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# REPRESENTATIONS OF QUANTUM HEISENBERG ALGEBRAS

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ABSTRACT. A Lie algebra $\mathcal{A}$ is called a *Heisenberg algebra* if its centre coincides with its derived algebra and is one-dimensional. When $\mathcal{A}$ is infinite-dimensional, Kac, Kazhdan, Lepowsky, and Wilson have proved that $\mathcal{A}$-modules that satisfy certain conditions are direct sums of a canonical irreducible submodule. This is an algebraic analogue of the Stone-von Neumann theorem. In this paper, we extract quantum Heisenberg algebras, $\mathcal{U}_q(\mathcal{A})$, from the quantum affine algebras whose vertex representations were constructed by Frenkel and Jing. We introduce the canonical irreducible $\mathcal{U}_q(\mathcal{A})$-module $M_q$ and a class $C_q$ of $\mathcal{U}_q(\mathcal{A})$-modules that are shown to have the Stone-von Neumann property. The only restriction we place on the complex number $q$ is that it is not a square root of 1. If $q_1$ and $q_2$ are not roots of unity, or are both primitive $m$-th roots of unity, we construct an explicit isomorphism between $\mathcal{U}_{q_1}(\mathcal{A})$ and $\mathcal{U}_{q_2}(\mathcal{A})$. If $q_1$ is a primitive $m$-th root of unity, $m$ odd, $q_2$ a primitive $2m$-th or a primitive $4m$-th root of unity, we also construct an explicit isomorphism between $\mathcal{U}_{q_1}(\mathcal{A})$ and $\mathcal{U}_{q_2}(\mathcal{A})$.

**Introduction.** Kac-Moody algebras—or affine Lie algebras—have proved to be far-reaching in physics and mathematics. Their representations provide solutions to models having conformal invariance in statistical mechanics and particle physics. The *basic representation* of Kac-Moody algebras was constructed by Kac and Frenkel using the vertex operators of string theory. Such a representation is built up from a canonical representation of an infinite-dimensional Heisenberg algebra. An infinite-dimensional Heisenberg Lie algebra over the field of complex numbers, $\mathbb{C}$, has generators $\{X_n : n \in \mathbb{Z} - \{0\}\}$ and relations

$$[X_m, X_n] = m\delta_{m+n,0}\gamma, \tag{1}$$

where $\gamma$ is central; see Section 2.2 of [4]. The relations in (1) are precisely the canonical commutation relations in quantum physics. The infinite-dimensional Heisenberg algebra can be realized as a subalgebra of the affine Kac-Moody algebra $A_1^{(1)}$.

Recently physicists and mathematicians have "quantized" certain Lie algebras, thus producing associative algebras called "$q$-algebras". The coefficients in the defining relations in these algebras often involve the classical $q$-integers $[m]_q$, defined in the Notations preceding Section 1. For instance one might consider instead of (1) the relations

$$[X_m, X_n] = [m]_q\delta_{m+n,0}\gamma$$

where again $\gamma$ is central but $[X_m, X_n]$ denotes $X_mX_n - X_nX_m$ in the associative algebra. If $q$ is a root of unity, $[m]_q$ can vanish for some integers, $m$.

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The problem of quantizing affine algebras was solved by Drinfeld in [1] and soon after that Frenkel and Jing constructed their vertex operator representations in [2]. Their approach is similar to that used in constructing the basic representation: they start with a canonical representation of a quantum Heisenberg algebra $\mathcal{U}_q(\mathcal{A})$ when $q$ is not a root of unity.

We begin by extracting the quantum Heisenberg algebra from [2] in the simplest rank one case. The relation we use is slightly more complicated than the one above. It is

$$[X_m, X_n] = ([2m]_q/m)\delta_{m+n,0}[m]_\gamma.$$

The effect of the coefficient $([2m]_q)/m$, as opposed to $[m]_q$, is that the algebra becomes commutative when $q$ is a fourth root of unity. The same coefficient also appears in the vertex operator construction of quantum affine algebras. Since we hope to develop quantum Heisenberg algebras in parallel to quantum affine algebras we use the coefficient $([2m]_q)/m$ instead of $[m]_q$.

In Section 1 we compute the centre of the quantum Heisenberg algebra, $\mathcal{U}_q(\mathcal{A})$. Consider the following families of Heisenberg algebras:

(i) $\mathcal{U}_q(\mathcal{A})$, $q$ is not a root of unity
(ii) $\mathcal{U}_q(\mathcal{A})$, $q$ is a fourth root of unity
(iii) $\mathcal{U}_q(\mathcal{A})$, $q$ is an $n$-th root of unity, $n \neq 4$.

We use the result on centres to show that if $j \neq k$ no algebra in family ($j$) is isomorphic to an algebra in family ($k$). We also show that the algebras in family (i) (respectively family (ii)) are isomorphic. We show that the algebras in family (iii) are isomorphic after a localization.

In Section 2 we turn to the representation theory of quantum Heisenberg algebras and establish a Stone-von Neumann theorem for these algebras. In the Lie algebra case there are two proofs available for generalization. The first one is in [6] and is in the context of affine algebras; the second is in [3] and [7]. We observed that the latter proof can be adapted to quantum Heisenberg algebras.

When $q$ is not a root of unity the canonical $\mathcal{U}_q(\mathcal{A})$-module is no longer irreducible. It has a maximal proper submodule. In order to formulate a Stone-von Neumann theorem in this case we build an induced module isomorphic to the irreducible quotient.

NOTATION. $\mathbb{C}, \mathbb{C}^\times$ stand respectively for complex numbers and nonzero complex numbers.

$\mathbb{Z} :=$ the set of integers. Let $q \in \mathbb{C}^\times, q^2 \neq 1$. For some domain of the natural logarithm define $t := 2\ln q$ and $[m]_q := (q^m - q^{-m})/t$.

Also if $q$ is a primitive $k$-th root of unity put

$$\tilde{k} := \begin{cases} k, & \text{if } k \text{ is odd,} \\ k/2 & \text{if } k = 2l,\ l \text{ odd,} \\ k/4 & \text{if } 4 \mid k. \end{cases}$$

We now have the conditions: $[2m]_q = 0 \leftrightarrow q^{4m} = 1 \leftrightarrow k \mid 4m \leftrightarrow m \in \tilde{k}\mathbb{Z} - \{0\}$.

1. **Quantum Heisenberg algebras $\mathcal{U}_q(\mathcal{A})$.** Let $\mathcal{X} = \{X_k : k \in \mathbf{Z} - \{0\}\}$ be a set of linearly independent symbols. The quantum Heisenberg algebra, $\mathcal{U}_q(\mathcal{A})$, is the associative algebra with generators

$$\mathcal{X} \cup \{\gamma, \gamma^{-1}\} \tag{2}$$

where $\gamma$ is a central symbol whose multiplicative inverse is $\gamma^{-1}$. The elements in $\mathcal{X}$ are related by

$$[X_m, X_n] = (1/mt^2)(q^{2m} - q^{-2m})\delta_{m+n,0}(\gamma^m - \gamma^{-m}) = ([2m]_q/m)\delta_{m+n,0}[m]_\gamma \tag{3}$$

where $\delta_{m+n,0}$ is Kronecker delta and $[m]_\gamma = (\gamma^m - \gamma^{-m})/t$ as in [5]. Equation (3) is the quantum analogue of (1). In all the representations that we will consider $\gamma^{\pm 1}$ will act as multiplication by $q^{\pm 1}$, in which case (3) becomes $[X_m, X_n] = ([2m]_q[m]_q)\delta_{m+n,0}$. As $q$ tends to 1 the right hand side of this equation approaches $2m$. Therefore the universal enveloping algebra of the Lie algebra $\mathcal{A}$ is the "limit" of $\mathcal{U}_q(\mathcal{A})$ as $q$ tends to 1.

If $q$ is a fourth root of unity then $[2m]_q = 0$ for all nonzero integers $m$. Hence $\mathcal{U}_q(\mathcal{A})$ is a commutative algebra. Except in Corollary 1.3(a) and Corollary 1.3(c) we shall assume in the sequel that $q$ is not a fourth root of unity.

We now establish some properties of $\mathcal{U}_q(\mathcal{A})$. We recall that all algebra maps fix elements of $\mathbf{C}$. All our algebra maps will also fix powers of $\gamma$.

PROPOSITION 1.1. *$\mathcal{U}_{q_1}(\mathcal{A}) \cong \mathcal{U}_{q_2}(\mathcal{A})$ by explicit formulae in the following cases:*
*(a) neither $q_1$ nor $q_2$ is a root of unity;*
*(b) both $q_1$ and $q_2$ are primitive $m$-th roots of unity.*
*(c) $q_1$ is a primitive $m$-th root of unity while $q_2$ is a primitive $(2m)$-th or a primitive $(4m)$-th root of unity, where $m$ is an odd positive integer.*

PROOF. (a) The assignment $\gamma^n \longmapsto \gamma^n$, $n \in \mathbf{Z}$; $X_{-n} \longmapsto X_{-n}$; $X_n \longmapsto \frac{t_2^2(q_1^{2n} - q_1^{-2n})}{t_1^2(q_2^{2n} - q_2^{-2n})} X_n$, $n > 0$, extends to an isomorphism from $\mathcal{U}_{q_1}(\mathcal{A})$ onto $\mathcal{U}_{q_2}(\mathcal{A})$

(b) If $q_1$ and $q_2$ are primitive $m$-th roots of unity (same $m$) then the resulting algebras are isomorphic via $X_k \longmapsto X_k$ for all nonzero integers $k \in \tilde{m}\mathbf{Z} - \{0\}$. All other assignments are as in (a).

(c) The images of $\gamma^n, n \in \mathbf{Z}$ and $X_{-n}$, $n > 0$ are the same as in (a) and (b), while $X_{nm} \longmapsto X_{nm}$, $n \in \mathbf{Z} - \{0\}$. Since $m$, $2m$ and $4m$ divide $4nm$, (3) is preserved by this assignment. Suppose $m$ does not divide $k, k > 0$; we have to decide where to map $X_k$. First we observe that both $(q_1^{2k} - q_1^{-2k})$ and $(q_2^{2k} - q_2^{-2k})$ are nonzero. If the former were zero then $q_1^{4k} = 1$. Since $q_1$ is a primitive $m$-th root of unity, this implies that $m$ divides $4k$. Since $m$ is odd, this contradicts "$m$ does not divide $k$". If $(q_2^{2k} - q_2^{-2k}) = 0$, then $2m \mid 4k$ and $4m \mid 4k$. Hence $m \mid k$, which has already been ruled out. So $X_k \longmapsto c_k X_k, k > 0$ and $m$ does not divide $k$, where $c_k = \frac{t_2^2(q_1^{2k} - q_1^{-2k})}{t_1^2(q_2^{2k} - q_2^{-2k})}$, yields a well-defined map. We now extend these formulae to an isomorphism from $\mathcal{U}_{q_1}(\mathcal{A})$ onto $\mathcal{U}_{q_2}(\mathcal{A})$. ∎

REMARK. If $m$ in Proposition 1.1(c) is even the map in our proof may take 0 to a nonzero element. We shall deal with the general case in Proposition 1.4, where unfortunately the isomorphism is not a $\mathbf{C}$-algebra isomorphism.

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PROPOSITION 1.2. *(a) The group of units, $U$, of $\mathcal{U}_q(\mathcal{A})$ is $\{\alpha\gamma^n : \alpha \in \mathbb{C} - \{0\}, n \in \mathbb{Z}\}$.*

*(b) If $q$ is not a root of unity, then the centre of $\mathcal{U}_q(\mathcal{A})$ is the subalgebra, $\mathcal{Z}$, generated by $\{\gamma, \gamma^{-1}\}$.*

*(c) If $q$ is a primitive $l$-th root of unity, then the centre of $\mathcal{U}_q(\mathcal{A})$ is the subalgebra $\mathcal{Z}$ generated by $\{\gamma, \gamma^{-1}\} \cup \{X_k : k \in \tilde{l}\mathbb{Z} - \{0\}\}$.*

PROOF. (a) The displayed set is an abelian group and is contained in $U$ because $\gamma$ and $\gamma^{-1}$ are central elements of the $\mathbb{C}$-algebra $\mathcal{U}_q(\mathcal{A})$. Let $L = \mathbb{C}[\gamma, \gamma^{-1}]$ be the ring of Laurent polynomials. Suppose $g \in \mathcal{U}_q(\mathcal{A}) - L$. Say $X_n$ occurs in a component of $g$ for some $n \in \mathbb{Z} - \{0\}$. By applying (3) we may put $g$ in the form $g = X_n^k g_1 + g_2$ where $g_1 \neq 0$, and if $X_n^j$ occurs with nonzero coefficient in $g_2$ then $j < k$. If $f \in \mathcal{U}_q(\mathcal{A})$ we treat it in the same way with respect to $X_n$, *i.e.*; $f = X_n^l f_1 + f_2$ where $f_1 \neq 0$ and if $X_n^j$ occurs with nonzero coefficient in $f_2$ then $j < l$. So, if $f$ has no $X_n$-term, $l$ is 0. No application of (3) to $gf$ can eliminate the component containing $X_n^{k+l}$. Therefore $gf \neq 1$. Hence $g$ is not in $U$. So we may assume that $g \in \mathbb{C}[\gamma, \gamma^{-1}]$. In that case if $g$ is a unit it must be in the displayed set.

(b) Since $\gamma$ and $\gamma^{-1}$ are central, the centre of $\mathcal{U}_q(\mathcal{A})$ contains $\mathcal{Z}$.

Let $f \in \mathcal{U}_q(\mathcal{A}) - \mathcal{Z}$. We shall show that $f$ is not in the centre of the algebra. Since $\gamma$ and $\gamma^{-1}$ are central we may assume that neither of them is the last symbol in any term of $f$. Since $f$ is not a constant, at least one of the terms in $f$ ends in $X_n$, $n$ some nonzero integer. By applying (3) successively we may assume that $X_n$ occurs as the last symbol in every term of $f$ in which it occurs. Let $f = f_1 X_n + f_2$ where $f_1 \neq 0$ and $f_2$ has no term involving $X_n$.

It suffices to show that $[f_1 X_n, X_{-n}] \neq 0$ since $[f_2, X_{-n}] = 0$, and so $[f, X_n] = [f_1 X_n, X_{-n}] \neq 0$. We may, by collecting terms with the same power of $X_n$, write $f_1 X_n = h_1 X_n^{s_1} + h_2 X_n^{s_2} + \cdots + h_t X_n^{s_t}, s_1 > s_2 > \cdots > s_t$, where $h_1 \neq 0$ and none of the $h_i$'s has an $X_n$-term. ($t$ could be 1.) For $1 \leq k \leq t$ we will agree that $h_k X_n^{s_k}$ has "$X_n$-degree" equal to $s_k$. Since $q$ is not a root of unity, it follows from (3) that $[f_k X_n^{s_k}, X_{-n}]$ is nonzero and has $X_n$-degree $(s_k - 1)$. Thus $[f_1 X_n, X_{-n}]$ has a nonzero term of $X_n$-degree $(s_1 - 1)$ and by maximality of $s_1$, no term can cancel it. Thus $[f_1 X_n, X_{-n}] \neq 0$ as required.

(c) If $k \in \tilde{l}\mathbb{Z}$ then $q^{4k} = 1$. So, from (3), we get that $X_k$ is in $\mathcal{Z}$. So the displayed set is in $\mathcal{Z}$. We now proceed as in the second paragraph of (b) with the same notation. Let $f \in \mathcal{U}_q(\mathcal{A}) - \mathcal{Z}$. So one of the terms in $f$ ends in $X_n$ where $n$ is not in $\tilde{l}\mathbb{Z}$. Since $q$ is a primitive $l$-th root of unity, this implies that $q^{2n} - q^{-2n} \neq 0$. We continue just as in (b) to deduce that $f$ is not in the centre of the algebra. ∎

As a consequence of Proposition 1.2 we get

COROLLARY 1.3. *(a) If $q$ is a primitive fourth root of unity, then $\mathcal{U}_q(\mathcal{A})$ is a commutative algebra.*

*(b) If $q_1$ is not a root of unity and $q_2$ is a root of unity, then $\mathcal{U}_{q_1}(\mathcal{A})$, and $\mathcal{U}_{q_2}(\mathcal{A})$ are nonisomorphic algebras.*

*(c) If $q_1$ is a fourth root of unity and $q_2$ is not a primitive fourth root of unity, then $\mathcal{U}_{q_1}(\mathcal{A})$, and $\mathcal{U}_{q_2}(\mathcal{A})$ are nonisomorphic algebras.*

PROOF. (a) follows from Proposition 1.2(c) and Equation (3).

(b) The two algebras have nonisomorphic centres.

(c) If $q_2$ is not a primitive fourth root of unity, then $X_1$ is not in the centre of the algebra by Proposition 1.2(c). So (c) follows from Corollary 1.3(a). ∎

We now continue from where we left off in Proposition 1.1. Since $S := \mathbb{C}[\gamma]-\{0\}$, the set of nonzero complex polynomials in $\gamma$, is a central multiplicative subset of $\mathcal{U}_q(\mathcal{A})$, we may localize the algebra at $S$, *i.e.*; we invert elements of $S$, and then regard the resulting algebra as a $\mathbb{C}(\gamma)$-algebra, where $\mathbb{C}(\gamma)$ is the field of rational functions. We still denote this localization of $\mathcal{U}_q(\mathcal{A})$ at $S$ by $\mathcal{U}_q(\mathcal{A})$. We then get

PROPOSITION 1.4. *If $q_1$ is a primitive m-th root of unity and $q_2$ is a primitive n-th root of unity then $\mathcal{U}_{q_1}(\mathcal{A})$, and $\mathcal{U}_{q_2}(\mathcal{A})$ are isomorphic $\mathbb{C}(\gamma)$-algebras.*

PROOF. Let

$$S_i = \{k \in \mathbb{Z} - \{0\} : q_i^{4k} = 1\}, \quad i = 1, 2.$$

We observe that for a fixed $i, k \in S_i \leftrightarrow -k \in S_i$. Since $S_1$ (respectively $S_2$) is the set of integral multiples of $\tilde{m}$ (respectively $\tilde{n}$) it follows that $S_i$ and its complement, $C_i$, in $Z - \{0\}$ are countably infinite.

Now $S_i = S_i^+ \cup S_i^-$, where $\pm$ stand for the positive and negative elements of $S_i$ respectively. We have already noted that

$$S_i^- = -S_i^+. \tag{4}$$

Using an analogous notation for $C_i$ we get from (4) that

$$C_i^- = -C_i^+.$$

Let $s^+ : S_1^+ \longrightarrow S_2^+$ be a bijective map. We note that $-s^+$ gives a bijection between $S_1^-$ and $S_2^-$ In this way we get a bijection, $s$, between $S_1$ and $S_2$. We obtain a bijection, $c$, between $C_1$ and $C_2$ using $c^+$ and $-c^+$, where $c^+$ is a bijection between $C_1^+$ and $C_2^+$.

As usual we are going to define the isomorphism $\phi$: $\mathcal{U}_{q_1}(\mathcal{A}) \longrightarrow \mathcal{U}_{q_2}(\mathcal{A})$ by specifying the images of the symbols in (2). In this case, $\gamma^n \longmapsto \gamma^n$, $n \in \mathbb{Z}$; $X_k \longmapsto X_{s(k)}$, if $k \in S_1$. The definition of $S_i$ then tells us that for $n, m$ in $S_1$ (3) is satisfied. Suppose $k \in C_1$. Then we define $X_k \longmapsto X_{c(k)}$, if $k < 0$. (So, $X_{c(k)} = X_{-c(-k)}$.)

If $k > 0$, we want to map $X_k$ to $\alpha_k X_{c(k)}$, where $\alpha_k$ is chosen in such a way that (3) is satisfied. To that end let

$$a_k := c(k) t_2^2 (q_1^{2k} - q_1^{-2k})(\gamma^k - \gamma^{-k})$$
$$b_k := k t_1^2 (q_2^{2c(k)} - q_2^{-2c(k)})(\gamma^{c(k)} - \gamma^{-c(k)}).$$

Since $k$ and $c(k)$ are in $C_1, C_2$ respectively, $a_k b_k \neq 0$.

Put $\alpha_k := a_k / b_k$.

This guarantees that (3) is satisfied. ∎

REMARK. We do not know if all the algebras in family (iii) of the introduction are isomorphic.

2. **The canonical module $M_q$ and the category $C_q$.** The algebra $\mathcal{U}_q(\mathcal{A})$ is $\mathbb{Z}$-graded in such a way that the degree of the monomial $\prod_{i=1}^{n} X_{k_i}$ is $\sum_{i=1}^{n} k_i$. We assign elements in the subalgebra generated by $\gamma$ and $\gamma^{-1}$ a degree of zero. Every $\mathcal{U}_q(\mathcal{A})$-module that we consider is $\mathbb{Z}$-graded with a grading that is compatible with the grading on $\mathcal{U}_q(\mathcal{A})$.

We now define the 'positive' and 'negative' components of the algebra, $\mathcal{U}_q(\mathcal{A})$. To that end, for $q$ not a root of unity let

$$\mathcal{A}_q^- = \{X_k : k < 0\} \tag{5}$$

while $\mathcal{B}_q = \mathcal{U}_q(\mathcal{A})^+$ is the subalgebra of $\mathcal{U}_q(\mathcal{A})$ generated by

$$\{X_k : k > 0\} \cup \{\gamma, \gamma^{-1}\}. \tag{6}$$

If $q$ is a primitive $n$-th root of unity, let

$$\mathcal{A}_q^- = \{X_k : k < 0, k \notin \tilde{n}\mathbb{Z}\} \tag{7}$$

while $\mathcal{B}_q = \mathcal{U}_q(\mathcal{A})^+$ is the subalgebra of $\mathcal{U}_q(\mathcal{A})$ generated by

$$\{X_k : k > 0\} \cup \{X_k : k < 0, k \in \tilde{n}\mathbb{Z}\} \cup \{\gamma, \gamma^{-1}\}. \tag{8}$$

Denote by $\mathcal{U}_q(\mathcal{A})^-$ the subalgebras of $\mathcal{U}_q(\mathcal{A})$, generated by the respective complements of (6) and (8) in (2); *i.e.*; (5) and (7). Both $\mathcal{U}(\mathcal{A})_q^-$ and $\mathcal{U}(\mathcal{A})_q^+$ are commutative algebras by (3). Therefore,

$$\mathcal{U}_q(\mathcal{A})^- \cong S(\mathcal{A}_q^-) \tag{9}$$

as $\mathbb{C}$-vector spaces, where $S$ denotes the symmetric algebra; which in this case is the commutative algebra of polynomial rings in the variables in $\mathcal{A}_q^-$.

Let $f = f(\mathbf{X}) \in \mathcal{U}_q(\mathcal{A})$. Since the right hand side of (3) is central, successive applications of (3) to $X_m X_n$ and the linearity of $\otimes$ allow us to write $f$ as a sum of terms of the form $f_- f_+$, where $f_+ \in \mathcal{B}_q$ and $f_- \in \mathcal{U}_q(\mathcal{A})^-$. Therefore we have the following Poincaré-Birkhoff-Witt decomposition:

$$\mathcal{U}_q(\mathcal{A}) = \mathcal{U}_q(\mathcal{A})^- \, \mathcal{U}_q(\mathcal{A})^+. \tag{10}$$

Let $\mathbb{C}v_+$ be the one-dimensional vector space spanned by the symbol $v_+$. We make $\mathbb{C}v_+$ a $\mathcal{B}_q$-module by setting

$$X_k v_+ = 0, \quad k > 0, \; \gamma v_+ = q v_+, \; \gamma^{-1} v_+ = q^{-1} v_+. \tag{11}$$

If $q$ is a primitive $n$-th root of unity we require in addition to (11) that

$$X_k v_+ = 0, \quad k < 0, \; k \in \tilde{n}\mathbb{Z} \tag{12}$$

Here comes our canonical irreducible module, $M_q$.

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Let $M_q = \mathcal{U}_q(\mathcal{A}) \otimes_{\mathcal{B}_q} \mathbb{C}v_+$. We get from (10), (11), and (12) the following isomorphism of $\mathbb{C}$-spaces.

$$M = M_q \cong S(\mathcal{A}_q^-). \tag{13}$$

If one transfers the $\mathcal{U}_q(\mathcal{A})$-module structure of $M_q$ to $S(\mathcal{A}_q^-)$, one sees from (3), (11), and (12) that $X_k$ acts on $M_q$ as multiplication by $X_k$ for $k < 0$, (if $q$ is a primitive $n$-th root of unity and $k < 0$ and $k \in \tilde{n}Z$ then $X_k$ is not in $M_q$ under the identification in (13) and $X_k$ acts on $M_q$ as multiplication by 0.) When $k > 0$, $X_k$ acts on $M_q$ as the unique derivation, $\delta_{X_k}$,

$$\delta_{X_k}(X_m) = \big(([2m]_q[m]_q)/m\big)\delta_{m+k,0} \tag{14}$$

where $\delta_{m+k,0}$ is Kronecker delta. The elements $\gamma, \gamma^{-1}$ act as multiplications by $q$ and $q^{-1}$ respectively on $M_q$.

In the proof of Theorem 2.4, the expression $([2m]_q[m]_q)/m$ is denoted by $q_{(m)}$. Thus for $l > 0$, $X_l$ acts as $q_{(l)}\frac{\partial}{\partial X_{-l}}$.

The proof of Proposition 2.1 is along the lines of that of Lemma 2.1 of [4].

PROPOSITION 2.1. *$M_q$ is an irreducible $\mathcal{U}_q(\mathcal{A})$-module.*

PROOF. By (13) we may consider $M_q$ as a polynomial ring. Let $f$ be any nonzero element of $M_q$. We have to show that the cyclic submodule $\mathcal{U}_q f$ is $M_q$. If $X_{-l}$, $l > 0$, is a term in $f$, then $[X_l, X_{-l}] \neq 0$, by (5), (7), (13), and (3). Therefore by differentiating $f$ using appropriate choices of $k$ in (14) we get that $\mathcal{U}_q f$ contains a nonzero constant. Using left multiplication by $X_{-k}$, $k > 0$, we get that $\mathcal{U}_q f = M_q$ as required. ∎

REMARK 2.2. If we had used (5) instead of (7) when $q$ is a primitive $n$-th root of unity the resulting $M = M_q$—the polynomial ring in the variables in $\mathcal{A}_q^-$ would not be an irreducible $\mathcal{U}_q$-module. Moreover, $M_q$ is not completely reducible and it has a maximal submodule $R$ that is not completely reducible.

PROOF. The complement of (7) in (5), *i.e.*; $\{X_k : k < 0, k \in \tilde{n}\mathbb{Z}\}$ generates a maximal submodule, $R$, of $M_q$: Suppose $g \in M_q$ but not in $R$. We want to show that the submodule of $M_q$ generated by $R + g$ is $M_q$. Every summand of $g$ with a term in $\{X_k : k < 0, k \in \tilde{n}\mathbb{Z}\}$ is in $R$. By subtracting such summands from $g$, we may assume that if $X_{-l}$, $l > 0$, is a term in $g$ then $[X_l, X_{-l}] \neq 0$ because $q_{(l)} \neq 0$. Then just as in the proof of Proposition 2.1, we get that the submodule of $M_q$ generated by $R + g$ is $M_q$.

Suppose $M_q$ is completely reducible. Then the submodule $R$ would also be completely reducible. In particular it would be a direct sum of cyclic submodules that are irreducible. However, if $f \in R, f \neq 0$, then it has a term $X_{-l}$, $l > 0$, $l \in \tilde{n}\mathbb{Z} - \{0\}$. By noting that $\mathcal{U}_q f \neq \mathcal{U}_q X_{-l} f$, we see that no nonzero cyclic submodule of $R$ is irreducible. ∎

A consequence of Remark 2.2 and Proposition 2.3 below is that $M/R$ is the unique (up to isomorphism) irreducible building block for a Stone-Von Neumann theorem. We constructed this irreducible module directly from the start by including the additional central elements of $\mathcal{U}_q(\mathcal{A})$ in the "$q$-Borel" subalgebra $\mathcal{B}_q$ because of the convenience of working with polynomial rings.

Let us return to $M_q$ as given in (13). We make $M_q$ a $\mathbb{Z}$-graded $\mathcal{U}_q(\mathcal{A})$-module with trivial positive component by setting the degree of nonzero constants to be 0, and the degree of a nonzero monomial $\prod_{i=1}^n X_{k_i}$ is $\sum_{i=1}^n k_i$.

We now introduce a category of $\mathcal{U}_q(\mathcal{A})$-modules, denoted by $C_q$, and defined as follows. A $\mathbb{Z}$-graded $\mathcal{U}_q(\mathcal{A})$-module $V = \bigoplus_{n\in\mathbb{Z}} V_n$ belongs to $C_q$ if

(i) $\gamma, \gamma^{-1}$ act as multiplication by $q$ and $q^{-1}$ respectively

(ii) There exists $N \in Z$ such that $V_n = 0$ for every $n > N$.

If $q$ is a primitive $n$-th root of unity we require, in addition to (i) and (ii),

(iii) $X_k v = 0$ for every $v \in V$ and every $k < 0$, $k \in \tilde{n}\mathbb{Z}$.

DEFINITION. A nonzero element, $v$, in a graded $\mathcal{U}_q(\mathcal{A})$-module $V$, is called a *vacuum vector* if $X_k v = 0$ for every positive integer $k$. The *vacuum space*, $\Omega_V$, of $V$ is the subspace of $V$ consisting of its vacuum vectors and 0.

We observe that if $N \in C_q$ then $\Omega_N$ is $\mathbb{Z}$-graded and contains a nonzero vacuum vector. From (13) we get that $\Omega_{M_q}$ is a one-dimensional vector space spanned by the constant polynomial 1.

PROPOSITION 2.3. *Every nonzero module $N$ in $C_q$ contains a submodule isomorphic to $M_q$. In particular $M_q$ is the unique (up to isomorphism) irreducible module in $C_q$.*

PROOF. Let $v$ be a vacuum vector in $N$. The map from $M_q$ to $N$ that takes $x$ to $xv$, $x \in S(\mathcal{A}_q^-)$ is readily checked to be a $\mathcal{U}_q(\mathcal{A})$-module map. Since $M_q$ is irreducible, the map is a module-isomorphism between $M_q$ and $\mathcal{U}_q(\mathcal{A})v$. The latter module is $N$ when $N$ is irreducible. ∎

Let $V$ be a module in $C_q$. As a vector space $\Omega_V = \bigoplus \mathbb{C}w_\nu$, $\nu$ ranging over some indexing set $J$. Let $\operatorname{Ind}\Omega_V = \mathcal{U}_q \otimes_{\mathcal{B}_q} (\bigoplus \mathbb{C}w_\nu)$. This is a $\mathcal{U}_q$-module in $C_q$. From the properties of $\otimes$, we get that $\Omega_{\operatorname{Ind}\Omega_V} = \Omega_{\operatorname{Ind}(\bigoplus \mathbb{C}w_\nu)}$. By Proposition 2.3, the latter module is isomorphic to $\Omega_{\bigoplus_J M_q} = \bigoplus_J \Omega_{M_q} = \bigoplus_J \mathbb{C}.1 = \Omega_V$. The equalities are actually natural identifications. Summarising, we have

$$\Omega_{\operatorname{Ind}\Omega_V} = \Omega_V. \tag{15}$$

In the proof of the next theorem we shall use (15) and the notation preceeding (15). The proof relies only on the fact that $M_q$ is a polynomial ring and so (13) permits a uniform proof for both $q$ a root of unity and $q$ not a root of unity.

THEOREM 2.4. *Every $\mathcal{U}_q(\mathcal{A})$-module $V$ in $C_q$ is a direct sum of copies of $M_q$.*

PROOF [3, THEOREM 1.7.3]. The prescription $u \otimes v \longmapsto uv$ defines a homomorphism $f$ from $\mathcal{U}_q \otimes_{\mathcal{B}_q} \Omega_V$ to $V$. Now, $\Omega_V$ embeds in $\operatorname{Ind}\Omega_V$ via $1 \otimes v$. Therefore, $f$ is injective on $\Omega_V$. Since every graded submodule of $\operatorname{Ind}\Omega_V$ is also in $C_q$, the kernel of $f$, if nontrivial, contains a vacuum vector, $x$. By (15), $x \in \Omega_V$. So the kernel of $f$ is 0.

If $V/\operatorname{Im} f \neq 0$, then it too contains a vacuum vector $w$ (say). So if $v$ is a representative of $w$ in $V$ we have that $v$ is not in $\operatorname{Im} f$, $X_i v \in \operatorname{Im} f$ for every positive integer $i$; there exists an integer $i_0$ such that $X_i v = 0$ for all $i > i_0$. The last statement follows from the fact that for $i$ large enough $X_i v$ is in $V_{N+1} = 0$. Since $f$ is injective, $\Omega_V \subseteq \mathrm{Im} f \subseteq V$. If we can find $t \in \mathrm{Im} f$ such that $X_i(t - v) = 0$ for all $i > 0$, then $t - v$ would be a vacuum vector in $V - \Omega_V$, a contradiction.

Since $f$ is injective we may assume that

$$\mathrm{Im} f = \bigoplus_J (\mathcal{U}_q \otimes_{\mathcal{B}_q} \mathbb{C} w_\nu). \tag{16}$$

So $\mathrm{Im} f$ satisfies the conclusion of the theorem. For each positive integer $i$ and $\nu \in J$ let $s_{i_\nu}$ be the component of $X_i v$ in $\mathrm{Ind}\, \mathbb{C} w_\nu$ with respect to the decomposition (16).

By (3), $[X_i, X_j] = 0$ for all positive integers $i, j$. Therefore, $X_i X_j v = X_j X_i v$. So for all $\nu \in J$, we have

$$X_i s_{j_\nu} = X_j s_{i_\nu}. \tag{17}$$

and $s_{i_\nu} = 0$ for all $i > i_0$. Since $X_1 v, \ldots, X_{i_0} v$ have nonzero component only in a finite subset of $J$ and $X_i v = 0$ for $i > i_0$ we conclude that there is a finite subset $J_0$ such that $s_{i_\nu} = 0$ unless $\nu \in J_0$ and $i \leq i_0$. We need only find $t_\nu \in \mathrm{Ind}\, \mathbb{C} w_\nu$ with $X_i t_\nu = s_{i_\nu}$ for $\nu \in J_0$. Then $t = \sum_{\nu \in J_0} t_\nu$ will have the property that $X_i(t - v) = 0$ for every positive integer $i$.

Fix $\nu \in J_0$ and, as in (13), consider $\mathrm{Ind}\, \mathbb{C} w_\nu$ as a polynomial ring. Recall from (14) that $X_i$ acts as $q_{(i)} \frac{\partial}{\partial X_{-i}}$. So (17) becomes

$$q_{(i)} \frac{\partial}{\partial X_{-i}} s_{j_\nu} = q_{(j)} \frac{\partial}{\partial X_{-j}} s_{i_\nu}. \tag{18}$$

Since $s_{i_\nu} = 0$ for $i > i_0$ we see that (18) is an equation in the polynomial ring, $S$, in only finitely many variables $X_{-i}$, $0 < i \leq i_0$. In order to invoke the result on conservative vector fields we make the change of variable: $X_i$ to $\frac{X_i}{q_{(i)}}$ for all $i > 0$. This is permissible since $q_{(i)} \neq 0$. So there exists $t_\nu \in S$ such that

$$X_i t_\nu = s_{i_\nu}$$

for $0 < i \leq i_0$, and hence for all positive integers $i$. ∎

CONCLUDING REMARKS. The fact that the Stone-Von Neumann theorem generalizes nicely to quantum Heisenberg algebras leads us to wonder whether other interesting results in Lie algebra theory also generalize. As an example we point that although there have been successful attempts at constructing quantum Virasoro algebras, we are not aware of any that admit oscillator representations, *i.e.*; representations that arise from Segal or Sugawara operators involving quantum Heisenberg algebras.

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