living a single year steadily declines after the age of 85.

If the graphic method be followed after the age of 85, it is found that according to the data of the Brighton Life Table the probability of living one year does not follow a regular course, in the case of males when the age 87 has been passed. The female curve also becomes irregular at the same age. It is better therefore to assume for these ages a steadily decreasing value of p_x until all the lives have become extinct. For the sake of exactitude the method of differences has been here adopted.

The logarithms of the p_x values for ages 55, 65, 75, and 85 are differenced, as shewn in the following example:

	1st difference	2nd difference	3rd difference
$\log p_{55} = \bar{1}{\cdot}9885151$	$-{\cdot}0092959$	$-{\cdot}0111731$	$-{\cdot}0404793$
$\log p_{65} = \bar{1}{\cdot}9792192$	$-{\cdot}0204690$	$-{\cdot}0516524$	
$\log p_{75} = \bar{1}{\cdot}9587502$	$-{\cdot}0721214$		
$\log p_{85} = \bar{1}{\cdot}8866288$			

In this example the sign of $\log p_{65}$ is first changed and then this log is added to $\log p_{55}$, giving the result $-{\cdot}0092959$. This is the first difference of the first term of the series. By similarly treating $\log p_{75}$ and $\log p_{85}$, we obtain the remaining first differences. By treating the first differences in the same way, the second differences are obtained, and so with the third difference.

The last differences opposite each of the series are then added together. Thus

$$\begin{array}{r} -\cdot 0404793 \\ -\cdot 0516524 \\ -\cdot 0721214 \\ \hline -\cdot 1642531 \end{array}$$

Adding this to the last term of the series

$$\begin{array}{r} \bar{1}\cdot 8866288 \\ -\cdot 1642531 \\ \hline \bar{1}\cdot 7223757 \end{array}$$

we obtain

This is the log of $\cdot 52769 = p_{95}$.

The value of p_{95} having been obtained as above, p_{85} and p_{95} are then plotted out on sectional paper, along with the p_x values of a sufficient number of preceding years to ensure a good sweep for the total p_x curve. The intermediate values of p from 85 to 95 are then read from the curve. The curve can be continued beyond 95 in a similar manner. The effect on p_x values produced by adopting the unmodified graphic method and the combination of the method of differences and graphic method for ages over 85 can be seen from the following table:—

Male probabilities of living one year (p_x).

Age	By graphic method	By method of differences
86	·74500	·74500
87	·73654	·71000
88	·75000	·68500
89	·74163	·66500
90	·75700	·64150
91	·76800	·62000
92	·78600	·59600
93	·79200	·57500
94	·70580	·54980
95	·77700	·52770
96	·81800	·50200
97	·79000	·47500
98	·78000	·44800
99	·60000	·42400

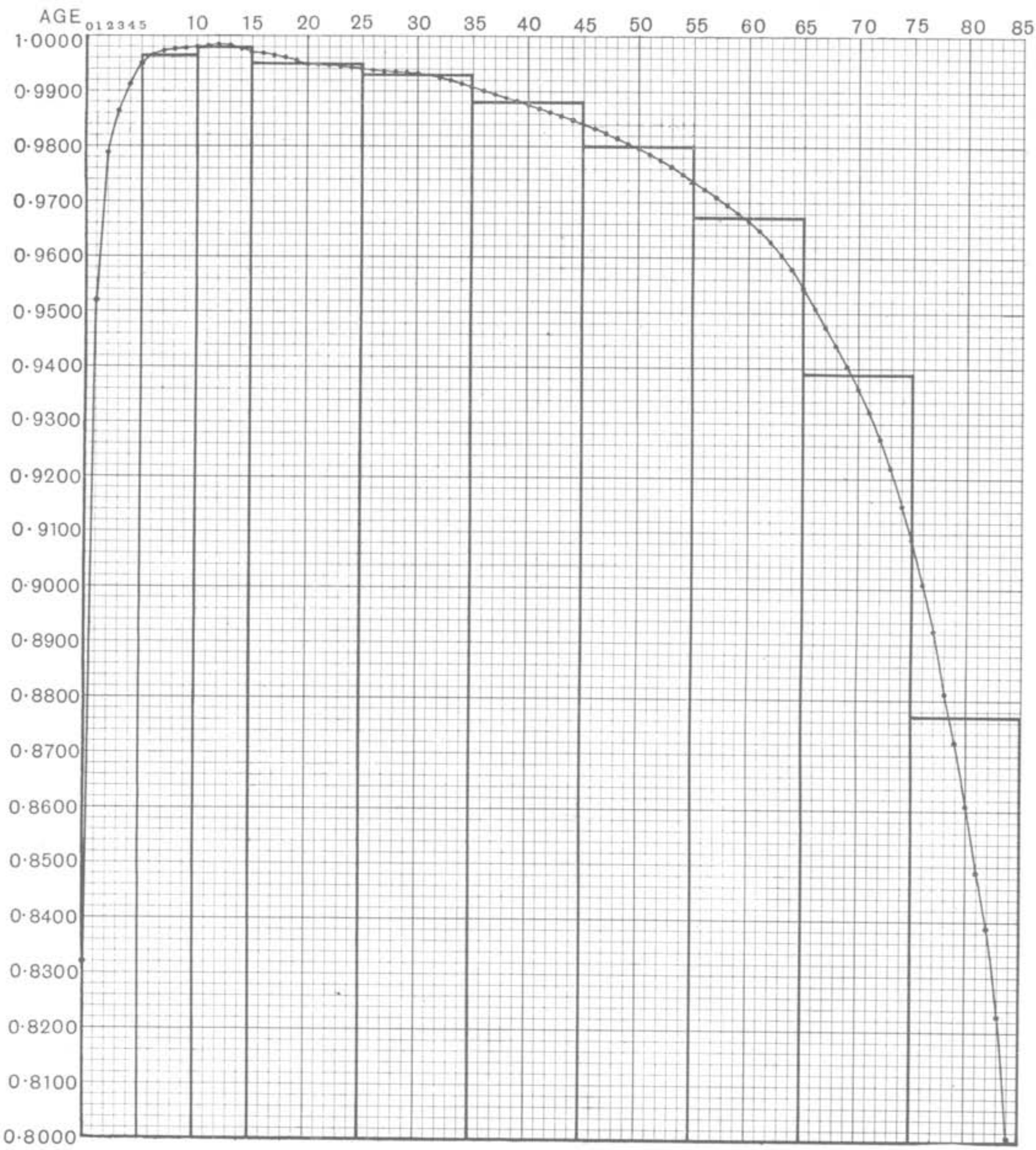
Probability of living one year.

The number of years of life at risk at each age and the number of deaths in the corresponding years of life being now known (Tables, pages 298, 301), the probability of living through one year can be stated.

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

PLATE XIV.

Probability of Living One Year at each Year of Age (curved line); and Average Probability of Living One Year for Age-Groups (horizontal lines).—Males.



https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

The mean population for each year of life (P_x) has been ascertained from the curve (Plate XII.), and the number of deaths during each year of life (d_x) has been similarly ascertained (Plate XIII.). For every year of life, except the first, it is assumed that the deaths are equally distributed throughout the year. Hence the number living at the end of the year x is equal to $P_x - \frac{1}{2}d_x$, and the number living at the beginning of the year is equal to $P_x + \frac{1}{2}d_x$, and probability of living from the age x to the age $x+1 = \frac{P_x - \frac{1}{2}d_x}{P_x + \frac{1}{2}d_x} = p_x$.

By this means p_x for every year of life except the first can be calculated. Thus—

$$\begin{aligned}\log p_{5-6} &= \log(14{,}400 - 33{\cdot}5) - \log(14{,}400 + 33{\cdot}5)\\ &= 4{\cdot}1573510 - 4{\cdot}1593717\\ &= \bar{1}{\cdot}9979793\end{aligned}$$

therefore $p_{5-6} = {\cdot}9953582.$

p_x for 1st Year of Life. The deaths in this year of life are very unequally distributed. Thus in Brighton in 1891–1900 out of 3036 male deaths under 1 year of age, 2142 occurred in the first and 894 in the second half year.

The probability for the first year of life is obtained by dividing the mean population *minus* the deaths during the second six months by the mean population *plus* the deaths during the first six months of life.

Thus $$p_{0-1} = \frac{P_0 - 894}{P_0 + 2142} = {\cdot}8319402.$$

Construction of Life Table.

Having obtained p_x for each year of life separately for the two sexes we can now build up the Life Table step by step. It is usual to start with 100,000 children at birth. In Brighton during 1891–1900 the births of male and female children were in the proportion of 50,614 to 49,386, making 100,000 of both sexes. The numbers 50,614 and 49,386 are therefore taken as the number at age 0 in the l_x column of the male and female Life Tables.

Starting with 50,614 male infants at birth, the number living at the end of one year is obtained by multiplying this number by the probability of surviving to the end of the first year.

Thus $50{,}614 \times {\cdot}8319402 = 42108.$

Similarly $42{,}108 \times {\cdot}9521333 = 40{,}092$, and so on.

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

In order to obtain the mean expectation of life for each individual we must ascertain the total number of years lived by the individuals under consideration, and divide the sum by the number of individuals living this total number of years. The l_x column of the Life Table, a few entries of which are appended, gives the necessary data for this calculation :

Age	Probability of living from age x to age $x+1$	Born and surviving at each age	Sum of the number living, or years of life lived at each age $x+1$ and upwards, to the last age in the Table	Expectation of Life at each age
	p_x	l_x	Σl_{x+1}	$E^0{}_x$
0	·8319402	50,614	2,248,526	44·92
1	·9521333	42,108	2,206,418	52·90
2	·9784324	40,092	2,166,326	54·53
3	·9866960	39,228	2,127,098	54·75
4	·9910755	38,706	2,088,392	54·46
5	·9953582	38,360	2,050,032	53·94
6	·9963215	38,182	2,011,850	53·19
7	·9968735	38,042	1,973,808	52·38
	⋮	⋮	⋮	⋮
94	·54980	51	56	1·60
95	·52770	28	28	1·50
96	·50220	15	13	1·37
97	·47500	8	5	1·13
98	·44800	4	1	·75
99	·42400	1	0	·50

Thus the 42,108 males surviving at the end of the first year of life out of 50,614 born will have each lived a complete year in the first year, or among them 42,108 years. Similarly 40,092 males will live another complete year each in the second year, or among them a further 40,092 complete years; similarly 39,228 complete years of life will be lived in the third year of life, 38,706 in the fourth year, and so on, until all the males started with become extinct in the 100th year of life.

It is evident therefore that the total number of complete years lived by the 50,614 males started with at birth will be

$$42{,}108 + 40{,}092 + 39{,}228 + \ldots\ldots 51 + 28 + 15 + 8 + 4 + 1 = 2{,}248{,}526$$

years. As this number of years is lived by 50,614 males, the number of complete years lived by each male

$$= \frac{2{,}248{,}526}{50{,}614} = 44{\cdot}42 \text{ years.}$$

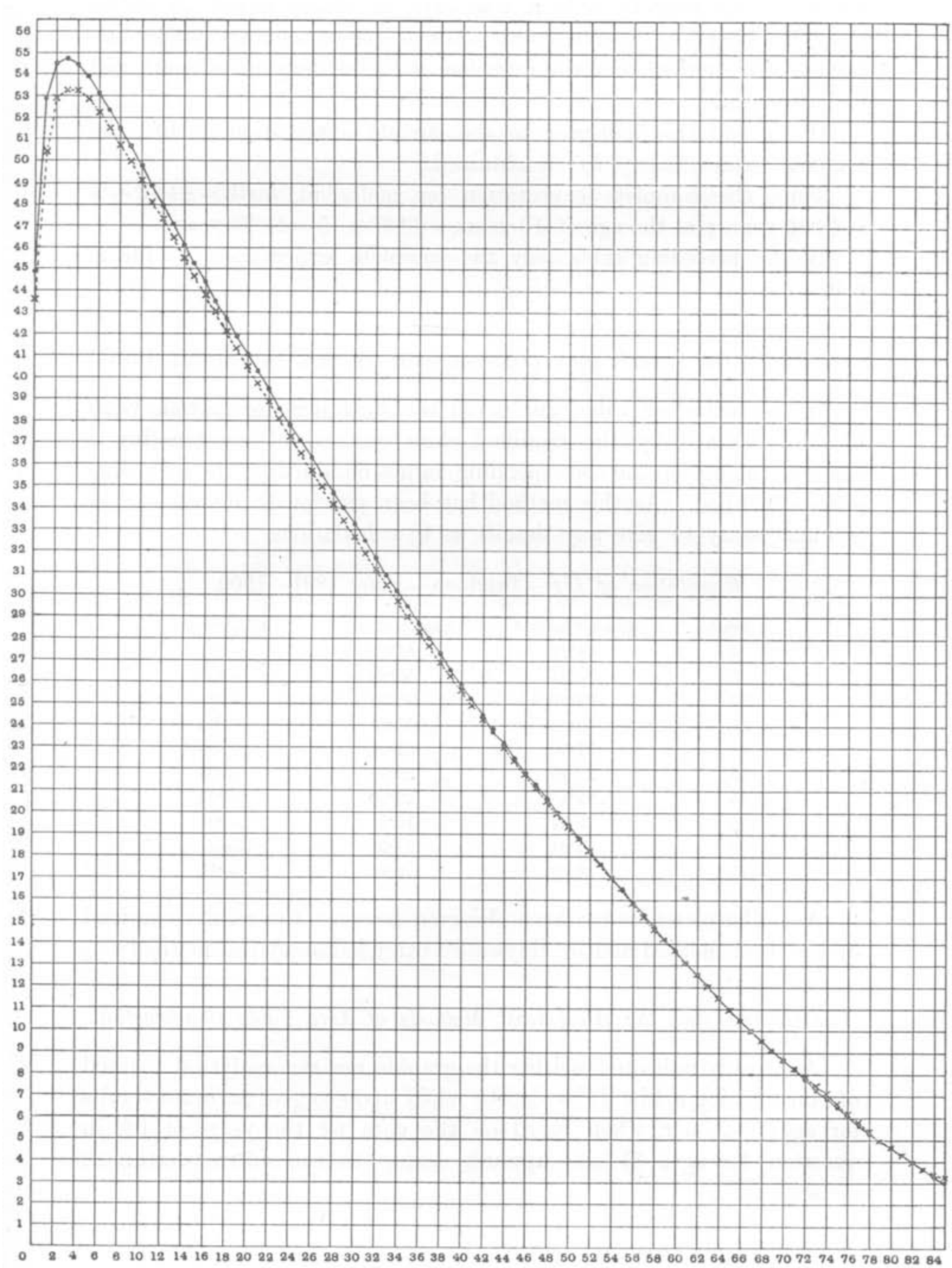
This result is known as the *curtate expectation of life.*

We have in the above remarks confined our attention to the

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

PLATE XV. EXPECTATION OF LIFE.—MALES.

1881–90 +--+--+--+ 1891–1900 ·—·—·—·—·



https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

complete years of life, and have not taken into account that portion of lifetime lived by each person in the year of his death. In some instances this may be only a few days, in others nearly an entire year; but it may be assumed with a fair degree of accuracy, taking one person with another, that the duration of life in the year of death will be half-a-year.

If we add this half-year to the curtate expectation of life, the *complete expectation of life* is obtained.

Thus, the complete expectation for males at birth $= 44{\cdot}42 + {\cdot}5 = 44{\cdot}92$ years; at the age of 10 years $= 49{\cdot}30 + {\cdot}5 = 49{\cdot}80$ years.

In the preceding table only the complete expectation of life is printed.

A Brighton Life Table constructed by the Short Method.

In the following table are given the E_x values for certain years of life obtained by the modified short method of constructing a life table, alongside the corresponding values obtained by the preceding graphic method. As this method has been previously described [6], it is unnecessary to give any details as to calculations.

Expectation of Life—Brighton—*Males* 1891—1900.

Age	Graphic method	Short method	Differences
0	44·92	44·89	− ·03
5	53·94	53·89	− ·05
10	49·80	49·75	− ·05
15	45·29	45·24	− ·05
25	37·12	37·06	− ·06
35	29·45	29·40	− ·05
45	22·45	22·50	− ·05
55	16·44	16·41	− ·03
65	11·01	11·01	*nil*
75	6·49	6·60	+ ·11

A similar closeness has been observed between the results obtained by the short method and by Hayward's extended analytical method [8].

The Graphic and the Analytical Methods of Life Table Construction.

As already indicated, neither the graphic nor the analytical method of constructing a Life Table can be relied upon to give accurate results for ages 0–5, nor, when based on the data for the years of life in question, for ages 85 and upwards, owing to the untrustworthiness at these ages of the census and death returns. Hence at the extremes of life special processes require to be adopted (pages 304 and 307). For intermediate ages the question arises whether the graphic or the analytical method is preferable.

1. *Difficulty of application and special knowledge required.* For many Medical Officers of Health the answer to the above question will be determined by the decision as to which method is the easier and more expeditious, unless it can be shewn that the easier process is markedly less accurate. On these grounds the graphic is undoubtedly preferable and should in our opinion be chosen. (By the phrase "analytical method" is meant the extended analytical method unless otherwise stated.) The above remark does not apply to the Short Method of Farr, as modified and improved by Dr Hayward. Nor does it apply to the combined graphic and analytical method described by Dr Hayward [9]. Both these methods involve less labour than the extended graphic method described in this paper. We therefore propose to compare certain results obtained by these three methods.

(*a*) This is done for the Modified Short Method on page 314. It will be noted that if the age 75 be excluded from consideration the differences in the expectation of life at quinquennial intervals of age never exceed ·06 part of a year, when the Modified Short Method is compared with the Graphic Method. The greater difference at age 75 is relatively unimportant, as expectations of life at this age are necessarily based on somewhat uncertain data.

(*b*) In Fig. 1 A and B the p_x values obtained by the graphic method for ages 5 to 15 are compared with the corresponding values obtained by the combined graphic and analytical method (grade 1)[1]. This latter method is based on and has been shown by Dr Hayward to give results closely approximating to those given by his improved extended analytical method. The ages 5–15 are the period of life at which the p_x values obtained by different applications of the analytical method have in actual experience shown the greatest discrepancies. It will be seen that the correspondence between the two curves is extremely close, as may be seen also by reference to the following table. The results obtained by the graphic and by the combined method are for the most part

[1] Strictly speaking the $\log p_x$ values should have been plotted out in Fig. 1. This was done and the results coincided with those shown in Fig. 1. The p_x values have been plotted out in this figure by preference, because the p_x curve brings out more clearly any differences between the results obtained by the two methods.

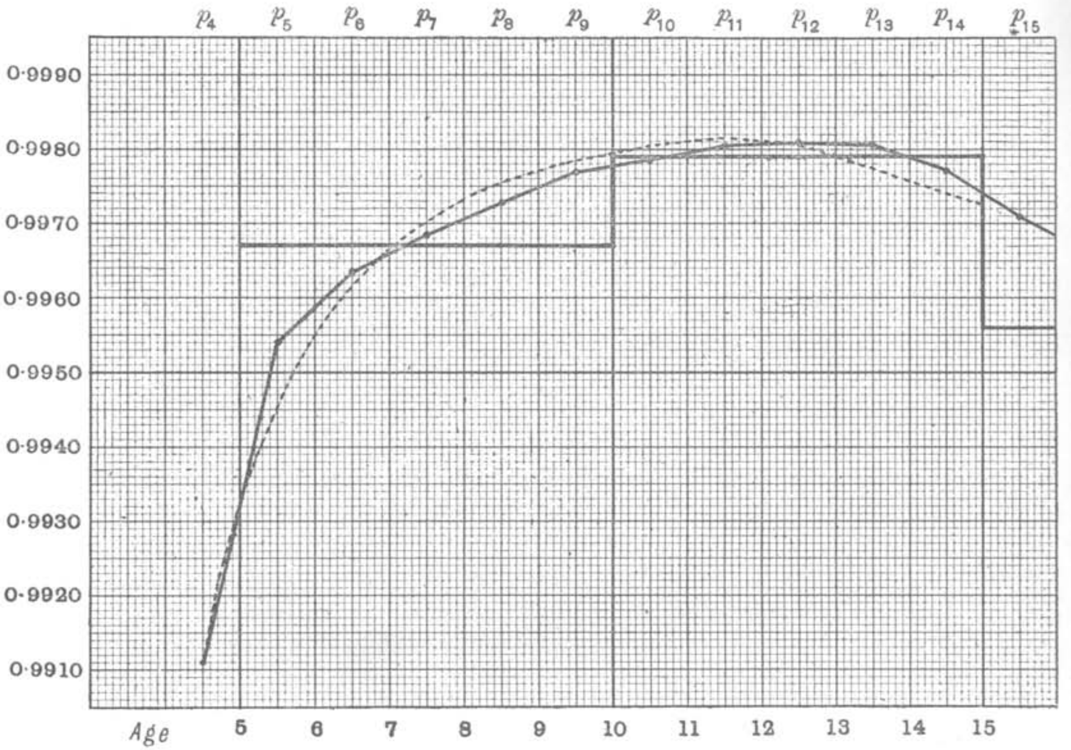


Fig. 1. A.

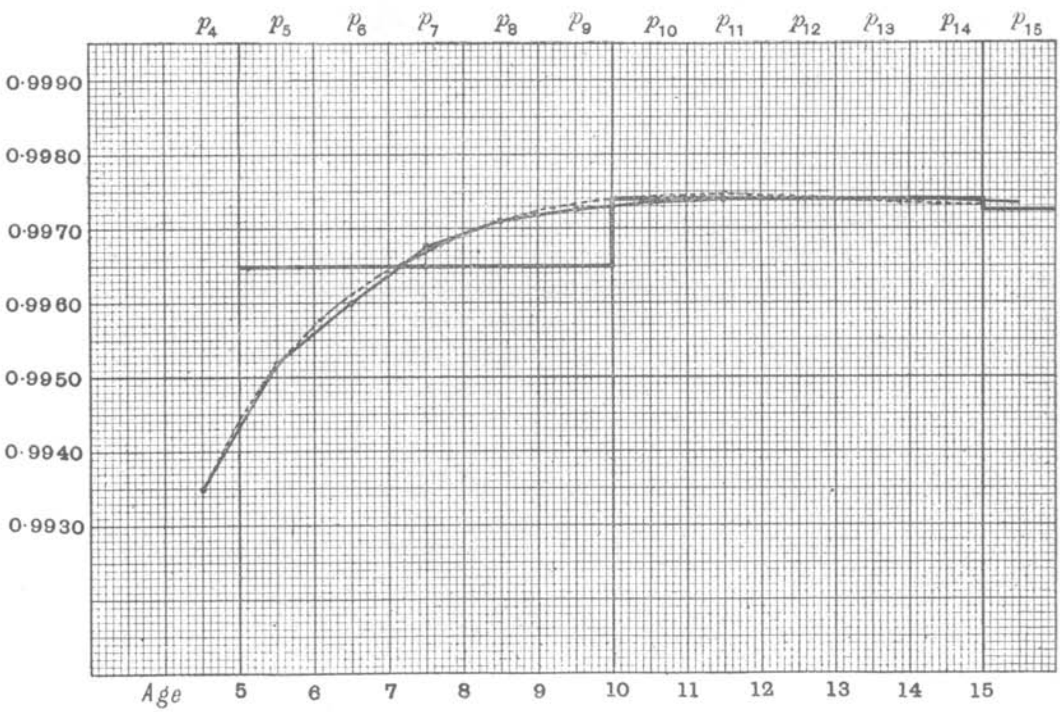


Fig. 1. B.

Fig. 1. Comparison of p_x values by Graphic Method and by Combined Method for ages 5—15.
A. Males. *B*. Females.

practically identical; a fact which strongly supports the value of the empirical rules for drawing the p_x curve for ages 5–15, and indirectly strongly confirms the accuracy of the extended analytical method which Dr Hayward after repeated trials has adopted.

Table comparing Probabilities of Life by the Graphic and Combined Methods.

p_x values.

	Males			Females		
Age	Combined Method	Graphic Method	Difference	Combined Method	Graphic Method	Difference
5	·99456	·99536	+ ·00080	·99518	·99519	+ ·00001
6	·99616	·99632	+ ·00016	·99614	·99600	− ·00014
7	·99708	·99687	− ·00021	·99671	·99675	+ ·00006
8	·99761	·99729	− ·00032	·99710	·99710	± ·00000
9	·99786	·99770	− ·00016	·99733	·99725	− ·00008
10	·99804	·99785	− ·00019	·99740	·99734	− ·00006
11	·99814	·99805	− ·00009	·99742	·99742	± ·00000
12	·99805	·99807	− ·00002	·99740	·99743	− ·00003
13	·99777	·99805	− ·00028	·99740	·99741	+ ·00001
14	·99740	·99766	− ·00026	·99740	·99739	− ·00001

From the facts already given it is clear that so far as Brighton is concerned the short method, the graphic method, and the combined method all give approximately identical results. Similarly for England and Wales Dr Hayward has obtained approximately identical results by the use of the short method and of his improved extended analytical method. The conclusion is inevitable that the results obtained by these four methods are practically correct, and that the choice as to which method shall be adopted is one to be determined largely by considerations of convenience and personal preference.

We have preferred the detailed graphic method, by which we have obtained results strictly comparable with those of 1881–90; but our experiments with the other two methods, which are decidedly less laborious, indicate that the differences of values of E_x are so small that one of the other methods might have been adopted without destroying the comparability of the 1891–1900 Brighton Life Table with that of 1881–90, or with other Life Tables constructed by the most modern analytical methods.

In the remarks which follow it must still be noted that we are comparing the graphic method with the extended analytical methods.

2. *Accuracy of Methods of Determination of Facts for Each Age.* In insurance experience the number of lives at risk and the number dying at each age are exactly known, but in national and district Life

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

Tables we are concerned with census returns and death returns, in both of which ages are often erroneously stated, with a special run on round numbers such as 10, 20, 30, etc. In passing it may be noted however that actuaries, who have more accurate data than those provided by the census and death returns of the whole country or of a particular town, appear generally to prefer the graphic method of applying their data. In local and national returns of population and deaths, the data are only available in age periods 0–5, 5–10......25–35, 35–45, etc., or where given for single years of life cannot be trusted. It is necessary therefore to adopt some method of interpolation for single years of life, and this can be done by graphic or analytical processes. In the analytical method this is done by the method of interpolation by finite differences. Thus if the age 35 is being dealt with the facts at ages 15, 25, 45, and 55, are taken into account if four orders of differences are being used. In the graphic method these ages are also taken into account. There is "only one curve for all the parallelograms" (Plate XII.) "not a curve for each[1]." Although in each parallelogram the accuracy of the curve is determined by the fact that the sum of the ordinates equals the area of the corresponding parallelogram, this equality might be secured by many other curves for the years of life in question than the one actually constructed. The possibility of this variation is almost without exception prevented by the fact that the part of the curve relating to the parallelogram in question must fit into the general sweep of the curve for neighbouring parallelograms. Hence in practice it is found that one curve and one curve only can as a rule be constructed in which (1) the sum of the ordinates shall equal the area of the parallelogram, and (2) the sweep of this part of the curve shall fit in with the general sweep of the neighbouring parts of the curve. The only exception is when the curve changes from an upward to a downward direction or *vice versa* (see footnote, p. 304). The accuracy of these results is confirmed by the facts that the p_x curve (Plate XIV.) and the E_x curve (Plate XV.) based on the original curves of population and deaths run smoothly, although with the exception referred to above there has been no "smoothing," the curves exactly representing the calculated results. This meets the main objection urged against the graphic method, that the curves will vary with the individuality of the draughtsman. The objection that the heights of ordinates are difficult to measure is more theoretical than practical, as the possible differences

[1] King, *op. cit.*

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

of reading throughout the greater part of life when an adequate scale is employed have no appreciable effect upon the E_x values. Near the end of life the error is obviated by drawing the curve on a larger scale.

3. *Diversity of Analytical Methods.* From the above facts it is clear that except at one or two points it is within narrow limits only practicable to secure one graphic curve interpreting the data. Can the same be said for analytical methods? In practice most discrepant and irreconcilable results are obtained when analytical methods are employed, owing to the fact that no one system of interpolation has been uniformly adopted. Two examples of this fact may be adduced.

(1) Dr Hayward on applying his new method of interpolation to the data of the Manchester Life Table for 1881–90 obtained the following results:

	New values	Values in the published Life Table	Differences of new from published values
E_0	35·10	34·71	+0·39
E_5	46·16	45·59	+0·41
E_{10}	42·86	42·75	+0·11
E_{15}	38·62	38·78	−0·16
E_{20}	34·63	34·62	+0·11

The considerable divergencies at the above ages depend upon different modes of interpolating the p_x values from age 5 to age 25. The corresponding discrepancy for p_x values is shown in Fig. 2, where the dotted line shows the values in the published Life Table, and the continuous line the values in the Life Table calculated by Dr Hayward.

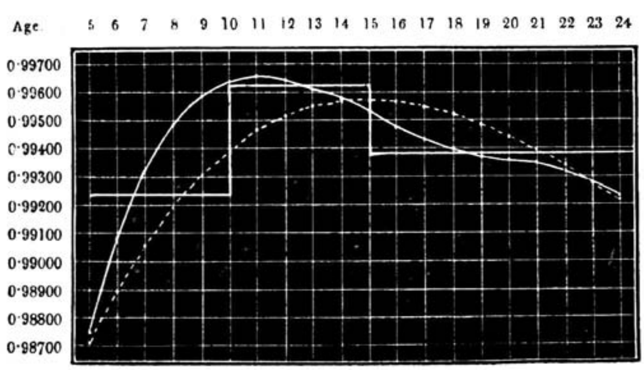


Fig. 2.

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

(2) A further instance may be quoted from Dr Hayward's paper [10]. In his diagram here reproduced (Fig. 3) the continuous line gives the p_x values for ages 5 to 24, as calculated by him, while the two dotted curves give p_x values derived from the same data by two other methods which have been employed in Life-Table construction. To take the instance of one year, $p_{10} = \cdot 9940$ by one method, and $\cdot 9984$ by another.

Fig. 3.

In contrast with the above examples of discrepant results obtained by the use of different analytical methods, the closeness of the correspondence between results obtained by the graphic method and by the most modern and accurate analytical methods may be adduced. We refer to the methods described by Dr Hayward [11] and employed by Mr Shirley Murphy [12] in the London Life Table for 1891–1900, which in the case of Brighton have been shown (p. 316, Fig. 1) to give results closely corresponding with those obtainable by the graphic method.

A further instance of the same close correspondence between results obtained in the Brighton Life Table constructed ten years ago by the graphic method, and corresponding results obtained by an extended and improved analytical method, is shown in the following table, in which the values obtained by the analytical method have been calculated by Dr Hayward.

Brighton (Males) Life Table, 1881–90.

	Graphic Method ("G") (Newsholme)	Analytical Method ("A") (Hayward)	Differences of "A" from "G"
E_0	43·59	43·56	− ·03
E_5	52·87	52·88	+ ·01
E_{10}	49·12	49·12	± ·00
E_{15}	44·67	44·66	− ·01
E_{20}	40·55	40·51	− ·04
E_{25}	36·51	36·51	± ·00
E_{35}	29·02	29·04	+ ·02
E_{45}	22·36	22·36	± ·00
E_{55}	16·48	16·46	− ·02
E_{65}	10·96	10·94	− ·02
E_{75}	6·64	6·58	− ·06
E_{85}	3·33	3·46	+ ·13
E_{95}	1·68	1·22	− ·46

The discrepancies are insignificant until after the age of 75, after which age they are unimportant.

4. *The analytical appeals to the graphic method in selecting the best method of interpolation.* With analytic methods of Life Table construction an indefinite number of applications of the process of analysis are available, and the various applications employed have been shewn above to give discrepant results, the differences sometimes amounting to as much as a year in the case of a particular E_x value. The particular mathematical process to be adopted is therefore "by no means a matter of indifference," and no guidance appears to be furnished by mathematics as to the special method to be adopted. Thus we find Dr Hayward, one of the ablest exponents of the analytical method, appealing in this embarrassment of choice, not to mathematics, but to the graphic method. Referring to the instance of Manchester already given, he constructed (11) eight different p_x curves for the years of life 5—15, by eight different applications of the method of interpolation, and based his choice from amongst these solely upon graphic considerations. In other words, he constructed an "ideal curve" representing the known facts as closely as possible, and then by a process of elimination selected the curve obtained by analytical methods which differed least from it. The fact that by this means he was able to arrive at an analytical method which as tested by our application of the graphic method gives correct results, does not appear

to us to detract from the force of the contention that analytical methods are inconclusive and give no guidance as to which analytical process gives results most in accordance with the facts. It appears preferable, since the "ideal curve," *i.e.* the curve nearest the known facts, forms the ultimate appeal to which the selection of analytical methods must be referred, to adopt the "ideal curve" itself rather than any approximation, however close, *i.e.*, to adopt the graphic method. The authors of other analytical Life Tables so far as we are aware have furnished no information as to the considerations guiding their choice of method; but since in the London Life Table for 1891—1900 the same series of u_x values has been adopted from which to calculate the critical p_x values[1] for ages 5—15 as that selected in the manner above described, the London Life Table may also be regarded as constructed by the application of methods based on graphic considerations.

From the preceding facts it is clearly unsafe to compare Life Tables constructed by different analytical methods. By so doing most erroneous deductions may be obtained. In the National Life Table for 1881—90 an improved method of construction was employed. In the Decennial Supplement of the Registrar-General for 1881—90[(13)] it is stated "for ages 45 and upwards the expectations of life are lower by the new table than by either of the others" (*i.e.*, for 1834—54 and for 1871—80). The following table shows in the first two columns the results indicated in the above quotation; and in the next two columns the results obtained by Dr Hayward[(14)] when the two Life Tables are constructed by a uniform method.

Expectations of Life.

Age	By Official Tables		Recalculated by a uniform method		Difference of *B* from *A*	Difference of *b* from *a*
	A 1871—80	*B* 1881—90	(*a*) 1871—80	(*b*) 1881—90		
35	28·64	28·91	28·40	28·87	+ ·27	+ ·47
45	22·07	22·06	21·88	22·04	− ·01	+ ·16
55	15·95	15·74	15·66	15·71	− ·21	+ ·05
65	10·55	10·31	10·21	10·24	− ·24	+ ·03
75	6·34	6·10	5·91	6·06	− ·24	+ ·15
85	3·56	3·29	3·15	3·32	− ·27	+ ·17
95	2·01	1·72	1·57	1·72	− ·29	+ ·15

[1] $u_x = 2P - d$ at ages x and upwards.

https://doi.org/10.1017/S002217240003285X Published online by Cambridge University Press

It will be seen that the expectation of life at ages 45 and upwards has not declined but actually increased. Hence until the authors of these tables agree upon the adoption of some one scheme of construction to be strictly followed in all instances, the full value of their work will not be secured, since one of the main functions of Life Tables is to enable accurate comparisons to be instituted for each sex between the vital conditions of populations differing from each other in geographical distribution, time, social status, sanitary condition, etc.

Summary as to Graphic and Analytical Methods.

(1) The graphic method is more easy of application and requires less mathematical knowledge than the extended analytical method.

(2) In our experience it produces smooth curves of p_x and E_x.

(3) The analytical in the selection of the best process of interpolation appeals to the graphic method.

(4) While either the analytical or graphic process may in unskilful or careless hands give erroneous results, owing to errors in working out, the analytical process, unlike the graphic, presents a wide choice of methods, which although accurately worked out, give incomparable results.

(5) The facts given in the table on page 321 and in Fig. 1 A and B and table on page 317 show that almost identical results are obtained by *the graphic method* and *by the improved extended analytical* method. We regard this as important testimony to the accuracy of each of these methods, and at the same time as indicating the inaccuracy of the analytical methods giving different results.

(6) As the results obtained by the modified short analytical method and by the combined analytical and graphic method approximate so closely to those obtained by the detailed graphic method, either of them may safely be adopted by those who cannot spare time for working out a detailed Life Table by the graphic method.

(7) Every published Life Table should give exact details of the method of its construction, in order that the comparability of data and results may be tested. Much confusion has arisen, and important errors in comparison between different Life Tables have been produced, by the use of different methods of construction and the non-publication of details of methods.

REFERENCES.

(1) *The Brighton Life Table* (Based on the Mortality of Ten Years 1881—90), and the *Elements of Vital Statistics*, 3rd Ed. page 265. A. Newsholme.

(2) "On the Method used by Milne in the Construction of the Carlisle Table of Mortality." G. King, F.I.A., *Journal Inst. Actuaries*, No. CXXXI. (Oct. 1883).

(3) "On the Construction and Use of Life Tables from a Public Health Point of View." T. E. Hayward, M.B. &c., *Journal of Hygiene*, Vol. II. No. 1, p. 1.

(4) "A Method for Estimating Mean Populations in the Last Intercensal Period." A. C. Waters, *Journal Statistical Society*, Vol. LXIV. Part II., 1901.

(5) "On the Construction of Life Tables, illustrated by a new Life Table of the Healthy Districts of England." W. Farr, M.D., F.R.S., *Philos. Trans.* 1860, reprinted in *Journal Inst. Actuaries*, Vol. IX. Part III., p. 122.

(6) Hayward. *Journal of Hygiene*, Vol. II. No. 1, p. 12.

(7) Do. page 34.

(8) Do. page 40.

(9) Do. *Journal of Hygiene*, Vol. II. No. 2, p. 207.

(10) Do. *Journal of Hygiene*, Vol. II. No. 1, page 28.

(11) Do. *Journal Royal Statistical Society*, Vol. LXII. Part III., 1899.

(12) The London Life Table (Based on the Mortality in the Decennium 1891—1900). Shirley F. Murphy.

(13) Decennial Supplement to the Report of the Registrar-General 1881—90. Dr Tatham, Vol. I. page ix.

(14) Hayward. *Journal Royal Statistical Society*, Vol. LXIV. Part IV. p. 640.