Bull. Aust. Math. Soc. **108** (2023), 504–510
doi:10.1017/S0004972723000047

ERDŐS PROPERTIES OF SUBSETS OF THE MAHLER SET S

TABOKA PRINCE CHALEBGWA and SIDNEY A. MORRIS

(Received 1 December 2022; accepted 30 December 2022; first published online 27 February 2023)

Abstract

Erdős proved that every real number is the sum of two Liouville numbers. A set W of complex numbers is said to have the Erdős property if every real number is the sum of two members of W. Mahler divided the set of all transcendental numbers into three disjoint classes S, T and U such that, in particular, any two complex numbers which are algebraically dependent lie in the same class. The set of Liouville numbers is a proper subset of the set U and has Lebesgue measure zero. It is proved here, using a theorem of Weil on locally compact groups, that if $m \in [0, \infty)$, then there exist $2^{\mathfrak{c}}$ dense subsets W of S each of Lebesgue measure m such that W has the Erdős property and no two of these W are homeomorphic. It is also proved that there are $2^{\mathfrak{c}}$ dense subsets W of S each of full Lebesgue measure, which have the Erdős property. Finally, it is proved that there are $2^{\mathfrak{c}}$ dense subsets W of S such that every complex number is the sum of two members of W and such that no two of these W are homeomorphic.

2020 *Mathematics subject classification*: primary 54C10; secondary 11J81, 28C15.

Keywords and phrases: Lebesgue measure, transcendental number, Mahler classification, S-number, Liouville number, Erdős properties.

1. Preliminaries

In 1844, Joseph Liouville proved the existence of transcendental numbers [1, 2]. He introduced the set $\mathcal{L}$ of real numbers, now known as Liouville numbers, and showed that they are all transcendental. A real number x is said to be a *Liouville number* if for every positive integer n, there exists a pair of integers (p, q) with $q > 1$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

In [8], in 1962, Paul Erdős proved that every real number is the sum of two Liouville numbers (and also, if nonzero, is the product of two Liouville numbers). He gave two proofs. One was a constructive proof. The other proof used the fact that the set $\mathcal{L}$ of all Liouville numbers is a dense G_δ-set in $\mathbb{R}$ and showed that every dense G_δ-set in $\mathbb{R}$ has this property.

The first author's research is supported by the Fields Institute for Research in Mathematical Sciences, via the Fields-Ontario postdoctoral Fellowship.

DEFINITION 1.1. A subset W of the set $\mathbb{C}$ of all complex numbers is said to have the *Erdős property* [7] if every real number is a sum of two numbers in W. A subset V of the set $\mathbb{C}$ of all complex numbers is said to have the *multiplicative Erdős property* if every positive real number is a product of two numbers in V.

By the theorem proved by Erdős mentioned above, the set $\mathcal{L}$ of all Liouville numbers has the Erdős property and the multiplicative Erdős property.

The classification of Mahler partitions the set $\mathbb{C}$ of all complex numbers into four sets denoted by A, S, T and U, characterised by the rate with which a nonzero polynomial with integer coefficients approaches zero when evaluated at a particular number. We follow the presentation in [4, Section 3]. While the definitions and results therein are stated and proved for real numbers, they carry over to the case of complex numbers.

Given a polynomial $P(X) \in \mathbb{C}[X]$, recall that the height of P, denoted by $H(P)$, is the maximum of the absolute values of the coefficients of P. Given a complex number ξ, a positive integer n, and a real number $H \geq 1$, we define the quantity

$$w_n(\xi, H) = \min\{|P(\xi)| : P(X) \in \mathbb{Z}[X],\ H(P) \leq H,\ \deg(P) \leq n,\ P(\xi) \neq 0\}.$$

Furthermore, we set

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log w_n(\xi, H)}{\log H}$$

and

$$w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

DEFINITION 1.2. Let ξ be a complex number. Then ξ is said to be

(i) an A-number if $w(\xi) = 0$;
(ii) an S-number if $0 < w(\xi) < \infty$;
(iii) a T-number if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any $n \geq 1$;
(iv) a U-number if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for all $n \geq n_0$, for some positive integer n_0.

REMARK 1.3. Note the following from [2, 4, 10, 11].

(i) The A-numbers are the algebraic numbers.
(ii) Each Liouville number is a real U-number.
(iii) If the complex numbers α and β are algebraically dependent then they are in the same Mahler class; for example, the numbers $\alpha, -\alpha, n\alpha$ are all in the same Mahler set, for $n \in \mathbb{N}$.
(iv) The two-dimensional Lebesgue measure of each of the sets A, U and T is zero; S has full two-dimensional Lebesgue measure, that is, its complement in $\mathbb{C}$ has zero two-dimensional Lebesgue measure.

https://doi.org/10.1017/S0004972723000047 Published online by Cambridge University Press

(v) The one-dimensional Lebesgue measure of each of the sets $A \cap \mathbb{R}$, $U \cap \mathbb{R}$, $\mathcal{L}$ and $T \cap \mathbb{R}$ is zero; $S \cap \mathbb{R}$ has full one-dimensional Lebesgue measure, that is, its complement in $\mathbb{R}$ has zero one-dimensional Lebesgue measure.
(vi) If a is an algebraic number with $a \neq 0$, then $\exp(a) \in S$.

André Weil (see [9, 16]) proved the following generalisation of the Steinhaus theorem [14].

THEOREM 1.4. *If G is a locally compact Hausdorff group and B is a subset of G of positive (left) Haar measure, then $BB^{-1} = \{b_1 b_2^{-1} : b_1, b_2 \in B\}$ contains an open neighbourhood of the identity.*

COROLLARY 1.5 (See [3, 13]).

(i) *If B is a subset of $\mathbb{R}$ of positive one-dimensional Lebesgue measure, then the set $B - B = \{b_1 - b_2 : b_1, b_2 \in B\}$ contains an open interval containing 0.*
(ii) *If D is a subset of $\mathbb{C}$ of positive two-dimensional Lebesgue measure, then the set $D - D = \{d_1 - d_2 : d_1, d_2 \in D\}$ contains an open disc with centre 0.*
(iii) *If E is a subset of the multiplicative group $\mathbb{C}^*$ of all nonzero complex numbers, where E has positive two-dimensional Lebesgue measure, then the set $EE^{-1} = \{e_1 \cdot e_2^{-1} : e_1, e_2 \in E\}$ contains an open disc with centre 1.*
(iv) *If F is a subset of the multiplicative group $\mathbb{R}^{>0}$ of all positive real numbers, then the set $FF^{-1} = \{f_1 \cdot f_2^{-1} : f_1, f_2 \in F\}$ contains an open interval containing 1.*

The following theorem was proved by Mahler [10]. (See [4, Section 3.5].)

THEOREM 1.6. *If a is an algebraic number with $a \neq 0$, then $\exp(a) \in S$.*

REMARK 1.7. It is an immediate consequence of Theorem 1.6 and Definition 1.2 that the number e and $\exp(a)$, for a any nonzero algebraic number, are not Liouville numbers.

As the set of nonzero algebraic numbers is a dense subset of $\mathbb{C}$ and exp is a continuous mapping from $\mathbb{C}$ onto $\mathbb{C} \setminus \{0\}$, we obtain the following corollary.

COROLLARY 1.8. *If $\mathbb{A}'$ is the set of all nonzero algebraic numbers, then $B = \exp(\mathbb{A}')$ is a countably infinite dense subset of S (and also a countably infinite dense subset of $\mathbb{C}$) and B has two-dimensional Lebesgue measure equal to zero.*

COROLLARY 1.9. *The set S is dense in the set of all transcendental numbers and in $\mathbb{C}$.*

2. The main results

DEFINITION 2.1. A subset W of $\mathbb{C}$ is said to have the *complex Erdős property* if every complex number is a sum of two members of W. A subset V of $\mathbb{C}$ is said to have the *complex multiplicative Erdős property* if every nonzero complex number is a product of two members of V.

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REMARK 2.2. Let $A \subseteq B \subseteq \mathbb{C}$ and $C \subseteq D \subseteq \mathbb{C}$. If A has the Erdős property, then so does B. If $B \subseteq \mathbb{R}$, the set of all real numbers, and A has the multiplicative Erdős property, then so does B. If C has the complex Erdős property, then so does D. If C has the complex multiplicative Erdős property, then so does D.

PROPOSITION 2.3

(i) *Let E be a subset of $\mathbb{C}$ of positive two-dimensional Lebesgue measure such that if $x \in E$, then $-x \in E$ and $nx \in E$, for all $n \in \mathbb{N}$. Then E has the complex Erdős property.*

(ii) *Let F be a subset of $\mathbb{R}$ of positive one-dimensional Lebesgue measure such that if $x \in F$, then $-x \in F$ and $nx \in F$, for all $n \in \mathbb{N}$. Then F has the Erdős property.*

(iii) *If G is a subset of $\mathbb{C}^*$ with positive two-dimensional Lebesgue measure such that if $x \in G$, then $x^{-1} \in g$ and $x^n \in G$, for all $n \in \mathbb{N}$, then G has the complex multiplicative Erdős property.*

(iv) *If H is a subset of $\mathbb{R}^{>0}$ with positive one-dimensional Lebesgue measure such that if $x \in F$ then $x^{-1} \in F$ and $x^n \in F$, for all $n \in \mathbb{N}$, then H has the multiplicative Erdős property.*

PROOF. By Corollary 1.5, $F - F$ contains a nonempty open interval I containing 0 in $\mathbb{R}$ and $E - E$ contains a nonempty open disc D with centre 0 in $\mathbb{C}$. As $E = -E$ and $F = -F$,

$$E + E = \bigcup_{n\in\mathbb{N}} n(E + E) = \bigcup_{n\in\mathbb{N}} n(E - E) = \bigcup_{n\in\mathbb{N}} nD = \mathbb{C} \quad \text{and}$$

$$F + F = \bigcup_{n\in\mathbb{N}} n(F + F) = \bigcup_{n\in\mathbb{N}} n(F - F) = \bigcup_{n\in\mathbb{N}} nI = \mathbb{R}.$$

So items (i) and (ii) are proved.

The proofs of items (iii) and (iv) for the multiplicative Erdős cases are analogous. □

The next result follows from Proposition 2.3 and Remark 1.3(iii), (iv) and (v).

COROLLARY 2.4. *The real Mahler set $S \cap \mathbb{R}$ has the Erdős property and the multiplicative Erdős property and the complex Mahler set S has the complex Erdős property and the complex multiplicative Erdős property.*

REMARK 2.5. In contradistinction with the result of Erdős that the set $\mathcal{L}$ of zero Lebesgue measure has the Erdős property and the results in [7] that certain subsets of Lebesgue measure zero of the set $\mathcal{L}$ have the Erdős property, we have in Corollary 2.4 and Theorem 2.6 sets of all Lebesgue measures having the Erdős property.

THEOREM 2.6. *Let $m \in [0, \infty)$. Then there exist $2^{\mathfrak{c}}$ dense subsets W of S each of two-dimensional Lebesgue measure m such that W has the Erdős property and no two of these W are homeomorphic. There also exist $2^{\mathfrak{c}}$ dense subsets of S which have full two-dimensional Lebesgue measure, and no two of these are homeomorphic.*

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PROOF. First consider the case $m \in (0, \infty)$. Let W be the set $(S \setminus [-1, 1]) \cap \mathbb{R}$. Then W is a set of positive one-dimensional Lebesgue measure and two-dimensional Lebesgue measure 0. By Proposition 2.3(ii), W has the Erdős property. By Remark 2.2, any set which contains W has the Erdős property. Let V_m be the intersection of S with any square in $\mathbb{C}$ of side $\sqrt{m}$, X any subset of $[-1, 1] \cap S$ and B the subset of S defined in Corollary 1.8. Then the set $V_m^* = W \cup V_m \cup X \cup B$ is a set of two-dimensional Lebesgue measure m which has the Erdős property. As $S \cap [-1, 1]$ has cardinality $\mathfrak{c}$, there are $2^{\mathfrak{c}}$ choices for X each resulting in a different set V_m^*. So for each $m \in (0, \infty)$, there are $2^{\mathfrak{c}}$ subsets of S of two-dimensional Lebesgue measure m and having the Erdős property. Each of these sets contains the set B, and so, by Corollary 1.8, is dense in S.

Next, consider the case $m = 0$. We simply choose $V_0 = \emptyset$ in the above proof.

Finally, consider the case of full two-dimensional Lebesgue measure. This time, we put $V_\infty = \mathbb{C} \setminus \mathbb{R}$ instead of V_m in our proof and we obtain the desired result.

In each of the three cases above, we use the Lavrentieff theorem [15, Theorem A8.5], which says that there are at most $\mathfrak{c}$ subspaces of $\mathbb{C}$ which are homeomorphic. Thus, there exist $2^{\mathfrak{c}}$ subsets of S of measure $m \in [0, \infty)$ which have the Erdős property, no two of which are homeomorphic, and $2^{\mathfrak{c}}$ subsets of S of full two-dimensional Lebesgue measure which have the Erdős property, no two of which are homeomorphic. □

THEOREM 2.7 (See [6, 12]). *There exist $2^{\mathfrak{c}}$ dense subsets W of S such that W has the complex Erdős property and no two of these W are homeomorphic.*

PROOF. Let D be a closed disc in $\mathbb{C}$ of radius one with centre 0. Put $W = S \setminus D$. By Proposition 2.3, W has the complex Erdős property. Therefore, any set which contains W has the complex Erdős property. Let X be any subset of $D \cap S$ and let B be as in Corollary 1.8. Then $X \cup W \cup B$ has the complex Erdős property and is dense in S. As the cardinality of $D \cap S$ is $\mathfrak{c}$, there are $2^{\mathfrak{c}}$ sets $X \cup W \cup B$, each a subset of S and having the complex Erdős property. As in the proof of Theorem 2.6, the Lavrentieff theorem implies that these subsets can be chosen so that no two are homeomorphic. □

PROPOSITION 2.8. *Let $S' = \exp(S \cap \mathbb{R}) \cap S$. Then S' has full Lebesgue measure in the set $\mathbb{R}^{>0}$ of positive real numbers.*

PROOF. Note that exp maps $\mathbb{R}$ onto $\mathbb{R}^{>0}$ and exp is Lipschitz continuous on bounded sets. As $S \cap \mathbb{R}$ has full Lebesgue measure in $\mathbb{R}$, $\mathbb{R} \setminus S$ has Lebesgue measure zero. Therefore, $\exp(\mathbb{R} \setminus S)$ has Lebesgue measure zero. As the map exp is surjective from $\mathbb{R}$ to $\mathbb{R}^{>0}$ and $\exp(\mathbb{R} \setminus S)$ has zero Lebesgue measure, $\exp(S \cap \mathbb{R})$ has full measure in $\mathbb{R}^{>0}$. Noting that any subset of a set of Lebesgue measure zero has Lebesgue measure zero, we see that the set $\exp(S \cap \mathbb{R}) \cap (\mathbb{R}^{>0} \setminus S)$ has Lebesgue measure zero. So $S' = \exp(S \cap \mathbb{R}) \cap S$ has full Lebesgue measure in $\mathbb{R}^{>0}$. □

PROPOSITION 2.9. *The set $\exp(S \cap \mathbb{R})$ has the multiplicative Erdős property and the set $\exp(S)$ has the complex multiplicative Erdős property.*

PROOF. We shall apply Proposition 2.3(iv) to the set $S' = \exp(S \cap \mathbb{R}) \cap S$. First observe that exp maps $\mathbb{R}$ onto $\mathbb{R}^{>0}$. Let $x \in S'$. Then $x \in S$ and $x = \exp(s)$, for some $s \in S \cap \mathbb{R}$. Now for any $n \in \mathbb{N}$, $ns \in S \cap \mathbb{R}$ and so $\exp(ns) \in \exp(S \cap \mathbb{R})$. However, $\exp(ns) = (\exp(s))^n = x^n$. As $x \in S \cap \mathbb{R}$, $x^n \in S \cap \mathbb{R}$. Thus, $x^n \in S'$. Also, $x \in S$ implies $x^{-1} \in S$. Further, $x = \exp(s)$ implies $x^{-1} = \exp(-s) \in \exp(S)$. So by Proposition 2.3(iv) and Proposition 2.8, $\exp(S \cap \mathbb{R}) \cap S$ has the multiplicative Erdős property. Thus, $\exp(S \cap \mathbb{R})$ has the multiplicative Erdős property.

As exp maps $\mathbb{C}$ onto $\mathbb{C} \setminus \{0\}$, an analogous argument shows that $\exp(S)$ has the complex multiplicative Erdős property. □

PROPOSITION 2.10. *Let* $S'' = S \cap (0, \infty)$. *Then* $\log(S'') \cap S$ *has full Lebesgue measure in* $\mathbb{R}$.

PROOF. Clearly $(\mathbb{R} \setminus S) \cap (0, \infty)$ has zero Lebesgue measure. As log is Lipschitz continuous on all closed bounded subintervals of $(0, \infty)$, $\log((\mathbb{R} \setminus S) \cap (0, \infty))$ has zero Lebesgue measure. Thus, $\log S''$ has full Lebesgue measure in $\mathbb{R}$. As $\mathbb{R} \setminus S$ has zero Lebesgue measure, $\log(S'') \cap S$ has full Lebesgue measure in $\mathbb{R}$. □

PROPOSITION 2.11. *The sets* $\log(S'')$ *and* $(\log(S'')) \cap S$ *have the Erdős property.*

PROOF. Observe that if $x \in \log(S'') \cap S$, then $x = \log y$, $y \in S''$ and $x \in S$. So $nx = n$, $\log y = \log(y^n)$ and clearly $y^n \in S''$, for any $n \in \mathbb{N}$. So $nx \in \log(S'') \cap S$. Also, $-x = \log(1/y)$ and $1/y \in S \cap (0, \infty)$. So $-x \in \log(S'') \cap S$. By Propositions 2.3 and 2.10 and Remark 2.2, $\log(S'')$ and $(\log(S'')) \cap S$ have the Erdős property. □

REMARK 2.12. Let X be a subset of the set of positive real numbers and the set $Y = \{\log x : x \in X\} \cap X$ such that every positive real number is a product of two numbers in Y. If r is any positive real number strictly greater than 1, we put $t = \log(r)$. As t is a positive real number, $t = y_1 \cdot y_2$, where $y_1, y_2 \in Y$. So $t = y_1 \cdot \log x_2 = \log x_2^{y_1}$, $x_2 \in X$. So $e^t = x_2^{y_1}$; that is, $r = x_2^{y_1}$. Thus,

every positive real number strictly greater than 1 *equals* a^b,

for $b = y_1 \in Y \subseteq X$ and $a = x_2 \in X$.

This should be contrasted with the Gelfond–Schneider theorem [2] which says that *for any algebraic numbers a and b, with* $a \neq 0, 1$ *and b not a rational number,* a^b *is a transcendental number.*

The authors do not know whether any interesting set X with the property stated above exists. Of course, $X = (0, \infty)$ is an example. However, we observe that Burger [5] proves that for any positive real number r, there exist Liouville numbers a and b such that $r = a^b$.

Acknowledgement

The authors thank the referee for helpful comments.

https://doi.org/10.1017/S0004972723000047 Published online by Cambridge University Press

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TABOKA PRINCE CHALEBGWA,
The Fields Institute for Research in Mathematical Sciences, 222 College Street,
Toronto, Ontario MST 3J1, Canada
e-mail: taboka@aims.ac.za

SIDNEY A. MORRIS, School of Engineering, IT and Physical Sciences,
Federation University Australia, PO Box 663, Ballarat, Victoria 3353, Australia
and
Department of Mathematical and Physical Sciences,
La Trobe University, Melbourne, Victoria 3086, Australia
e-mail: Morris.sidney@gmail.com

https://doi.org/10.1017/S0004972723000047 Published online by Cambridge University Press