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Cyclic Surgery Between Toroidal Surgeries

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Abstract. We show that there is an infinite family of hyperbolic knots such that each knot admits a cyclic surgery m whose adjacent surgeries $m - 1$ and $m + 1$ are toroidal. This gives an affirmative answer to a question asked by Boyer and Zhang.

1 Introduction

For a knot K in the 3-sphere S^3 and an integer m, $K(m)$ denotes the closed orientable 3-manifold obtained by m-Dehn surgery on K. If $K(m)$ has a finite cyclic fundamental group, then this surgery is called a *cyclic surgery*. By the solution of the spherical space form conjecture [7], this is equivalent to $K(m)$ being a lens space. It is a challenging problem to determine all hyperbolic knots that admit a cyclic surgery.

Motivated by a result in [2], Boyer and Zhang asked if there is a hyperbolic knot K that admits a cyclic surgery with slope m, but neither of $K(m - 1)$ and $K(m + 1)$ is an irreducible non-Haken manifold [2, Question (2)]. In the paragraph after the question, they conjecture that the answer is negative. However, the purpose of this paper is to answer it in the affirmative.

Theorem 1.1 *There exist infinitely many hyperbolic knots K in S^3, each of which admits a cyclic surgery with slope m such that both $K(m-1)$ and $K(m+1)$ are irreducible and toroidal.*

Recall that a closed orientable 3-manifold is said to be *Haken* if it is irreducible and contains an incompressible surface and *toroidal* if it contains an incompressible torus.

2 Construction

Let $n \geq 2$ be an integer. Consider the tangle $B_n = (B^3, t)$ as illustrated in Figure 2.1. Here, the vertical rectangle with label n ($-n$, resp.) means n vertical right-handed (left-handed, resp.) half-twists.

For a rational tangle α, $B_n(\alpha)$ denotes the knot or link in S^3 obtained by filling the central sphere with the rational tangle α. In fact, we use only the four rational tangles illustrated in Figure 2.2. (We follow the convention of [5] for the parameterization of rational tangles.)

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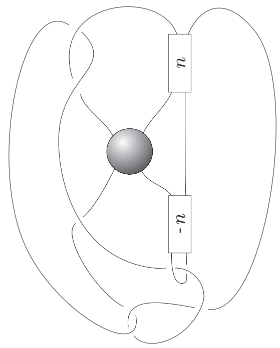

Figure 2.1

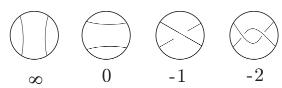

Figure 2.2

Let $M[r,s]$ denote a Montesinos tangle consisting of two rational tangles associated with rational numbers r and s, respectively.

Lemma 2.1 *The tangle B_n has the following properties.*

(i) $B_n(\infty)$ *is the trivial knot.*
(ii) $B_n(0)$ *is the union of two Montesinos tangles.*
(iii) $B_n(-1)$ *is the 2-bridge knot or link corresponding to* $-(10n^2+17n+7)/(10n+11)$.
(iv) $B_n(-2)$ *is the union of two Montesinos tangles.*

Proof It is straightforward to verify that $B_n(\infty)$ is the trivial knot and that $B_n(-1)$ is the 2-bridge knot or link corresponding to $-(10n^2+17n+7)/(10n+11)$.

By Figure 2.3, $B_n(0)$ is decomposed into the Montesinos tangles $M[1/2, 1/n]$ and $M[-2/3, (2n+1)/(2n+3)]$. Similarly, $B_n(-2)$ is decomposed into the Montesinos tangles $M[1/2, -(n+1)/(n+2)]$ and $M[1/2, -2/(2n+1)]$ as shown in Figure 2.4. ■

Let $\widetilde{B}_n(\alpha)$ denote the double branched cover of S^3 branched over $B_n(\alpha)$.

Lemma 2.2 (i) $\widetilde{B}_n(\infty)$ *is the* 3*-sphere.*
(ii) $\widetilde{B}_n(-1)$ *is the lens space* $L(10n^2+17n+7, -10n-11)$.
(iii) $\widetilde{B}_n(0)$ *and* $\widetilde{B}_n(-2)$ *are irreducible toroidal manifolds.*

Proof (i) and (ii) follow immediately from Lemma 2.1. For (iii), recall that the double branched cover of a Montesinos tangle is a Seifert fibered manifold over the disk with two exceptional fibers, which is irreducible and whose boundary torus is incompressible. Thus we have the conclusion. ■

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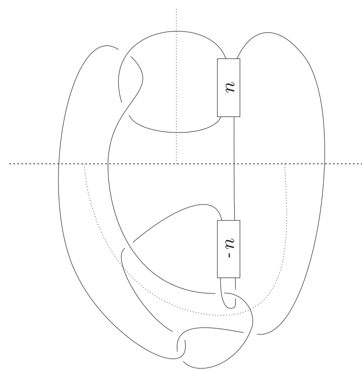

Figure 2.3

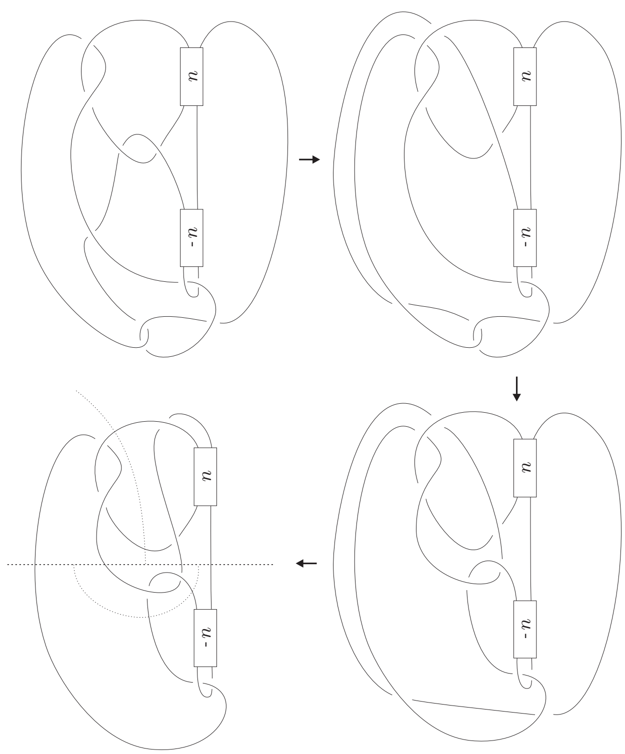

Figure 2.4

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We remark that $\widetilde{B}_n(0)$ (and $\widetilde{B}_n(-2)$) contains a separating incompressible torus.

By Lemma 2.2(i), the lift of B_n in $\widetilde{B}_n(\infty) = S^3$ gives a knot exterior. Since a knot is uniquely determined by its exterior [6], we can define a knot K_n whose exterior is the lift of B_n.

Lemma 2.3 *The knot K_n satisfies the following properties.*

(i) *K_n admits a cyclic surgery with integer slope m for some m.*
(ii) *$K_n(m-1)$ and $K_n(m+1)$ are irreducible toroidal manifolds, each containing a separating incompressible torus.*

Proof This is a direct consequence of Lemma 2.2. ■

It follows from Lemma 2.2(ii) that $m = \pm(10n^2 + 17n + 7)$. By using an explicit description of K_n, we can see that in fact m is $10n^2 + 17n + 7$.

Lemma 2.4 *K_n is hyperbolic.*

Proof It suffices to show that K_n is neither a torus knot nor a satellite knot. First, K_n is not a torus knot because no torus knot yields a separating incompressible torus by Dehn surgery [8]. Suppose that K_n is a satellite knot. Since K_n has a cyclic surgery, it is the $(2, 2pq+\varepsilon)$-cable of a (p, q)-torus knot, where $\varepsilon = \pm 1$ [1,10–12]. In particular, the cyclic surgery corresponds to an integer $4pq + \varepsilon$, which is adjacent to the slope $4pq+2\varepsilon$ of the cabling annulus. Thus $K_n(4pq+2\varepsilon)$ is reducible, so K_n is not a satellite knot by Lemma 2.3(ii). ■

Proof of Theorem 1.1 Lemmas 2.3 and 2.4 show that the knot K_n has the desired properties. By [3], any hyperbolic knot admits at most two cyclic surgeries, and if there are two, then they correspond to consecutive integers. Thus the K_n's are mutually distinct. ■

It is possible to give an explicit description of the knot K_n as in [4]. Attaching the ∞-tangle to B_n, we have the trivial knot U $(= B_n(\infty))$. The core ξ of the ∞-tangle (that is, the trivial straight arc connecting the two strings) has its endpoints on U. Then the lift of ξ in the double branched cover, which is S^3 again, of S^3 branched over U gives K_n. As the simplest example, we show K_2 in Figure 2.5. Although there is one negative crossing, it will be canceled by a positive crossing in the 2-full twists. Thus K_2 has the form of a closed positive braid. Then K_2 is fibered and the Seifert algorithm on the diagram gives a minimal genus Seifert surface [9]. Since the braid has six strings and 67 positive crossings, the genus is equal to 31. For K_2, 81-surgery gives the lens space $L(81, -31)$, and 80-, 82-surgeries yield irreducible toroidal manifolds.

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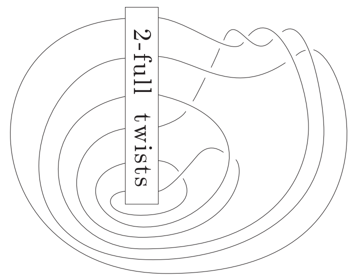

Figure 2.5

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