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AN ASYMPTOTIC ESTIMATE FOR THE BERNOULLI AND EULER NUMBERS

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1. **Introduction.** We derive here simple asymptotic estimates for both the Euler and Bernoulli numbers. The derivations follow easily from known results, but I am unable to find them elsewhere in the literature. C. Jordan [1, p. 245 and p. 303] gives some related inequalities. Other properties of these two classical sets of numbers may be found in [1], [3] and [4].

2. **The Bernoulli case.** It is well known (see e.g. [2]) that the Fourier series expansion of $B_{2k}(x)$, the Bernoulli polynomial of degree $2k$ is

(1) $$B_{2k}(x)=\frac{(-1)^{k+1}2(2k)!}{(2\pi)^{2k}}\sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^{2k}} \qquad (k=1,2,\ldots)$$

valid for $0\le x\le 1$. Setting $x=\frac{1}{4}$ in (1) yields

$$B_{2k}(\tfrac{1}{4})=\frac{(-1)^{k+1}2(2k)!}{(4\pi)^{2k}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2k}}$$

Hence,

(2) $$|B_{2k}(\tfrac{1}{4})|\sim\frac{2(2k)!}{(4\pi)^{2k}} \qquad (k\to\infty)$$

Using Stirling's formula and the known result (see, e.g. [4], p. 22) $B_{2k}(\frac{1}{4})=-2^{-2k}(1-2^{1-2k})B_{2k}$ we get

(3) $$|B_{2k}(\tfrac{1}{4})|\sim 4\sqrt{\pi k}\left(\frac{k}{2\pi e}\right)^{2k} \qquad (k\to\infty)$$

(Note that the coefficient $\sqrt{8\pi k}$ appearing in [2], p. 537 is incorrect and should read $4\sqrt{\pi k}$).

Taking the $2k$th root in (3) yields

(4) $$|B_{2k}|^{1/2k}\sim\frac{k}{\pi e} \qquad (k\to\infty).$$

A short table helps to illustrate the rate at which the two quantities are

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approaching each other. Let $S_k = |B_{2k}|^{1/2k}$ and $T_k = k/\pi e$

k	S_k	T_k	S_k/T_k
5	0.77258	0.58550	1.31952
10	1.36829	1.17100	1.16848
15	1.96175	1.75649	1.11685
20	2.55351	2.34199	1.09031
25	3.14414	2.92749	1.07401
30	3.73399	3.51299	1.06291

3. **The Euler case.** The Fourier series expansion of $B_{2k+1}(x)$, the Bernoulli polynomial of degree $2k+1$ is

$$(5)\qquad B_{2k+1}(x)=\frac{(-1)^{k+1}2(2k+1)!}{(2\pi)^{2k+1}}\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n^{2k+1}}\qquad (k=1,2,\ldots)$$

valid for $0\le x\le 1$. Setting $x=\frac{1}{4}$ in (5) yields

$$(6)\qquad B_{2k+1}(\tfrac{1}{4})=\frac{(-1)^k 2(2k+1)!}{(2\pi)^{2k+1}}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^{2k+1}}$$

Therefore,

$$(7)\qquad |B_{2k+1}(\tfrac{1}{4})|\sim\frac{2(2k+1)!}{(2\pi)^{2k+1}}\qquad (k\to\infty).$$

It is known (see e.g. [4], p. 29) that

$$(8)\qquad B_{2k+1}(\tfrac{1}{4})=\frac{-(2k+1)E_{2k}}{4^{2k+1}}.$$

Therefore, from (7) and (8) we get

$$(9)\qquad |E_{2k}|\sim\frac{2^{2k+2}(2k)!}{\pi^{2k+1}}\qquad (k\to\infty)$$

Using Stirling's formula in (9) we have

$$(10)\qquad |E_{2k}|\sim\frac{8\sqrt{k}}{\sqrt{\pi}}\left(\frac{4k}{\pi e}\right)^{2k}\qquad (k\to\infty)$$

Taking the $2k$th root in (10) yields

$$(11)\qquad |E_{2k}|^{1/2k}\sim\frac{4k}{\pi e}\qquad (k\to\infty)$$

A second short table here illustrates the rate at which these two quantities are approaching each other. Let $w_k = |E_{2k}|^{1/2k}$ and $y_k = 4k/\pi e$.

k	w_k	y_k	w_k/y_k
5	2.95357	2.34199	1.26133
10	5.35096	4.68399	1.14239
15	7.73128	7.02598	1.10038
20	10.0994	9.36797	1.07807
25	12.4635	11.7100	1.06435
30	14.8240	14.0520	1.05494

References

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