Can. J. Math., Vol. XXIX, No. 1, 1977, pp. 65–70

AN ADDENDUM TO IDEALS AND HIGHER DERIVATIONS IN COMMUTATIVE RINGS

WEI-EIHN KUAN

Let $A = k[x_1, \ldots, x_n]$ be a finitely generated ring over a field k, and let $\mathfrak{H}_k(A)$ be the set of all k-higher derivations of A. In [**1**], we obtained some results on prime ideals in A which are differential under $\mathfrak{H}_k(A)$. In this Addendum similar results are proved for a complete local ring $k[[x_1, \ldots, x_n]]$, a homomorphic image of the formal power series ring over a field k; that is, algebraic varieties in [**1**] are replaced by algebroid varieties. We start with some technical remarks.

Remark 1. Let A' be a ring and $A \subset A'$ be a subring. Let $A'[[t]]$ be the ring of formal power series in t over A'. Let $d \in \operatorname{Hom}(A, A'[[t]])$ such that for each $a \in A$, $d(a) = a + d_1(a)t + d_2(a)t^2 + d_3(a)t^3 + \ldots + d_n(a)t^n + \ldots$. Then $\{d_i\}_{i=0}^{\infty}$, where $d_0 = \mathrm{id}_A$, is a higher derivation in $\mathfrak{H}(A, A')$. Conversely every higher derivation $\{d_i\}_{i=0}^{\infty} \in \mathfrak{H}(A, A')$ defines a ring homomorphism $d \in \operatorname{Hom}(A, A'[[t]])$ such that $d(a) = a + d_1(a)t + \ldots + d_n(a)t^n + \ldots$ for each $a \in A$. Let $\mathfrak{A} \subset A$, $\mathfrak{A}' \subset A'$ be ideals such that $\mathfrak{A}A' \subset \mathfrak{A}'$. Consider A, A' as topological rings with the $\mathfrak{A}$-adic and $\mathfrak{A}'$-adic topologies respectively. It follows from [**3**, Lemma 1, p. 334] that $\{d_i\}_{i=0}^{\infty} \in \mathfrak{H}(A, A')$ is continuous, i.e., every d_i is a continuous map.

Remark 2. Let $k[[x_1, \ldots, x_n]]$ be a formal power series ring over a field k. Let $\{\Delta_i\}_{i=0}^{\infty}$ be a higher derivation in $\mathfrak{H}_k(k[[x_1, \ldots, x_n]])$. Let $f(x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]]$; then

$$\Delta_1(f(x_1, \ldots, x_n)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) \cdot \Delta_1(x_i)$$

and for $i \geqq 2$,

$$\Delta_i(f(x_1, \ldots, x_n)) = \sum_{j=1}^{n} A_{ij}(x_1, \ldots, x_n)\Delta_i(x_j) + B_i(x_1, \ldots, x_n; \Delta_1(x_1), \ldots, \Delta_{i-1}(x_n))$$

where $A_{ij} \in k[[x_1, \ldots, x_n]]$ and

$$B_i \in k[[x_1, \ldots, x_n]][\{\Delta_l(x_j) | l = 1, 2, \ldots, i-1 \text{ and } j = 1, 2, \ldots, n\}].$$

Indeed let $\Delta : k[[x_1, \ldots, x_n]] \Rightarrow k[[x_1, \ldots, x_n]][[t]]$ be the ring homomorphism

Received December 15, 1975 and in revised form, September 10, 1976.

given rise by $\{\Delta_i\}_{i=1}^{\infty}$. Then

$$\Delta_i(x_1^{i_1}\ldots x_n^{i_n}) = \text{coefficient of } t^i \text{ in } (x_1 + \Delta_1(x_1)t + \ldots)^{i_1}\ldots(x_n + \Delta_1(x_n)t + \ldots)^{i_n}.$$

Thus,

$$\Delta_i(x_1^{i_1}\ldots x_n^{i_n}) = \sum_{j=1}^{n} a_{ij}^{i_1\ldots i_n}\Delta_i(x_j) + \sum_{\substack{1\leqq l_k\leqq i-1\\ 1\leqq j_k\leqq n}} a_{l_1j_1l_2j_2}^{i_1\ldots i_n}\Delta_{l_1}(x_{j_1})\Delta_{l_2}(x_{j_2}) + \ldots + \sum_{\substack{1\leqq j_k\leqq n\\ 1\leqq l_k\leqq i-1}} a_{l_1j_1\ldots l_{i-1}j_{i-1}}^{i_1\ldots i_n}\Delta_{l_1}(x_{j_1})\ldots\Delta_{l_{i-1}}(x_{j_{i-1}})$$

where

$a_{ij}^{i_1\ldots i_n}$ are monomials of degree $(i_1 + \ldots + i_n) - 1$,

$a_{l_1j_1l_2j_2}^{i_1\ldots i_n}$ are monomials of degree $(i_1 + \ldots + i_n) - 2$, and

$a_{l_1j_1\ldots l_{i-1}j_{i-1}}^{i_1\ldots i_n}$ are monomials of degree $(i_1 + \ldots + i_n) - (i - 1)$.

Thus, if $f(x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]]$, $f(x_1, \ldots, x_n) = \sum a_{i_1\ldots i_n}x_1^{i_1}\ldots x_n^{i_n}$, where $\{i_1 + i_2 + \ldots + i_n\}$ is monotone increasing.

$$\begin{aligned}\Delta_i f(x_1, \ldots, x_n) &= \sum a_{i_1\ldots i_n}\Delta_i(x_1^{i_1}\ldots x_n^{i_n})\\ &= \sum a_{i_1\ldots i_n}\Bigg(\sum_{j=1}^{n} b_{ij}^{i_1\ldots i_n}\Delta_i(x_j) + \sum_{\substack{1\leqq l_k\leqq i-1\\ 1\leqq j_k\leqq n}} b_{l_1j_1l_2j_2}^{i_1\ldots i_n}\Delta_{l_1}(x_{j_1})\Delta_{l_2}(x_{j_2}) + \ldots\\ &\qquad + \sum_{\substack{1\leqq l_k\leqq i-1\\ 1\leqq j_k\leqq n}} b_{l_1j_1\ldots l_{i-1}j_{i-1}}^{i_1\ldots i_n}\Delta_{l_1}(x_{j_1})\ldots\Delta_{l_{i-1}}(x_{j_{i-1}})\Bigg)\\ &= \sum_{j=1}^{n} A_{ij}\Delta_i(x_j) + \sum_{\substack{1\leqq l_k\leqq i-1\\ 1\leqq j_k\leqq n}} A_{l_1j_1l_2j_2}\Delta_{l_1}(x_{j_1})\Delta_{l_2}(x_{j_2}) + \ldots\\ &\qquad + \sum_{\substack{1\leqq l_k\leqq i-1\\ 1\leqq j_k\leqq n}} A_{l_1j_1\ldots l_{i-1}j_{i-1}}\Delta_{l_1}(x_{j_1})\ldots\Delta_{l_{i-1}}(x_{j_{i-1}}),\end{aligned}$$

where $A_{ij}, \ldots, A_{l_1j_1\ldots l_{i-1}j_{i-1}} \in k[[x_1, \ldots, x_n]]$.

Remark 3. Let $k[[x_1, \ldots, x_n]]$ be a complete local ring over a field k. Let $\{\delta_i\}_{i=0}^{\infty}$ be a higher derivation in $\mathfrak{H}_k(k[[x_1, \ldots, x_n]])$. Then for $f(x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]]$, $\delta_i f(x_1, \ldots, x_n) = \sum_{j=1}^{n} A_{ij}\delta_i(x_j) + B_i$ where

$$A_{ij} \in k[[x_1, \ldots, x_n]]$$

and $B_i \in k[[x_1, \ldots, x_n]][\{\delta_l(x_j) | l = 1, 2, \ldots (i - 1); j = 1, 2, \ldots, n\}]$. Let

https://doi.org/10.4153/CJM-1977-005-x Published online by Cambridge University Press

$k[[X_1, \ldots, X_n]]$ be the formal power series ring. Let π be the canonical surjection from $k[[X_1, \ldots, X_n]]$ to $k[[x_1, \ldots, x_n]]$ with $\mathfrak{A}$ as its kernel. Then δ can be lifted to a higher derivation $\Delta = \{\Delta_i\}_{k=0}^{\infty}$ of $k[[X_1, \ldots, X_n]]$ such that $\mathfrak{A}$ is differential under Δ. In fact let $\Delta_i : k[X_1, \ldots, X_n] \rightarrow k[[X_1, \ldots, X_n]]$ be a k-linear map such that $\Delta_0(f(X)) = f(X)$ for all $f(X) \in k[X_1, \ldots, X_n]$, $\pi(\Delta_i(X_j)) = \delta_i(x_j)$ for all $i = 1, 2, \ldots, j = 1, 2, \ldots, n$ subjected to Leibnitz's rule. Then $\{\Delta_i\}_{k=0}^{\infty}$ is a higher derivation: $k[X_1, \ldots, X_n] \rightarrow k[[X_1, \ldots, X_n]]$. Since the $(X_1, \ldots, X_n)$-adic completion of $k[X_1, \ldots, X_n]$ is $k[[X_1, \ldots, X_n]]$ therefore it follows from [**2**, Proposition 2, p. 41] that $\{\Delta_i\}_{i=0}^{\infty}$ can be extended to $k[[X_1, \ldots, X_n]]$. Moreover $\Delta_i(\mathfrak{A}) \subset \mathfrak{A}$. Since

$$\Delta_i(f(X_1, \ldots, X_n)) = \textstyle\sum_{j=1}^{n} A_{ij}\Delta_i(X_j) + B_i \text{ where } A_{ij} \in k[[X_1, \ldots, X_n]]$$

and $B_i \in k[[X_1, \ldots, X_n]][\{\Delta_l(X_j) | l = 1, 2, \ldots, i-1, j = 1, 2, \ldots, n\}]$, it follows that $\delta_i(f(x_1, \ldots, x_n)) = \pi(\Delta_i f(X_1, \ldots, X_n)) = \sum \pi(A_{ij}) \cdot \pi\Delta_i(X_j) + \pi(B_i) = A_{ij}(x_1, \ldots, x_n)\delta_i(x_j) + B_i(x_1, \ldots, x_n; \delta_1(x_1), \ldots, \delta_{i-1}(x_n))$ where $A_{ij}(x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]]$ and $B_i(x_1, \ldots, x_n; \delta_1(x_1), \ldots, \delta_{i-1}(x_n)) \in k[[x_1, \ldots, x_n]]\,[\{\delta_l(x_j) | l = 1, 2, \ldots, i-1, j = 1, 2, \ldots, n\}]$.

LEMMA 2′. *Let $\mathfrak{D} = k[[x_1, \ldots, x_n]]$ be a complete local ring over a field k, let $\mathfrak{p}$ be a prime ideal in $\mathfrak{D}$, and let $\mathfrak{N} = \{x \in \mathfrak{D} | xr = 0$ for some $r \in \mathfrak{D} - \mathfrak{p}\}$. If $\delta = \{\delta_i\} \in \mathfrak{H}_k(\mathfrak{D}_\mathfrak{p})$, then for every positive integer m, there exist $k_1, \ldots, k_m \in \mathfrak{D}/\mathfrak{N} - \mathfrak{p}/\mathfrak{N}$ such that $\{\delta, k_1\delta_1, \ldots, k_m\delta_m\}$ is a higher derivation of rank m from $\mathfrak{D}/\mathfrak{N}$ to itself.*

Proof. Let $\mathfrak{D}/\mathfrak{N} = k[[\bar{x}_1, \ldots, \bar{x}_n]] = k[[\bar{x}]]$ where $\bar{x}_i = x_i + \mathfrak{N}$. By the definition of localization, $\mathfrak{D}/\mathfrak{N}$ is a subring of $\mathfrak{D}_\mathfrak{p}$. Let m be a positive integer. For each $j = 1, 2, \ldots, m$, $\delta_j(\bar{x}_i) \in \mathfrak{D}_\mathfrak{p}$, say $\delta_j(\bar{x}_i) = u_{ij}(\bar{x})/v_{ij}(\bar{x})$. Let $d_j = \Pi_{i=1}^{n} v_{ij}(\bar{x})$, then $d_j \in \mathfrak{D}/\mathfrak{N} - \mathfrak{p}/\mathfrak{N}$. Set $t_j = d_1{}^j d_2{}^j \ldots d_{j-1}{}^j d_j$ for $j = 1, 2, \ldots, m$, we claim that, as additive group homomorphisms $\{t_j\delta_j\}_{j=0}^{m} \in$ Hom $(\mathfrak{D}/\mathfrak{N})$. Suffice to check $t_j\delta_j(f) \in \mathfrak{D}/\mathfrak{N}$. Let $f \in \mathfrak{D}/\mathfrak{N}$, and let $f = \sum a_{i_1 \ldots i_n}\bar{x}_1{}^{i_1} \ldots \bar{x}_n{}^{i_n}$, $a_{i_1 \ldots i_n} \in k$, $t_j\delta_j(f) = t_j(\sum_{i=1}^{n} A_{ji}\delta_j(\bar{x}_i) + B_j)$ where $A_{ji} \in \mathfrak{D}/\mathfrak{N}$ and $B_j \in \mathfrak{D}/\mathfrak{N}[\{\delta_l(\bar{x}_i) | i = 1, 2, \ldots, n,\ \ l = 1, 2, \ldots, j-1\}]$. Thus $t_jA_{ji}\delta_j(\bar{x}_i) \in \mathfrak{D}/\mathfrak{N}$. Since $B_j = \mathfrak{D}/\mathfrak{N}$-linear combination of power products of $\delta_l(\bar{x}_i)$ involving at most j of $\delta_l(x_i)$ counting repeated ones for $i = 1, 2, \ldots, n; l = 1, 2, \ldots, j-1$. Thus $t_jB_j = d_1{}^j d_2{}^j \ldots d_{j-1}{}^j d_j B_j \in \mathfrak{D}/\mathfrak{N}$ also. Hence $t_j\delta_j(f) \in \mathfrak{D}/\mathfrak{N}$. Now set $k_i = (t_1{}^m \ldots t_{m-1}{}^m t_m)^i$, $\{\delta_0, k_1\delta_1, \ldots, k_m\delta_m\} \in \mathfrak{H}_k(\mathfrak{D}/\mathfrak{N})$.

THEOREM 3′. *Let $\mathfrak{D} = k[[x_1, \ldots, x_n]]$ be a complete local ring over a field k, let $\mathfrak{p}$ be a prime ideal in $\mathfrak{D}$. If $\mathfrak{p}$ is differential under all k-higher derivation of finite rank m for all m, then $\mathfrak{p}\mathfrak{D}_\mathfrak{p}$ is differential under all k-higher derivations of finite or infinite rank.*

Proof. Let $\mathfrak{N}$ be the kernel of the canonical surjection $\mathfrak{D} \rightarrow \mathfrak{D}_\mathfrak{p}$, and let $\mathfrak{D}/\mathfrak{N} = k[[\bar{x}_1, \ldots, \bar{x}_n]]$, $\mathfrak{D}/\mathfrak{N}$ is a subring of $\mathfrak{D}_\mathfrak{p}$. Let $(0) = \bigcap_{i=1}^{s} \mathfrak{q}_i$ be a primary decomposition of the zero ideal in $\mathfrak{D}$. Suppose $\mathfrak{q}_i \subset \mathfrak{p}$ for $i = 1, 2, \ldots, t$ and $\mathfrak{q}_i \not\subset \mathfrak{p}$ for $i > t$. Then $\mathfrak{N} = \bigcap_{i=1}^{t} \mathfrak{q}_i$.

Suppose $\mathfrak{p}\mathfrak{D}_\mathfrak{p}$ is not differential under $\mathfrak{H}_k(\mathfrak{D}_\mathfrak{p})$, then there exists a higher derivation $\{\delta_i\} \in \mathfrak{H}_k(\mathfrak{D}_\mathfrak{p})$ such that $\delta_m(\mathfrak{p}\mathfrak{D}_\mathfrak{p}) \not\subset \mathfrak{p}\mathfrak{D}_\mathfrak{p}$ for some $m \geqq 1$. Suppose m is the least index such that $\delta_m(\mathfrak{p}\mathfrak{D}_\mathfrak{p}) \not\subset \mathfrak{p}\mathfrak{D}_\mathfrak{p}$. $\{\delta_0, \delta_1, \ldots, \delta_m\} \in \mathfrak{H}_k(\mathfrak{D}_\mathfrak{p})$ and is of rank m. It follows from Lemma 2′, there exists $t_0, t_1, \ldots, t_m \in \mathfrak{D}/\mathfrak{N} - \mathfrak{p}/\mathfrak{N}$ such that $\{t_i\delta_i\}_{i=1}{}^m \in \mathfrak{H}_k(\mathfrak{D}/\mathfrak{N})$. For $j < m$, $t_j\delta_j(\mathfrak{p}/\mathfrak{N}) \subset \mathfrak{p}\mathfrak{D}_\mathfrak{p} \cap \mathfrak{D}/\mathfrak{N} = \mathfrak{p}/\mathfrak{N}$ and $\delta_m(\mathfrak{p}\mathfrak{D}_\mathfrak{p}) \not\subset \mathfrak{p}\mathfrak{D}_\mathfrak{p}$ yields $t_m\delta_m(\mathfrak{p}/\mathfrak{N}) \not\subset \mathfrak{p}/\mathfrak{N}$. Let $\mathfrak{D}/\mathfrak{N} = k[[\bar{x}_1, \ldots, \bar{x}_n]]$ and $\bar{\pi}$ be the canonical surjection from the formal power series ring $k[[X_1, \ldots, X_n]]$ to $\mathfrak{D}/\mathfrak{N}$. $\{t_0\delta_0, \ldots, t_m\delta_m\}$, can be lifted to

$$\{\Delta_0, \Delta_1, \ldots, \Delta_m\} \in \mathfrak{H}_k(k[[X_1, \ldots, X_n]]),$$

according to Remark 3. Let $\mathfrak{N}'$ be the pre-image in $k[[X_1, \ldots, X_n]]$ of $\mathfrak{N}$. Then $\mathfrak{N}'$ is $\{\Delta_i\}_{i=1}^m$ − differential, i.e. $\Delta_i(\mathfrak{N}') \subset \mathfrak{N}'$. Let $\mathfrak{p}'$ and $\mathfrak{q}_i'$ be the pre-image of $\mathfrak{p}$ and $\mathfrak{q}_i$ in $k[[X_1, \ldots, X_n]]$ for $i = 1, 2, \ldots, s$ respectively. Then $\mathfrak{N}' = \mathfrak{q}_1' \cap \ldots \cap \mathfrak{q}_t'$. Let $d \in (\bigcap_{i=t+1} \mathfrak{q}_i') - \mathfrak{p}'$, $\{d^i\Delta_i\}_{i=0}^m$ is a higher derivation on $k[[X_1, \ldots, X_n]]$. Let $\mathfrak{A}' = \bigcap_{i=1}^{s} \mathfrak{q}_i'$. Then $\mathfrak{A}' \subset \mathfrak{N}'$ and $d^i\Delta_i(\mathfrak{A}') \subset d^i \cdot \mathfrak{N}' \subset (\mathfrak{q}_1' \cap \ldots \cap \mathfrak{q}_t') \cap (\mathfrak{q}_{t+1}' \cap \ldots \cap \mathfrak{q}_s') = \mathfrak{A}'$. Hence $\mathfrak{A}'$ is $\{d^i\Delta_i\}$-differential. Thus $\{d^i\Delta_i\}_{i=1}^m$ induces a higher derivation $\{\overline{d^i\Delta_i}\}_{i=1}^m$ on $k[[X_1, \ldots, X_n]]/\mathfrak{A}' = \mathfrak{D}$. Since $d_m\delta_m(\mathfrak{p}/\mathfrak{N}) \not\subset \mathfrak{p}/\mathfrak{N}$, therefore $\Delta_m(\mathfrak{p}') \not\subset \mathfrak{p}'$. Thus $\overline{d^m\Delta_m}(\mathfrak{p}) \not\subset \mathfrak{p}$, i.e. $\mathfrak{p}$ is not $\{\overline{d^i\Delta_i}\}_{i=1}^m$-differential, a contradiction to the hypothesis.

Let k be a field $X_1, \ldots, X_n$ indeterminates over k, $\Sigma = k((X_1, \ldots, X_n))$ = quotient field of $k[[X_1, \ldots, X_n]]$. Let $u = \{u_{ij} | i = 1, 2, \ldots, n; j = 1, 2, \ldots, \infty\}$ and t be indeterminates over Σ. The mapping $q : k[[X_1, \ldots, X_n]] \to k[[X_1, \ldots, X_n]][u][[t]]$ defined by the substitution $X_i \to X_i + \sum_{j=1}^{\infty} u_{ij}t^j$ is a continuous k-homomorphism. Let

$$a \in k[[X_1, \ldots, X_n]], a = \Sigma\, a_{i_1,\ldots,i_n} X_1^{i_1} \ldots X_n^{i_n}.$$

Then

$$q(a) = \sum a_{i_1\ldots i_n}\left(X_1 + \sum_{j=1}^{\infty} u_{1j}t^j\right)^{i_1} \ldots \left(X_n + \sum_{j=1}^{\infty} u_{nj}t^j\right)^{i_n}$$
$$= \sum a_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n} + q_1(a)t + \ldots + q_j(a)t^j + \ldots,$$

where $q_j(a) \in k[[X_1, \ldots, X_n]][u]$. Set q_0 = identity map on $k[[X_1, \ldots, X_n]]$. Then, by Remark 1, $\{q_j\}_{j=1}^{\infty} \in \mathfrak{H}_k(k[[X_1, \ldots, X_n]], k[[X_1, \ldots, X_n]][u])$, and q_j's are continuous for all j. By Remark 2, $q_j(k[[X_1, \ldots, X_n]]) \subset k[[X_1, \ldots, X_n]][\{u_{il} | l = 1, 2, \ldots, j-1, \ i = 1, 2, \ldots, n\}]$. Let $\mathfrak{D} = k[[x_1, \ldots, x_n]]$ be a complete local domain over a field k, $\Sigma = k((x_1, \ldots, x_n))$ its quotient field. Let $\bar{u} = \{\bar{u}_{ij} \in \Sigma[i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, \infty\}$ be a collection of elements in Σ. Let $u_j = \{u_{il} | l = 1, 2, \ldots, j;\ i = 1, 2, \ldots, n\}$ and let $\bar{u}_j = \{\bar{u}_{il} | l = 1, 2, \ldots, j;\ i = 1, 2, \ldots, n\}$. Let

$$\pi^{(j)} : k[[X_1, \ldots, X_n]][u_j] \to k[[x_1, \ldots, x_n]][\bar{u}_j] \subset \Sigma$$

be the canonical k-homomorphism such that $\pi^{(j)}(X_i) = x_i$ and $\pi^{(j)}(u_{il}) =$

$\bar{u}_{il}$ for $l = 1, 2, \ldots, j$, and $i = 1, 2, \ldots, n$. Let $f(X_1, \ldots, X_n) \in k[[X_1, \ldots, X_n]]$. We say $\bar{u} = \{\bar{u}_{ij}\}$ is a set of solutions of $q_j(f) = 0$ if and only if $\pi^{(j)}(q_j(f)) = 0$. The notations $\pi^{(j)}$, q_j are to be used in the following.

LEMMA 3'. *Let $\mathfrak{D} = k[[x_1, \ldots, x_n]]$ be a complete local domain over a field k, Σ its quotient field. Let $\mathfrak{A} = (f_1, \ldots, f_r) \subset k[[X_1, \ldots, X_n]]$ be the kernel of the canonical homomorphism $k[[X_1, \ldots, X_n]] \to \mathfrak{D}$. If $\delta = \{\delta_i\} \in \mathfrak{H}_k(\Sigma, \Sigma)$ then the set $\{\bar{u}_{ij} \in \Sigma | \bar{u}_{ij} = \delta_j(x_i),\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, \infty\}$ is a set of solutions of the equation*

(3') $\quad q_j(f_m) = 0, \quad m = 1, 2, \ldots, r, j = 1, 2, \ldots, \infty.$

Conversely, if a subset $\{\bar{u}_{ij} | i = 1, 2, \ldots, n, j = 1, 2, \ldots, \infty\}$ of Σ is a family of solutions of (3'), then there is a higher derivation $\delta = \{\delta_j\} \in \mathfrak{H}_k(\Sigma, \Sigma)$ such that $\delta_j(x_i) = \bar{u}_{ij}$ for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, \infty$.

Proof: $f_m(x_1, \ldots, x_n) = 0$ for $m = 1, 2, \ldots, r$. Since $\delta = \{\delta_j\} \in \mathfrak{H}_k(\Sigma, \Sigma)$, $\delta_j(f_m) = 0$. By Remark 2, $0 = \delta_i(f_m) = \sum_{j=1}^{n} A_{mji}(x_1, \ldots, x_n)\delta_j(x_i) + B_m$ where $A_{mji}(x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]]$ and $B_m \in k[[x_1, \ldots, x_n]][\{\delta_l(x_i) | l = 1, 2, \ldots, i-1, i = 1, 2, \ldots, n\}] \subset \Sigma$. Therefore $\{\delta_j(x_i) | i = 1, 2, \ldots, n, j = 1, 2, \ldots, \infty\}$ solves the system $q_j(f_m) = 0$ in Σ.

Conversely, if $\{\bar{u}_{ij} | i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, \infty\} \subset \Sigma$ form a family of solutions to the system $q_j(f_m) = 0$. Then we can find a $\delta = \{\delta_j\} \in \mathfrak{H}(\Sigma, \Sigma)$ such that $\delta_j(x_i) = \bar{u}_{ij}$ as follows: For $g(X_1, \ldots, X_n) \in k[[X_1, \ldots, X_n]]$, and for $j \geqq 1$, set $\delta_j(g(x_1, \ldots, x_n)) = \pi^{(j)}(q_j(g(X_1, \ldots, X_n))$, where $\pi^{(j)}$, q_j are the same as defined in the preceding. δ_j is well defined and $\{\delta_j\} \in \mathfrak{H}_k(k[[x_1, \ldots, x_n]], \Sigma)$ and $\delta_j(x_i) = \bar{u}_{ij}$. By [**2**, Lemma 2, p. 35], $\{\delta_j\}$ can be extended to Σ.

The following theorem shows that simple algebroid sub-varieties of an algebroid variety yield non-differential ideals.

THEOREM 4'. *Let $\mathfrak{D} = k[[x_1, \ldots, x_n]]$ be a complete local ring containing a field k. Let $\mathfrak{p} \subset \mathfrak{D}$ be a non-minimal prime ideal such that $\mathfrak{D}_\mathfrak{p}$ is a regular local ring. Then $\mathfrak{p}\mathfrak{D}_\mathfrak{p}$ is not differential under $\mathfrak{H}_k(\mathfrak{D}_\mathfrak{p}, \mathfrak{D}_\mathfrak{p})$.*

Proof. Let $\mathfrak{N}$ be the kernel of the canonical homomorphism $\mathfrak{D} \Rightarrow \mathfrak{D}_\mathfrak{p}$. Let $\hat{\mathfrak{D}}_\mathfrak{p}$ be the completion of $\mathfrak{D}_\mathfrak{p}$, then it is well known from [**5**, Corollary, p. 307] that $\hat{\mathfrak{D}}_\mathfrak{p} = K[[t_1, \ldots, t_r]]$, a formal power series ring over a field K with $K \cong \mathfrak{D}_\mathfrak{p}/\mathfrak{p}\mathfrak{D}_\mathfrak{p}$. Without loss of generality, we may assume K contains k. It follows from Lemma 3', by taking $\bar{u} = \{\bar{u}_{ij} | \bar{u}_{11} = 1$ and $\bar{u}_{ij} = 0$ for $i = 1, 2, \ldots, r$ and $j = 2, 3, \ldots, \infty\}$, and noting that $\mathfrak{A}$ in the Lemma 3' is the zero ideal, that there exists a higher derivation $\{\delta_j\}_{j=0}^{\infty} \in \mathfrak{H}_K(\hat{\mathfrak{D}}_\mathfrak{p}, \hat{\mathfrak{D}}_\mathfrak{p})$ such that $\delta_j(t_i) = \bar{u}_{ij}$. Let $\mathfrak{D}/\mathfrak{N} = k[[\bar{x}_1, \ldots, \bar{x}_n]]$, we may assume $t_1 \in \mathfrak{D}/\mathfrak{N}$. Let $\mathfrak{A} = (f_1, \ldots, f_s)k[[X_1, \ldots, X_n]]$ be the kernel of the canonical homomorphism $k[[X_1, \ldots, X_n]] \Rightarrow k[[\bar{x}_1, \ldots, \bar{x}_n]]$. Then the system $\sum f_{mi}'(\bar{x})\delta_1(x_i) = 0$, $\sum t_{1i}'(\bar{x})\delta_1(x_i) - 1 = 0$ where $f_{mi}' = \partial f_m/\partial X_i$, $t_{1i}' = \partial t_1(X)/\partial X_i$ with $t_1(X)$

https://doi.org/10.4153/CJM-1977-005-x Published online by Cambridge University Press

being a representative of t_1 in $k[[X_1, \ldots, X_n]]$, and for each $j = 2, 3, \ldots, \infty$ the linear system $\sum_{j=1}^{n} A_{mji} d_j(\bar{x}_i) + B_m = 0$ where $m = 1, 2, \ldots, s$, $A_{mji} \in k[[\bar{x}_1, \ldots, \bar{x}_n]]$ and

$$B_m \in k[[\bar{x}_1, \ldots, x_n]][\{\delta_l(\bar{x}_i) | l = 1, 2, \ldots, (j-1), i = 1, 2, \ldots, n\}]$$

have solution set $= \{\delta_j(x_i) | i = 1, 2, \ldots, n; j = 1, 2, \ldots, \infty\}$ in $\hat{\mathfrak{D}}_{\mathfrak{p}}$. Thus by [**4**, Lemma p. 39], the linear system $q_1(f_m) = 0, m = 1, 2, \ldots, s$ and $q_1(t_1(X)) - 1 = 0$, and for each $i = 2, \ldots, \infty$ the linear system $q_j(f_m) = 0$ $m = 1, 2, \ldots, s$ have solutions set $\bar{u} = \{\bar{u}_{ij} | i = 1, 2, \ldots, n; j = 1, 2, \ldots, \infty\} \subset \mathfrak{D}_{\mathfrak{p}}$. Thus it follows from Lemma 3′ that there is a higher derivation $\{\delta_j'\}_{j=0}^{\infty} \in \mathfrak{H}_k(\mathfrak{D}/\mathfrak{N}, \mathfrak{D}_{\mathfrak{p}})$, such that $\delta_j'(t_1) = 1$. Extending $\{\delta'\}_{j=0}^{\infty}$ to $\mathfrak{D}_{\mathfrak{p}}$ we have thus a higher derivation $\{\delta'\}_{n=0}^{\infty} \in \mathfrak{H}_k(\mathfrak{D}_{\mathfrak{p}})$ such that $\delta_1'(\mathfrak{p}\mathfrak{D}_{\mathfrak{p}}) \not\subset \mathfrak{p}\mathfrak{D}_{\mathfrak{p}}$.

I would like to thank the referee for his comments.

Added in proof. W. C. Brown and the author have noted that Example 3 in [**1**] is incorrect. A correct example is as follows: Let k denote a field of characteristic two. Set $\mathcal{O} = k[[X]]$, the power series ring in one indeterminate X over k. We can define a higher derivative $D = \{\delta_i\}$ on $\mathcal{O}$ by setting $\delta_i(X) = 1$ for all $i \geq 1$. Then $\delta_2(X^2) = 1$, but there exists no subring $\mathcal{O}_1 \subset \mathcal{O}$ such that X^2 is analytically independent over $\mathcal{O}_1$ and $\mathcal{O} = \mathcal{O}_1[[X^2]]$.

Thus, the conjecture mentioned in [**1**] before Example 3 is false even for regular local rings.

References

1. W. Brown and W. Kuan, *Ideals and higher derivations in commutative rings*, Can. J. Math. *24* (1972), 400–415.
2. N. Heerema, *Convergent higher derivations on local rings*, Trans. Amer. Math. Soc. *132* (1968), 31–44.
3. W. Kuan, *A note on a lemma of Zariski and higher derivations*, Proc. Amer. Math. Soc. *42* (1974), 333–340.
4. A. Seidenberg, *Differential ideals in rings of finitely generated type*, Amer. Jour. Math. *89* (1967), 22–42.
5. O. Zariski and P. Samuel, *Commutative algebra*, Vol. 2 (New York, 1960).

Michigan State University,
East Lansing, Michigan