PERIODIC ORBITS OF ARBITRARY INCLINATIONS AND ECCENTRICITIES IN THE GENERAL 3-BODY PROBLEM.

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ABSTRACT

Usual periodic orbits have periods of the order of magnitude of a few revolutions. However if we consider much longer periods it is possible to find, for three given masses, periodic orbits in any neighbourhood of arbitrary given initial eccentricities and inclinations provided that the distance of the outer body is sufficiently large with respect to the mutual distance of the two inner bodies.

INTRODUCTION

Euler and Lagrange found the first periodic orbits of the 3-body problem. Later, using the symmetries of the problem, Poincaré described a very general way for the construction of periodic orbits (Poincaré 1892 - 1893 - 1899) and considered them as our essential key for the understanding of that problem. He formulated this conjecture : "In the phase space of the 3-body problem the set of periodic orbits is dense in the set of bounded orbits". Since then many families of periodic orbits have been found both in the restricted case and in the general case and a method is proposed here for the research of periodic orbits of arbitrary inclinations and eccentricities, but of very long period.

NOTATIONS

Let us use the ordinary Jacobi decomposition of the 3-body motion into the "inner motion" (i.e. the relative motion of the two nearest point-mass m_1 and m_2) and the slower outer motion (i.e. the motion of the third point-mass m_3 with respect to $O_{1.2}$ the center of mass of m_1 and m_2). The usual osculating elements of the inner and outer orbits will be:

$$a\,,\, e\,,\, i\,,\, \Omega\,,\, \omega\,,\, M\,, \qquad a_3\,,e_3\,,\, i_3\,,\, \Omega_3\,, \omega_3\,,\, M_3 \qquad (1)$$

(semi-major axis, eccentricity, inclination, longitude of the node, argument of the pericenter, mean anomaly; with subscripts 3 for the orbit of m_3).

R. L. Duncombe (ed.), Dynamics of the Solar System, 29–32.

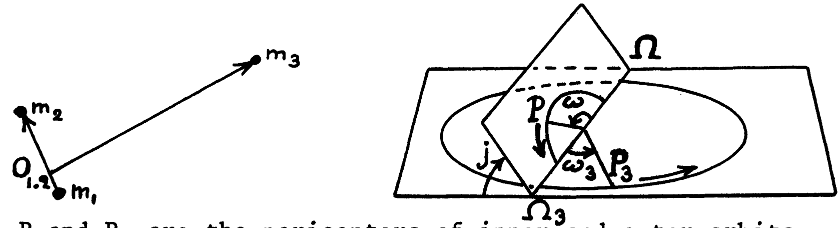

Figure 1 - P and P_3 are the pericenters of inner and outer orbits.

We will choose the polar reference direction in the direction of angular momentum, the line of nodes is then in the equatorial plane, the angles ω and ω_3 are also the angles between the line of nodes and the pericenters directions (fig. 1), the mutual inclination j is equal to i + i_3 and the longitudes Ω and Ω_3 are related by $\Omega_3 = \Omega + \pi$

1. PERIODIC ORBITS

The Poincaré conditions of symmetry are :

$$0 = \sin M = \sin M_3 = \sin 2\omega = \sin(\omega_3 - \omega) \qquad (2)$$

If they are satisfied at some instant t_1 there is a past-future symmetry : the three mutual distances are even function of time with respect to t_1, if they are also satisfied at some other instant t_2 the two symmetries imply the periodicity with the period 2 ($t_2 - t_1$). Hence starting from initial conditions satisfying (2) we shall look for a second passage at these conditions and thus we shall look for initial a, e, i, a_3, e_3, i_3 leading to such a second passage.

2. FIRST ORDER SECULAR APPROXIMATION

If the ratio $(m_1 + m_2 + m_3)\, a^3 / (m_1 + m_2)\, a_3^3 (1 - e_3^2)^3$ is small the use of Delaunay's variables and a proper Von Zeipel transformation (Marchal 1977) lead to "secular elements" a_s, e_s, i_s, Ω_s, ω_s M_s and a_T, e_T, i_T, Ω_T, ω_T, M_T very near to the corresponding osculating elements a, --, M, a_3, -- M_3 but with much smaller short period variations. The "secular mutual inclination" j_s is equal to $i_s + i_T$ and is near $j = i + i_3$

The symmetries of the transformation lead to symmetry conditions :

$$0 = \sin M_s = \sin M_T = \sin 2\omega_s = \sin(\omega_T - \omega_s) \qquad (3)$$

and in the first order approximation the 3-body problem is integrable : a_s, a_T, e_T, dM_s/dt, dM_T/dt are constant.
Ω_s, Ω_T $(= \Omega_s + \pi)$ and ω_T are ignorable and given by final quadratures.
i_s and i_T are given interms of a_s, e_s, a_T, e_T, j_s by :

$$0 \leq i_s \leq j_s \leq \pi \quad ; \quad 0 \leq i_T \leq j_s \leq \pi \quad ; \quad i_s + i_T = j_s \qquad (4)$$

$$\sin i_T / \sin i_s = m_1 m_2 m_3^{-1} (m_1 + m_2)^{-3/2} (m_1 + m_2 + m_3)^{1/2} \left[a_s (1 - e_s^2) / a_T (1 - e_T^2) \right]^{1/2} \qquad (5)$$

Finally the three remaining elements e_s, ω_s, j_s are related by :

$$(1 - e_s^2)(1 + \sin^2 j_s) + 5 e_s^2 \sin^2 j_s \sin^2 \omega_s = Z = \text{constant} \qquad (6)$$

$$\cos j_s = A (1 - e_s^2)^{-1/2} - B (1 - e_s^2)^{1/2} \qquad (7)$$

(A and B being two constants : $B = 0.5\, m_1 m_2 m_3^{-1} (m_1 + m_2)^{-3/2} (m_1 + m_2 + m_3)^{1/2} \left[a_s / a_T (1 - e_T^2) \right]^{1/2}$

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and the variations of e_s, ω_s, j_s are given by the final quadrature :

$$de_s/dt = K\, e_s (1-e_s^2)^{1/2} \sin^2 j_s \sin 2\omega_s \tag{8}$$

with, G being the constant of the law of universal attraction :

$$K = 1.875\, G^{1/2} m_3 (m_1+m_2)^{-1/2} \left[a_s/a_T^2 (1-e_T^2)\right]^{3/2} = \text{constant} \tag{9}$$

It leads to motions in which e_s, j_s, ω_s, $d\Omega_s/dt$, $d\omega_T/dt$ are generally periodic (with the same period much larger than that of the mean anomalies M_s and M_T) but sometimes the secular motion is asymptotic (either to $e_s = 0$ or to $j_s = \pi$ or to $\sin\omega_s = 0$ or to $\cos\omega_s = 0$ for proper values of the integrals A, B, Z; for instance the motion is asymptotic to $e_s = 0$ if $Z = 2-(A-B)^2$; $1+AB \geqslant B$ and $3 \geqslant (A-B)(5A-3B)$). With that secular motion it is easy to look for periodic orbits by the Poincaré method, i.e. to determine initial conditions satisfying the Poincaré symmetry conditions (3) and leading to a second passage at these conditions, these "approximated periodic orbits" are even dense everywhere in the region of interest. However we have only studied a first order approximation, let us consider now the upper order effects.

3. ANALYSIS OF THE UPPER ORDER EFFECTS

Let us use M_T as parameter of description instead of the time, it gives:
A) dM_s/dM_T is very near to the ratio of the mean angular motions n_s/n_T, that is $\left[(m_1+m_2)a_T^3/(m_1+m_2+m_3)a_s^3\right]^{1/2}$ or, in terms of the "quasi integrals" B and e_T , we can write :

$$dM_s/dM_T = \left[m_1^3 m_2^3 (m_1+m_2+m_3)/8 m_3^3 (m_1+m_2)^4 B^3 (1-e_T^2)^{3/2}\right].\left(1+O(\varepsilon_1)\right) \tag{10}$$

with : $\varepsilon_1 = \left[m_1 m_2 a_s^2 a_T + m_3 (m_1+m_2) a_s^3\right] / \left[a_T^3 (1-e_T^2)^3 . \inf\{e_s, e_T\}.(m_1+m_2)^2\right]$ (11)

Note that, since by hypothesis $(m_1 + m_2 + m_3)\, a^3 / (m_1 + m_2)$. $a_3^3\,(1 - e_3^2)^3$ is small, the ratio dM_s/dM_T is large.
B) With $x = 1 - e_s^2$ we obtain similarly :

$$dx/dM_T = \pm 12 B^3 m_1^{-3} m_2^{-3} m_3^4 (m_1+m_2)^4 (m_1+m_2+m_3)^{-2} \left[\{Z-2x+(A-Bx)^2\}\{(5-Z)x-3x^2+(4x-5)(A-Bx)^2\} + O(\varepsilon_2)\right]^{1/2} \tag{12}$$

with the $\pm$ sign $= -\,\mathrm{sign}(e_s') = \mathrm{sign}\,(-\sin 2\omega_s + O(\varepsilon_2))$ and :

$$\varepsilon_2 = \left[a_s/a_T e_T (1-e_T^2)^{3/2}\right].\sup\left\{\left[|m_2-m_1|/(m_1+m_2)\right];\left[(m_1+m_2+m_3)a_s/(m_1+m_2)a_T(1-e_T^2)\right]^{1/2}\right\} \tag{13}$$

On the other hand, with $\varepsilon_3 = \varepsilon_2 (1-e_T^2)^{-1/2} \geqslant \varepsilon_2$:

$$d\omega_T/dM_T = 6B^3 m_1^{-3} m_2^{-3} m_3^4 (m_1+m_2)^4 (m_1+m_2+m_3)^{-2}\left[B(5-5Z+2x)+(Z-x)(A+Bx)\left[x-(A-Bx)^2\right]^{-1} + O(\varepsilon_3)\right] \tag{14}$$

And finally, with $\varepsilon_4 = \varepsilon_2\, e_T (1-e_s^2)^{-1/2} / \inf\{e_s ; e_T\}$:

$$\frac{d\omega_s}{dM_T} = \frac{6B^3 m_3^4 (m_1+m_2)^4 x^{1/2}}{m_1^3 m_2^3 (m_1+m_2+m_3)^2}\left[2+2AB-2B^2x+\frac{\left[Z-2x+(A-Bx)^2\right]\left[(A-Bx)(A+Bx-2Bx^2)-x^2\right]}{x(1-x)\left[x-(A-Bx)^2\right]}+O(\varepsilon_4)\right] \tag{15}$$

Note that : A) dx/dM_T, $d\omega_T/dM_T$, $d\omega_s/dM_T$ are small, of the order of dM_T/dM_s or even smaller. The three-body motion of interest is the composition of two slowly perturbed Keplerian motions.

B) If we neglect the error terms and the small variations of the "quasi integrals" e_T , A,B,Z, the equation (12) is a quadrature and we obtain the integrable system of the previous section.

C) e_T only appears in the error terms and in dM_s/dM_T . Hence, taking account of the continuity of the problem and of the small variations and the independance of e_T , A,B,Z, if we consider an "approximate periodic orbit" of the previous section we can obtain a true periodic orbit by a modification of the initial values of e_T^2, A,B,Z of the order of sup (ε_1, ε_3, ε_4) provided that the three modifications δA, δB and δZ imply independant final modifications $\delta\omega_T$ and $\delta\omega_s$ in the integration of (14) and (15) without error terms

(as it is everywhere the case except on some submanifolds).
More precisely let us put $\varepsilon = \sup (\varepsilon_1 , \varepsilon_3 , \varepsilon_4)$ in the interval (t_1 , t_2):

$$\varepsilon = \frac{a_S}{a_T (1-e_T^2)^{5/2}} \cdot \sup\left\{\frac{|m_2-m_1|}{m_1+m_2} ; \left[\frac{(m_1+m_2+m_3)a_S}{(m_1+m_2)a_T(1-e_T^2)}\right]^{1/2}\right\} \cdot \sup\left\{\frac{1}{e_T\sqrt{1-e_T^2}} ; \frac{1}{e_S\sqrt{1-e_T^2}} ; \frac{1}{e_S\sqrt{1-e_S^2}}\right\} \quad (16)$$

(t_1 and t_2 are the Poincaré instants of symmetry; e_S varies in (t_1, t_2)).
Then, if ε is small with respect to 1 and if the above condition of independance of $\delta\omega_S$ and $\delta\omega_T$ is respected, we obtain in the vicinity of the "approximate periodic orbit" of interest a one non trivial parameter family of true periodic orbits (of the same $\Delta M_S, \Delta M_T, \Delta\omega_S, \Delta\omega_T$ per period) inside the region defined by :

$$\delta Z \; ; \; \delta(A - Bx) \; ; \; \delta B/B \; ; \; \delta(e_T^2) = O(\varepsilon) \quad (17)$$

That is, with the ordinary osculating parameters a, e, a_3, e_3 and j (mutual inclination) in the region defined by :

$$\delta(\mathrm{Ln}\,(a_3/a)) \; ; \; \delta(e^2) \; ; \; \delta(e_3^2) \; ; \; \delta(\cos j) = O(\varepsilon) \quad (18)$$

Note 1 - The true periodic orbits, obtained by this method are sometimes called "relative periodic orbits" or "periodic orbits in a rotating frame of reference". The absolute periodicity must take account of the motion of the elements Ω and Ω_3(with $\Omega_3 = \Omega + \pi$).
In general the absolute periodic orbits are dense along the one-parameter families of relative periodic orbits, they correspond to rational values of $\Delta\Omega/2\pi$ during one period of the relative motion.
Note 2 - ε is infinite when e_S goes to one, hence the method doesn't work for oscillating orbits of the second kind (Marchal 1977) in which the bodies m_1 and m_2 have an infinite number of approaches as close as desired (but they don't have a strict collision). These orbits fill a set of positive measure of phase space, one of them has been integrated by Hadjidemetriou (Hadjidemetriou 1977).
Note 3 - The relation (18) doesn't imply the denseness of periodic orbits even in the regions of small ε and the Poincaré conjecture remain open. However, if we consider arbitrary orientations, we can select any small open set of the 11-dimension space of elements (a, e, i, Ω, ω, M, e_3, i_3, Ω_3, ω_3, M_3) : that set is crossed by periodic orbits for all sufficiently large values of a_3.

CONCLUSION

The research of periodic orbits of very long period has led to many new families of periodic orbits of arbitrary inclinations, eccentricities and orientations (for any 3 given masses, both in the restricted and in the general case), the Poincaré conjecture on periodic orbits remain open but it is likely true.

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