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ON OVERTWISTED CONTACT SURGERIES

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Abstract

In this paper, we obtain a new result for overtwisted contact $(+1/n)$-surgery. We also give a counterexample to a conjecture by James Conway on overtwistedness of manifolds obtained by contact surgery.

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1. Introduction

Contact surgeries have long been an essential tool in the study of contact 3-manifolds. This paper is concerned with the behaviour of contact structures under contact $(+1/n)$- and contact $(+n)$-surgeries along Legendrian knots where the surgery slope is measured with respect to the contact framing of the Legendrian knot.

Let L be a null homologous oriented Legendrian knot in a closed contact 3-manifold (M, ξ) with cooriented contact structure ξ. For the remainder of the paper, let $\mathrm{tb}(L)$ denote the Thurston–Bennequin invariant of L, $\mathrm{rot}(L)$ the rotation number of L and $\chi(L)$ the Euler characteristic of any Seifert surface of L.

The goal of this paper is to study the conditions on L and the surgery coefficient r under which tightness is preserved or new overtwistedness is created. Wand's fundamental result states that all contact r-surgeries, for $r < 0$, preserve tightness [12]. On the other hand, a contact surgery on a tight contact 3-manifold with surgery coefficient $r > 0$ can yield tight or overtwisted contact structures. Contact $(+1/n)$-surgeries along stabilised Legendrian knots always yield overtwisted contact structures [11]; however, such a stabilisation restriction is not necessary in this paper.

The first result gives a condition under which contact $(+1/n)$-surgery along L results in an overtwisted manifold.

THEOREM 1.1. *Let L be a null homologous oriented Legendrian knot in a tight contact 3-manifold. If $\mathrm{tb}(L) < 0$, $n\,\mathrm{tb}(L) + 1 \neq 0$ and $|\mathrm{rot}(L)| > -\chi(L)$, then for any positive integer n, contact $(+1/n)$-surgery along L is overtwisted.*

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In particular, if $\mathrm{tb}(L) < 0$ and $|\mathrm{rot}(L)| > -\chi(L)$, then contact $(+1)$-surgery along L is overtwisted. Note that $(+1)$-surgery in a tight contact 3-manifold is not necessarily overtwisted; for example, a single contact $(+1)$-surgery in the tight 3-sphere along the $\mathrm{tb} = -1$ Legendrian unknot yields the tight and Stein fillable $S^1 \times S^2$ [3].

Conjecture 6.13 of James Conway [2] states that if L is a null homologous Legendrian knot with $\mathrm{tb}(L) \leq -2$, then contact $(+n)$-surgery along L is overtwisted for any positive integer $n < |\mathrm{tb}(L)|$. Conway explains that the conjecture, if true, would allow him to remove the bounds on the rotation number from his results in [2, Theorems 6.7 and 6.10]. As a corollary of his theorems he shows that for every genus g and positive integer $n \geq 2$, there is negative integer t (which depends on the bound on the rotation number) such that if L is a null homologous Legendrian knot of genus g and $\mathrm{tb}(L) \leq t$, then contact $(+n)$-surgery is overtwisted.

In this paper, we give a counterexample to Conway's conjecture.

Theorem 1.2. *There exists a null homologous Legendrian knot L in a tight contact 3-manifold with* $\mathrm{tb}(L) \leq -2$ *where contact $(+n)$-surgery along L is tight, for any positive integer $n < |\mathrm{tb}(L)|$.*

Note that contact $(+1/n)$-surgeries are unique, while this is not true for general surgery slopes. For contact $(+n)$-surgeries there are exactly two choices of contact structures on the resulting manifold that fit to the contact surgery. Theorem 1.2 is true for both choices.

We assume that the reader is familiar with the elements of contact topology. The reader may refer to [5–7] for the fundamentals of contact structures and Legendrian knots.

2. Proof of the theorems

Proof of Theorem 1.1. Let L be a null homologous oriented Legendrian knot with $\mathrm{tb}(L) = \mathrm{tb} < 0$, $|\mathrm{rot}(L)| = |\mathrm{rot}| > -\chi(L)$ and $n\,\mathrm{tb} + 1 \neq 0$. Using an algorithm from [3] that turns a rational contact surgery into a sequence of contact (± 1)-surgeries, contact $(+1/n)$-surgery along L is equivalent to contact $(+1)$-surgeries along n successive push-offs $L_1, L_2, \ldots, L_n$ of L. Let L' be the $(n+1)$th push-off of L and let L^* denote the surgery dual knot which is the image of L'. We will show that L^* violates a generalisation of Bennequin's inequality [1, Theorem 2.1], which holds only for knots in tight contact 3-manifolds. To do so, we compute the rational Thurston–Bennequin invariant $\mathrm{tb}_{\mathbb{Q}}(L^*)$ and the rational rotation number $\mathrm{rot}_{\mathbb{Q}}(L^*)$. For the definition of rational classical invariants, the knot L^* needs to be a torsion element in the first homology, and this is the case if $n\,\mathrm{tb} + 1 \neq 0$.

Since $(+1/n)$-surgery along L is a topological $(n\,\mathrm{tb} + 1)/n$-surgery, the homological order r of L^* is $|n\,\mathrm{tb} + 1|$. Note that if Σ is a rational minimal-genus Seifert surface of L^*, then topologically it is the image of a minimal-genus Seifert surface of L and hence $\chi(\Sigma) = \chi(L)$.

Following [2, Lemma 6.4], which extends [8, Lemma 2] and [10, Lemma 6.6] to more general contact 3-manifolds, the linking matrix $\mathbf{M}$ is the $(n \times n)$ matrix

$$\mathbf{M} = \begin{pmatrix} \mathrm{tb}+1 & \mathrm{tb} & \mathrm{tb} & \dots & \mathrm{tb} \\ \mathrm{tb} & \mathrm{tb}+1 & \mathrm{tb} & \dots & \mathrm{tb} \\ \vdots & \mathrm{tb} & \mathrm{tb}+1 & \dots & \vdots \\ \mathrm{tb} & \vdots & \vdots & \ddots & \mathrm{tb} \\ \mathrm{tb} & \mathrm{tb} & \dots & \mathrm{tb} & \mathrm{tb}+1 \end{pmatrix},$$

and $\det(\mathbf{M}) = n\,\mathrm{tb} + 1$. The extended matrix $\mathbf{M}_0$ is the $(n+1) \times (n+1)$ matrix $\mathbf{M}_0 = \begin{pmatrix} 0 & \mathrm{tb} \\ \mathrm{tb} & \mathbf{M} \end{pmatrix}$ and $\det(\mathbf{M}_0) = -n\,\mathrm{tb}^2$. Then we compute

$$\mathrm{tb}_{\mathbb{Q}}(L^*) = \mathrm{tb}(L') + \frac{\det \mathbf{M}_0}{\det \mathbf{M}} = \mathrm{tb} + \frac{-n\,\mathrm{tb}^2}{n\,\mathrm{tb}+1} = \frac{\mathrm{tb}}{n\,\mathrm{tb}+1}$$

and

$$\begin{aligned} \mathrm{rot}_{\mathbb{Q}}(L^*) &= \mathrm{rot}(L') - \left\langle \begin{pmatrix} \mathrm{rot}(L_1) \\ \mathrm{rot}(L_2) \\ \vdots \\ \mathrm{rot}(L_n) \end{pmatrix}, \mathbf{M}^{-1} \begin{pmatrix} \ell k(L', L_1) \\ \ell k(L', L_2) \\ \vdots \\ \ell k(L', L_n) \end{pmatrix} \right\rangle \\ &= \mathrm{rot} - \left\langle \begin{pmatrix} \mathrm{rot} \\ \mathrm{rot} \\ \vdots \\ \mathrm{rot} \end{pmatrix}, \mathbf{M}^{-1} \begin{pmatrix} \mathrm{tb} \\ \mathrm{tb} \\ \vdots \\ \mathrm{tb} \end{pmatrix} \right\rangle \\ &= \mathrm{rot} - \left\langle \begin{pmatrix} \mathrm{rot} \\ \mathrm{rot} \\ \vdots \\ \mathrm{rot} \end{pmatrix}, \begin{pmatrix} \mathrm{tb}/(n\,\mathrm{tb}+1) \\ \mathrm{tb}/(n\,\mathrm{tb}+1) \\ \vdots \\ \mathrm{tb}/(n\,\mathrm{tb}+1) \end{pmatrix} \right\rangle \\ &= \mathrm{rot} - \frac{n\,\mathrm{rot}\,\mathrm{tb}}{n\,\mathrm{tb}+1} = \frac{\mathrm{rot}}{n\,\mathrm{tb}+1}. \end{aligned}$$

Since L is a Legendrian knot in a tight contact 3-manifold, Bennequin's inequality $\mathrm{tb}(L) + |\mathrm{rot}(L)| \le -\chi(L)$ holds for L. If $\mathrm{tb}(L) < 0$ and $|\mathrm{rot}(L)| > -\chi(L) \ge 0$, then $\mathrm{tb}(L) + |\mathrm{rot}(L)| \le -\chi(L)$ or $-\mathrm{tb}(L) \ge \chi(L) + |\mathrm{rot}(L)|$. We compute

$$\begin{aligned} \mathrm{tb}_{\mathbb{Q}}(L^*) + |\mathrm{rot}_{\mathbb{Q}}(L^*)| &= \frac{\mathrm{tb}}{n\,\mathrm{tb}+1} + \left| \frac{\mathrm{rot}}{n\,\mathrm{tb}+1} \right| = \frac{-\mathrm{tb}}{|n\,\mathrm{tb}+1|} + \frac{|\mathrm{rot}|}{|n\,\mathrm{tb}+1|} \\ &\ge \frac{\chi(L) + 2|\mathrm{rot}|}{|n\,\mathrm{tb}+1|} \ge \frac{-\chi(L)}{|n\,\mathrm{tb}+1|}, \end{aligned}$$

since $|\mathrm{rot}| > -\chi(L)$. Thus, the surgery dual knot L^* violates a generalisation of Bennequin's inequality. Hence L^* is a knot in an overtwisted contact 3-manifold. □

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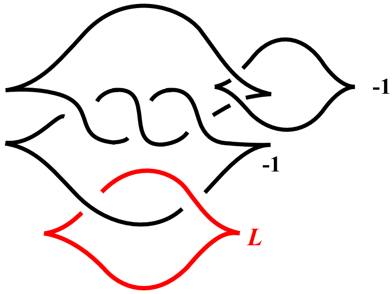

FIGURE 1. The Legendrian knot L has $\mathrm{tb} = -3$ in the surgered manifold.

REMARK 2.1. To compute the rational classical invariants, one can use the formulas for contact $(+1/n)$-surgeries from [9, Section 8.3] for $\mathrm{tb}_{\mathbb{Q}}$ and from [4, Theorem 4.3] for $\mathrm{rot}_{\mathbb{Q}}$.

LEMMA 2.2. *Let L be a Legendrian knot in a contact* 3*-manifold* (M, ξ)*. If contact* $(+1)$*-surgery along L is tight, then for relatively prime integers* $p, q > 0$ *with* $q < p$*, contact* $(+p/q)$*-surgery along L is tight.*

In particular, if contact $(+1)$-surgery along L is tight, then contact $(+n)$-surgery along L is tight for any positive integer n. Contact r-surgery is in general not unique. Lemma 2.2 is true for all choices of tight contact structures on the new glued-in solid tori.

PROOF. By Ding *et al.* [3], contact $(+p/q)$-surgery along L is equivalent to contact $(+1)$-surgery along L and a contact $(-p/(p-q))$-surgery along its push-off, say L'. By [3] again, contact $(-p/(p-q))$-surgery along L' is equivalent to a sequence of (-1)-surgeries along push-offs of L' with some additional zigzags. If contact $(+1)$-surgery along L is tight, then by [12] the remaining (-1)-surgeries result in a tight contact 3-manifold. □

PROOF OF THEOREM 1.2. Let L be the Legendrian knot in Figure 1. By [3], a single contact $(+1)$-surgery along a standard Legendrian unknot produces the unique tight and Stein fillable contact structure on $S^1 \times S^2$. A further two contact (-1)-surgeries in Figure 1 produce a Stein fillable and hence a tight contact structure. Thus, contact $(+1)$-surgery along L is tight. Then, by Lemma 2.2 contact $(+n)$-surgery along L is tight for any $n \geq 2$.

We now check that $\mathrm{tb}(L) = -3$ where L is seen as a Legendrian knot in the surgered manifold. For this we may use the tb formula from the proof of Theorem 1.1. Consider the linking matrix $\mathbf{M}$ and the extended linking matrix $\mathbf{M}_0$:

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}, \quad \mathbf{M}_0 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -2 \end{pmatrix}.$$

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The Thurston–Bennequin invariant of L in the unsurgered manifold is denoted by tb_0 and here $\mathrm{tb}_0 = -1$. Then in the surgered manifold,

$$\mathrm{tb}(L) = \mathrm{tb}_0 + \frac{\det \mathbf{M}_0}{\det \mathbf{M}} = -1 + \frac{2}{-1} = -3. \qquad \square$$

REMARK 2.3. One can alternatively replace the knot L in Figure 1 by any knot having $\mathrm{tb} \leq -1$ in $(S^3, \xi_{\mathrm{std}})$ where contact $(+1)$-surgery is tight; the knot L in Figure 1 is the simplest choice. The same proof applies for all such knots and each one gives a new counterexample to Conway's conjecture.

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