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PROJECTIVE AND INJECTIVE HOPF ALGEBRAS OVER THE DYER-LASHOF ALGEBRA

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ABSTRACT The purpose of this paper is to discuss the existence, structure, and properties of certain projective and injective Hopf algebras in the category of Hopf algebras that support the structure one expects on the homology of an infinite loop space As an auxiliary project, we show that these projective and injective Hopf algebras can be realized as the homology of infinite loop spaces associated to spectra obtained from Brown-Gitler spectra by Spanier-Whitehead duality and Brown-Comenetz duality, respectively We concentrate mainly on indecomposable projectives and injectives, and we work only at the prime 2

The mod 2 homology $H_*X = H_*(X, \mathbb{F}_2)$ of an infinite loop space X is a commutative, cocommutative Hopf algebra with an action by Dyer-Lashof operations Q^i, which increase degree, and Steenrod operations Sq^j, which lower degree. These ingredients of structures are not independent, but related by Cartan formulas, instability requirements, and Adem and Nishida relations. We give the explicit formulas in Section 1; the reference is Chapter 1 of [3]. Let $\mathcal{AR}$ be the category of such Hopf algebras.

Then $\mathcal{AR}$ is an abelian category and we say $H \in \mathcal{AR}$ is projective if whenever $H_2 \longrightarrow H_1$ is a surjection in $\mathcal{AR}$, and $H \longrightarrow H_1$ is any map in $\mathcal{AR}$, there is a factoring

$$\begin{array}{ccc} H & \longrightarrow & H_2 \\ \big\downarrow{=} & & \big\downarrow \\ H & \longrightarrow & H_1. \end{array}$$

For example, $H_*\Omega^\infty\Sigma^\infty S^n$ is not projective (if $n \geq 2$) because the action by the Steenrod operations in $H_*\Omega^\infty\Sigma^\infty S^n$ is trivial. However, $H_*\Omega^\infty\Sigma^\infty S^n$ does have a projective cover.

THEOREM 1. *(1) There exists a projective object $\Gamma(n) \in \mathcal{AR}$ and a surjection $g\colon \Gamma(n) \longrightarrow H_*\Omega^\infty\Sigma^\infty S^n$ in $\mathcal{AR}$ so that if $H \longrightarrow H_*\Omega^\infty\Sigma^\infty S^n$ is any surjection in $\mathcal{AR}$ with H projective, then there is a factoring*

$$\begin{array}{ccc} H & \xrightarrow{f} & \Gamma(n) \\ \big\downarrow & & \big\downarrow g \\ H & \longrightarrow & H^*\Omega^\infty\Sigma^\infty S^n \end{array}$$

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with f split surjective.

(2) There is a spectrum $T(n)$ so that $H_*\Omega^\infty T(n) \cong \Gamma(n)$ and if X is any spectrum, the map

$$[T(n), X] \to \mathrm{Hom}_{\mathcal{AR}}\big(\Gamma(n), H_*\Omega^\infty X\big)$$

given by $f \mapsto \Omega^\infty f_*$ is onto for all n and an isomorphism if n is even.

This theorem is not new, although this formulation might be different: these results are implicit in [5]; furthermore, $T(n)$ is Spanier-Whitehead dual to a Brown-Gitler spectrum [2] and, thus, has been heavily studied.

$\Gamma(n)$ is simply described, at least as an algebra. First

$$H_*\Omega^\infty\Sigma^\infty S^n \cong \mathbb{F}_2[Q^I(\iota_n)]$$

where $\iota_n \in H_n\Omega^\infty\Sigma^\infty S^n$ is the non-zero class, $Q^I(\iota_n) = Q^{i_1}\cdots Q^{i_s}(\iota_n)$ is an iterated Dyer-Lashof operation with $i_t \leq 2i_{t+1}$ and $i_1 - i_2 - \cdots - i_s < n$, and $\mathbb{F}_2[\cdot]$ denotes the polynomial algebra. Since what prevents $H_*\Omega^\infty\Sigma^\infty S^n$ from being projective is the lack of Steenrod operations, it is not surprising that

$$\Gamma(n) \cong \mathbb{F}_2[Q^I(\iota_n\,\mathrm{Sq}^J)].$$

where $\iota_n \in \Gamma(n)_n$, $\mathrm{Sq}^J = \mathrm{Sq}^{j_1}\cdots\mathrm{Sq}^{j_k}$ is an iterated admissible Steenrod operation with $2j_1 \leq n$, and $Q^I = Q^{i_1}\cdots Q^{i_s}$ is an iterated Dyer-Lashof operation with $i_t \leq 2i_{t+1}$ and $i_1 - i_2 - \cdots - i_s \geq n - j_1 - \cdots - j_k$. The action of the Dyer-Lashof and Steenrod operations on $\Gamma(n)$ is determined by the axioms for infinite loop spaces; the coalgebra structure, however, is complicated and could be called a "Witt vector diagonal" as suggested by Segal (see [13]).

An object $H \in \mathcal{AR}$ is injective if, whenever $H_1 \to H_2$ is an injection in $\mathcal{AR}$, and $H_1 \to H$ is any map, then there is a factoring

$$\begin{array}{ccc} H_1 & \longrightarrow & H_2 \\ \downarrow & & \downarrow \\ H & \xrightarrow{=} & H \end{array}$$

For example, $H_*K\big(\mathbb{Z}/(2), n\big)$ is not injective because the action by the Dyer-Lashof operations is trivial. However, $H_*K\big(\mathbb{Z}/(2), n\big)$ has an injective hull.

THEOREM 2. *(1) There exists an injective object $H(n) \in \mathcal{AR}$ and an injection $g\colon H_*K\big(\mathbb{Z}/(2), n\big) \to H(n)$ in $\mathcal{AR}$ so that if $H_*K\big(\mathbb{Z}/(2), n\big) \to H$ is an injection with H injective in $\mathcal{AR}$, then there is an factoring*

$$\begin{array}{ccc} H^*K\big(\mathbb{Z}/(2), n\big) & \xrightarrow{g} & H(n) \\ \downarrow & & \downarrow f \\ H & \longrightarrow & H \end{array}$$

with f split injective.

(2) There is a spectrum $D(n)$ so that $H_*\Omega^\infty D(n) \cong H(n)$ and, for any spectrum X, the map

$$[X, D(n)] \longrightarrow \operatorname{Hom}_{\mathcal{AR}}\big(H_*\Omega^\infty X, H(n)\big)$$

is an isomorphism if n is even

The existence of $H(n)$ is nearly formal, using the special adjoint functor theorem, as put forth in [8] For any abelian group, G, let $G^{\mathbb{Q}/\mathbb{Z}} = \operatorname{Hom}(G, \mathbb{Q}/\mathbb{Z})$ Since $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, the functor $(\ )^{\mathbb{Q}/\mathbb{Z}}$ is exact We will show that the functor on $\mathcal{AR}$

$$H \longmapsto \operatorname{Hom}_{\mathcal{AR}}\big(\Gamma(n), H\big)^{\mathbb{Q}/\mathbb{Z}}$$

is representable, that is, there exists $H(n) \in \mathcal{AR}$ and a natural isomorphism

$$\operatorname{Hom}_{\mathcal{AR}}\big(H, H(n)\big) \cong \operatorname{Hom}_{\mathcal{AR}}\big(\Gamma(n), H\big)^{\mathbb{Q}/\mathbb{Z}}$$

Then $H(n)$ is automatically injective

Given this description it should not be surprising that $D(n)$ is Brown-Comenetz dual to a Brown-Gitler spectrum Indeed, if E is a spectrum with associated homology theory $E_*(-)$, and if $E_n = E_nS^0$ is a finite group for each n, then the functor

$$X \longmapsto (E_*X)^{\mathbb{Q}/\mathbb{Z}}$$

defines a cohomology theory, and the representing spectrum is the Brown-Comenetz dual of E See [1]

What is not formal is finding a description of $H(n)$ The difficulty lies in the fact that $H(n)_0 \not\cong \mathbb{F}_2$, so that all the conveniences of connected Hopf algebras are not available to us

The idea is simple—one has to add enough Dyer-Lashof operations to $H_*K\big(\mathbb{Z}/(2), n\big)$ to make this Hopf algebra "divisible" by Dyer-Lashof operations The method is this we first consider the category $\mathcal{UR}$ of modules that are at once modules over the Dyer-Lashof operations and the Steenrod operations, subject to the Nishida relations and instability criteria Being a category of modules, $\mathcal{UR}$ has a set of describable injectives $G(n)$, $n \geq 0$ Then we consider a category $\mathcal{ARC}$ of coalgebras over the Dyer-Lashof and Steenrod operations There will be a functor $V\ \mathcal{UR} \longrightarrow \mathcal{ARC}$ right adjoint to the forgetful functor, and we will have an isomorphism of coalgebras in $\mathcal{ARC}$

$$H(n) \cong V\big(G(n)\big) \tag{3}$$

But even the existence of V poses a problem, again we use the special adjoint functor theorem

In Section 1 we recall the work of May and others on the structure of the homology of infinite loop spaces Sections 2 and 3 are devoted to constructing and studying projectives and injectives in various relevant abelian categories Section 4 is devoted to Theorem 1 1 and its ramifications Section 5 is devoted to Theorem 2 1 and the isomorphism 3 Section 6 is spent on the homotopy theory of Theorems 1 and 2 We close Section 6 with a few paragraphs on the relationship between this work and past work. Finally, there is an appendix on the special adjoint functor theorem—an extremely useful result for producing adjoints and showing functors are representable.

And I should make a remark on what is not proved here. Various families of projectives and injectives are given in the category $\mathcal{AR}$ of Hopf algebras over the Dyer-Lashof operations. It would be nice to have a list of all such; in particular, it would be nice to know where the families here fit into some overall scheme. I know of no way of addressing this problem. Specifically, I know of no way of reducing these questions to similar ones answered by Jean Lannes and Lionel Schwartz (see *Topology* 28, pp. 153–169) for the category of unstable cohomology modules over the Steenrod algebra.

1. **Recollections on the homology of infinite loop spaces.** The homology of infinite loop spaces supports a great deal of structure; in this section we recapitulate that structure. The standard reference is the first chapter of [3]. In the process we define our notation.

If X is a space, then $H_*X \cong H_*(X, \mathbb{F}_2)$ is an object in the category of unstable coalgebras over $\mathcal{A}$. Call the category of such $\mathcal{CA}$. Thus $C \in \mathcal{CA}$ is a graded $\mathbb{F}_2$ vector space equipped with a coassociative, cocommutative comultiplication

$$\psi \colon C \longrightarrow C \otimes C$$

and equipped with a right action of the Steenrod algebra

$$(\cdot)\,\mathrm{Sq}^i \colon C_n \longrightarrow C_{n-i}$$

that is unstable in the sense that for all $x \in C_n$

$$x\,\mathrm{Sq}^i = \begin{cases} 0 & \text{if } 2i > n \\ \xi x & \text{if } 2i = n. \end{cases} \tag{1.1}$$

Here $\xi \colon C_{2i} \longrightarrow C_i$ is the shift map defined by

$$C_{2i} \xrightarrow{\psi} (C \otimes C)_{2i}^{\Sigma_2} \xrightarrow{\rho} C_i$$

where ψ is the comultiplication, and $\rho(x \otimes x) = x$, $\rho(x \otimes y + y \otimes x) = 0$. Note that (1.1) and the fact that $x\,\mathrm{Sq}^0 = x$ has strong implications for the structure of C_0. In fact, define

$$X_0 \subseteq C_0$$

to be the set of set-like elements; that is $x \in X_0$ if and only if $x \neq 0$ and

$$\psi(x) = x \otimes x.$$

The requirement that $x\,\mathrm{Sq}^0 = x$ implies that $\mathbb{F}_2(X_0) \cong C_0$, where $\mathbb{F}_2(X_0)$ is the vector space with basis X_0. The proof of this statement requires some fairly deep facts about the structure of coalgebras; these facts are available, for example, in [14].

There is an abelian category associated to $\mathcal{CA}$. Let $\mathcal{U}$ be the category of unstable modules over the Steenrod algebra. Thus $\mathcal{U}$ is the full-subcategory of the category of right modules over $\mathcal{A}$ given by the condition that for all $x \in M_m$,

$$x\,\mathrm{Sq}^i = 0 \quad \text{if } 2i > m. \tag{1.2}$$

Then there is forgetful functor $\mathcal{CA} \longrightarrow \mathcal{U}$. Here and hereafter we adopt the convention of leaving forgetful functors unlabeled.

PROPOSITION 1.3. *The forgetful functor $\mathcal{CA} \to \mathcal{U}$ has a right adjoint*

$$V: \mathcal{U} \longrightarrow \mathcal{CA}.$$

Since this is not obvious, nor in the literature, we provide a proof in the appendix.

If X is an infinite loop space H_*X is a commutative Hopf algebra with conjugation—an abelian group object in $\mathcal{CA}$. But H_*X supports more structure than that. We define a category of Hopf algebras $\mathcal{AR} \subseteq \mathcal{CA}$ by saying $H \in \mathcal{CA}$ is in $\mathcal{AR}$ if

(1.4.1) H is an abelian group object in $\mathcal{CA}$;

(1.4.2) there are Dyer-Lashof operations

$$Q^i: H_n \longrightarrow H_{n+i}, \quad i \geq 0,$$

so that

(i) $Q^i(x) = 0$ if $i < n$ and $Q^n(x) = x^2$, $Q^i(1) = 0$ for all $i \neq 0$;

(ii) $Q^i(xy) = \sum_{a+b=i} Q^a(x)Q^b(y)$

(iii) $\psi Q^i(x) = \sum_{a+b=i} Q^a(x_j) \otimes Q^b(y_j)$ where $\psi(x) = \sum x_j \otimes y_j$

(iv) if $i > 2j$,

$$Q^iQ^j = \sum_t \binom{t-j-1}{2t-i} Q^{i+j-t}Q^t$$

(v) $Q^i(x)\,\mathrm{Sq}^j = \sum_t \binom{i-j}{j-2t} Q^{i-j+t}(x\,\mathrm{Sq}^t)$.

It is a consequence of (i) and (iv) that for every $x \in H_n$ any composition of Dyer-Lashof operations applied to x can be written as a sum of elements of the form

$$Q^I(x) = Q^{i_1} \cdots Q^{i_s}(x)$$

with $i_k \leq 2i_{k+1}$ for all k and $e(I) = i_1 - i_2 - \cdots - i_s \geq n = \deg(x)$. Q^I satisfying these requirements will be called an *allowable monomial* of excess greater than or equal to n. Morphisms in $\mathcal{AR}$ commute with the Q^i.

PROPOSITION 1.5 ([3]). *If X is an infinite loop space, then $H_*X \in \mathcal{AR}$*

There are various abelian categories associated to $\mathcal{AR}$. To define them we first make an algebra out of the Dyer-Lashof operations. Let $R(-\infty)$ be the free non-commutative graded $\mathbb{F}_2$ algebra an symbols Q^i divided by the ideal of relations given in (1.4.2)(iv). Note that since these relations are homogeneous, we could bigrade $R(-\infty)$ if we wished.

The subspace spanned by $\{Q^I : e(I) < n\}$ is an ideal $B(n) \subseteq R(-\infty)$. Let $R(n) = R(-\infty)/B(n)$ and set $R(0) = R$ to be the Dyer-Lashof algebra.

An $R(-\infty)$ module M is *n-allowable* if and only if $Q^ix = 0$ for all $x \in M$ when $i < |x| + n$. Thus for a non-negatively graded n-allowable $R(-\infty)$ module M, the $R(-\infty)$-action factors through an $R(n)$ action. Write $\mathcal{R}_n$ for the abelian category of n-allowable $R(-\infty)$ modules.

We are particularly interested in the cases $n = 0$ and $n = 1$. The primitive element functor defines a functor $P\colon \mathcal{AR} \longrightarrow \mathcal{R}_0$; the indecomposables functor defines a functor $Q\colon \mathcal{AR} \longrightarrow \mathcal{R}_1$.

$R = R(0)$ admits a Hopf algebra structure with cocommutative diagonal given by the requirement that

$$\Delta Q^i = \sum_{a+b=i} Q^a \otimes Q^b.$$

There is more structure than this, however. We form the semi-tensor product of R and the Steenrod algebra $\mathcal{A}$. As a vector space this is $R \otimes \mathcal{A}$, and the multiplication is determined by the multiplications in R and $\mathcal{A}$ and the formula

$$(1 \otimes \mathrm{Sq}^j)(Q^i \otimes 1) = \sum_t \binom{i-j}{j-2t} Q^{i-j+t} \otimes \mathrm{Sq}^t.$$

See (1.4.2). Denote this algebra by $R \odot \mathcal{A}$.

Let $\mathcal{UR}$ be the category of $R \odot \mathcal{A}$ modules determined by the requirement that there be forgetful functors

$$\mathcal{UR} \longrightarrow \mathcal{R}_0$$

and

$$\mathcal{UR} \longrightarrow \mathcal{U}.$$

Thus $M \in \mathcal{UR}$ is a left 0-allowable $R(-\infty)$ module, a right unstable $\mathcal{A}$ module and the Nishida relations (1.4.2)(v) hold. The co-augmentation defines a functor

$$J\colon \mathcal{AR} \longrightarrow \mathcal{UR}.$$

Finally, the diagonal formula (1.4.2)(iii) defines a category of coalgebras over the Dyer-Lashof algebra. In fact, we define a category $\mathcal{ARC} \subseteq \mathcal{CA}$ by the requirements that $C \in \mathcal{ARC}$ if and only if C satisfies (1.4.2)(i), (iii), (iv), and (v). Morphisms in $\mathcal{ARC}$ must commute with the Dyer-Lashof operations. An analog to Proposition 1.3 is given by:

PROPOSITION 1.6. *The forgetful functor $\mathcal{ARC} \longrightarrow \mathcal{UR}$ has a right adjoint $V\colon \mathcal{UR} \longrightarrow \mathcal{ARC}$: in fact, V covers the functor of Proposition 1.3 in the sense that there is a diagram*

$$\begin{array}{ccc} \mathcal{UR} & \xrightarrow{V} & \mathcal{ARC} \\ \downarrow & & \downarrow \\ \mathcal{U} & \xrightarrow{V} & \mathcal{CA}. \end{array}$$

This is also proved in the appendix.

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2. **Projective and injective objects.** The purpose of this section is to show that the various abelian categories of the previous section have interesting projective and injective objects. Our primary goal is to understand such objects in $\mathcal{U}\mathcal{R}$—unstable modules over the semi-tensor product of the Dyer-Lashof and Steenrod algebras.

To begin, we consider the Steenrod algebra. The forgetful functor from $\mathcal{U}$ to the category of all modules over the Steenrod algebra has a right adjoint Ω^∞ given by the formula

$$(2.1)\qquad (\Omega^\infty M)_n = \{x \in M_n : x\,\mathrm{Sq}^i = 0 \text{ if } 2i > n\}.$$

This defines a class of injective objects in $\mathcal{U}$ and proves that the forgetful functor $\mathcal{U} \longrightarrow n\mathbb{F}_2$ (= graded vector spaces) has a right adjoint K. In particular, if $\Sigma^n\mathbb{F}_2$ is the vector space of dimension one concentrated in degree n, let $K(n) = K(\Sigma^n\mathbb{F}_2)$. Then

$$(2.2)\qquad \mathrm{Hom}_{\mathcal{U}}\big(M, K(n)\big) \cong (M_n)^*$$

where $(\cdot)^*$ denote the dual and

$$K(n) \cong QH_*K\big(\mathbb{Z}/(2), n\big).$$

These are the indecomposables in the homology of the indicated Eilenberg-MacLane space.

The forgetful functor $\mathcal{U} \longrightarrow n\mathbb{F}_2$ has a left adjoint also. Indeed, if $V \in n\mathbb{F}_2$ and $M \in \mathcal{U}$, the functor

$$M \longrightarrow \mathrm{Hom}_{n\mathbb{F}_2}(V, M)$$

is exact and preserves sums. Since $\mathcal{U}$ is an abelian category with enough injectives, this functor is representable: there exists a $J(V) \in \mathcal{U}$ so that

$$\mathrm{Hom}_{\mathcal{U}}\big(J(V), M\big) \cong \mathrm{Hom}_{n\mathbb{F}_2}(V, M).$$

The assignment $V \longrightarrow J(V)$ is natural, so J becomes the adjoint. In particular, if $J(n) = J(\Sigma^n\mathbb{F}_2)$, then

$$(2.3)\qquad \mathrm{Hom}_{\mathcal{U}}\big(J(n), M\big) \cong M_n.$$

$J(n)$ is the "dual Brown-Gitler module". See [11]. $J(V)$ is, of course, projective.

We next turn to $\mathcal{R}_0$-the category of 0-allowable modules over the Dyer-Lashof algebra. the forgetful functor

$$\mathcal{R}_0 \longrightarrow R(-\infty) - \text{modules}$$

has a left adjoint as given by

$$(2.4)\qquad a_0M = M/\{Q^ix : i < |x|, x \in M\}.$$

See [10]. This defines a projective class in $\mathcal{R}_0$ and shows that the forgetful functor $\mathcal{R}_0 \longrightarrow n\mathbb{F}_2$ has a left adjoint P. If $P(n) = P(\Sigma^n\mathbb{F}_2)$ then

$$(2.5)\qquad P(n) \cong PH_*\Omega^\infty\Sigma^\infty S^n \quad \text{if } n \geq 1.$$

and

$$P(0) \subseteq H_*\Omega^\infty\Sigma^\infty S^0$$

is defined by

$$P(0) = \mathrm{Span}\{Q^{i_1}\cdots Q^{i_s}[1] : s \geq 0\}.$$

where

$$[1] \in H_0\Omega^\infty\Sigma^\infty S^0 \cong F_2[\mathbb{Z}]$$

is the element corresponding to $1 \in \mathbb{Z}$.

P can be used to define a functor

$$P'\colon \mathcal{U} \longrightarrow \mathcal{UR}$$

also left adjoint to the forgetful functor. P' covers P or, more concretely,

$$(2.6) \qquad P'(M) \cong P(M)$$

as 0-allowable modules over the Dyer-Lashof operations, and the $\mathcal{A}$-module structure on $P'(M)$ is determined by the $\mathcal{A}$-module structure on M and the Nishida relations. This functor is called $D(\cdot)$ in [3].

The composite functor $F = P' \circ J\colon n\mathbb{F}_2 \longrightarrow \mathcal{UR}$ is left adjoint to the forgetful functor and defines a projective class in $\mathcal{UR}$. Most striking perhaps is $F(n) = F(\Sigma^n\mathbb{F}_2)$ which has the property that

$$(2.7) \qquad \mathrm{Hom}_{\mathcal{UR}}\big(F(n), M\big) \cong M_n.$$

A basis for $F(n)$ is given by monomials of the form

$$Q^I(\iota_n\,\mathrm{Sq}^J)$$

where $\iota_n \in F(n)_n$ corresponds to the identity, $I = (i_1, \ldots, i_s)$, $s \geq 0$, is allowable in the sense that $i_t \leq 2i_{t+1}$ for all t, $J = (j_1, \ldots, j_k)$, $k \geq 0$ is admissible in the usual sense, $2j_1 \leq n$, and

$$e(I) + \ell(J) \geq n.$$

Here $\ell(J) = j_1 + \cdots + j_k$ is the dimension of Sq^J and $e(I)$ is the usual excess.

Note that $F(0) \cong P(0), F(1) \cong P(1)$, but $F(2) \cong P(2) \oplus P(1)$ in $\mathcal{R}_0$.

We next construct injectives in $\mathcal{UR}$ and $\mathcal{R}_0$, using the method that was used to produce projectives in $\mathcal{U}$. We proceed more slowly, as this is the point of this section. The idea is this: let $V \in n\mathbb{F}_2$. Then the functor an $\mathcal{UR}$ given by

$$M \longmapsto \mathrm{Hom}_{n\mathbb{F}_2}(M, V)$$

is exact and sends sums to products. Therefore the following lemma applies.

LEMMA 2 8 *Let E $\mathcal{UR} \to \mathbb{F}_2$ vector spaces be a contravariant functor that sends sums to products and surjections to injections (left exact) Then E is representable there exists $M_E \in \mathcal{UR}$ and a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{UR}}(M, M_E) \xrightarrow{\sim} E(M)$$

PROOF This is the same argument as in [11] or [7] Define a graded vector space M_E by

$$(M_E)_n = E\big(F(n)\big)$$

Then the operations Q^i and Sq^j define maps respectively

$$F(n+i) \longrightarrow F(n)$$
$$F(n-j) \longrightarrow F(n)$$

by $\iota_{n+i} \longmapsto Q^i(\iota_n)$ and $\iota_{n-j} \longmapsto \iota_n \mathrm{Sq}^j$ These define operations

$$Q^i \ (M_E)_n \longrightarrow (M_E)_{n+i}$$
$$\mathrm{Sq}^j \ (M_E)_n \longrightarrow (M_E)_{n-j}$$

and the naturality of the definitions implies $M_E \in \mathcal{UR}$ By definition

$$\mathrm{Hom}_{\mathcal{UR}}\big(F(n), M_E\big) \cong E\big(F(n)\big)$$

Since E sends sums to products and is left exact, E sends colimits to limits and

$$\mathrm{Hom}_{\mathcal{UR}}\big(F(V), M_E\big) \cong E\big(F(V)\big)$$

for all $V \in n\mathbb{F}_2$ The result then follows from left exactness by considering a projective resolution of an arbitrary $M \in \mathcal{UR}$ This lemma is a variation on the special adjoint functor theorem See the appendix

So saying, apply this lemma to, for $V \in n\mathbb{F}_2$

$$E(M) \cong \mathrm{Hom}_{n\mathbb{F}}(M, V)$$

Let $M_E = G(V)$ The assignment $V \longrightarrow G(V)$ is natural in V and the isomorphism of Lemma 2 9

$$\mathrm{Hom}_{\mathcal{UR}}\big(M, G(V)\big) \cong \mathrm{Hom}_{n\mathbb{F}}(M, V) \tag{2 9}$$

demonstrates that $G(\ )$ is right adjoint to the forgetful functor

Therefore G defines a class of injectives in $\mathcal{UR}$ Of particular interest is $G(n) = G(\Sigma^n\mathbb{F}_2)$, which has the property that

$$\mathrm{Hom}_{\mathcal{UR}}\big(M, G(n)\big) \cong (M_n)^* \tag{2 10}$$

and, in particular, that

(2. 11) $$G(n)_k \cong \mathrm{Hom}_{\mathcal{U}\mathcal{R}}\big(F(k), G(n)\big) \cong \big(F(k)_n\big)^*$$

Since $F(k)_n \neq 0$ for infinitely many k, $G(n)$ is not finite.

The associated category $\mathcal{R}_0$ also has an interesting class of injectives. For $\mathcal{R}_0$—the category of 0-allowable modules over the Dyer-Lashof algebra—the appropriate analog of Lemma 2.8 goes through using P (see (2.5)) instead of F. So there is a right adjoint $G_0: n\mathbb{F}_2 \longrightarrow \mathcal{R}_0$ to the forgetful functor. Note that

$$G_0(n)_k \cong \mathrm{Hom}_{\mathcal{R}_0}\big(P(k), G_0(n)\big) \cong \big(P(k)_n\big)^* = 0$$

if $k > n$. Thus $G_0(n)$ is bounded above. We will discuss the structure of $G(n)$ and $G_0(n)$ more in the next section.

3. **The Structure of $G(n)$ and $G_0(n)$.** We use a technique pioneered by Haynes Miller [11] to give a complete description of $G_0(n)$ and then $G(n)$. The key point is that it is easier to understand all $G(n)$ at once because together they form a ring. A key input to the calculation is Madsen's calculation of the dual of the Dyer-Lashof algebra. To begin with $G_0(n)$, recall that

$$\mathrm{Hom}_{\mathcal{R}_0}\big(M, G_0(n)\big) \cong (M_n)^*.$$

Hence

$$G_0(n)_k \cong \mathrm{Hom}_{\mathcal{R}_0}\big(P(k), G_0(n)\big) \cong \big(P(k)_n\big)^*$$

Since $P(k)_n = 0$ for $n < k$ and

$$P(k)_k \cong \mathbb{F}_2$$

generated by the universal element

$$G_0(n)_k = 0 \qquad \text{for } k > n$$

and

$$G_0(n)_n = \mathbb{F}_2$$

generated by the dual of the universal element. Thus there is a unique non-zero class in $\big(G_0(n) \otimes G_0(m)\big)^*_{n+m}$ and, hence, a unique map in $\mathcal{R}_0$.

$$\mu: G_0(n) \otimes G_0(m) \longrightarrow G_0(n+m).$$

Or put another way, the map

$$P(n+m) \longrightarrow P(n) \otimes P(m)$$

classifying $\iota_n \otimes \iota_m$ yields, via the isomorphism above, the same product. Thus if we define a bigraded object $G_0(\cdot)$ with

(3. 1) $$G_0(\cdot)_{p,q} = G_0(p)_q,$$

then $G_0(\cdot)$ has a commutative, associative bigraded multiplication. (We discuss the existence of a unit below.) Furthermore, the Dyer-Lashof operations

$$Q^\iota\colon G_0(p)_q \longrightarrow G_0(p)_{q+\iota}$$

commute with this multiplication in the sense that

$$\mu\big(Q^\iota(x \otimes y)\big) = Q^\iota\big(\mu(x \otimes y)\big)$$

But since

$$\mu\big(Q^\iota(x \otimes y)\big) = \mu\Big(\sum_{\iota_1+\iota_2=\iota} Q^{\iota_1}x \otimes Q^{\iota_2}y\Big)$$

we have, in $G_0(\cdot)$,

$$Q^\iota(xy) = \sum_{\iota_1+\iota_2=\iota} Q^{\iota_1}x \cdot Q^{\iota_2}y.$$

This actually splits up a bit further. Since the relations among the Dyer-Lashof operations are homogeneous, we may write

$$P(n) \cong \bigoplus_{k\geq 0} P(n,k).$$

where

$$P(n,k) = \operatorname{Span}\{Q^{\iota_1}\cdots Q^{\iota_k}(\iota_n) \in P(n)\}.$$

The diagonal map $P(n+m) \to P(n) \otimes P(m)$ respects this splitting, so we get maps

$$P(n+m,k) \longrightarrow P(n,k) \otimes P(m,k).$$

Thus, if we define, $G_0(\cdot,k)$ by

$$G_0(\cdot,k)_{p,q} = [P(q,k)_p]^* \tag{3.2}$$

we have

$$G_0(\cdot) \cong \prod_{k\geq 0} G_0(\cdot,k) \tag{3.3}$$

and the splitting respects the multiplication. Finally, since

$$Q^\iota\colon G_0(p)_q \longrightarrow G_0(p)_{q+\iota}$$

is defined by dualizing that map

$$P(q+i) \longrightarrow P(q)$$

that classifies $Q^\iota(\iota_q)$ and this map induces a map

$$P(q+i,k) \longrightarrow P(q,k+1)$$

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we see that

$$Q^{\imath}\colon G_0(\cdot,k) \longrightarrow G_0(\cdot,k-1)$$

and this commutes with the isomorphism above.

We compute $G_0(\cdot,k)$. For this recall the Dickson algebra D_n: If

$$P_n = \mathbb{F}_2[x_1,\ldots,x_n] \cong H^*B\big(\mathbb{Z}(2)\big)^n$$

is the polynomial algebra on n-elements of degree one, then $G\ell_n(\mathbb{F}_2)$ acts on P_n and

$$D_n \cong \mathbb{F}_2[x_1,\ldots,x_n]^{G\ell_n(\mathbb{F}_2)}$$

D_n is an algebra; in fact, there are elements

$$Q_{n,\imath} \in D_n \quad 0 \le i \le n-1$$

of degree $2^n - 2^{\imath}$ so that

$$D_n \cong \mathbb{F}_2[Q_{n,0},\ldots,Q_{n,n-1}].$$

Now, consider $G_0(\cdot,k)_{*,0} \subseteq G_0(\cdot,k)$. This is closed under the multiplication.

LEMMA 3.4. *There is an isomorphism of algebras*

$$G_0(\cdot,k)_{*,0} \cong D_k,$$

where $Q_{k,\imath}$ is dual to

$$Q^{I_{k\,\imath}}(\iota_0) \in P(0,k)_{2^k-2^{\imath}}$$

where

$$I_{k,\imath} = \big(2^{\imath-1}(2^{k-\imath}-1), 2^{\imath-2}(2^{k-\imath}-1),\ldots,2^{k-\imath}-1, 2^{k-\imath-1}, 2^{k-\imath-2},\ldots,1\big) \quad i>0$$
$$I_{k,0} = (2^{k-1},2^{k-2},\ldots,1).$$

The unit $1 \in G_0(\cdot,k)_{0,0}$ is dual to $Q^{I_{k\,k}}(\iota_0)$ where

$$I_{k,k} = (0,0,\ldots,0).$$

PROOF. $G_0(\cdot,k)_{*,0} \cong P(0,k)^*$. But $P(0) \cong R_0$ and

$$R_0 \cong \bigoplus_{k\ge 0} R_0[k].$$

Here $P(0,k) \cong R_0(k)$ and $R_0[k]^*$ is computed in [9] and [3].

Lemma 3.4 implies that $G_0(\cdot,k)$ is an algebra over the Dickson algebra.

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PROPOSITION 3 5 *As algebras over D_k there is an isomorphism*

$$G_0(,k) \cong D_k[t_{1\ k}]$$

where $t_{1\ k} \in G_0(,k)_{2^k\ 1}$ is the unique non-zero element, that is, $t_{1\ k}$ is dual to

$$Q^{2^k\ 1} \qquad Q^1(\iota_1) \in P(1)_{2^k}$$

Thus

$$G_0(,k) \cong \mathbb{F}_2[Q_{k\ 0}, \qquad , Q_{k\ k\ \ 1}, t_{1\ k}]$$

PROOF Let $I(k,q)$ be the set of all allowable sequences

$$I = (\iota_1, \qquad , \iota_k)$$

with

$$e(I) = \iota_1 - \iota_2 - \qquad - \iota_k \geq q$$

The function $I(k,q) \longrightarrow P(q,k)$ sending

$$I \longmapsto Q^I(\iota_q)$$

defines a bijection between $I(k,q)$ and the allowable basis for $P(q,k)$ Define

$$f \quad I(q,k) \longrightarrow I(q-1,k)$$

by

$$f(\iota_1, \qquad , \iota_k) = (\iota_1 - 2^{k\ \ 1}, \iota_2 - 2^{k\ \ 2}, \qquad , \iota_k - 1)$$

f is a bijection, hence

$$f^q \quad I(q,k) \longrightarrow I(0,k)$$

is a bijection The result now follows from the calculation that

$$P(q,k) \longrightarrow P(1,k) \otimes P(q-1,k)$$

sends

$$Q^I(\iota_q) \longrightarrow Q^{2^k\ 1} \qquad Q^1(\iota_1) \otimes Q^{f(I)}(\iota_{q\ \ 1}) + \sum x_j \otimes y_j$$

where $|x_j| > 2^k$

We next describe the action of the Dyer-Lashof operations

$$Q^i \quad G_0(,k) \longrightarrow G_0(,k-1)$$

PROPOSITION 3 6 *The following formulas hold*

$$Q^0 Q_{k\ i} = Q^2_{k\ \ 1\ i\ \ 1} \quad \text{if } i \neq 0$$

$$Q^0 Q_{k\ 0} = 0$$

$$Q^0 t_{1\ k} = 0$$

$$Q^1 t_{1\ k} = t^2_{1\ k\ \ 1}$$

$$Q^1 Q_{k\ i} = t_{1\ k\ \ 1} Q_{k\ \ 1\ i} \quad \text{where } Q_{k\ \ 1\ k\ \ 1} = 1$$

All others are zero on the generators; this action may be extended to all of $G_0(\cdot, k)$ by the Cartan formula.

PROOF. $Q^j Q_{k,\iota} = 0 = Q^j t_{1,k}$ for $j > 1$ for dimensional reasons. To compute $Q^0 Q_{k,\iota}$ we proceed as follows. Note that, by counting degrees

$$Q^0 Q_{k,0} = 0$$

and

$$Q^0 Q_{k,\iota} = \alpha Q^2_{k-1,\iota-1}$$

where $\alpha = 0$ or 1. To find α, we note that it's necessary to compute

$$P(0, k-1) \longrightarrow P(0, k)$$

where $Q^I(\iota_0) \to Q^I Q^0(\iota_0)$. Now $Q^2_{k-1,\iota-1}$ is dual to $Q^{J_\iota}(\iota_0)$ where

$$J_\iota = \left(2^{\iota-1}(2^{k-\iota} - 1), 2^{\iota-2}(2^{k-\iota} - 1), \ldots, 2(2^{k-\iota} - 1), 2^{k-\iota}, 2^{k-\iota-1}, \ldots, 2\right).$$

Hence we must compute $Q^{J_\iota} Q^0(\iota_0) = Q^{I_{k,\iota}}(\iota_0)$.

But this follows by repeated application of the formula

$$Q^{2^j} Q^{2^{j-1}-1} = Q^{2^j-1} Q^{2^{j-1}}.$$

The other formulas are similar, and are left to the reader.

In discussing $G_0(\cdot)$ itself, we note that since $D_k = 0$ in degrees less than 2^{k-1} and since $t_{1,k} \in G_0(\cdot)_{(1,2^k)}$, we have

$$G_0(\cdot)_{(p,q)} \cong \prod G_0(\cdot, k)_{(p,q)}$$

and, unless $(p, q) = (0, 0)$, only finitely many of the product terms are non-zero. This is an isomorphism of algebras and modules over the Dyer-Lashof algebra.

To extend these results for $\mathcal{UR}$, we use the fact that

$$G(k)_n \cong \mathrm{Hom}_{\mathcal{UR}}\left(F(n), G(k)\right) \cong F(n)^*_k$$

and the evident universal maps

$$F(n+m) \longrightarrow F(n) \otimes F(m)$$

to give the bigraded object $G(\cdot)$ where

$$G(\cdot)_{(p,q)} = G(p)_q$$

a commutative associative multiplication. As before

$$F(n) \cong \oplus_k F(n, k)$$

where

$$F(n, k) \cong \mathrm{Span}\{Q^{\iota_1} \cdots Q^{\iota_k}(\iota_n \, \mathrm{Sq}^I)\}$$

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and, since, the Nishida relations preserve length, this is a splitting of right $\mathcal{A}$-modules. Furthermore the diagonal $F(n+m) \longrightarrow F(n) \otimes F(m)$ maps preserve this splitting and we obtain a splitting of algebras over $\mathcal{A}$:

$$G(n) \cong \prod_k G(n,k).$$

Note that the action of the Dyer-Lashof operations and the Steenrod squares both obey the Cartan formula.

Now, recall $F(n) = P'\big(J(n)\big)$ where $P'\colon \mathcal{U} \longrightarrow \mathcal{U}\mathcal{R}$ is left adjoint to the forgetful functor and $\operatorname{Hom}_{\mathcal{U}}\big(J(n), M\big) \cong M_n$. There is a unique non-zero map in $\mathcal{U}$, $J(n) \longrightarrow \Sigma^n\mathbb{F}_2$ and this induces a map

$$F(n) \longrightarrow P'(\Sigma^n\mathbb{F}_2).$$

But since P' covers the adjoint $P\colon n\mathbb{F}_2 \longrightarrow \mathcal{R}_0$, we have an isomorphism of $\mathcal{R}_0$ modules $P'(\Sigma^n\mathbb{F}_2) \cong P(n)$. Thus we get a morphism of $\mathcal{R}_0$ modules $F(n) \longrightarrow P(n)$ and, hence, an algebra map

$$G_0(\cdot) \longrightarrow G(\cdot).$$

By specializing, we get a map of algebras

$$G_0(\cdot,k) \longrightarrow G(\cdot,k).$$

Define elements $\xi_{k,j} \in G(\cdot,k)_{2^k,2^j}$ by letting $\xi_{k,j}$ be dual to

$$Q^{2^{k-1}} \cdots Q^1(\iota_{2^j}\operatorname{Sq}^{2^{j-1}} \cdots \operatorname{Sq}^1) \in F(2^j,k)_{2^k}, \quad j \geq 0.$$

Note that $\xi_{k,j}$ is the unique non-zero element of $G(n,k)$ in this bidegree and that $\xi_{k,0} = t_{k,1}$.

PROPOSITION 3.7. *$G(\cdot,k)$ is the free bigraded commutative algebra over $G_0(\cdot,k)$ on the elements $\xi_{k,j}$, $j \geq 1$; or*

$$\begin{aligned} G(\cdot,k) &\cong G_0(\cdot,k)[\xi_{k,j}, j \geq 0]/(\xi_{k,0} + t_{k,1}) \\ &\cong \mathbb{F}_2[Q_{k,i}, \xi_{k,j}] \end{aligned}$$

where $0 \leq i \leq k-1$ and $j \geq 0$. The action of the Steenrod algebra is determined by the Cartan formula and the equations

$$\begin{aligned} \xi_{k,j}\operatorname{Sq}^{2^{j-1}} &= \xi_{k,j-1} \quad \text{for } j \geq 1. \\ \xi_{k,j}\operatorname{Sq}^{2^i} &= 0 \quad \text{for } i \neq j-1 \text{ or } j = 0. \end{aligned}$$

It is possible to prove this using Milnor's method for computing the structure of the dual Steenrod algebra but perhaps the following is more enlightening. Let

$$K\colon n\mathbb{F}_2 \longrightarrow \mathcal{U}$$

be the right adjoint to the forgetful functor.

LEMMA 3.8. *$G(n) \cong K\big(G_0(n)\big)$ in $\mathcal{U}$. In particular*

$$G(n,k) \cong K\big(G_0(n,k)\big).$$

PROOF. We have that

$$\begin{aligned} \operatorname{Hom}_{\mathcal{U}}\big(M, G(n)\big) &\cong \operatorname{Hom}_{\mathcal{U}\mathcal{R}}\big(P'(M), G(n)\big) \\ &\cong \operatorname{Hom}_{\mathbb{F}_2}\big(P'(M)_n, \mathbb{F}_2\big) \\ &\cong \prod_k \operatorname{Hom}_{\mathbb{F}_2}\big(P(k)_n \otimes M_k, \mathbb{F}_2\big) \\ &\cong \prod_k \operatorname{Hom}_{\mathbb{F}_2}\Big(M_k, \operatorname{Hom}_{\mathbb{F}_2}\big(P(k)_n, \mathbb{F}_2\big)\Big) \\ &\cong \prod_k \operatorname{Hom}_{\mathbb{F}_2}\big(M_k, G_0(n)_k\big) \\ &\cong \operatorname{Hom}_{n\mathbb{F}_2}\big(M, G_0(n)\big). \end{aligned}$$

These are natural isomorphisms, so the result follows.

LEMMA 3.9. *The multiplication $G(\cdot,k)\otimes G(\cdot,k) \longrightarrow G(\cdot,k)$ is adjoint to the composite*

$$G(\cdot,k)\otimes G(\cdot,k) \cong K\big(G_0(\cdot,k)\big)\otimes K\big(G_0(\cdot,k)\big) \xrightarrow{\varepsilon\otimes\varepsilon} G_0(\cdot,k)\otimes G_0(\cdot,k) \longrightarrow G_0(\cdot,k).$$

where $\varepsilon\colon K(-) \longrightarrow (-)$ is the counit of the adjunction.

PROOF. This follows from the definitions.

Thus we are in a very general situation: if R is any bigraded $\mathbb{F}_2$-algebra, we can define a new $\mathbb{F}_2$ algebra $K(R)$ by

$$K(R)_{s,*} = K(R_{s,*})$$

with multiplication given by the adjoint to

$$K(R_{s,*})\otimes K(R_{t,*}) \xrightarrow{\varepsilon\otimes\varepsilon} R_{s,*}\otimes R_{t,*} \longrightarrow R_{s+t,*}.$$

LEMMA 3.10. *(1) Let $R \cong \mathbb{F}_2[x_{p,1}]$ where the bidegree of $x_{p,1}$ is $(p,1)$. Then*

$$K(R) \cong R[\xi_i : i \geq 0]/(\xi_0 + x_{p,1})$$

where the bidegree of ξ_i is $(p, 2^i)$ and

$$\xi_i \operatorname{Sq}^{2^{i-1}} = \xi_{i-1} \quad \text{for } i \geq 1$$

and

$$\xi_i \operatorname{Sq}^{2^j} = 0 \quad \text{for } j \neq i-1 \text{ or } i = 0.$$

(2) Let R^1 be a bigraded $\mathbb{F}_2$ algebra so that $R^1_{*,t} = 0$ for $t > 0$. Let R^2 be a bigraded $\mathbb{F}_2$ algebra of finite type. Then

$$R^1 \otimes K(R^2) \cong K(R^1 \otimes R^2).$$

PROOF. For (1) notice that the unit $\eta\colon (-) \longrightarrow K(-)$ of the adjunction gives $K(R)$ the structure of an R algebra. Let $K(q) \cong K(\Sigma^q\mathbb{F}_2) \cong QH_*K\big(\mathbb{Z}/(2), q\big)$. Then

$$K(R_{sp,*}) \cong K(s)$$

and

$$K(R_{s,*}) = 0 \quad \text{if } s \not\equiv 0 \bmod p.$$

Furthermore the multiplication in $K(R)$ is given by the unique maps

$$K(s) \otimes K(t) \longrightarrow K(s+t)$$

that are non-zero in degree $s+t$. The result now follows from the familiar calculation for Eilenberg-Maclane spaces. See [15].

For (2) one uses the fact that if V is a finite vector space in degree zero and W is a graded vector space of finite type, then the natural map $V \otimes K(W) \longrightarrow K(V \otimes W)$ is an isomorphism.

PROOF OF PROPOSITION 3.7. By the proceding lemmas,

$$G(\cdot, k) \cong G_0(\cdot, k)[\xi_i : i \geq 0]/(t_{k,1} + \xi_0).$$

where the bidegree of ξ_i is $(2^k, 2^i)$. Since $G(\cdot, k)_{(2^k,2^i)}$ is of dimension one over $\mathbb{F}_2$, $\xi_i = \xi_{k,i}$ and the result follows,

3.1 *The action of the Dyer-Lashof algebra on* $G(\cdot)$. The counit

$$\varepsilon\colon G(\cdot) \cong K\big(G_0(\cdot)\big) \longrightarrow G_0(\cdot)$$

is an algebra map and commutes with the action of the Dyer-Lashof algebra.

Thus, by 3.6, we have

$$\begin{aligned} Q^0 Q_{k,i} &= Q^2_{k-1,k-1} \quad \text{if } i \neq 0 \\ Q^0 Q_{k,0} &= 0 \\ Q^0 t_{1,k} &= 0 \\ Q^1 t_{1,k} &= t^2_{1,k-1} \\ Q^1 Q_{k,1} &= t_{1,k-1} Q_{k-1,i} \end{aligned}$$

where $Q_{k-1,k-1} = 1$. This follows from the fact that ε is an isomorphism in the appropriate bidegree. In theory, the action of the Dyer-Lashof operations on $G(\cdot)$ follow from these, the Cartan formula, and the Nishida relations. In practice, however, they can act in a complicated manner. For example, the reader is invited to verify that

$$\begin{aligned} Q^{2^j}(\xi_{k,i}) &= \xi_{k-1,i}\xi_{k-1,j}, \quad j \geq i \geq 0 \\ Q^{2^j}(Q_{k,i}) &= \xi_{k-1,j} Q_{k-1,i}, \quad j \geq 0, \quad i \geq 0. \end{aligned}$$

where $Q_{k-1,k-1} = 1$. However, there are other non-zero operations.

4. **Projective Hopf algebras.** It is the purpose of this and the next section to show that the structure of projectives and injectives uncovered in the category of modules $\mathcal{UR}$ can be used to produce projectives and injectives in the category $\mathcal{AR}$ of allowable Hopf algebras over the Dyer-Lashof algebra. In addition, we will see in Section 6 that these Hopf algebras can be realized as the homology of infinite loop spaces.

First, we discuss the projectives. For $M \in \mathcal{UR}$, define a graded commutative algebra S_*M by letting S be the symmetric algebra functor and

$$S_*M = S(M)/\big(x^2 + Q^{|x|}(x)\big).$$

Here $|x|$ is the degree of x. Then it is possible for S_*M to be the algebra of an object in $\mathcal{AR}$; indeed, if $C \in \mathcal{AR}C$ is a coalgebra over the Dyer-Lashof algebra, then the inclusion

$$C \longrightarrow C \otimes C \longrightarrow S_*C \otimes S_*C$$

extends to a unique algebra map $S_*C \longrightarrow S_*C \otimes S_*C$ making $S_*C \in \mathcal{AR}$. Thus $S_*\colon \mathcal{AR}C \longrightarrow \mathcal{AR}$ is a left adjoint. Let $F\colon n\mathbb{F}_2 \longrightarrow \mathcal{UR}$ be left adjoint to the forgetful functor.

THEOREM 4.1. *Let $V \in n\mathbb{F}_2$ be of finite type and $V_0 = 0$. Then there is a commutative coproduct*

$$S_*F(V) \longrightarrow S_*F(V) \otimes S_*F(V)$$

so that $S_*F(V) \in \mathcal{AR}$ and $S_*F(V)$ is projective in $\mathcal{AR}$.

PROOF. This is proved exactly as in [5], Section 1.

REMARK 4.2. The hypothesis that $V_0 = 0$ can be removed either algebraically—using some sort of group completion—or topologically as follows. Write

$$V \cong V^+ \oplus V_0$$

where $(V^+)_0 = 0$ and V_0 concentrated in degree 0. In the category of spectra, choose a wedge of spheres so that $H_* \vee_\alpha S^0_\alpha \cong V_0$. Then

$$\hat{S}_*F(V) = S_*F(V^+) \otimes H_*\Omega^\infty(\vee_\alpha S^0_\alpha)$$

satisfies the conclusions of (3.1). There is an inclusion

$$S_*F(V) \longrightarrow \hat{S}_*F(V)$$

and a coalgebra structure on $S_*F(V)$ so that this morphism is a group completion in the category of coalgebras.

REMARK 4.3. Notice that since V is of finite type, we can—by choosing a basis—write V as a sum of one-dimensional vector spaces $\Sigma^n\mathbb{F}_2$ for various n. Then $F(V)$ is a sum of $F(\Sigma^n\mathbb{F}_2) = F(n)$ for various n and, hence, $S_*F(V)$ is isomorphic to a tensor product of algebras of the form $S_*F(n)$ for various n. Furthermore, combining the description of $F(n)$ given in Lemma 2.8 with the definition of S_* we see that for $n > 0$

$$S_*F(n) \cong \mathbb{F}_2[Q^I(\iota_n \operatorname{Sq}^J)]$$

with $e(I)+\ell(J)>n$ Thus $S_*F(V)$ is understood as an algebra Let $\Gamma(n)=S_*F(n)$ with a Hopf algebra structure from Theorem 4 1 If n is odd ι_n is primitive, so $\Gamma(n)$ is primitively generated If n is even the coalgebra structure an $S_*F(n)$ is less well understood, but is unique up to isomorphism See Remark 4 8 below

To exploit these results, let V $\mathcal{UR} \longrightarrow \mathcal{ARC}$ be the right adjoint to the forgetful functor There is a morphism in $\mathcal{UR}$

$$V(M)\otimes V(M)\longrightarrow M\times M$$

given by $x\otimes y\longmapsto \big(x\varepsilon(y),\varepsilon(x)y\big)$ Thus we get a morphism in $\mathcal{ARC}$

$$V(M)\otimes V(M)\longrightarrow V(M\times M)$$

Composing with the addition map $M\times M\longrightarrow M$, yields a commutative associative product $V(M)\otimes V(M)\longrightarrow V(M)$ in $\mathcal{ARC}$ Thus $V(M)$ has a natural Hopf algebra structure over the Dyer-Lashof algebra, however, $V(M)$ may fail to be in $\mathcal{AR}$ because we don't necessarily have $Q^n(x)=x^2$ for x of degree n Therefore, we need a further hypothesis

Let $\mathcal{UR}_0\subseteq\mathcal{UR}$ be the full sub-category stipulated by the condition that $M\in\mathcal{UR}_0$ if and only if for all $x\in M_n$, $Q^n(x)=0$

PROPOSITION 4 4 *Let $M\in\mathcal{UR}_0$ Then there is a natural (in M) Hopf algebra structure on $V(M)$ so that $V(M)\in\mathcal{AR}$ and the functor*

$$V\ \ \mathcal{UR}_0\longrightarrow\mathcal{AR}$$

is right adjoint to the indecomposables functor

PROOF As with all our right adjoints, this proved in the appendix See Corollary A 6

COROLLARY 4 5 *Let H_1,H_2 be two Hopf algebras in $\mathcal{AR}$ so that*

$$H_1\cong S_*\Gamma(V)\cong H_2$$

as algebras over $\mathcal{R}$ and the Steenrod algebra Then $H_1\cong H_2$ in $\mathcal{AR}$

PROOF Note $QH_1\cong QH_2$ Consider the diagram

$$\begin{array}{ccc} H_2 & \xrightarrow{g} & H_1 \\ \big\downarrow & & \big\downarrow f \\ H_2 & \xrightarrow{g'} & V(QH_1) \end{array}$$

where the arrows f and g are adjoint to the isomorphisms $QH_i\cong QH_1$ The arrow g' exists because H_2 is projective It is an isomorphism because it is an isomorphism on indecomposables, and both algebras are free

Now let $\Gamma(n)=S_*F(n)$ $(n\geq 1)$ with any choice of $\mathcal{AR}$ Hopf algebra structure This choice is unique up to an isomorphism in $\mathcal{AR}$

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COROLLARY 4.6. *Let $M \in \mathcal{UR}_0$. Then*

$$\mathrm{Hom}_{\mathcal{AR}}\big(\Gamma(n), V(M)\big) \cong M_n.$$

PROOF.

$$\begin{aligned}\mathrm{Hom}_{\mathcal{AR}}\big(\Gamma(n), V(M)\big) &\cong \mathrm{Hom}_{\mathcal{UR}_0}\big(Q\Gamma(n), M\big)\\ &\cong \mathrm{Hom}_{\mathcal{UR}}\big(F(n), M\big) \cong M_n.\end{aligned}$$

PROPOSITION 4.7. *$\Gamma(n)$ is the projective cover of $H_*\Omega^\infty\Sigma^\infty S^n$.*

PROOF. If $n = 0$ or 1, $\Gamma(n) \cong H_*\Omega^\infty\Sigma^\infty S^n$; so assume $n > 2$. Note that

$$H_n\Omega^\infty\Sigma^\infty S^n \cong [QH_*\Omega^\infty\Sigma^\infty S^n]_n \cong \mathbb{F}_2.$$

We may use Corollary 4.6 and the fact that $\Gamma(n)$ is projective to get a diagram

$$\begin{array}{ccc}\Gamma(n) & \xrightarrow{g} & H_*\Omega^\infty\Sigma^\infty S^n\\ \big\downarrow{=} & & \big\downarrow\\ \Gamma(n) & \xrightarrow{f} & V(QH_*\Omega^\infty\Sigma^\infty S^n)\end{array}$$

where f is adjoint to the non-zero class in $H_n\Omega^\infty\Sigma^\infty S^n$. Thus g is surjective. Suppose $H \in \mathcal{AR}$ is projective and there is a surjective map $H \longrightarrow H_*\Omega^\infty\Sigma^\infty S^n$. Then, because H is projective, there is a diagram

$$\begin{array}{ccc}H & \xrightarrow{\varphi} & \Gamma(n)\\ \big\downarrow{=} & & \big\downarrow{g}\\ H & \longrightarrow & H_*\Omega^\infty\Sigma^\infty S^n\end{array}$$

Since $\Gamma(n) = S_*F(n)$ as algebras, φ must be surjective.

REMARK 4.8. The diagonal on $\Gamma(n)$ can be described by understanding the functor

$$\mathrm{Hom}_{\mathcal{AR}}\big(\Gamma(n), \cdot\big) \longrightarrow \text{groups}.$$

Write $n = 2^s(2t+1)$ and let

$$B = \mathbb{Z}_{(2)}[x_0, x_1, \ldots]$$

be the polynomial algebra over the integers localized at 2 on indeterminants x_i of degree $2^i(2t+1)$. Let

$$w_k(x) = x_0^{2^k} + 2x_1^{2^{k-1}} + \cdots + 2^k x_k$$

be the Witt polynomial of degree $2^k(2t+1)$. There is a unique coproduct

$$\Delta\colon B \longrightarrow B \otimes B$$

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so that $w_k(x)$ is primitive for every k. With this coproduct, B becomes a bicommutative Hopf algebra. This coproduct is called "the Witt vector diagonal" because, if R is a $\mathbb{Z}_{(2)}$ algebra and if we ignore the grading, then the group

$$\mathrm{Hom}_{\mathrm{algebras}}(B,R)$$

is the Witt vectors in R. Now let

$$B(n) = \mathbb{Z}_{(2)}[x_0,\ldots,x_s] \subseteq B$$

be the sub-Hopf algebra on the first s indeterminants. Recall the $n = 2^s(2t+1)$. Then set

$$C(n) = \mathbb{F}_2 \otimes B(n).$$

Then for $H \in \mathcal{AR}$ we have

$$\mathrm{Hom}_{\mathcal{AR}}\big(\Gamma(n),H\big) = \mathrm{Hom}_{\mathcal{H}}\big(C(n),H\big)$$

where $\mathcal{H}$ is the category of bicommutative Hopf algebras over $\mathbb{F}_2$. In particular, the identity $\Gamma(n) \longrightarrow \Gamma(n)$ corresponds to a map in $\mathcal{H}$

$$C(n) \longrightarrow \Gamma(n)$$

that determines the coproduct on $\Gamma(n)$. These thoughts are addressed in detail in [5].

5. **Injective Hopf algebras.** It is the purpose of this section to prove to following result.

THEOREM 5.1. *There is an injective Hopf algebra $H(n) \in \mathcal{AR}$, $n \geq 0$, so that there is an isomorphism in $\mathcal{AR}C$ of coalgebras*

$$H(n) \xrightarrow{\cong} V\big(G(n)\big).$$

In particular, $H(n)$ is isomorphic as a coalgebra over the Steenrod algebra to the tensor product of the homologies of various Eilenberg-MacLane spaces. The strategy of the proof is to first prove the existence of $H(n)$ and only then to establish an isomorphism of $H(n)$ with $V\big(G(n)\big)$.

To dispense with the case $n = 0$, let

$$\mathbb{Z}/(2^\infty) = \operatorname{colim} \mathbb{Z}/(2^n)$$

be the 2-torsion in the divisible group $\mathbb{Q}/\mathbb{Z}$, and let $H\mathbb{Z}/(2^\infty)$ be the Eilenberg-Maclane spectrum with $\pi_0 H\mathbb{Z}/(2^\infty) \cong \mathbb{Z}/(2^\infty)$. Then

$$H(0) = H_*\Omega^\infty H\mathbb{Z}/(2^\infty) = \mathbb{F}_2[\mathbb{Z}/(2^\infty)]$$

satisfies the conclusions of Theorem 5.1 for $n = 0$.

As a bit of notation, if G is an abelian group, let

$$G^{\mathbb{Q}/\mathbb{Z}} = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, the functor

$$G \longrightarrow G^{\mathbb{Q}/\mathbb{Z}}$$

is exact.

Now define a functor, for $n \geq 1$,

$$D_n \colon \mathcal{AR} \longrightarrow \text{Abelian groups}.$$

by

$$D_n(H) = \mathrm{Hom}_{\mathcal{AR}}\big(\Gamma(n), H\big)^{\mathbb{Q}/\mathbb{Z}}$$

where $\Gamma(n)$ is the projective Hopf algebra of the last section.

PROPOSITION 5.2. *$D_n(\cdot)$ is representable; that is, there is an object $H(n) \in \mathcal{AR}$ so that*

$$\mathrm{Hom}_{\mathcal{AR}}\big(H, H(n)\big) \cong D_n H.$$

This result will be proved below, and this result defines $H(n)$.

COROLLARY 5.3. *$H(n)$ is injective in $\mathcal{AR}$.*

PROOF. This is a consequence of the fact that $\Gamma(n)$ is projective and that $\mathrm{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ is exact.

To prove Proposition 5.2, we use the special adjoint functor theorem—see Corollary A.2. Thus we must verify some facts about the category $\mathcal{AR}$.

LEMMA 5.4. *The category $\mathcal{AR}$ has all (small) colimits.*

PROOF. Suppose we have shown that $\mathcal{AR}$ has arbitrary coproducts. Then for any functor $F \colon I \longrightarrow \mathcal{AR}$, for a small category I, we may define

$$\operatorname{colim} F = \operatorname*{colim}_{i \in I} H_i$$

to be the coequalizer in the diagram

$$\coprod_{f \in \mathrm{Mor}(I)} H_f \underset{d_1}{\overset{d_0}{\rightrightarrows}} \coprod_{i \in I} H_i \longrightarrow \operatorname*{colim}_{i \in I} H_i$$

where, in the left coproduct, H_f = domain of f, and where d_0, d_1 are defined on H_f by $d_0|_{H_f} = id$ and $d_1|_{H_f} = f$. Since $\mathcal{AR}$ is abelian, the coequalizer is the cokernel of the difference between d_0 and d_1.

To show $\mathcal{AR}$ has arbitrary coproducts, let X be a set and

$$\{H_x \colon x \in X\}$$

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an X-indexed collection of objects in $\mathcal{AR}_*$. If X is finite

$$\coprod_{x\in X} H_x = \bigotimes_{x\in X} H_x.$$

If X is arbitrary, write $X = \operatorname{colim} Y$ where we have $Y \subseteq X$ finite. Then set

$$\coprod_{x\in X} H_x = \operatorname*{colim}_{Y\subseteq X} \coprod_{x\in Y} H_x.$$

where this colimit is as vector spaces, and inclusions are defined by unit maps $\mathbb{F}_2 \longrightarrow H_x$.

LEMMA 5.5. *$\mathcal{AR}_*$ has a set of generating objects.*

PROOF. In the argument before A.4 we showed that the objects $P'(C) \in \mathcal{AR}_*C$ with $C \in \mathcal{CA}$ finite as vector spaces generate $\mathcal{AR}_*C$. $P'\colon \mathcal{CA} \longrightarrow \mathcal{AR}_*C$ is left adjoint to the forgetful functor. Also, if $C \in \mathcal{AR}_*C$, $S_*C \in \mathcal{AR}_*$. Thus the objects $S_*P'(C)$ with $C \in \mathcal{CA}$ generate $\mathcal{AR}_*$.

REMARK. These generators are familiar objects. Specifically, if X is a space,

$$H_*\Omega^\infty\Sigma^\infty X \cong S_*P'H_*X.$$

PROOF OF PROPOSITION 5.2. Since an object in $\mathcal{AR}_*$ has only a set of quotient objects, Corollary A.2, Lemma 5.4, and Lemma 5.5 will imply Proposition 5.2 if we can show that

$$D_n(\cdot) = \operatorname{Hom}_{\mathcal{AR}_*}\big(\Gamma(n), \cdot\big)^{\mathbb{Q}/\mathbb{Z}}$$

sends colimits to limits. Since $(\cdot)^{\mathbb{Q}/\mathbb{Z}}$ is exact we need only show

$$\operatorname{Hom}_{\mathcal{AR}_*}\big(\Gamma(n), \cdot\big)$$

preserves colimits. The fact that $\Gamma(n)$ is projective and the description of colimits given in the proof of Lemma 5.4 demonstrate that we need only show

$$\operatorname{Hom}_{\mathcal{AR}_*}\Big(\Gamma(n), \coprod_{x\in X} H_x\Big) \cong \coprod_{x\in X} \operatorname{Hom}_{\mathcal{AR}_*}\big(\Gamma(n), H_x\big).$$

There is an obvious natural map

$$\coprod_{x\in X} \operatorname{Hom}_{\mathcal{AR}_*}\big(\Gamma(n), H_x\big) \longrightarrow \operatorname{Hom}_{\mathcal{AR}_*}\Big(\Gamma(n), \coprod_{x\in X} H_x\Big)$$

and this map is always an injection. To prove that it is a surjection, note that there is a class $j_n \in \Gamma(n)_n$ so that a morphism $f\colon \Gamma(n) \longrightarrow H$ in $\mathcal{AR}_*$ is determined by $f(j_n)$. Thus the evident map

$$\operatorname*{colim}_{Y\subseteq X} \operatorname{Hom}_{\mathcal{AR}_*}\Big(\Gamma(n), \coprod_{x\in Y} H_x\Big) \longrightarrow \operatorname{Hom}_{\mathcal{AR}_*}\Big(\Gamma(n), \coprod_{x\in X} H_x\Big)$$

where the colimit is over $Y \subseteq X$ finite is onto. But if Y is finite

$$\coprod_{x\in Y} H_x = \bigotimes_{x\in Y} H_x \cong \prod_{x\in Y} H_x$$

so

$$\coprod_{x\in Y} \mathrm{Hom}\big(\Gamma(n), H_x\big) \cong \mathrm{Hom}\Big(\Gamma(n), \coprod_{x\in Y} H_x\Big).$$

This proves Proposition 5.2.

The rest of the section will be devoted to producing an isomorphism in $\mathcal{AR}C$ from $V\big(G(n)\big)$ to $H(n)$. We begin by showing that $V\big(G(n)\big)$ is the injective hull of a familiar object. Let $V\big(K(n)\big) \in \mathcal{AR}C$ be the homology of $K\big(\mathbb{Z}/(2), n\big)$.

LEMMA 5.6. *There is a unique non-zero map in* $\mathcal{AR}C$

$$i\colon V\big(K(n)\big) \longrightarrow V\big(G(n)\big)$$

and this morphism is an injection.

PROOF. Consider:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{AR}C}\Big(V\big(K(n)\big), V\big(G(n)\big)\Big) &\cong \mathrm{Hom}_{\mathcal{UR}}\Big(V\big(K(n)\big), G(n)\Big)\\ &\cong V\big(K(n)\big)^*_n.\end{aligned}$$

But $V\big(K(n)\big)^*_n \cong H^nK\big(\mathbb{Z}/(2), n\big) \cong \mathbb{F}_2$. Let i be the morphism corresponding to the non-zero element of $\mathbb{F}_2$. Since

$$i\colon V\big(K(n)\big)_n \longrightarrow V\big(G(n)\big)_n$$

is one-to-one, i must be one-to-one.

PROPOSITION 5.7. $V\big(G(n)\big)$ *is the injective hull of* $V\big(K(n)\big)$.

This is equivalent to saying that if we have a one-to-one morphism $f\colon V\big(K(n)\big) \to C$ in $\mathcal{AR}C$ and C is injective in $\mathcal{AR}C$ then we may complete the diagram

$$\begin{array}{ccc} V\big(K(n)\big) & \xrightarrow{\ i\ } & V\big(G(n)\big)\\ \big\downarrow f & & \big\downarrow f'\\ C & \xrightarrow{\ =\ } & C\end{array}$$

in such a manner that f' is injective. This requires some lemmas.

First let $G_0(n) \subseteq G(n)$ be as in Section 3. Thus

$$G(n) \cong K\big(G_0(n)\big)$$

where $K\colon \mathcal{U} \to \mathcal{UR}$ is right adjoint to the forgetful functor. Hence, $V\big(G(n)\big) \cong V\big(KG_0(n)\big)$ is, as a coalgebra over the Steenrod algebra, the homology of an Eilenberg-Maclane space. There is a canonical inclusion

$$G_0(n) \subseteq V\big(G(n)\big).$$

This could be called the image of the Hurewicz homomorphism.

LEMMA 5.8. *Let $0 \neq x \in G_0(n)_k$. Then there is an element $\theta \in \mathcal{R}(-\infty)_{n-k}$ so that $0 \neq \theta(x) \in G_0(n)_n \cong \mathbb{F}_2$.*

PROOF. Recall that $G_0(n)_k \cong \operatorname{Hom}_{\mathcal{R}_0}\big(P(k), G_0(n)\big) \cong P(k)_n^*$ and $\theta\colon G_0(n)_k \longrightarrow G(n)_n$ is defined by dualizing the map $f_\theta\colon P(n) \longrightarrow P(k)$ given by $f_\theta(\iota_n) = \theta\iota_k$. But the map $R(-\infty)_{n-k} \longrightarrow P(k)_n$ given by $\theta \longmapsto \theta\iota_k$ is onto; hence if $0 \neq x \in G(n)_k \cong P(k)_n^*$, choose $\theta \in \mathcal{R}(-\infty)$ so that $\langle x, \theta\iota_k\rangle \neq 0$. Then

$$\langle \theta x, \iota_n\rangle = \langle f_\theta^* x, \iota_n\rangle = \langle x, f_\theta(\iota_n)\rangle = \langle x, \theta\iota_k\rangle \neq 0,$$

so we have $\theta x \neq 0$.

LEMMA 5.9. *Given a diagram in $\mathcal{AR}C$.*

$$\begin{array}{ccc} V\big(K(n)\big) & \xrightarrow{i} & V\big(G(n)\big) \\ \downarrow f & & \downarrow f' \\ C & \xrightarrow{=} & C \end{array}$$

with $0 \neq f\colon V\big(K(n)\big)_n \longrightarrow C_n$. Then both f and f' must be injections.

PROOF. f is immediately one-to-one. To show f' is one-to-one it is sufficient to prove the composite

$$G_0(n) \xrightarrow{\subseteq} V\big(G(n)\big) \xrightarrow{f'} C$$

is one-to-one. By hypothesis, if $y \in G_0(n)_n$ is the non-zero class $f'(y) \neq 0$. If $0 \neq x \in G_0(n)_k$, choose $\theta \in \mathcal{R}(-\infty)$ so that $\theta x = y$. Then

$$\theta f'(x) = f'(\theta x) = f'(y) \neq 0,$$

so $f'(x) \neq 0$.

PROOF OF PROPOSITION 5.7. If $f\colon V\big(K(n)\big) \longrightarrow K$ is an injective morphism in $\mathcal{AR}C$ and C is injective in $\mathcal{AR}C$, then we automatically obtain a diagram

$$\begin{array}{ccc} V\big(K(n)\big) & \xrightarrow{i} & V\big(G(n)\big) \\ \downarrow f & & \downarrow f' \\ C & \xrightarrow{=} & C \end{array}$$

because i is an injection. Then f' is an injection by Lemma 5.9.

Now let $H(n)$ be as in Proposition 5.2. Since the forgetful functor $\mathcal{AR} \longrightarrow \mathcal{AR}C$ has a left adjoint $H(n)$ is injective in $\mathcal{AR}C$. Furthermore

$$\begin{aligned} \operatorname{Hom}_{\mathcal{AR}}\Big(V\big(K(n)\big), H(n)\Big) &\cong \operatorname{Hom}_{\mathcal{AR}}\Big(\Gamma(n), V\big(K(n)\big)\Big)^{\mathbb{Q}/\mathbb{Z}} \\ &\cong K(n)_n^* \end{aligned}$$

by Corollary 4.6. Thus there is a unique non-zero Hopf algebra map

$$f\colon V\big(K(n)\big) \longrightarrow H(n)$$

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and this map is non-zero in degree n. Thus there is an injection in $\mathcal{AR}_*C$ $f'\colon V\big(G(n)\big) \longrightarrow H(n)$ making the following diagram commute

$$\begin{array}{ccc} V\big(K(n)\big) & \xrightarrow{\iota} & V\big(G(n)\big) \\ \downarrow f & & \downarrow f' \\ H(n) & \xrightarrow{=} & H(n). \end{array}$$

Thus we must show f' is an isomorphism. Now f' defines an injective natural transformation

$$f'_*\colon \mathrm{Hom}_{\mathcal{AR}_*C}\Big(D, V\big(G(n)\big)\Big) \longrightarrow \mathrm{Hom}_{\mathcal{AR}_*C}\big(D, H(n)\big)$$

and f' is an isomorphism if and only if f'_* is a natural isomorphism. Thus we must show f'_* is a surjection for all $D \in \mathcal{AR}_*C$.

It is sufficient to prove this when D is among the generators for the category $\mathcal{AR}_*C$; that is, when $D = P'(C)$, with $C \in \mathcal{CA}$ a finite coalgebra. In this case

$$\mathrm{Hom}_{\mathcal{AR}_*C}\Big(P'(C), V\big(G(n)\big)\Big) \cong P'(C)^*_n$$

is a finite dimensional $\mathbb{F}_2$ vector space; therefore, it is sufficient to show that

$$\mathrm{Hom}_{\mathcal{AR}_*C}\big(P'(C), H(n)\big)$$

has the same cardinality as $P'(C)^*_n$. Note that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{AR}_*C}\big(P'(C), H(n)\big) &\cong \mathrm{Hom}_{\mathcal{AR}_*}\big(S_*P'(C), H(n)\big) \\ &\cong \mathrm{Hom}_{\mathcal{AR}_*}\big(\Gamma(n), S_*P'(C)\big)^{\mathbb{Q}/\mathbb{Z}} \end{aligned}$$

To proceed, filter $S_*P'(C)$ by powers of two; that is,

$$F^sS_*P'(C) = \{x^{2^s} : x \in S_*P'(C)\}.$$

One easily computes that $F^sS_*P'(C)$ is a sub-$\mathcal{AR}_*$ Hopf algebra of $F^{s-1}S_*P'(C)$.

To explicitly compute with this filtration, filter $P'(C)$ by

$$F^sP'(C) = \{Q^{2^sn}Q^{2^{s-1}n}\cdots Q^n(x) : x \in P'(C)_n\}.$$

LEMMA 5.10. *$F^sP'(C) \subseteq P'(C)$ is a sub-object in $\mathcal{AR}_*C$ and*

$$F^sS_*P'(C) \cong S_*F^sP'(C).$$

PROOF. First consider the case $s = 1$. The formula

$$Q^kQ^n(x) = \begin{cases} Q^{k+t}Q^t(x) & k = 2t \\ 0 & k = 2t+1 \end{cases}$$

$$Q^n(x)\,\mathrm{Sq}^k = \begin{cases} Q^{n-t}(x\,\mathrm{Sq}^t) & k = 2t \\ 0 & k = 2t+1 \end{cases}$$

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show that $F^1P'(C) \subseteq P'(C)$ is a sub-object in $\mathcal{UR}$. The fact that it's a sub-object in $\mathcal{AR}C$ follows from the Cartan formula and the fact that $Q^j(y) = 0$ if $j < \deg(y)$. For general s, use the fact that

$$F^sP'(C) = \underbrace{F^1 \cdots F^1}_{s} P'(C).$$

That $F^sS_*P(C) \cong S_*F^sP(C)$ is a consequence of the fact that

$$x^2 = Q^n(x)$$

if $n = \deg(x)$.

Next, let $E_0S_*P'(C)$ be the associated graded (in the abelian category $\mathcal{AR}$) of this filtration. Since $\Gamma(n)$ is projective,

$$E^0 \operatorname{Hom}_{\mathcal{AR}}\big(\Gamma(n), S_*P(C)\big) \cong \operatorname{Hom}_{\mathcal{AR}}\big(\Gamma(n), E_0S_*P'(C)\big)$$

Lemma 5.10 implies that

$$E_0^sS_*P'(C) \cong S_*E_0^sP'(C).$$

LEMMA 5.11. $\operatorname{Hom}_{\mathcal{UR}}\big(\Gamma(n), S_*E_0^sP'(C)\big) \cong [E_0^sP'(C)]_n$.

PROOF. This follows from Corollary 4.6 once we demonstrate the following fact. Let $M \in \mathcal{AR}C$ have the property that $Q^n(x) = 0$ for all $x \in M$. Then there is an isomorphism of Hopf algebras $S_*M \xrightarrow{\cong} V(M)$. To see this, note that $QS_*M \cong M$. Hence, by Proposition 4.4, there is a map of Hopf algebras $S_*M \longrightarrow V(M)$. Since both Hopf algebras have the property that $x^2 = 0$ for all x in the augmentation ideal, and since this map is an isomorphism on indecomposables, it must be an isomorphism.

PROOF OF THEOREM 5.1. Lemma 5.11 implies that

$$f'_*\colon \operatorname{Hom}_{\mathcal{AR}C}\Big(P'(C), V\big(G(n)\big)\Big) \longrightarrow \operatorname{Hom}_{\mathcal{AR}C}\big(P'(C), H(n)\big)$$

is an isomorphism for all $C \in \mathcal{CA}$ finite. Because the inclusion

$$f'\colon V\big(G(n)\big) \longrightarrow H(n)$$

is split ($V\big(G(n)\big)$ is injective), this implies f' is an isomorphism. This completes the proof.

COROLLARY 5.12. *$H(n)$ is the injective hull of $V\big(K(n)\big)$ in $\mathcal{AR}$.*

PROOF. Let $H \in \mathcal{AR}$ be injective and suppose $f\colon V\big(K(n)\big) \longrightarrow H$ is one-to-one. Then we have a diagram of Hopf algebras in $\mathcal{AR}$

$$\begin{array}{ccc} V\big(K(n)\big) & \longrightarrow & H(n) \\ \big\downarrow f & & \big\downarrow f' \\ H & \xrightarrow{=} & H \end{array}$$

By Theorem 5.1 and Lemma 5.9, f' is one-to-one as coalgebras, hence an injection in $\mathcal{AR}$.

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6. **Realizing injective Hopf algebras.** The homotopy theory behind the algebra of the previous sections is straightforward. For example, the projective case is well-known:

THEOREM 6.1. *There is a two-complete spectrum $T(n), n \geq 0$ so that*
(1) $H_*T(n) \cong J(n)$
(2) $H_*\Omega^\infty T(n) \cong \Gamma(n)$ and $\sigma_*\colon H_*\Sigma^\infty\Omega^\infty T(n) \longrightarrow H_*T(n)$ is onto
(3) If n is even, the morphism

$$[T(n), x] \longrightarrow \mathrm{Hom}_{\mathcal{AR}}\big(H_*\Omega^\infty T(n), H_*\Omega^\infty X\big)$$
$$f \longmapsto \Omega^\infty f_*$$

is an isomorphism. This map is only onto if n is odd.

This is proved in [5], with previous results along these lines in [4] and [6]. $T(n)$ is unique up to homotopy equivalence. The injective case is handled by the following result.

THEOREM 6.2. *There is a spectrum $D(n)$, $n \geq 0$ so that*
(1) $H_*\Omega^\infty D(n) \cong H(n)$.
(2) If n is even

$$[X, D(n)] \longrightarrow \mathrm{Hom}_{\mathcal{AR}}\big(H_*\Omega^\infty X, H_*\Omega^\infty D(n)\big)$$

is an isomorphism.

To begin the proof, note that the functor on finite spectra

$$X \longrightarrow [T(n), X]^{\mathbb{Q}/\mathbb{Z}}$$

satisfies the requirements of Brown representability. Thus, let $D(2n)$ be the spectrum so that

$$[X, D(2n)] \cong [T(2n), X]^{\mathbb{Z}/(2^\infty)}$$

and let $D(2n-1) = \Sigma^{-1}D(2n)$.

Certainly, we have

$$\begin{aligned}[X, D(2n)] &\cong [T(2n), X]^{\mathbb{Q}/\mathbb{Z}}\\ &\cong \mathrm{Hom}_{\mathcal{AR}}\big(H_*\Omega^\infty T(2n), H_*\Omega^\infty X\big)^{\mathbb{Q}/\mathbb{Z}}\\ &\cong \mathrm{Hom}_{\mathcal{AR}}\big(H_*\Omega^\infty X, H(n)\big)\end{aligned}$$

so Theorem 6.2(2) holds if Theorem 6.2(1) holds, at least for finite X. To extend to all X, note that if X is finite and $n > 0$,

$$\mathrm{Hom}_{\mathcal{AR}}\big(H_*\Omega^\infty T(2n), H_*\Omega^\infty X\big)$$

is a finite 2 group. Thus for general X we write $X = \operatorname{colim} X_\alpha$ as the filtered colimit of finite spectra and compute that

$$[X, D(2n)] \cong \lim[X_\alpha, D(2n)]$$

because lim is exact on finite groups. Thus

$$\begin{aligned}[X, D(2n)] &\cong \lim[X_\alpha, D(2n)] \\ &\cong \lim \operatorname{Hom}_{\mathcal{AR}}\big(H_*\Omega X_\alpha, H(2n)\big) \\ &\cong \operatorname{Hom}_{\mathcal{AR}}\big(H_*\Omega^\infty X, H(2n)\big)\end{aligned}$$

Thus Theorem 6.2(2) holds for all X, if Theorem 6.2(1) holds.

To prove Theorem 6.2(1), note that the identity $D(2n) \longrightarrow D(2n)$ yields a map

$$H_*\Omega^\infty D(2n) \xrightarrow{\varphi} H(2n).$$

and the isomorphism above sends $f\colon X \longrightarrow D(2n)$ to

$$H_*\Omega^\infty X \xrightarrow{\Omega^\infty f_*} H_*\Omega^\infty D(2n) \xrightarrow{\varphi} H(2n).$$

Since $H(2n) \cong V\big(G(2n)\big)$ as coalgebras and $G(n)$ is injective as an unstable $\mathcal{A}$ module, there is an Eilenberg-Maclane space Y so that $H_*Y \cong H(2n)$ as coalgebras in $\mathcal{CA}$ and a map $g\colon \Omega^\infty D(2n) \longrightarrow Y$ so that $g_* = \varphi$. But since

$$\begin{aligned}\pi_k\Omega^\infty D(2n) &\cong [S^k, D(2n)] \\ &\cong \operatorname{Hom}_{\mathcal{AR}}\big(H_*\Omega^\infty\Sigma^\infty S^k, H(2n)\big) \\ &\cong \pi_k Y\end{aligned}$$

g is a weak-equivalence and hence g_* is an isomorphism. Thus Theorem 6.2(1) holds for n even. For the rest

$$H_*\Omega^\infty D(2n-1) \cong H_*\Omega\Omega^\infty D(2n)$$

and the result follows because $\Sigma G_0(2n-1) = G_0(2n)_+$. See Proposition 3.5.

REMARKS ON PREVIOUS WORK. (1) The proof of Theorem 6.2 is one way to understand Brown and Gitler's original construction of Brown-Gitler spectra [2]. Let $B(n)$ be the $2n$-th suspension of the Spanier-Whitehead dual of $T(2n)$:

$$B(n) \cong \Sigma^{2n} F\big(T(2n), S^0\big). \tag{6.3}$$

Then for a spectrum X

$$[T(2n), X] \cong [S^{2n}, B(n) \wedge X] \cong B(n)_{2n}X.$$

So

$$[X, D(2n)] \cong [T(2n), X]^{\mathbb{Q}/\mathbb{Z}} \cong \big(B(n)_{2n}X\big)^{\mathbb{Q}/\mathbb{Z}}.$$

This demonstrates that $D(2n)$ is the $2n$-th suspension of the Brown-Comenetz dual of $B(n)$ [1]:

$$D(2n) \cong \Sigma^{2n} cB(n).$$

This last equation is actually Brown and Gitler's definition of $B(n)$—they constructed $D(2n)$. One easily computes from 6.3 that

$$H^*B(n) \cong A/A\{\chi\,\mathrm{Sq}^i; i > n\}.$$

Since $H^*B(n)$ is finite one can conclude from the work of Brown and Comenetz that $H_*D(2n) = 0$. Since $H_*\Omega^\infty D(2n) \neq 0$, $D(2n)$ is not a connective spectrum.

Furthermore, the unique non-trivial map $B(n) \longrightarrow K\mathbb{Z}/(2)$ is Brown-Comenetz dual to the unique non-trivial map $\Sigma^{2n}K\mathbb{Z}/(2) \longrightarrow D(2n)$. Since $\Omega^\infty D(2n)$ is a product of Eilenberg-Maclane spaces, we have that for any space X,

$$H^{2n}X \cong [X, \Sigma^{2n}K\mathbb{Z}/(2)] \cong \left[X, K\left(\mathbb{Z}/(2), 2n\right)\right] \longrightarrow [X, \Omega^\infty D(2n)] \xrightarrow{\cong} D(2n)^0X$$

is an inclusion, and by, examining the delooping of $\Omega^\infty D(2n)$, that $H^{2n+1}X \longrightarrow D(2n)^1X$ is also an inclusion. One easily concludes that $B(n)_qX \longrightarrow H_qX$ is onto for $q \leq 2n+1$.

(2) Others have studied injective and projective objects in the category of Hopf algebras over the Steenrod operations—that is, forget the Dyer-Lashof operations. There the indecomposable projective objects are related to cohomology theories constructed by Segal. See the article by Steiner [13]. This is where the phrase "Witt vector diagonal" arises. See also [5] for more on this point. This was pointed out to me by Tom Hunter.

Appendix: the special adjoint functor theorem. This paper makes repeated use of the special adjoint functor theorem—we used it explicitly in Section 5 and implicitly in Section 2. Thus, in this appendix, we make the statement of this theorem and a further application to producing right adjoints of forgetful functors out of categories of coalgebras.

Let $\mathcal{C}$ be a category—which, for the purposes of this paper, carries the implicit assumption that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set for all objects X and Y of $\mathcal{C}$. Recall that a morphism $f\colon X \longrightarrow Y$ is $\mathcal{C}$ is epi if whenever we have $g_1, g_2\colon Y \longrightarrow Z$ so that $g_1f = g_2f$, then $g_1 = g_2$. For all the categories in this paper categorical epimorphisms are set theoretic surjections.

Next recall that a *quotient object* of an object x in $\mathcal{C}$ is an equivalence class of epis $f\colon X \longrightarrow Y$ where $f_1\colon X \longrightarrow Y_1$ is equivalent to f if there is an invertible morphism $\theta\colon Y \longrightarrow Y_1$ so that $f_1 = \theta f$. If the epis of $\mathcal{C}$ are set-theoretic surjections, then X can have only a set of quotient objects.

Finally, recall that a set of objects $\mathcal{A}$ of $\mathcal{C}$ is called a *generating set* if whenever $f, g\colon X \longrightarrow Y$ are two morphisms in $\mathcal{C}$ so that $f \neq g$, there is an $A \in \mathcal{A}$ and $\theta\colon A \longrightarrow X$ in $\mathcal{C}$ so that $f\theta \neq g\theta$. The elements of $\mathcal{A}$ "separate" the morphisms of $\mathcal{C}$.

With this notation, we give the special adjoint functor theorem.

THEOREM A.1 (SAFT). *Suppose $\mathcal{C}$ is a category that*

(1) has all small colimits,

(2) has a generating set, and

(3) every object of $\mathcal{C}$ has only a set of quotient objects.

Then a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ has a right adjoint if and only if F preserves colimits.

This is due to Freyd, among others, and a proof can be found in [8], Section V.8. An immediate corollary is the following.

COROLLARY A.2. *Let $\mathcal{C}$ be category satisfying the hypotheses Theorem A.1 and let $\mathcal{D}$ be a category so that there is a forgetful functor*

$$\mathcal{D} \longrightarrow \text{Sets}$$

with a left adjoint. Then any contravariant functor

$$E\colon \mathcal{C} \longrightarrow \mathcal{D}$$

that sends colimits to limits is representable; that is, there exists $Y_E \in \mathcal{C}$ and a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(X, Y_E) \xrightarrow{\cong} E(X).$$

In this result $\mathcal{D}$ might be the category of groups, abelian groups, vector spaces over a field, or a category of modules. This is the result used in Sections 2 and 5.

As an application let $\mathcal{CA}$ be the category of coalgebras over the Steenrod algebra. $\mathcal{CA}$ has all colimits, $\mathcal{CA}$ has a generating set consisting of all coalgebras that are finite as graded vector spaces, and every object in $\mathcal{CA}$ has only finite many quotient objects.

Since the forgetful functor $\mathcal{CA} \to \mathcal{U}$ preserves colimits and we have:

COROLLARY A.3. *The forgetful functor $\mathcal{CA} \to \mathcal{U}$ has a right adjoint*

$$V\colon \mathcal{U} \longrightarrow \mathcal{CA}.$$

To handle the forgetful functor $\mathcal{AR}_*\mathcal{C} \to \mathcal{UR}_*$ for allowable coalgebras over the Dyer-Lashof algebra to allowable modules over the Dyer-Lashof algebra, we first must establish some notation.

The forgetful functor $\mathcal{UR}_* \to \mathcal{U}$ has a left adjoint P'. (See (2.6)). If $C \in \mathcal{CA}$, then the composite

$$C \xrightarrow{\psi} C \otimes C \to P'(C) \otimes P'(C)$$

gives, by adjointness, a coproduct

$$P'(C) \xrightarrow{\psi} P'(C) \otimes P'(C)$$

making $P'(C) \in \mathcal{AR}_*\mathcal{C}$. Or, said differently, $P'\colon \mathcal{CA} \to \mathcal{AR}_*\mathcal{C}$ is left adjoint to the forgetful functor. Thus $\mathcal{AR}_*\mathcal{C}$ has a generating set consisting of coalgebras $P'(C)$ where $C \in \mathcal{CA}$ is finite as a graded vector space.

COROLLARY A.4. *The forgetful functor $\mathcal{AR}_*\mathcal{C} \to \mathcal{UR}_*$ has a right adjoint.*

$$V'\colon \mathcal{UR}_* \longrightarrow \mathcal{AR}_*\mathcal{C}.$$

Actually, little new has been done, for we have the following result.

PROPOSITION A 5 *Let $M \in \mathcal{UR}$ Then there is an isomorphism in $\mathcal{CA}$*

$$V'(M) \cong V(M)$$

PROOF Let $C \in \mathcal{CA}$ Then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{U}}(C, M) &\cong \mathrm{Hom}_{\mathcal{UR}}\big(P(C), M\big)\\ &\cong \mathrm{Hom}_{\mathcal{ARC}}\big(P(C), V'(M)\big)\\ &\cong \mathrm{Hom}_{\mathcal{CA}}\big(C, V'(M)\big)\end{aligned}$$

This proves the result

Finally, we come to the result asserted in Section 4

COROLLARY A 6 *The indecomposables functor Q $\mathcal{AR} \longrightarrow \mathcal{UR}_0$ has a right adjoint V'' $\mathcal{UR}_0 \longrightarrow \mathcal{AR}$ and for $M \in \mathcal{UR}_0$ there is an isomorphism in $\mathcal{CA}$*

$$V''(M) \cong V(M)$$

PROOF In Section 5 we constructed the colimits in $\mathcal{AR}$, Q preserves colimits We also showed that $S_*P'(C)$, $C \in \mathcal{CA}$ finite, generate $\mathcal{AR}$ Thus V'' exists Copy the proof of Proposition A 5 to show $V''(M) \cong V(M)$

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