BULL. AUSTRAL. MATH. SOC. MOS 20F30, 20E15
VOL. 6 (1972), 287-306.

The subnormal structure of metanilpotent groups

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Let G be a group with a normal nilpotent subgroup N such that G/N is periodic and nilpotent. If $G(p)/N$ is the Sylow p-subgroup of G/N and $Q(p)$ is the maximal p-radicable subgroup of N, it is shown that G has a bound on the subnormal indices of its subnormal subgroups if and only if there is a positive integer c such that $G(p)/Q(p)$ is nilpotent of class at most c, for all primes p. It is also shown that if G is a periodic metanilpotent group and Q is its maximal radicable abelian normal subgroup then G has a bound on its subnormal indices if and only if there is a positive integer c such that for all primes p the Sylow p-subgroups of G/Q are nilpotent of class at most c.

1. Introduction

A subgroup H of a group G is subnormal in G if H can be connected to G by a finite chain of subgroups each of which is normal in its successor. If such chains exist then there is one of minimal length; the number of strict inclusions in this chain is called the subnormal index (or defect) of H in G. Groups in which every subnormal subgroup has subnormal index at most one are precisely those groups in which normality is transitive, and have been studied in the context of finite groups by Best and Taussky, [2], Gaschütz, [6], and Zacher, [20]; in the case of infinite soluble groups a similar study has been made by Robinson in [12]. Soluble groups in which every subnormal subgroup has subnormal index at

Received 5 November 1971.

https://doi.org/10.1017/S0004972700044506 Published online by Cambridge University Press

most n, for an arbitrary positive integer n, have been studied by Robinson, with the added restriction that the groups be standard wreath products of two nilpotent groups, in [16], and by McDougall, with a restriction to p-groups, in [11]. In this paper we are primarily concerned with metanilpotent groups in which every subnormal subgroup has subnormal index at most n, for some positive integer n.

In common with many other investigations of this type, the basis for our results is a lemma dealing with a much simplified situation. However before we can discuss this we need some additional terminology. Let π be a non-empty set of primes. A group G is said to be quasi-π-radicable if, for each π-number k, G is generated by the k-th powers of its elements. A group is π-reduced if it has no non-trivial quasi-π-radicable subgroups. If a group G is an extension of a p-reduced nilpotent group by a cyclic p-group, and has the property that the intersection of any set of subnormal subgroups is again subnormal, then G is nilpotent (Lemma 3.5). In §3 we use this key result to obtain a characterization of nilpotent-by-(periodic nilpotent) groups which have a bound for the subnormal indices of their subnormal subgroups. By Theorem E of [16] the standard unrestricted wreath product of an arbitrary torsion-free abelian group with itself has the property that every subnormal subgroup has subnormal index at most two. If we take the torsion-free group to be infinite cyclic we see that, in contrast to Lemma 3.5, an extension of a reduced (that is, π-reduced where π is the set of all primes) abelian group by a cyclic group can have a bound on its subnormal indices and yet not be nilpotent. Thus a different approach will be needed to deal with arbitrary metanilpotent groups which have a bound on their subnormal indices, and any characterization is likely to be extremely complex.

Results analogous to those of §3 are sought in §4 in connection with groups in which the subnormal subgroups form a complete lattice, since such groups have the property that arbitrary intersections of subnormal subgroups are subnormal and thus Lemma 3.5 is still available. Theorem 4.1 gives necessary conditions for the subnormal subgroups to form a complete lattice, but Theorem 4.3 shows that these conditions are not sufficient. Conversely Theorem 4.4 gives sufficient conditions which, as shown in Theorem 4.6, are not necessary. It seems likely that somewhere between these two sets of conditions lies a set of necessary and sufficient conditions, but we have been unable to locate it. If G is a periodic metanilpotent group and Q is its maximal radicable abelian normal subgroup, then our characterization can be rephrased in terms of the Sylow p-subgroups of G/Q. Thus in a group in which Q is trivial we have the surprising result that a knowledge of the Sylow p-subgroups suffices for us to decide whether the group has subnormal subgroups of arbitrary subnormal index.

2. Preliminaries

Our aim in this section is to summarise some elementary facts about subnormal subgroups and about quasi-π-radicability. Most of these facts will be used in §3 and §4, but no explicit mention will be made in these later sections.

2.1 Subnormal subgroups. The concept of subnormal subgroup can be approached via the idea of the standard series of a subgroup. The standard series of H in G is the series $\{H_i : i \geq 0\}$, where $H_0 = G$ and H_i is the normal closure of H in H_{i-1}. Then H is subnormal in G if and only if $H_n = H$ for some non-negative integer n. If we let $[H, K]$ denote the commutator of two subgroups H and K in a group, and define γHK^i inductively by:- $\gamma HK^0 = H$, $\gamma HK^i = [\gamma HK^{i-1}, K]$, then the n-th term of the standard series of H in G can be written as $H\gamma GH^n$. Thus H is subnormal in G if and only if γGH^i is contained in H for some non-negative integer i. If γGH^r is contained in H but γGH^{r-1} is not contained in H then H has subnormal index r in G.

A group G has the subnormal intersection property if the intersection of an arbitrary collection of subnormal subgroups of G is subnormal in G. A group has the subnormal join property if the join of an arbitrary collection of subnormal subgroups is subnormal.

Robinson has proved that every subnormal subgroup of a group G has subnormal index at most n if and only if $H_n = H_{n+1}$ for all subgroups H of G, and that a group has the subnormal intersection property if and only if the standard series of every subgroup becomes stationary after

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finitely many terms (Lemma 2 of [*14*]). He also shows (Lemma 8.1 of [*13*]) that a group has the subnormal join property if and only if the join of every ascending chain of subnormal subgroups is itself subnormal.

The subnormal join property, the subnormal intersection property, and the property of having bounded subnormal indices, are all inherited by subnormal subgroups and homomorphic images.

A group has the property that its subnormal subgroups form a complete lattice if and only if it has the subnormal join property and the subnormal intersection property. A group which has a bound on its subnormal indices has both these properties and hence its subnormal subgroups form a complete lattice. A fuller treatment of this material, with examples, is given in Chapter 3 of [*18*].

2.2 Quasi-π-radicability. A group is said to be π-radicable (where π is a non-empty set of primes) if, for every π-number k, each element of the group can be expressed as a k-th power of some element of the group. Clearly π-radicability implies quasi-π-radicability, and Černikov, [*4*], has shown that for ZA-groups the converse holds. It is easy to see that a group generated by quasi-π-radicable subgroups is itself quasi-π-radicable. This fact, together with Černikov's result, shows that every nilpotent group has a unique maximal π-radicable subgroup, which is therefore characteristic. If π is the set of all primes we use the terms radicable and reduced, instead of π-radicable and π-reduced.

It is easy to prove that an extension of a quasi-π-radicable group by a quasi-π-radicable group is itself quasi-π-radicable. Therefore the factor group of a nilpotent group by its maximal π-radicable subgroup can have no π-radicable non-trivial subgroups, and is therefore π-reduced.

Robinson proves in [*15*] (Lemma 2.2) that a subnormal periodic radicable abelian subgroup of a group commutes with any subnormal periodic nilpotent subgroup. Therefore in any periodic group G the join of all the subnormal radicable abelian subgroups is the unique maximal radicable abelian normal subgroup of G.

2.3 Notation. We will use $\langle X\rangle$, where X is a set of elements of a group G, to mean the subgroup of G generated by the elements in X. For any positive integer n, the symbol G^n will denote the subgroup of

G generated by the n-th powers of the elements of G.

3. Metanilpotent groups with bounded subnormal indices

In order to obtain the main results of this section we need a series of preliminary lemmas.

LEMMA 3.1. *Let* θ *be an automorphism of order* p^r *of an abelian group* A, *where* p *is a prime and* r *is a non-negative integer. Let* $a\psi = a^{-1}a\theta$ *for all elements* a *of* A. *Then* $A\psi^{p^r} \leq A^p$.

Proof. If we denote the identity endomorphism of A by 1 then in the ring of endomorphisms of A we have $\psi = \theta - 1$. Therefore

$$\psi^{p^r} = (\theta-1)^{p^r} = \theta^{p^r} + (-1)^{p^r} + pL(\theta) ,$$

where $L(\theta)$ is some polynomial in θ. Since θ has order p^r we have

$$\psi^{p^r} = 1 + (-1)^{p^r} + pL(\theta) .$$

If $p = 2$ this reduces to $\psi^{2^r} = 2\big(1+L(\theta)\big)$; and if $p \neq 2$ we have $\psi^{p^r} = pL(\theta)$.

Let b be an arbitrary element of $A\psi^{p^r}$. Then $b = a\psi^{p^r}$ for some element a of A. Substituting for ψ^{p^r} we have

$$b = a^2a^{2L(\theta)} \quad \text{if } p = 2 ,$$

$$b = a^{pL(\theta)} \quad \text{if } p \neq 2 .$$

Hence by putting $c = aa^{L(\theta)}$ $(p = 2)$ or $c = a^{L(\theta)}$ $(p \neq 2)$ we have, for all primes p, that $b = c^p$. This proves the lemma.

COROLLARY 3.2. *If in the situation of Lemma* 3.1 *there is a subgroup* B *of* A *with* $B\psi = B$, *then* B *is* p-*radicable.*

Proof. Since $B\psi = B$ it follows that $B\theta \leq B$ and hence, since θ has finite order, that $B\theta = B$. Therefore θ restricted to B is an

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automorphism of order p^t, where $0 \leq t \leq r$. Hence by the lemma $B\psi^{p^t} \leq B^p$. However $B\psi^{p^t} = B$, and so B is p-radicable.

LEMMA 3.3. *Let* $G = \langle x, A \rangle$, *where* A *is a normal abelian subgroup of* G. *Suppose that* $\left[x^{p^n}, A\right] = 1$ *for some prime* p *and non-negative integer* n. *Let* $A_0 = A$ *and define* A_{i+1} *inductively by:-* $A_{i+1} = [A_i, x]$. *If* $A_k = A_{k+1}$ *for some* k, *and* A *is* p*-reduced, then* G *is nilpotent.*

Proof. Conjugation by x gives an automorphism of A whose order divides p^n. Putting $a\psi = a^{-1}a^x$ for all elements a of A we have $A_k\psi = A_{k+1} = A_k$. Therefore by Corollary 3.2, A_k is p-radicable. But A is p-reduced and so A_k is trivial. Therefore A lies in the k-th term of the upper central series of G, and so G is nilpotent.

LEMMA 3.4. *If* π *is a non-empty set of primes and* N *is a nilpotent* π*-reduced group, then* N/Z *is* π*-reduced, where* Z *is the centre of* N.

Proof. If N is abelian there is nothing to prove, so we may assume N is not abelian. Let H/Z be the maximal π-radicable subgroup of N/Z. Then H is a normal subgroup of N and is π-reduced. We will prove that H is abelian. Suppose on the contrary that H is not abelian. Then there is an element y of H which is in the second term of the upper central series of H but is not central. Let A be the centre of H. We define a homomorphism θ of H/A into H by:-

$$(hA)\theta = [h, y] \text{ for all } h \text{ in } H .$$

Since A contains Z, the image of H/A under θ is π-radicable, and so, since H is π-reduced, must be trivial. Thus y commutes with all h in H and is therefore central. This is a contradiction and so H is abelian.

Since H is abelian it is a proper subgroup of N. Let x be an element of N not in H. We define a map ψ of H into H by:-

$$h\psi = [h, x] \text{ for all } h \text{ in } H .$$

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Since H is abelian and normal in N, ψ is a homomorphism of H into H. The kernel of ψ contains Z and so $H\psi$ is π-radicable. Since N is π-reduced we have $[h, x] = 1$ for all h in H and all x in N, so that H is contained in the centre of N. Therefore $H = Z$ and N/Z is π-reduced.

LEMMA 3.5. *Let $G = \langle x, N\rangle$ where N is a p-reduced normal nilpotent subgroup of G and G/N has order p^n. If G has the subnormal intersection property then G is nilpotent.*

Proof. We use induction on the nilpotent class of N. If this class is zero, that is if N is trivial, then the result is clearly true. Let A be the centre of N. By Lemma 3.4, N/A is p-reduced and so by the induction hypothesis G/A is nilpotent. Therefore $\langle x, A\rangle/A$ is subnormal in G/A and hence $\langle x, A\rangle$ has the subnormal intersection property. Since x^{p^n} lies in N we have $\left[x^{p^n}, A\right] = 1$. Let $M = \langle x\rangle \cap A$. Then M is a central subgroup of $\langle x, A\rangle$. The standard series of $\langle xM\rangle$ in $\langle x, A\rangle/M$ is easily seen to be $\{\langle x\rangle A_i/M : i \geq 0\}$, where A_i is defined as in Lemma 3.3 $\left(A_0 = A,\ A_{i+1} = [A_i, x]\right)$. Since $\langle x, A\rangle/M$ has the subnormal intersection property, $\langle x\rangle A_t/M = \langle x\rangle A_{t+1}/M$ for some t. Therefore A_tM/M is contained in $\langle xM\rangle A_{t+1}M/M$, and since this latter group is a split extension it follows that $A_tM = A_{t+1}M$. Therefore

$$A_{t+1} = [A_t, x] = [A_tM, x] = [A_{t+1}M, x] = [A_{t+1}, x] = A_{t+2} .$$

Thus we can apply Lemma 3.3 to deduce that $\langle x, A\rangle$ is nilpotent. G is therefore the join of a normal nilpotent subgroup N and a subnormal nilpotent subgroup $\langle x, A\rangle$, and hence must be nilpotent (Lemma 4.5 of [13]).

LEMMA 3.6. *Let N be a nilpotent group with upper central series $\{Z_i : 0 \leq i \leq c\}$. Let π be a non-empty set of primes. Define R_i inductively by:- $R_0 = 1$, R_i/R_{i-1} is the maximal π-radicable subgroup of the centre of N/R_{i-1}. Then R_i is the maximal π-radicable subgroup of Z_i and in particular $R_c = R_{c+1}$.*

Proof. We use induction on i . The case $i = 0$ is trivial. Suppose R_{i-1} is the maximal π-radicable subgroup of Z_{i-1} . By definition $[R_i, N] \leq R_{i-1}$, and since R_{i-1} is a subgroup of Z_{i-1} it follows that R_i is a subgroup of Z_i . Since R_i is π-radicable it remains to prove that it is the maximal π-radicable subgroup of Z_i . Let R be the maximal π-radicable subgroup of Z_i . Then the commutator of R and N is contained in Z_{i-1} .

Let us assume that R/R_{i-1} is abelian. For fixed g in N the mapping τ_g which sends rR_{i-1} to $[r, g]R_{i-1}$, for all r in R , is a homomorphism of R/R_{i-1} into Z_{i-1}/R_{i-1} . But R/R_{i-1} is π-radicable and Z_{i-1}/R_{i-1} is π-reduced, and so the kernel of τ_g must be the whole of R/R_{i-1} . It follows that R/R_{i-1} is contained in the centre of N/R_{i-1} . However R_i/R_{i-1} is the maximal π-radicable subgroup of the centre of N/R_{i-1} and so $R = R_i$. Thus it remains to show that R/R_{i-1} is indeed abelian.

π-radicability is a tensorial property (in the sense of [17]) and so is inherited by each factor of the lower central series of R (Theorem 1 of [17]). Since R is nilpotent it follows that R' , the derived group of R , is π-radicable. But since R' is contained in Z_{i-1} it follows that R' is in fact contained in R_{i-1} . Thus R/R_{i-1} is abelian and the proof of the lemma is complete.

LEMMA 3.7. *Let* N *be a nilpotent normal subgroup of a group* G *, with* G/N *a* π*-group for some non-empty set of primes* π *. Let* $Q(\pi)$ *be the maximal* π*-radicable subgroup of* N *. If* N *has nilpotent class* c *and* S *is any subnormal subgroup of* G *then the subnormal index of* S *in* $SQ(\pi)$ *is at most* c *.*

Proof. Let $R_0, R_1, R_2, \ldots, R_c$ be chosen as in Lemma 3.6. Since each R_i is normal in G the groups SR_i are subnormal in G and, since R_{i+1} normalises $SR_i \cap N$, it follows that $SR_i/SR_i \cap N$ is a subnormal

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subgroup of $SR_{i+1}/SR_i \cap N$. However $SR_i/SR_i \cap N$ is a π-group and $R_{i+1}(SR_i \cap N)/SR_i \cap N$ is π-radicable and abelian. Therefore by Lemma 4 of [16], $SR_i/SR_i \cap N$ is normal in $SR_{i+1}/SR_i \cap N$. Hence SR_i is normal in SR_{i+1} for $0 \leq i \leq c-1$. By Lemma 3.6, $R_c = Q(\pi)$ and so the subnormal index of S in $SQ(\pi)$ is at most c .

We now have enough lemmas at our disposal to make the proof of the main theorem relatively short.

THEOREM A. *Let N be a normal nilpotent subgroup of a group G such that G/N is periodic and nilpotent. For any prime p let $Q(p)$ be the maximal p-radicable subgroup of N and let $G(p)/N$ be the Sylow p-subgroup of G/N . Then G has a bound on the subnormal indices of its subnormal subgroups if and only if there is a positive integer c such that $G(p)/Q(p)$ is nilpotent of class at most c , for all primes p .*

Proof. Suppose that every subnormal subgroup of G has subnormal index at most n . Then each group $G(p)/Q(p)$ has the same property. $N/Q(p)$ is p-reduced and $G(p)/N$ is a nilpotent p-group. Therefore, if x is any element of $G(p)$, the group $\langle x, N\rangle/Q(p)$ has the subnormal intersection property, and so by Lemma 3.5, $\langle x, N\rangle/Q(p)$ is nilpotent. It follows that $\langle x, Q(p)\rangle/Q(p)$ is subnormal in $G(p)/Q(p)$. This means that every cyclic subgroup of $G(p)/Q(p)$ is subnormal. Since $G(p)/Q(p)$ also has the subnormal join property it follows that every subgroup of $G(p)/Q(p)$ is subnormal. Hence in $G(p)/Q(p)$ every subgroup is subnormal with subnormal index at most n , and so by a result of Roseblade (Corollary to Theorem 1 of [19]), $G(p)/Q(p)$ is nilpotent of class at most $f(n)$. Putting $c = f(n)$ we have one half of the theorem.

Conversely, suppose that there is a positive integer c such that $G(p)/Q(p)$ is nilpotent of class at most c for all primes p . Let the nilpotent class of N be d . We will show that if S is subnormal in G then the subnormal index of S in G is at most $(d+1)c + \frac{1}{2} d(d+3)$. Let us denote this expression by $f(d)$. We will proceed by induction on d . If $d = 0$ then G is nilpotent of class at most c and the result is trivial. Let Z be the centre of N . Then $Q(p)Z/Z$ is contained in the maximal p-radicable subgroup of N/Z . Since $G(p)/Q(p)Z$ is nilpotent of class at most c it follows that the hypotheses are satisfied by G/Z .

Therefore by the induction hypothesis every subnormal subgroup of G/Z has subnormal index at most $f(d-1)$. In particular if S is subnormal in G then SZ/Z has subnormal index at most $f(d-1)$ in G/Z . Thus SZ has subnormal index at most $f(d-1)$ in G . It remains to find a bound on the subnormal index of S in SZ .

$S/S\cap N$ is a subnormal subgroup of $SZ/S\cap N$. Since $S/S\cap N$ is periodic and nilpotent we can write S as the product of its normal subgroups S_p , where $S_p/S\cap N$ is the Sylow p-subgroup of $S/S\cap N$. Since S_pN/N is isomorphic to $S_p/S_p\cap N$, which is a p-group, S_p is contained in $G(p)$. However $G(p)/N$ is a p-group and $Q(p)$ is the maximal p-radicable subgroup of N , and so by Lemma 3.7 the subnormal index of S_p in $S_pQ(p)$ is at most d . But $G(p)$ is normal in G and $G(p)/Q(p)$ is nilpotent of class at most c . Therefore the subnormal index of S_p in G is at most $c + d + 1$. Thus the subnormal index of $S_p/S\cap N$ in $SZ/S\cap N$ is at most $c + d + 1$. Since $SZ/S\cap N$ is the join of a nilpotent normal subgroup $Z(S\cap N)/S\cap N$ and a subnormal nilpotent subgroup $S/S\cap N$ it must be nilpotent (Lemma 4.5 of [13]). Hence the terms of the standard series of $S/S\cap N$ in $SZ/S\cap N$ all lie in the torsion-subgroup of $SZ/S\cap N$ (apart from the 0-th term, which is $SZ/S\cap N$ itself), and so each term is the product of the corresponding terms of the standard series of the subgroups $S_p/S\cap N$. Therefore the subnormal index of $S/S\cap N$ in $SZ/S\cap N$ is at most $c + d + 1$.

Putting the two pieces together it follows that the subnormal index of S in G is at most $f(d-1) + c + d + 1$. But

$$f(d-1) = dc + \tfrac{1}{2}(d-1)(d+2) = dc + \tfrac{1}{2}(d^2+d-2) .$$

Therefore

$$\begin{aligned} f(d-1) + c + d + 1 &= (d+1)c + \tfrac{1}{2}(d^2+d-2+2d+2) \\ &= (d+1)c + \tfrac{1}{2}\, d(d+3) \\ &= f(d) . \end{aligned}$$

Thus the subnormal index of S in G is at most $f(d)$.

COROLLARY. *Let* N *be a nilpotent normal subgroup of a group* G *, with* G/N *periodic and nilpotent. Let* $Q(p)$ *be the maximal* p*-radicable*

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subgroup of N *and let* $G(p)/N$ *be the Sylow* p*-subgroup of* G/N . *If there is a positive integer* c *such that* $G(p)/Q(p)$ *is nilpotent of class at most* c , *for all primes* p , *then the subnormal index in* G *of any subnormal subgroup is at most* $f(d)$, *where* d *is the nilpotent class of* N *and* $f(d) = (d+1)c + \frac{1}{2}d(d+3)$.

For periodic metanilpotent groups the following characterization is probably more useful, since it involves Sylow subgroups.

THEOREM B. *Let* G *be a periodic metanilpotent group, and let* Q *be the maximal radicable abelian normal subgroup of* G . *Then* G *has a bound on its subnormal indices if and only if there is a positive integer* c *such that for all primes* p *the Sylow* p*-subgroups of* G/Q *are nilpotent of class at most* c .

Proof. There is a normal nilpotent subgroup B of G such that G/B is nilpotent. By Fitting's Theorem, BQ is nilpotent. Thus if we put $N = BQ$ we have that N is a normal nilpotent subgroup of G and G/N is nilpotent. The result will follow from Theorem A if we can show that, for any prime p , the Sylow p-subgroups of G/Q are nilpotent of class at most c if and only if $G(p)/Q(p)$ is nilpotent of class at most c (where $G(p)$ and $Q(p)$ are as defined in Theorem A). $Q(p)$, since it is periodic and nilpotent, is the direct product of a p'-group (where p' denotes the set of primes other than p) with a radicable p-group P . But by a result of Černikov, [3], a radicable nilpotent p-group must be abelian, and so P is contained in Q . Therefore $Q(p)/Q$ is a p'-group. Also, since $Q(p)$ contains all the p'-elements of N , $G(p)/Q(p)$ is a p-group.

Let $G(p)/Q(p)$ be nilpotent of class at most c , and let S/Q be a Sylow p-subgroup of G/Q . Since $S/Q \cap Q(p)/Q = 1$ we know that $S \cap Q(p) = Q$. But $SQ(p)/Q(p)$ is isomorphic to $S/S \cap Q(p) = S/Q$, and since $SQ(p)/Q(p)$ is a subgroup of $G(p)/Q(p)$ it follows that S/Q is nilpotent of class at most c .

Conversely, suppose that the Sylow p-subgroups of G/Q are nilpotent of class at most c . Let $\{x_1, \ldots, x_r\}$ be any finite set of elements of $G(p)$. Since $G(p)/Q(p)$ is a p-group it follows that $\langle x_1, \ldots, x_r\rangle Q(r)/Q(p)$ is a finite p-group. Therefore

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$\langle x_1, \ldots, x_r \rangle Q(p)/Q$ splits over $Q(p)/Q$, since the latter is a p'-group (Theorem 3 of [5]). Thus we have $\langle x_1, \ldots, x_r \rangle Q(p) = S_r Q(p)$, where $S_r \cap Q(p) = Q$. Then S_r/Q is a finite p-group and so by assumption has nilpotent class at most c. Therefore every finitely generated subgroup of $G(p)/Q(p)$ is nilpotent of class at most c, and so $G(p)/Q(p)$ is nilpotent of class at most c. This completes the proof of the theorem.

As an easy application of Theorem B and the corollary to Theorem A we have:-

COROLLARY. *Let G be a quasi-radicable metabelian group satisfying the minimal condition for normal subgroups. Then every subnormal subgroup of G has subnormal index at most four.*

Proof. By a result of Baer, [*1*], soluble groups satisfying the minimal condition for normal subgroups are periodic. By Corollary 3.3 of [*10*] the Sylow p-subgroups of G are abelian for all primes p. Thus we have the situation of Theorem B with $c = 1$. Combining Theorem B with the corollary to Theorem A we have that every subnormal subgroup of G has subnormal index at most $f(d)$. Since $d = 1$ and $c = 1$ we find by substitution in the formula that $f(d)$ is then four.

4. Groups in which the subnormal subgroups form a complete lattice

In this section we attempt to find analogues of the previous theorems for the class of groups in which the subnormal subgroups form a complete lattice. A group in this class will have the subnormal intersection property and so Lemma 3.5 can be used.

THEOREM 4.1. *Let N be a nilpotent normal subgroup of a group G, with G/N periodic and nilpotent. For each prime p let $G(p)/N$ be the Sylow p-subgroup of G/N and let $Q(p)$ be the maximal p-radicable subgroup of N. If the subnormal subgroups of G form a complete lattice then $G(p)/Q(p)$ has the property that every subgroup is subnormal, for all primes p.*

Proof. Since the subnormal intersection property is inherited by subnormal subgroups and homomorphic images, $G(p)/Q(p)$ has the subnormal intersection property. Thus for any element x of $G(p)$ we can apply

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Lemma 3.5 to $\langle x, N\rangle/Q(p)$. It follows that $\langle x, Q(p)\rangle/Q(p)$ is subnormal in $G(p)/Q(p)$. Since $G(p)/Q(p)$ also has the subnormal join property, every subgroup of $G(p)/Q(p)$, being the join of its cyclic subgroups, must be subnormal.

As in §3 we can restate this result for periodic groups.

THEOREM 4.2. *Let G be a periodic metanilpotent group and let Q be the maximal radicable abelian normal subgroup of G . If the subnormal subgroups of G form a complete lattice then in every Sylow p-subgroup of G/Q each subgroup is subnormal, for all primes p .*

The proof is similar to the corresponding part of the proof of Theorem B, and so will be omitted.

We now give an example to show that the converse to Theorem 4.2 is false.

THEOREM 4.3. *There is a periodic metabelian group which is locally nilpotent but has neither the subnormal intersection property nor the subnormal join property, although in each Sylow p-subgroup every subgroup is subnormal.*

Proof. Let H_p denote the p-group constructed by Heineken and Mohamed in [8]. Then H_p is a metabelian group in which every subgroup is subnormal, and in which the set of subnormal indices is unbounded. Let H be the direct product over all primes p of the groups H_p . Let S_p be a subnormal subgroup of H_p of subnormal index p , and let T_p be the direct product of S_p with all H_q , where $q \neq p$. Then S_p and T_p are subnormal of index p in H . Let T be the join of the subgroups S_p (actually it is just their direct product). It is also the intersection of the subgroups T_p . If T were subnormal of index n in H then each S_p would be subnormal of index at most n in H_p , contrary to the construction for $p > n$. Hence T is not subnormal in H , and so H does not have either the subnormal intersection property or the subnormal join property.

It follows from Theorem 4.3 that the condition that every subgroup of

$G(p)/Q(p)$ be subnormal, for all primes p, is not sufficient to guarantee that the subnormal subgroups of G form a complete lattice. Thus we have a necessary condition which is not sufficient. We next exhibit a set of sufficient conditions which, however, are not necessary.

THEOREM 4.4. *Let N be a normal nilpotent subgroup of a group G with G/N periodic and nilpotent. Let $G(p)/N$ be the Sylow p-subgroup of G/N and let $Q(p)$ be the maximal p-radicable subgroup of N. If there is a positive integer c such that for almost all primes p, $G(p)/Q(p)$ is nilpotent of class at most c, and if for the finitely many exceptional primes p every subgroup of $G(p)/Q(p)$ is subnormal, then the subnormal subgroups of G form a complete lattice.*

Proof. By the remarks made in §2 it will suffice to show that if S is the union of an ascending chain $\{S_i ; i < \rho\}$, where ρ is a limit ordinal, of subnormal subgroups of G, and if T is the intersection of an arbitrary family $\{T_i ; i \in I\}$ of subnormal subgroups of G, then S and T are again subnormal in G.

We proceed by induction on the class of N. If N has class zero then $N = 1$ and the result is trivial. Let Z be the centre of N. For each prime p, $Q(p)Z/Z$ is a p-radicable subgroup of N/Z, and it follows easily that G/Z inherits the properties of G. Therefore by the induction hypothesis we may assume that G/Z satisfies the conclusion of the theorem. In particular, since $SZ = \bigcup_{i<\rho} S_iZ$, it follows that SZ is subnormal in G. Thus we need only show that S is subnormal in SZ or, equivalently, that $S/S\cap N$ is subnormal in $SZ/S\cap N$. Each of the groups $S_iZ/S_i\cap N$ is nilpotent, since it is the join of a subnormal nilpotent group $S_i/S_i\cap N$ and a normal abelian subgroup $Z(S_i\cap N)/S_i\cap N$ (Lemma 4.5 of [13]). Hence $SZ/S\cap N$ is locally nilpotent, since it is the union of the subgroups $S_iZ(S\cap N)/S\cap N$. Since $S/S\cap N$ is periodic it is contained in the torsion-subgroup of $SZ/S\cap N$. Let $S(p)/S\cap N$ be the Sylow p-subgroup of $S/S\cap N$. Then if $S(p, i)/S_i\cap N$ is the Sylow p-subgroup of $S_i/S_i\cap N$ it follows easily that $S(p)$ is the union of the $S(p, i)$. If the nilpotent class of N is d then application of Lemma 3.7 to $S(p, i)N$ shows that $S(p, i)$ has subnormal index at most d in $S(p, i)Q(p)$. Hence $S(p)$

has subnormal index at most d in $S(p)Q(p)$. Let π be the set of primes for which $G(p)/Q(p)$ is nilpotent of class at most c , so that π' is a finite set (π' is the complement of π in the set of all primes). Then if $p \in \pi$ it follows that $S(p)$ is subnormal in G and has subnormal index at most $c + d + 1$. If $p \in \pi'$ then every subgroup of $G(p)/Q(p)$ is subnormal and so $S(p)$ is subnormal in G . As there are only finitely many primes in π' we can find an integer r such that the subnormal index of $S(p)$ in G is at most r , for all primes p . Thus $S(p)/S\cap N$ has subnormal index at most r in $SZ/S\cap N$. Since $SZ/S\cap N$ is locally nilpotent it follows that the n-th term of the standard series of $S/S\cap N$ in $SZ/S\cap N$ is the direct product of the n-th terms of the standard series of the $S(p)/S\cap N$, for $n \geq 1$. Therefore $S/S\cap N$ is subnormal in $SZ/S\cap N$ with subnormal index at most r .

Similarly we can consider $T/T\cap N$. Let $T(p)/T\cap N$ be the Sylow p-subgroup of $T/T\cap N$. If $T(p, i)/T_i\cap N$ is the Sylow p-subgroup of $T_i/T_i\cap N$ then $T(p) = \bigcap_i T(p, i)$. By an argument similar to that above we can prove that each $T(p)$ is subnormal in G . Since T is the join of all the groups $T(p)$ it follows from the first part of the proof that T is subnormal in G . Thus the proof is complete.

As before we can try to rewrite the theorem for periodic groups in terms of Sylow p-subgroups. However we need another assumption, namely that G/N is countable.

THEOREM 4.5. *Let G be a periodic group with a normal nilpotent subgroup N such that G/N is a countable nilpotent group. Let Q be the maximal radicable subgroup of N . Let π be a set of primes such that its complement π' is finite. If there is a positive integer c such that the Sylow p-subgroups of G/Q are nilpotent of class at most c for all p in π , and if every subgroup of each Sylow p-subgroup of G/Q is subnormal for all p in π' , then the subnormal subgroups of G form a complete lattice.*

Proof. It is sufficient to show that the hypotheses of the theorem imply those of Theorem 4.4.

Let $Q(p)$ be the maximal p-radicable subgroup of N , and let $G(p)/N$ be the Sylow p-subgroup of G/N . Since Q is contained in each

https://doi.org/10.1017/S0004972700044506 Published online by Cambridge University Press

$Q(p)$ we can assume $Q = 1$. Therefore $Q(p)$ is the p'-subgroup of N and $G(p)/Q(p)$ is a p-group. The centraliser in $G(p)$ of $Q(p)$ contains the Sylow p-subgroup of N, and so the product of $Q(p)$ and its centraliser in $G(p)$ contains N. Therefore by Theorem 3 of [5], since G/N is countable, $G(p) = SQ(p)$, where $S \cap Q(p) = 1$. Thus S is a p-group and hence is contained in a Sylow p-subgroup of G. If $p \in \pi$ then S is nilpotent of class at most c, so that $G(p)/Q(p)$ is nilpotent of class at most c. If $p \in \pi'$ then every subgroup of $G(p)/Q(p)$ is subnormal. Hence the result.

REMARK. In the previous theorem it would suffice to have $G(p)/N$ countable for all p in π'. The cases where p is in π could be dealt with as in the proof of Theorem B.

Theorem 4.5 gives sufficient conditions for the subnormal subgroups of a countable periodic metanilpotent group to form a complete lattice. We conclude with an example to show that these conditions are not necessary.

THEOREM 4.6. *There is a countable periodic metabelian group which has a Sylow p-subgroup of nilpotent class p for infinitely many primes p, but in which the subnormal subgroups form a complete lattice.*

Proof. Let p be any prime. We put $p(0) = p$ and construct an infinite sequence of primes as follows. Suppose $p(0), p(1), \ldots, p(n)$ are already chosen. By Dirichlet's Theorem (Theorem 15 of Hardy and Wright, [7]) the sequence $\{1+mp(0)p(1) \ldots p(n)\}$ where m varies over the positive integers, contains infinitely many primes. We choose $p(n+1)$ to be one such prime. Then $p(n+1) > p(n)$ and $p(n+1)$ is congruent to 1 modulo $p(m)$ for all $m \le n$.

If $m < n$ then the cyclic group of order $p(n)$ has an automorphism of order $p(m)$. This automorphism takes a generator to a power $\theta(n, m)$ of itself. Let X be the direct product of groups X_i, where X_i is a cyclic group of order $p(i)$ generated by an element $x(i)$, for all non-negative integers i. Let A_i be the direct product of $p(i)$ copies of the cyclic group of order $p(i)$. Let A be the direct product of the groups A_i taken over all non-negative integers i. We show that X can be considered as a group of automorphisms of A. We define an action of

X on A_i as follows:-

let X_i act on A_i via the right regular representation;

let X_j act trivially on A_i for $j > i$;

for all a in A_i and all $j < i$ let $a^{x(j)} = a^{\theta(i,j)}$.

It is easy to see that this makes X into a group of automorphisms of A . Thus we can form the natural split extension $G = AX$, where A is normal in G and $A \cap X = 1$. Clearly G is a countable periodic metabelian group.

Each subgroup A_iX_i is a Sylow $p(i)$-subgroup of G . But A_iX_i is isomorphic to the standard wreath product of the cyclic group of order $p(i)$ with itself, and so by a result of Liebeck (Theorem 5.1 of [9]) is nilpotent of class $p(i)$. It remains to show that the subnormal subgroups of G form a complete lattice. Let S be a subnormal subgroup of G . Suppose SA contains $x(j)$ for some j . Then $[A, x(j)]$ is generated by the elements $a^{-1}a^{x(j)}$ for all a in A . Taking a in A_i , where $i > j$, then $a^{x(j)} = a^{\theta(i,j)}$ and so $a^{-1}a^{x(j)} = a^{\theta(i,j)-1}$. $a^{\theta(i,j)-1}$ generates the same cyclic subgroup as a . Therefore $[A, x(j)]$ contains A_i for all $i > j$. Thus $\gamma A\langle x(j)\rangle^t$ contains A_i for all $i > j$ and all $t > 0$. Also, since A is abelian, $\gamma AS^t = \gamma A(SA)^t$. Therefore γAS^t contains A_i for all $i > j$, and since γAS^t is contained in S for some t it follows that S contains A_i for all $i > j$.

Let T be the union of an ascending chain $\{S_k ; k < \rho\}$, for some limit ordinal ρ , of subnormal subgroups of G . If T is contained in A then it is subnormal in G as required. If not, then for some k the group S_kA is strictly larger than A , and so contains some $x(j)$. Thus by the above remarks S_k contains A_i for all $i > j$. The product of all the A_i with $i > j$ is a normal subgroup $A(j)$ of G . It is easily seen that $G/A(j)$ satisfies the conditions of Theorem 4.5 and so has the subnormal join property. Since $S/A(j)$ is the union of the $S_\lambda/A(j)$, $\lambda \geq k$, it follows that $S/A(j)$ is subnormal in $G/A(j)$, and hence that G has the subnormal join property.

If S is the intersection of subnormal subgroups S_k of G $(k \in L)$ then either S is contained in A and so is subnormal in G, or A is a proper subgroup of SA. If the latter is the case then SA contains $x(j)$ for some j. Therefore S_kA contains $x(j)$ for all k, and so as above S_k contains A_i for all $i > j$ and for all k. Consequently S contains A_i for all $i > j$, and as before we may pass to the factor group $G/A(j)$. But $G/A(j)$ satisfies the hypotheses of Theorem 4.5 and so S is subnormal in G. Therefore G has the subnormal intersection property, and the proof is complete.

REMARK. If we let K be the direct product of the groups A_iX_i for all non-negative i, we can show as in Theorem 4.3 that K has neither the subnormal intersection property nor the subnormal join property. Thus it seems likely that to get necessary and sufficient conditions for the subnormal subgroups to form a complete lattice we need to look somewhere other than just at the groups $G(p)/Q(p)$. In other words a characterization in terms of the $G(p)/Q(p)$ alone seems unlikely in the light of our examples.

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