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ON A QUESTION OF BOURAS CONCERNING WEAK COMPACTNESS OF ALMOST DUNFORD–PETTIS SETS

JIN XI CHEN✉ and LEI LI

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Abstract

We give a positive answer to the question of Bouras ['Almost Dunford–Pettis sets in Banach lattices', *Rend. Circ. Mat. Palermo* (2) **62** (2013), 227–236] concerning weak compactness of almost Dunford–Pettis sets in Banach lattices. That is, every almost Dunford–Pettis set in a Banach lattice E is relatively weakly compact if and only if E is a *KB*-space.

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Let E be a Banach lattice. Recall that a norm bounded subset A of E is said to be *L-weakly compact* if $\|x_n\| \to 0$ for every disjoint sequence (x_n) contained in the solid hull of A. It is well known that every L-weakly compact set is a relatively weakly compact set, but the converse does not hold in general. In an L-space, L-weakly compact sets and relatively weakly compact sets coincide. More generally, every relatively weakly compact subset of E is L-weakly compact if and only if E has the positive Schur property (cf. [4, Corollary 3.6.8]). Here, we say that a Banach lattice E has the *positive Schur property* if every weakly null sequence with positive terms in E is norm null, equivalently, if every disjoint weakly null sequence in E is norm null.

Following Bouras [2], a bounded subset A of the Banach lattice E is called an *almost Dunford–Pettis set* if every disjoint weakly null sequence (f_n) of E' converges uniformly to zero on A, that is, $\sup_{x\in A} |f_n(x)| \to 0$. In [2], Bouras showed that every L-weakly compact set in a Banach lattice E is necessarily almost Dunford–Pettis. Also, every relatively weakly compact set in E is almost Dunford–Pettis if and only if E has the weak Dunford–Pettis property. If every almost Dunford–Pettis set in E is relatively weakly compact (in particular, L-weakly compact), then E is a *KB*-space [2, Theorem 2.10]. Conversely, if E is an L-space, or a dual *KB*-space or the norm of E'' is order continuous, then every almost Dunford–Pettis set in E is L-weakly compact

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(and hence relatively weakly compact). Note that in those three cases E is a KB-space. Therefore, Bouras naturally posed the following question.

QUESTION [2]. Does the assumption 'E is a KB-space' imply that each almost Dunford–Pettis set in E is relatively weakly compact (respectively L-weakly compact)?

In this short note, we give an affirmative answer to the question with respect to weak compactness of almost Dunford–Pettis sets. That is, every almost Dunford–Pettis set in a Banach lattice E is relatively weakly compact if and only if E is a KB-space.

Our notation is standard. For Banach lattice theory, we refer the reader to the monographs [1, 4] and also the original paper of Bouras [2].

THEOREM. *Let E be a Banach lattice. Every almost Dunford–Pettis set in E is relatively weakly compact if and only if E is a KB-space.*

PROOF. For the proof of the 'only if' part, see [2, Theorem 2.10]. We only need to prove the 'if' part. To this end, let E be a KB-space and let A be an almost Dunford–Pettis set in E. We have to show that A is relatively weakly compact.

First, we claim that the solid hull, $\mathrm{sol}(A)$, of A is likewise an almost Dunford–Pettis set. Otherwise, there would exist a disjoint weakly null sequence $(f_n) \subseteq E'$ such that $\sup_{x\in \mathrm{sol}(A)} |f_n(x)| > \varepsilon_0$ for some $\varepsilon_0 > 0$ and all $n \in \mathbb{N}$. From this, we can find two sequences $(x_n) \subseteq A$ and $(y_n) \subseteq \mathrm{sol}(A)$ satisfying

$$|y_n| \le |x_n|, \quad 0 < \varepsilon_0 < |f_n(y_n)| \le |f_n|(|x_n|)$$

for each n. Again, by [1, Theorem 1.23], there exists a sequence $(g_n) \subseteq E'$ such that

$$|g_n| \le |f_n|, \quad |g_n(x_n)| > \varepsilon_0.$$

Clearly, (g_n) is a disjoint sequence in E'. From the weak convergence of (f_n), it follows that $g_n \xrightarrow{w} 0$ (cf. [1, Theorem 4.34]). Since A is almost Dunford–Pettis,

$$\varepsilon_0 < |g_n(x_n)| \le \sup_{x\in A} |g_n(x)| \to 0,$$

which is impossible. This proves that $\mathrm{sol}(A)$ is an almost Dunford–Pettis set.

Now we can assume without loss of generality that A is a solid almost Dunford–Pettis set in E. Let $\rho_A(f) := \sup_{x\in A}\langle |f|, |x|\rangle$ for each $f \in E'$. Clearly, ρ_A is a lattice seminorm on E' and $\rho_A(f) = \sup_{x\in A} |f(x)|$. Let (f_n) be an arbitrary order bounded disjoint sequence in E'. Since $f_n \xrightarrow{w} 0$ and A is almost Dunford–Pettis, we have $\rho_A(f_n) \to 0$. Therefore, by [4, Theorem 2.3.3], for every $0 \le f \in E'$, A is approximately order bounded with respect to ρ_f, that is to say, for every $\varepsilon > 0$ there exists $0 \le x \in E$ such that

$$A \subseteq [-x, x] + \varepsilon B_{\rho_f}, \tag{$*$}$$

where $B_{\rho_f} = \{x \in E : \rho_f(x) = f(|x|) \le 1\}$.

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Note that the w^*-closure, $\overline{A}^{w^*}$, of A in E'' is w^*-compact. To prove that A is relatively weakly compact in E, it suffices to show that $\overline{A}^{w^*} \subseteq E$. Since E is a *KB*-space, we have $E = (E')'_n$ (cf. [1, Theorem 4.60]). Here, $(E')'_n$ is a band of E'' consisting of all order continuous linear functionals on E'. Thus, we have to show that every element x'' of $\overline{A}^{w^*}$ is an order continuous linear functional on E' or, equivalently, $|x''| \in (E')'_n$. To this end, let (f_α) be an arbitrary decreasing net in $(E')^+$ with $f_\alpha \downarrow 0$. It is enough to show that $\langle |x''|, f_\alpha \rangle \downarrow 0$. For each α,

$$\langle |x''|, f_\alpha \rangle = \sup\{|\langle x'', g \rangle| : |g| \leq f_\alpha\} \leq \sup\{|\langle g, x \rangle| : |g| \leq f_\alpha, x \in A\} \leq \rho_A(f_\alpha).$$

To finish the proof, we need only prove that $\rho_A(f_\alpha) \downarrow 0$. Indeed, it may be assumed that there exists an element $f \in E'$ such that $0 \leq f_\alpha \leq f$ for all α. Let $\varepsilon > 0$ be fixed. By (∗), we have $\rho_A(f_\alpha) \leq f_\alpha(x) + \varepsilon$ for all α. Note that $f_\alpha(x) \downarrow 0$. It follows that $\inf_\alpha \rho_A(f_\alpha) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\inf_\alpha \rho_A(f_\alpha) = 0$, as desired. □

In a *KB*-space without the weak Dunford–Pettis property (for example, ℓ_p with $1 < p < \infty$), almost Dunford–Pettis sets and relatively weakly compact sets cannot coincide.

There is another topic which is included in the question of Bouras.

(Q1) Is every almost Dunford–Pettis set in a *KB*-space L-weakly compact?

We are not able to answer this question and it remains open. However, we can make some comments on (Q1).

In [3], Chen *et al.* introduced the class of almost limited sets in Banach lattices. A norm bounded subset A of a Banach lattice E is said to be an *almost limited set* if every disjoint weak* null sequence (f_n) of E' converges uniformly to zero on A. Clearly, every almost limited set in E is an almost Dunford–Pettis set. Almost limited sets and L-weakly compact sets coincide in E if and only if the norm of E is order continuous [3, Theorem 2.6(2)]. Since the norm of a *KB*-space is order continuous, (Q1) is the same as the following question.

(Q1′) Do almost Dunford–Pettis sets and almost limited sets coincide in a *KB*-space?

Let us recall that a Banach lattice E has the positive Schur property if and only if every relatively weakly compact subset of E is L-weakly compact. We know that every Banach lattice with the positive Schur property is a *KB*-space with the weak Dunford–Pettis property. In 1994, Wnuk [5] asked a question which currently is still open.

(Q2) Does every *KB*-space with the weak Dunford–Pettis property have the positive Schur property?

In view of the Theorem of the present paper, we know that in a *KB*-space with the weak Dunford–Pettis property almost Dunford–Pettis sets and relatively weakly compact sets coincide. Hence, if the answer to (Q1) is positive, then the answer to (Q2) is likewise positive. Equivalently, if the answer to (Q2) is negative, then

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there exists a *KB*-space where we can find an almost Dunford–Pettis set which is not *L*-weakly compact.

We formulate the above discussion in the following proposition.

PROPOSITION. *Let E be a KB-space. Consider the following conditions:*

(1) *every almost Dunford–Pettis set in E is L-weakly compact;*
(2) *every almost Dunford–Pettis set in E is almost limited;*
(3) *E has the positive Schur property.*

Then we have $(3) \implies (1) \iff (2)$.

If, in addition, E has the weak Dunford–Pettis property, then $(1) \iff (3)$.

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References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators (reprint of the 1985 original)* (Springer, Dordrecht, 2006).
[2] K. Bouras, 'Almost Dunford–Pettis sets in Banach lattices', *Rend. Circ. Mat. Palermo* (2) **62** (2013), 227–236.
[3] J. X. Chen, Z. L. Chen and G. X. Ji, 'Almost limited sets in Banach lattices', *J. Math. Anal. Appl.* **412** (2014), 547–553.
[4] P. Meyer-Nieberg, *Banach Lattices (Universitext)* (Springer, Berlin, 1991).
[5] W. Wnuk, 'Banach lattices with the weak Dunford–Pettis property', *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia* **42** (1994), 227–236.

JIN XI CHEN, Department of Mathematics, Southwest Jiaotong University, Chengdu 610031, PR China
e-mail: jinxichen@home.swjtu.edu.cn

LEI LI, School of Mathematical Sciences, Nankai University, Tianjin 300071, PR China
e-mail: leilee@nankai.edu.cn