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Separability axioms and metric-like functions

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We call a topological space *completely regular* if points and closed sets can be separated by continuous real-valued functions, and we call a topological space *Tychonoff* or T_{3a} if it is T_1 and completely regular. It is well known that the Tychonoff spaces are precisely the gauge spaces, that is the topological spaces whose topologies are induced by separating families of pseudometrics. V.G. Boltjanskiĭ has given analogous characterizations for T_0, T_1, T_2, regular, and regular-T_1 spaces in terms of families of "metric-like" functions. The purpose of this note is to fill in a case missing in Boltjanskiĭ's paper, the completely regular spaces, and to relate Boltjanskiĭ's work to results on quasi-uniformization by William J. Pervin and results about semi-gauge spaces by J.V. Michalowicz.

1. Introduction

Since Boltjanskiĭ's paper [2] is couched in the language of topological semifields we begin with terminology and notation relating to these structures. (More details can be found in [1].)

DEFINITION 1. If Δ is a non-empty set then R^Δ, the set of real-valued functions on Δ, is called a *Tychonoff semifield* if:

(i) R^Δ is given the usual ring structure (pointwise operations);

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(ii) R^Δ is given coordinatewise partial order ($f \leq g$ iff $f(q) \leq g(q)$ for all q in Δ, where $f(q)$ is the q-th component of f);

(iii) R^Δ is given the Tychonoff topology.

REMARK. Because of (ii), the operations $f \vee g$ and $f \wedge g$ are carried out componentwise in R^Δ.

DEFINITION 2. If X is a non-empty set and R^Δ is a Tychonoff semifield then a *quasimetric* on X over R^Δ (or an R^Δ-*quasimetric* on X) is a mapping $\rho : X \times X \to R^\Delta$ such that:

1. $\rho(x, x) = 0$ for all x;
2. $\rho(x, y) \geq 0$ for all x, y;
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all x, y, z.

An R^Δ-quasimetric on X satisfying

4. $\rho(x, y) = \rho(y, x)$ for all x, y,

is called an R^Δ-*pseudometric* or R^Δ-*gauge* on X, and an R^Δ-pseudometric on X satisfying

5. $\rho(x, y) = 0$ implies $x = y$,

is called an R^Δ-*metric* on X.

In the event that Δ is a one-point set these definitions reduce to the ordinary definition of quasimetrics, pseudometrics (gauges), and metrics as defined in [7].

If ρ is an R^Δ-quasimetric (pseudometric) on X then for each q in Δ the mapping $\rho_q : X \times X \to R$ defined by

(1) $$\rho_q(x, y) = \rho(x, y)(q)$$

is a quasimetric (pseudometric) on X in the ordinary sense. Moreover, if ρ is an R^Δ-metric then $\{\rho_q : q \in \Delta\}$ is a family of pseudometrics which is *separating* in the sense that for each pair of distinct points x and y

in X , there exists an element q in Δ such that $\rho_q(x, y) \neq 0$. Conversely, if $\{\rho_q : q \in \Delta\}$ is a family of pseudometrics on X , then the mapping $\rho : X \times X \to R^\Delta$ defined by (1) is an R^Δ-pseudometric on X . Moreover, if the family $\{\rho_q : q \in \Delta\}$ is separating then ρ is an R^Δ-metric on X . Thus, every (separating) family of pseudometrics is equivalent to a *single* (metric) pseudometric over a suitable Tychonoff semifield.

REMARK. In [2] Boltjanskiĭ refers to a quasimetric over R^Δ as a semimetric over R^Δ .

Let $\rho : X \times X \to R^\Delta$ be an R^Δ-quasimetric. If $x \in X$ and if U is an arbitrary Tychonoff neighborhood of zero in R^Δ , then we denote by $\tilde{U}$ the set of all points (x, y) in $X \times X$ such that $\rho(x, y) \in U$ and we denote by $\tilde{U}(x)$ the set of all points y in X such that $\rho(x, y) \in U$. It is easy to verify that the family of sets $\tilde{U}$ is a base for a quasi-uniformity on X and that the family of sets $\tilde{U}(x)$ is a neighborhood base at x for the topology induced by this quasiuniformity. This topology is called the *natural topology* or the ρ-*quasimetric topology*.

REMARK. It is straightforward to verify that the quasiuniformity and topology generated by a single semifield quasimetric $\rho : X \times X \to R^\Delta$ are precisely the same as the quasiuniformity and topology induced in the usual way by the family of real-valued quasimetrics $\{\rho_q\}_{q \in \Delta}$, where $\rho_q(x, y) = \rho(x, y)(q)$.

2. Separability axioms and metrization

Consider the following two standard results.

THEOREM 1 ([6], [7]). *Every topological space is quasi-uniformizable.*

THEOREM 2 ([3], [7], [8]). *A topological space is* T_{3a} *if and only if it is a separated uniformizable space.*

Because every quasi-uniformity arises from a family of real-valued quasimetrics and because every family of real-valued quasimetrics is

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equivalent to a single quasimetric over a suitable Tychonoff semifield, Theorem 1 states that every topology arises from a quasi-metric over a suitable Tychonoff semifield. Thus Theorem 1 can be rephrased in the form:

THEOREM 1'. *Every topological space is quasi-metrizable over some Tychonoff semifield.*

Similarly, every separated uniformity arises from a separating family of real-valued pseudometrics and every separating family of real-valued pseudometrics is equivalent to a single metric over a suitable Tychonoff semifield. Thus Theorem 2 can be rephrased in the form:

THEOREM 2'. *A topological space is* T_{3a} *if and only if it is metrizable over some Tychonoff semifield.*

Theorems 1' and 2' suggest the possibility that for each separation condition T_i , $i = 0, 1, 2, 3, 3a$, there may exist a class Q_i of (semifield) quasimetrics yielding a "metrization theorem" of the form:

METRIZATION THEOREM 3. *A topological space has property* T *if and only if it is "metrizable" over some Tychonoff semifield by quasimetrics in class* Q .

Classes Q_i , $i = 0, 1, 2, 3, 3a$, were given by Boltjanskiĭ in [2]. In addition, he specified a class Q of quasimetrics yielding Metrization Theorem 3 for regular spaces. (For us, T_3 means regular plus T_1 .) In Table 1 below we have listed Boltjanskiĭ's results using his notation, 4°, 5°, 6°, 7°, 8°, and 9° , to designate the following axioms which are imposed on $\rho : X \times X \to R^\Delta$ in addition to the axioms for a quasimetric:

4°. $\rho(x, y) = 0$ implies $x = y$;

5°. $\rho(x, y) \vee \rho(y, x) = 0$ implies $x = y$;

6°. $\rho(x, y) \wedge \rho(y, x) = 0$ implies $x = y$;

7°. $\rho(x, z) + \rho(y, z) \geq \rho(x, y) \wedge \rho(y, x)$ for all x, y, z ;

8°. for any $x \in X$ and $q \in \Delta$, there exists an element $q^* \in \Delta$ such that

$$\rho(x, z)(q^*) + \rho(y, z)(q^*) \geq \rho(x, y)(q)$$

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for all y and z ;

9°. $\rho(x, y) = \rho(y, x)$ for all x, y .

Table 1

Class of topological spaces having property T	Axioms for a class Q of quasimetrics yielding Metrization Theorem 3
All topological spaces	No additional axioms
T_0-spaces	5°
T_1-spaces	6°
T_2-spaces	6°, 7°
regular spaces	8°
T_3-spaces	6°, 8°
T_{3a}-spaces	4°, 9°

In [2] our T_{3a} spaces are called completely regular, and the class of spaces we call completely regular was not considered. To fill this omission we will prove the following theorem.

THEOREM 4. *A topological space is completely regular if and only if it is pseudometrizable over some Tychonoff semifield.*

REMARK. This result adds to Table 1 the line:

Property T	Class Q
completely regular	9° .

Proof. Assume (X, τ) is a completely regular topological space and let Δ be the set of all continuous functions from X into $[0, 1]$. Define

$$\rho : X \times X \to R^{\Delta}$$

by

$$\rho(x, y)(q) = |q(x)-q(y)|$$

for all $q \in \Delta$. It is easy to check that ρ is a pseudometric over R^{Δ} . We now show that τ is the ρ-topology. For $q \in \Delta$ let π_q be the q-th projection map on R^{Δ} . Then $U = \pi_q^{-1}(-\varepsilon, \varepsilon)$ is a (Tychonoff) open neighborhood of zero in R^{Δ} and

$$\begin{aligned} U(x) &= \{y : \rho(x, y) \in U\} = \{y : \rho(x, y)(q) < \varepsilon\} \\ &= \{y : |q(x)-q(y)| < \varepsilon\} \\ &= q^{-1}(I) , \end{aligned}$$

where I is the open interval $(q(x)-\varepsilon, q(x)+\varepsilon)$. Thus $\tilde{U}(x)$ is τ-open since q is τ-continuous. Since U is an arbitrary subbasis neighborhood of zero in R^{Δ}, it follows that for any neighborhood V of zero in R^{Δ} , the set $\tilde{V}(x)$ will be τ-open. Thus the ρ-topology is weaker than the τ-topology. But if G is any τ-open set and $x \in G$ then by complete regularity there exists a function q in Δ such that $q(x) = 0$ and $q(G^c) = 1$. Thus

$$x \in \tilde{U}(x) \subset G$$

where

$$U = \pi_q^{-1}(-1, 1) .$$

But $\tilde{U}(x)$ is ρ-open so that G is ρ-open; thus the τ and ρ topologies are equivalent.

Conversely, if X is pseudometrizable over some Tychonoff semifield R^{Δ} , then its topology is induced by some pseudometric $\rho : X \times X \to R^{\Delta}$ or equivalently it is induced by the family of real-valued pseudometrics $\{\rho_q : q \in \Delta\}$, where $\rho_q(x, y) = \rho(x, y)(q)$. Thus X is uniformizable and consequently completely regular ([3], [7], [8]).

3. Concluding remarks

While our formulation of Theorem 4 is apparently new, the essential content of the theorem was studied in a different setting by Michalowicz [4], who proved that every completely regular space is a semi-gauge space. Thus this note serves two purposes. It remedies the omission of the completely regular case in [2] and at the same time shows that the isolated result of Michalowicz in [4] is part of the more general pattern of results

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obtained in [2] by Boltjanskiĭ.

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