4

Non-Abelian strings

In this chapter we discuss a particular class of $\mathcal{N} = 2$ supersymmetric gauge theories in which non-Abelian strings were found. One can pose the question: what is so special about these models that makes an Abelian Z_N string become non-Abelian? Models we will dwell on below have both gauge and flavor symmetries broken by the condensation of scalar fields. The common feature of these models is that some global diagonal combination of color and flavor groups survive the breaking. We consider the case when this diagonal group is $\mathrm{SU}(N)_{C+F}$, where the subscript $C + F$ means a combination of global color and flavor groups. The presence of this unbroken subgroup is responsible for the occurrence of the orientational zero modes of the string which entail its non-Abelian nature.

Clearly, the presence of supersymmetry is not important for the construction of non-Abelian strings. In particular, while here we focus on the BPS non-Abelian strings in $\mathcal{N} = 2$ supersymmetric gauge theories, in Chapter 5 we review non-Abelian strings in $\mathcal{N} = 1$ supersymmetric theories and in Chapter 6 in non-supersymmetric theories.

4.1 Basic model: $\mathcal{N} = 2$ SQCD

The model we will deal with derives from $\mathcal{N} = 2$ SQCD with the gauge group $\mathrm{SU}(N + 1)$ and $N_f = N$ flavors of the fundamental matter hypermultiplets which we will call quarks [3]. At a generic point on the Coulomb branch of this theory, the gauge group is broken down to $\mathrm{U}(1)^N$. We will be interested, however, in a particular subspace of the Coulomb branch, on which the gauge group is broken down to $\mathrm{SU}(N)\times\mathrm{U}(1)$. We will enforce this regime by a special choice of the quark mass terms.

The breaking $\mathrm{SU}(N + 1) \to \mathrm{SU}(N)\times\mathrm{U}(1)$ occurs at the scale m which is supposed to lie very high, $m \gg \Lambda_{\mathrm{SU}(N+1)}$, where $\Lambda_{\mathrm{SU}(N+1)}$ is the scale of the $\mathrm{SU}(N + 1)$ theory. Correspondingly, the masses of the gauge bosons from the

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SU$(N+1)$/SU$(N)\times$U(1) sector and their superpartners, are very large – proportional to m – and so are the masses of the $(N+1)$-th color component of the quark fields in the fundamental representation. We will be interested in the phenomena at the scales $\ll m$. Therefore, our starting point is in fact the SU$(N)\times$U(1) model with $N_f = N$ matter fields in the fundamental representation of SU(N), as it emerges after the SU$(N+1) \to$ SU$(N)\times$U(1) breaking. These matter fields are also coupled to the U(1) gauge field.

The field content of SU$(N)\times$U(1) $\mathcal{N}=2$ SQCD with N flavors is as follows. The $\mathcal{N}=2$ vector multiplet consists of the U(1) gauge field A_μ and the SU(N) gauge field A^a_μ, (here $a=1,\ldots,N^2-1$), and their Weyl fermion superpartners $(\lambda^1_\alpha, \lambda^2_\alpha)$ and $(\lambda^{1a}_\alpha, \lambda^{2a}_\alpha)$, plus complex scalar fields a, and a^a. The latter are in the adjoint representation of SU(N). The spinorial index of λ's runs over $\alpha = 1, 2$. In this sector the global SU$(2)_R$ symmetry inherent to the $\mathcal{N}=2$ model at hand manifests itself through rotations $\lambda^1 \leftrightarrow \lambda^2$.

The quark multiplets of the SU$(N)\times$U(1) theory consist of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and the Weyl fermions ψ^{kA} and $\tilde{\psi}_{Ak}$, all in the fundamental representation of the SU(N) gauge group. Here $k=1,\ldots,N$ is the color index while A is the flavor index, $A=1,\ldots,N$. Note that the scalars q^{kA} and $\bar{\tilde{q}}^{kA}$ form a doublet under the action of the global SU$(2)_R$ group.

Then the original SU$(N+1)$ theory is perturbed by adding a small mass term for the adjoint matter, via the superpotential $\mathcal{W} = \mu\,\mathrm{Tr}\,\Phi^2$. Generally speaking, this superpotential breaks $\mathcal{N}=2$ down to $\mathcal{N}=1$. The Coulomb branch shrinks to a number of isolated $\mathcal{N}=1$ vacua [2, 3, 126, 143, 144]. In the limit of $\mu \to 0$ these vacua correspond to special singular points on the Coulomb branch in which N monopoles/dyons or quarks become massless. The first $(N+1)$ of these points (often referred to as the Seiberg–Witten vacua) are always at strong coupling. They correspond to $\mathcal{N}=1$ vacua of the pure SU$(N+1)$ gauge theory.

The massless quark points – they present vacua of a distinct type, to be referred to as the quark vacua – may or may not be at weak coupling depending on the values of the quark mass parameters m_A. If $m_A \gg \Lambda_{\mathrm{SU}(N+1)}$, the quark vacua do lie at weak coupling. Below we will be interested only in these quark vacua assuming that the condition $m_A \gg \Lambda_{\mathrm{SU}(N+1)}$ is met.

In the low-energy SU$(N)\times$U(1) theory, which is our starting point, the perturbation $\mathcal{W} = \mu\,\mathrm{Tr}\,\Phi^2$ can be truncated, leading to a crucial simplification. Indeed, since the $\mathcal{A}$ chiral superfield, the $\mathcal{N}=2$ superpartner of the U(1) gauge field,[1]

$$\mathcal{A} \equiv a + \sqrt{2}\lambda^2\theta + F_a\,\theta^2, \tag{4.1.1}$$

[1] The superscript 2 in Eq. (4.1.1) is the global SU$(2)_R$ index of λ rather than λ squared.

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it not charged under the gauge group SU(N)×U(1), one can introduce a superpotential linear in $\mathcal{A}$,

$$\mathcal{W}_{\mathcal{A}} = -\frac{N}{2\sqrt{2}}\,\xi\,\mathcal{A}. \tag{4.1.2}$$

Here we expand Tr Φ^2 around its vacuum expectation value (VEV), and truncate the series keeping only the linear term in $\mathcal{A}$. The truncated superpotential is a Fayet–Iliopoulos (FI) F-term.

Let us explain this in more detail. In $\mathcal{N} = 1$ supersymmetric theory with the gauge group SU(N)×U(1) one can add the following FI term to the action [145] (we will call it the FI D-term here):

$$\xi_3\, D \tag{4.1.3}$$

where D is the D-component of the U(1) gauge superfield. In $\mathcal{N} = 2$ SUSY theory the field D belongs to the SU(2)$_R$ triplet, together with the F components of the chiral field $\mathcal{A}$, (F and $\bar{F}$). Namely, let us introduce a triplet F_p ($p = 1, 2, 3$) using the relations [2]

$$\begin{aligned} D &= F_3, \\ F_{\mathcal{A}} &= \frac{1}{\sqrt{2}}\,(F_1 + i\,F_2), \\ \bar{F}_{\mathcal{A}} &= \frac{1}{\sqrt{2}}\,(F_1 - i\,F_2). \end{aligned} \tag{4.1.4}$$

Now, the generalized FI term can be written as

$$S_{\text{FI}} = -\frac{N}{2}\int d^4x \sum_p \xi_p F_p. \tag{4.1.5}$$

Comparing this with Eq. (4.1.2) we identify

$$\begin{aligned} \xi &= (\xi_1 - i\xi_2), \\ \bar{\xi} &= (\xi_1 + i\xi_2) \end{aligned} \tag{4.1.6}$$

This is the reason why we refer to the superpotential (4.1.2) as to the FI F-term.

A remarkable feature of the FI term is that it does *not* break $\mathcal{N} = 2$ supersymmetry [127, 35]. Keeping higher order terms of the expansion of μ Tr Φ^2 in powers of $\mathcal{A}$ would inevitably explicitly break $\mathcal{N} = 2$. For our purposes it is crucial

[2] Attention: The index p is an SU(2)$_R$ index rather than the color index!

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that the model we will deal with is *exactly* $\mathcal{N}=2$ supersymmetric. This ensures that the flux tube solutions of the model are BPS-saturated. If higher order terms in $\mathcal{A}$ are taken into account, $\mathcal{N}=2$ supersymmetry is broken down to $\mathcal{N}=1$ and strings are no longer BPS, generally speaking. The superconductivity in the model becomes of type I [35].

4.1.1 SU(N)×U(1) $\mathcal{N}=2$ QCD

The bosonic part of our SU(N)×U(1) theory has the form [131]

$$S=\int d^4x\left[\frac{1}{4g_2^2}\left(F^a_{\mu\nu}\right)^2+\frac{1}{4g_1^2}\left(F_{\mu\nu}\right)^2+\frac{1}{g_2^2}\left|D_\mu a^a\right|^2+\frac{1}{g_1^2}\left|\partial_\mu a\right|^2\right.$$
$$\left.+\left|\nabla_\mu q^A\right|^2+\left|\nabla_\mu\bar{\tilde{q}}^A\right|^2+V(q^A,\tilde{q}_A,a^a,a)\right]. \tag{4.1.7}$$

Here D_μ is the covariant derivative in the adjoint representation of SU(N), and

$$\nabla_\mu=\partial_\mu-\frac{i}{2}A_\mu-iA^a_\mu T^a. \tag{4.1.8}$$

We suppress the color SU(N) indices, and T^a are the SU(N) generators normalized as

$$\mathrm{Tr}\,(T^aT^b)=(1/2)\,\delta^{ab}.$$

The coupling constants g_1 and g_2 correspond to the U(1) and SU(N) sectors, respectively. With our conventions, the U(1) charges of the fundamental matter fields are $\pm 1/2$.

The potential $V(q^A,\tilde{q}_A,a^a,a)$ in the action (4.1.7) is a sum of D and F terms,

$$V(q^A,\tilde{q}_A,a^a,a)=\frac{g_2^2}{2}\left(\frac{i}{g_2^2}f^{abc}\bar{a}^b a^c+\bar{q}_A T^a q^A-\tilde{q}_A T^a\bar{\tilde{q}}^A\right)^2$$
$$+\frac{g_1^2}{8}\left(\bar{q}_A q^A-\tilde{q}_A\bar{\tilde{q}}^A-N\xi_3\right)^2$$
$$+2g_2^2\left|\tilde{q}_A T^a q^A\right|^2+\frac{g_1^2}{2}\left|\tilde{q}_A q^A-\frac{N}{2}\xi\right|^2$$
$$+\frac{1}{2}\sum_{A=1}^{N}\left\{\left|(a+\sqrt{2}m_A+2T^a a^a)q^A\right|^2\right.$$
$$\left.+\left|(a+\sqrt{2}m_A+2T^a a^a)\bar{\tilde{q}}^A\right|^2\right\}. \tag{4.1.9}$$

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Here f^{abc} stand for the structure constants of the SU(N) group, and the sum over the repeated flavor indices A is implied.

The first and second lines represent D terms, the third line the $F_{\mathcal{A}}$ terms, while the fourth and the fifth lines represent the squark F terms. Using the $\mathrm{SU}(2)_R$ rotations we can always direct the FI parameter vector ξ_p in a given direction. Below in most cases we will align the FI F-term to make the parameter ξ real. In other words,

$$\xi_3 = 0, \quad \xi_2 = 0, \quad \xi = \xi_1. \tag{4.1.10}$$

4.1.2 The vacuum structure and excitation spectrum

Now we briefly review the vacuum structure and the excitation spectrum in our basic SU(N)×U(1) model. As was mentioned, the underlying $\mathcal{N}=2$ SQCD with the gauge group SU($N+1$) has a variety of vacua [143, 144, 140]. In addition to N strong coupling vacua which exist in pure gauge theory, there is a number of the so-called r quark vacua, where r is the number of the quark flavors which develop VEV's in the given vacuum. We will limit ourselves[3] to a particular isolated vacuum, with the maximal possible value of r,

$$r = N.$$

The vacua of the theory (4.1.7) are determined by the zeros of the potential (4.1.9). The adjoint fields develop the following VEV's:

$$\langle\Phi\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} m_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & m_N \end{pmatrix}, \tag{4.1.11}$$

where we defined the scalar adjoint matrix as

$$\Phi = \frac{1}{2}\, a + T^a\, a^a. \tag{4.1.12}$$

For generic values of the quark masses, the SU(N) subgroup of the gauge group is broken down to $\mathrm{U}(1)^{N-1}$. However, for a special choice

$$m_1 = m_2 = \cdots = m_N, \tag{4.1.13}$$

[3] There are singular points on the Coulomb branch of the underlying SU($N+1$) theory where more than N quark flavors become massless. These singularities are the roots of Higgs branches [143, 144, 140].

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which we will be mostly interested in in this section, the SU(N)×U(1) gauge group remains classically unbroken. In fact, the common value m of the quark masses determines the scale of breaking of the SU($N+1$) gauge symmetry of the underlying theory down to SU(N)×U(1) gauge symmetry of our benchmark low-energy theory (4.1.7).

If the value of the FI parameter is taken real we can exploit gauge rotations to make the quark VEV's real too. Then in the case at hand they take the color-flavor locked form

$$\langle q^{kA}\rangle = \langle \bar{\tilde{q}}^{kA}\rangle = \sqrt{\frac{\xi}{2}}\begin{pmatrix} 1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 1 \end{pmatrix},$$
$$k = 1, \dots, N, \quad A = 1, \dots, N, \tag{4.1.14}$$

where we write down the quark fields as an $N \times N$ matrix in the color and flavor indices. This particular form of the squark condensates is dictated by the third line in Eq. (4.1.9). Note that the squark fields stabilize at non-vanishing values entirely due to the U(1) factor represented by the second term in the third line.

The vacuum field (4.1.14) results in the spontaneous breaking of both gauge and flavor SU(N)'s. A diagonal global SU(N) survives, however,

$$\mathrm{U}(N)_{\mathrm{gauge}} \times \mathrm{SU}(N)_{\mathrm{flavor}} \to \mathrm{SU}(N)_{C+F}. \tag{4.1.15}$$

Thus, a color-flavor locking takes place in the vacuum. A version of this pattern of the symmetry breaking was suggested long ago [146].

Let us move on to the issue of the excitation spectrum in this vacuum [35, 131]. The mass matrix for the gauge fields (A^a_μ, A_μ) can be read off from the quark kinetic terms in Eq. (4.1.7). It shows that all SU(N) gauge bosons become massive, with one and the same mass

$$M_{\mathrm{SU}(N)} = g_2\sqrt{\xi}. \tag{4.1.16}$$

The equality of the masses is no accident. It is a consequence of the unbroken SU$(N)_{C+F}$ symmetry (4.1.15).

The mass of the U(1) gauge boson is

$$M_{\mathrm{U}(1)} = g_1\sqrt{\frac{N}{2}\,\xi}. \tag{4.1.17}$$

Thus, the theory is fully Higgsed. The mass spectrum of the adjoint scalar excitations is the same as the one for the gauge bosons. This is enforced by $\mathcal{N} = 2$.

What is the mass spectrum of the quark excitations? It can be read off from the potential (4.1.9). We have $4N^2$ real degrees of freedom of quark scalars q and $\tilde{q}$. Out of those N^2 are eaten up by the Higgs mechanism. The remaining $3N^2$ states split in three plus $3(N^2 - 1)$ states with masses (4.1.17) and (4.1.16), respectively. Combining these states with the massive gauge bosons and the adjoint scalar states we get [35, 131] one long $\mathcal{N} = 2$ BPS multiplet (eight real bosonic plus eight fermionic degrees of freedom) with mass (4.1.17) and $N^2 - 1$ long $\mathcal{N} = 2$ BPS multiplets with mass (4.1.16). Note that these supermultiplets come in representations of the unbroken $\mathrm{SU}(N)_{C+F}$ group, namely, the singlet and the adjoint representations.

To conclude this section we want to discuss quantum effects in the theory (4.1.7). At a high scale m the $\mathrm{SU}(N+1)$ gauge group is broken down to $\mathrm{SU}(N)\times\mathrm{U}(1)$ by condensation of the adjoint fields if the condition (4.1.13) is met. The $\mathrm{SU}(N)$ sector is asymptotically free. The running of the corresponding gauge coupling, if non-interrupted, would drag the theory into the strong coupling regime. This would invalidate our quasiclassical analysis. Moreover, strong coupling effects on the Coulomb branch would break $\mathrm{SU}(N)$ gauge subgroup (as well as the $\mathrm{SU}(N)_{C+F}$ group) down to $\mathrm{U}(1)^{N-1}$ by the Seiberg–Witten mechanism [2]. No non-Abelian strings would emerge.

A possible way out was proposed in [143, 144]. One can add more flavors to the theory making $N_f > 2N$. Then the $\mathrm{SU}(N)$ sector is not asymptotically free and does not evolve into the strong coupling regime. However, the ANO strings in the multiflavor theory (on the Higgs branches) become semilocal strings [147] and confinement is lost (see Section 4.7).

Here we take a different route assuming the FI parameter ξ to be large,[4]

$$\xi \gg \Lambda_{\mathrm{SU}(N)}. \tag{4.1.18}$$

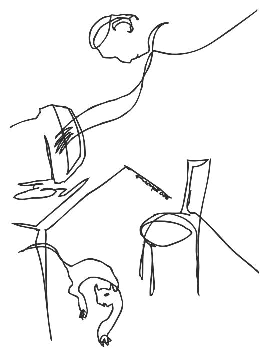
This condition ensures weak coupling in the $\mathrm{SU}(N)$ sector because the $\mathrm{SU}(N)$ gauge coupling does not run below the scale of the quark VEV's which is determined by ξ. More explicitly,

$$\frac{8\pi^2}{g_2^2(\xi)} = N \ln \frac{\sqrt{\xi}}{\Lambda_{\mathrm{SU}(N)}} \gg 1. \tag{4.1.19}$$

[4] We discuss this important issue in more detail at the end of Section 4.9.

Alternatively one can say that

$$\Lambda^N_{SU(N)} = \xi^{N/2} \exp\left(-\frac{8\pi^2}{g_2^2(\xi)}\right) \ll \xi^{N/2}. \qquad (4.1.20)$$



4.2 Z_N Abelian strings

Strictly speaking, $\mathcal{N} = 2$ SQCD with the gauge group SU($N + 1$) does not have stable flux tubes. They are unstable due to monopole–antimonopole pair creation in the SU($N+1$)/SU(N)$\times$U(1) sector. However, at large m these monopoles become heavy. In fact, there are no such monopoles in the low-energy theory (4.1.7) (where they can be considered as infinitely heavy). Therefore, the theory (4.1.7) has stable string solutions. When the perturbation μ Tr Φ^2 is truncated to the FI term (4.1.2), the theory enjoys $\mathcal{N} = 2$ supersymmetry and has BPS string solutions [127, 35, 148, 140, 131]. Note that here we discuss magnetic flux tubes. They are formed in the Higgs phase of the theory upon condensation of the squark fields and lead to confinement of monopoles.

Now, let us briefly review the BPS string solutions [140, 130, 131] in the model (4.1.7). We will consider the case of equal quark mass terms (4.1.13) when the global SU(N)$_{C+F}$ group is unbroken. First we review the Abelian solutions for Z_N strings and then, in Section 4.3 show that in the limit $m_1 = m_2 = \cdots = m_N \equiv m$ they acquire orientational moduli.

In fact, the Z_N Abelian strings considered below are just partial solutions of the vortex equations (see Eq. (4.2.10) below). In the equal mass limit (4.1.13) the global SU(N)$_{C+F}$ group is restored and the general solution for the non-Abelian string gets a continuous moduli space isomorphic to CP($N - 1$). The Z_N strings are just N discrete points on this moduli space.

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In the generic case of unequal quark masses, the $\mathrm{SU}(N)_{C+F}$ group is explicitly broken, and the continuous moduli space of the string solutions is lifted. Only the Z_N Abelian strings survive this breaking. We will dwell on the case of generic quark masses in Section 4.4.4.

It turns out that the string solutions do not involve the adjoint fields a and a^a. The BPS strings are "built" from gauge and quark fields only. Therefore, in order to find the classical solution, in the action (4.1.7) we can set the adjoint fields to their VEV's (4.1.11). This is consistent with equations of motion. Of course, at the quantum level the adjoint fields start fluctuating, deviating from their VEV's.

We use the *ansatz*

$$q^{kA} = \bar{\tilde{q}}^{kA} = \frac{1}{\sqrt{2}}\varphi^{kA} \tag{4.2.1}$$

reducing the number of the squark degrees of freedom to one complex field for each color and flavor. With these simplifications the action of the model (4.1.7) becomes

$$S = \int d^4x \left\{ \frac{1}{4g_2^2}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{4g_1^2}\left(F_{\mu\nu}\right)^2 \right. \\ \left. + |\nabla_\mu \varphi^A|^2 + \frac{g_2^2}{2}\left(\bar{\varphi}_A T^a \varphi^A\right)^2 + \frac{g_1^2}{8}\left(|\varphi^A|^2 - N\xi\right)^2 \right\}, \tag{4.2.2}$$

while the VEV's of the squark fields (4.1.14) are

$$\langle\varphi\rangle = \sqrt{\xi}\,\mathrm{diag}\,\{1, 1, \ldots, 1\}. \tag{4.2.3}$$

Since the spontaneously broken gauge U(1) is a part of the model under consideration, the model supports conventional ANO strings [36], in which one can discard the $\mathrm{SU}(N)_{\mathrm{gauge}}$ part of the action altogether. The topological stability of the ANO string is due to the fact that $\pi_1(\mathrm{U}(1)) = Z$.

These are not the strings we are interested in. At first sight, the triviality of the homotopy group, $\pi_1(\mathrm{SU}(N)) = 0$, implies that there are no other topologically stable strings. This impression is false. One can combine the Z_N center of $\mathrm{SU}(N)$ with the elements $\exp(2\pi i k/N)$ $\in$U(1) to get a topologically stable string solution possessing both windings, in $\mathrm{SU}(N)$ and U(1). In other words,

$$\pi_1\big(\mathrm{SU}(N) \times \mathrm{U}(1)/Z_N\big) \neq 0. \tag{4.2.4}$$

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It is easy to see that this nontrivial topology amounts to selecting just one element of φ, say, φ^{11}, or φ^{22}, etc, and make it wind, for instance,[5]

$$\varphi_{\text{string}} = \sqrt{\xi}\,\mathrm{diag}(1, 1, \ldots, e^{i\alpha}), \quad x \to \infty. \tag{4.2.5}$$

Such strings can be called elementary; their tension is $1/N$ th of that of the ANO string. The ANO string can be viewed as a bound state of N elementary strings.

More concretely, one of the Z_N string solutions (a progenitor of the non-Abelian string) can be written as follows [131]:

$$\varphi = \begin{pmatrix} \phi_2(r) & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \phi_2(r) & 0 \\ 0 & 0 & \ldots & e^{i\alpha}\phi_1(r) \end{pmatrix},$$
$$A_i^{\mathrm{SU}(N)} = \frac{1}{N}\begin{pmatrix} 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -(N-1) \end{pmatrix}(\partial_i\alpha)\big[-1 + f_{NA}(r)\big],$$
$$A_i^{\mathrm{U}(1)} = \frac{I}{2}A_i = \frac{I}{N}\,(\partial_i\alpha)\big[1 - f(r)\big], \quad A_0^{\mathrm{U}(1)} = A_0^{\mathrm{SU}(N)} = 0, \tag{4.2.6}$$

where $i = 1, 2$ labels the coordinates in the plane orthogonal to the string axis, r and α are the polar coordinates in this plane, and I is the unit $N \times N$ matrix. Other Z_N string solutions are obtained by permutations of the rotating flavor.

The profile functions $\phi_1(r)$ and $\phi_2(r)$ determine the profiles of the scalar fields, while $f_{NA}(r)$ and $f(r)$ determine the SU(N) and U(1) fields of the string solutions, respectively. These functions satisfy the following rather obvious boundary conditions:

$$\begin{aligned} &\phi_1(0) = 0, \\ &f_{NA}(0) = 1, \quad f(0) = 1, \end{aligned} \tag{4.2.7}$$

at $r = 0$, and

$$\begin{aligned} &\phi_1(\infty) = \sqrt{\xi}, \quad \phi_2(\infty) = \sqrt{\xi}, \\ &f_{NA}(\infty) = 0, \quad f(\infty) = 0 \end{aligned} \tag{4.2.8}$$

at $r = \infty$.

[5] As explained below, α is the angle of the coordinate $\vec{x}_\perp$ in the perpendicular plane.

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Now, let us derive the first-order equations which determine the profile functions, making use of the Bogomol'nyi representation [5] of the model (4.2.2). We have

$$T = \int d^2x \left\{ \left[\frac{1}{\sqrt{2}g_2} F_3^{*a} + \frac{g_2}{\sqrt{2}} \left(\bar{\varphi}_A T^a \varphi^A \right) \right]^2 + \left[\frac{1}{\sqrt{2}g_1} F_3^{*} + \frac{g_1}{2\sqrt{2}} \left(|\varphi^A|^2 - N\xi \right) \right]^2 + \left| \nabla_1 \varphi^A + i \nabla_2 \varphi^A \right|^2 + \frac{N}{2} \xi\, F_3^* \right\}, \tag{4.2.9}$$

where

$$F_3^* = F_{12} \text{ and } F_3^{*a} = F_{12}^a,$$

and we assume that the fields in question depend only on the transverse coordinates x_i, $i = 1, 2$.

The Bogomol'nyi representation (4.2.9) leads us to the following first-order equations:

$$\begin{aligned} &F_3^* + \frac{g_1^2}{2} \left(\left|\varphi^A\right|^2 - N\xi \right) = 0, \\ &F_3^{*a} + g_2^2 \left(\bar{\varphi}_A T^a \varphi^A \right) = 0, \\ &(\nabla_1 + i\nabla_2)\varphi^A = 0. \end{aligned} \tag{4.2.10}$$

Once these equations are satisfied, the energy of the BPS object is given by the last surface term in (4.2.9). Note that the representation (4.2.9) can be written also with the opposite sign in front of the flux terms. Then we would get the Bogomol'nyi equations for the anti-string.

For minimal winding we substitute the *ansatz* (4.2.6) in Eqs. (4.2.10) to get the first-order equations for the profile functions of the Z_N string [140, 131],

$$\begin{aligned} r\frac{d}{dr}\phi_1(r) - \frac{1}{N}\big(f(r) + (N-1) f_{NA}(r) \big)\phi_1(r) = 0, \\ r\frac{d}{dr}\phi_2(r) - \frac{1}{N}\big(f(r) - f_{NA}(r) \big)\phi_2(r) = 0, \\ -\frac{1}{r}\frac{d}{dr} f(r) + \frac{g_1^2 N}{4}\left[(N-1)\phi_2(r)^2 + \phi_1(r)^2 - N\xi \right] = 0, \\ -\frac{1}{r}\frac{d}{dr} f_{NA}(r) + \frac{g_2^2}{2}\left[\phi_1(r)^2 - \phi_2(r)^2 \right] = 0. \end{aligned} \tag{4.2.11}$$

These equations present a Z_N-string generalization of the Bogomol'nyi equations for the ANO string [5] (see also (3.2.19) and (C.13)). They were solved numerically

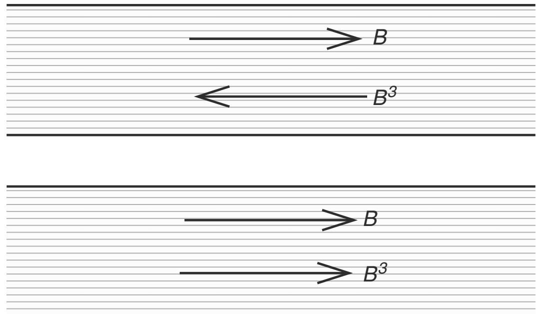

Figure 4.1. Two distinct Z_2 strings in U(2) theory.

for the U(2) case (i.e. $N = 2$) in [131]. Clearly, the solutions to the first-order equations automatically satisfy the second-order equations of motion.

The tension of this elementary Z_N string is

$$T_1 = 2\pi\,\xi. \tag{4.2.12}$$

Since our string is a BPS object, this result is exact and has neither perturbative nor nonperturbative corrections. Note that the tension of the ANO string is N times larger; in our normalization

$$T_{\rm ANO} = 2\pi\, N\,\xi. \tag{4.2.13}$$

Clearly, the *ansatz* (4.2.6) admits permutations, leading to other Z_N string solutions of type (4.2.6). They can be obtained by changing the position of the "winding" field in Eq. (4.2.6). Altogether we have N elementary Z_N strings. For instance, if $N = 2$ (i.e. the gauge group is SU(2)×U(1)), we have two distinct Z_2 strings differing by the orientation of the flux of the U(1) magnetic field with respect to that of the third isocomponent of the SU(2) magnetic field, see Fig. 4.1.

Of course, the first-order equations (4.2.11) can be also obtained using supersymmetry. We start from the supersymmetry transformations for the fermion fields in the theory (4.1.7),

$$\begin{aligned}
\delta\lambda^{f\alpha} &= \frac{1}{2}(\sigma_\mu\bar\sigma_\nu\epsilon^{\,f})^\alpha F_{\mu\nu} + \epsilon^{\alpha p}F^m(\tau^m)_p^{\,f} \;+\cdots,\\
\delta\lambda^{af\alpha} &= \frac{1}{2}(\sigma_\mu\bar\sigma_\nu\epsilon^{\,f})^\alpha F^a_{\mu\nu} + \epsilon^{\alpha p}F^{am}(\tau^m)_p^{\,f} \;+\cdots,\\
\delta\bar{\tilde\psi}{}^{kA}_{\dot\alpha} &= i\sqrt{2}\;\bar{\nabla}\!\!\!/_{\dot\alpha\alpha}q_f^{kA}\epsilon^{\alpha f} \;+\cdots,\\
\delta\bar\psi_{\dot\alpha Ak} &= i\sqrt{2}\;\bar{\nabla}\!\!\!/_{\dot\alpha\alpha}\bar q_{fAk}\epsilon^{\alpha f} \;+\cdots
\end{aligned} \tag{4.2.14}$$

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Here $f = 1, 2$ is the $\mathrm{SU}(2)_R$ index and $\lambda^{f\alpha}$ and $\lambda^{af\alpha}$ are the fermions from the $\mathcal{N} = 2$ vector supermultiplets of the U(1) and SU(2) factors, respectively, while q^{kAf} denotes the $\mathrm{SU}(2)_R$ doublet of the squark fields q^{kA} and $\bar{\tilde{q}}^{Ak}$ in the quark hypermultiplets. The parameters of the SUSY transformations in the microscopic theory are denoted as $\epsilon^{\alpha f}$. Furthermore, the F terms in Eq. (4.2.14) are

$$F^1 + iF^2 = i\,\frac{g_1^2}{2}\left(\mathrm{Tr}\,|\varphi|^2 - N\xi\right), \quad F^3 = 0 \tag{4.2.15}$$

for the U(1) field, and

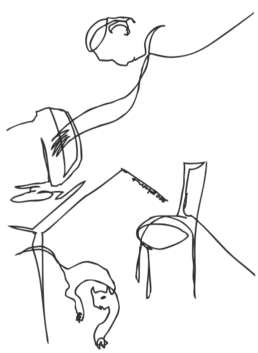
$$F^{a1} + iF^{a2} = i\,g_2^2\,\mathrm{Tr}\left(\bar{\varphi}T^a\varphi\right), \quad F^{a3} = 0 \tag{4.2.16}$$

for the SU(N) field. The dots in (4.2.14) stand for terms involving the adjoint scalar fields which vanish on the string solution (in the equal mass case) because the adjoint fields are given by their vacuum expectation values (4.1.11).

In Ref. [35] it was shown that four supercharges selected by the conditions

$$\epsilon^{12} = -\epsilon^{11}, \quad \epsilon^{21} = \epsilon^{22} \tag{4.2.17}$$

act trivially on the BPS string. Imposing the conditions (4.2.17) and requiring the left-hand sides of Eqs. (4.2.14) to vanish[6] we get, upon substituting the *ansatz* (4.2.6), the first-order equations (4.2.11).



[6] If, instead of (4.2.17), we required other combinations of the SUSY transformation parameters to vanish (changing the signs in (4.2.17)) we would get the anti-string equations, with the opposite direction of the gauge fluxes.

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4.3 Elementary non-Abelian strings

The elementary Z_N strings in the model (4.1.7) give rise to non-Abelian strings provided the condition (4.1.13) is satisfied [130, 131, 132, 133]. This means that, in addition to trivial translational moduli, they have extra moduli corresponding to spontaneous breaking (on the string) of a non-Abelian symmetry acting in the bulk theory in the infrared. Indeed, while the "flat" vacuum (4.1.14) is $\mathrm{SU}(N)_{C+F}$ symmetric, the solution (4.2.6) breaks this symmetry[7] down to $\mathrm{U}(1)\times\mathrm{SU}(N-1)$ (at $N > 2$). This ensures the presence of $2(N-1)$ orientational moduli.

To obtain the non-Abelian string solution from the Z_N string (4.2.6) we apply the diagonal color-flavor rotation preserving the vacuum (4.1.14). To this end it is convenient to pass to the singular gauge where the scalar fields have no winding at infinity, while the string flux comes from the vicinity of the origin. In this gauge we have

$$\varphi = U \begin{pmatrix} \phi_2(r) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \phi_2(r) & 0 \\ 0 & 0 & \dots & \phi_1(r) \end{pmatrix} U^{-1},$$

$$A_i^{\mathrm{SU}(N)} = \frac{1}{N} U \begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(N-1) \end{pmatrix} U^{-1} \left(\partial_i \alpha\right) f_{NA}(r),$$

$$A_i^{\mathrm{U}(1)} = -\frac{1}{N} \left(\partial_i \alpha\right) f(r), \quad A_0^{\mathrm{U}(1)} = A_0^{\mathrm{SU}(N)} = 0, \tag{4.3.1}$$

where U is a matrix $\in \mathrm{SU}(N)_{C+F}$. This matrix parametrizes orientational zero modes of the string associated with flux rotation in $\mathrm{SU}(N)$. Since the diagonal color-flavor symmetry is not broken by the VEV's of the scalar fields in the bulk (color-flavor locking) it is physical and has nothing to do with the gauge rotations eaten by the Higgs mechanism. The orientational moduli encoded in the matrix U are *not* gauge artifacts.

The orientational zero modes of a non-Abelian string were first observed in [130, 131]. In Ref. [130] a general index theorem was proved which shows that the dimension of elementary string moduli space is $2N = 2(N-1) + 2$ where 2 stands for translational moduli while $2(N-1)$ is the dimension of the internal moduli space.[8] In Ref. [131] the explicit solution for the non-Abelian string which we review here was found and explored.

[7] At $N = 2$ the string solution breaks SU(2) down to U(1).

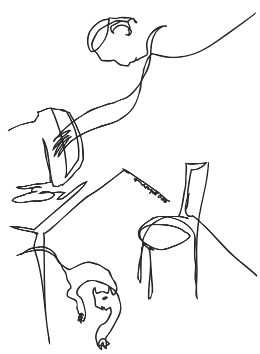
[8] The index theorem in [130] deals with more general multiple strings. It was shown that the dimension of the moduli space of the k-string solution is $2kN$.

In fact, non-translational zero modes of strings were discussed earlier in a U(1)×U(1) model [149, 150], and somewhat later, in more contrived models, in Ref. [151]. (The latter paper is entitled "Zero modes of non-Abelian vortices"!) It is worth emphasizing that, along with some apparent similarities, there are drastic distinctions between the "non-Abelian strings" we review here and the strings that were discussed in the 1980s. In particular, in the example treated in Ref. [151] the gauge group is not completely broken in the vacuum, and, therefore, there are massless gauge fields in the bulk. If the unbroken generator acts non trivially on the string flux (which is proportional to a broken generator) then it can and does create zero modes. Infrared divergence problems ensue immediately.

In the case we treat here the gauge group is completely broken (up to a discrete subgroup Z_N). The theory in the bulk is fully Higgsed. The unbroken group $\mathrm{SU}(N)_{C+F}$, a combination of the gauge and flavor groups, is global. There are no massless fields in the bulk.

It is possible to model the example considered in [151] if we gauge the unbroken global symmetry $\mathrm{SU}(N)_{C+F}$ of the model (4.1.7) with respect to yet another gauge field B_μ.

Let us also note that a generalization of the non-Abelian string solutions in six-dimensional gauge theory with eight supercharges was carried out in [152] while the non-Abelian strings in strongly coupled vacua were considered in [153].



4.4 The world-sheet effective theory

The non-Abelian string solution (4.3.1) is characterized by two translational moduli (the position of the string in the (1,2) plane) and $2(N-1)$ orientational moduli. Below we review the effective two-dimensional low-energy theory on the string world sheet. As usual, the translational moduli decouple and we focus on the internal dynamics of the orientational moduli. Our string is a 1/2-BPS state in $\mathcal{N}=2$

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supersymmetric gauge theory with eight supercharges. Thus it has four supercharges acting in the world sheet theory. This means that we have extended $\mathcal{N}=2$ supersymmetric effective theory on the string world sheet. This theory turns out to be a two-dimensional CP($N-1$) model [130, 131, 132, 133]. In Section 4.4 we will first present a derivation of this theory and then discuss the underlying physics.

4.4.1 Derivation of the CP($N-1$) model

Now, following Refs. [131, 132, 154], we will derive the effective low-energy theory for the moduli residing in the matrix U in the problem at hand. As is clear from the string solution (4.3.1), not each element of the matrix U will give rise to a modulus. The SU($N-1$)×U(1) subgroup remains unbroken by the string solution under consideration; therefore the moduli space is

$$\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-1)\times \mathrm{U}(1)} \sim \mathrm{CP}(N-1). \tag{4.4.1}$$

Keeping this in mind we parametrize the matrices entering Eq. (4.3.1) as follows:

$$\frac{1}{N}\left\{U\begin{pmatrix} 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots\\ 0 & \dots & 1 & 0\\ 0 & 0 & \dots & -(N-1)\end{pmatrix}U^{-1}\right\}_p^l = -n^l n_p^* + \frac{1}{N}\delta_p^l\,, \tag{4.4.2}$$

where n^l is a complex vector in the fundamental representation of SU(N), and

$$n_l^* n^l = 1, \tag{4.4.3}$$

($l, p = 1, \dots, N$ are color indices). As we will show below, one U(1) phase will be gauged away in the effective sigma model. This gives the correct number of degrees of freedom, namely, $2(N-1)$.

With this parametrization the string solution (4.3.1) can be rewritten as

$$\begin{aligned}
\varphi &= \frac{1}{N}\big[(N-1)\phi_2+\phi_1\big] + (\phi_1-\phi_2)\left(n\cdot n^* - \frac{1}{N}\right),\\
A_i^{\mathrm{SU}(N)} &= \left(n\cdot n^* - \frac{1}{N}\right)\varepsilon_{ij}\,\frac{x_j}{r^2}\,f_{NA}(r),\\
A_i^{\mathrm{U}(1)} &= \frac{1}{N}\varepsilon_{ij}\,\frac{x_j}{r^2}\,f(r),
\end{aligned} \tag{4.4.4}$$

where for brevity we suppress all SU(N) indices. The notation is self-evident.

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Assume that the orientational moduli are slowly varying functions of the string world-sheet coordinates x_k, $k = 0, 3$. Then the moduli n^l become fields of a $(1+1)$-dimensional sigma model on the world sheet. Since n^l parametrize the string zero modes, there is no potential term in this sigma model.

To obtain the kinetic term we substitute our solution (4.4.4), which depends on the moduli n^l, in the action (4.2.2), assuming that the fields acquire a dependence on the coordinates x_k via $n^l(x_k)$. In doing so we immediately observe that we have to modify our solution: we have to include in it the $k = 0, 3$ components of the gauge potential which are no longer vanishing. In the CP(1) case, as was shown in [132], the potential A_k must be orthogonal (in the SU(2) space) to the matrix (4.4.2), as well as to its derivatives with respect to x_k. Generalization of these conditions to the CP$(N-1)$ case leads to the following *ansatz*:

$$A_k^{\mathrm{SU}(N)} = -i\left[\partial_k n\ \cdot n^* - n\ \cdot \partial_k n^* - 2n\ \cdot n^*(n^*\partial_k n)\right]\rho(r), \quad \alpha = 0, 3, \quad (4.4.5)$$

where we assume the contraction of the color indices inside the parentheses,

$$(n^*\partial_k n) \equiv n_l^*\partial_k n^l,$$

and introduce a new profile function $\rho(r)$.

The function $\rho(r)$ in Eq. (4.4.5) is determined through a minimization procedure [131, 132, 154] which generates ρ's own equation of motion. Now we will outline its derivation. But at first we note that $\rho(r)$ vanishes at infinity,

$$\rho(\infty) = 0. \quad (4.4.6)$$

The boundary condition at $r = 0$ will be determined shortly.

The kinetic term for n^l comes from the gauge and quark kinetic terms in Eq. (4.2.2). Using Eqs. (4.4.4) and (4.4.5) to calculate the SU(N) gauge field strength we find

$$\begin{aligned} F_{ki}^{\mathrm{SU}(N)} &= \left(\partial_k n\ \cdot n^* + n\cdot\ \partial_k n^*\right)\varepsilon_{ij}\,\frac{x_j}{r^2}\,f_{NA}\big[1-\rho(r)\big] \\ &\quad + i\big[\partial_k n\ \cdot n^* - n\cdot\ \partial_k n^* - 2n\ \cdot n^*(n^*\partial_k n)\big]\,\frac{x_i}{r}\,\frac{d\rho(r)}{dr}. \quad (4.4.7)\end{aligned}$$

In order to have a finite contribution from the term Tr F_{ki}^2 in the action we have to impose the constraint

$$\rho(0) = 1. \quad (4.4.8)$$

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Substituting the field strength (4.4.7) in the action (4.2.2) and including, in addition, the quark kinetic term, after rather straightforward but tedious algebra we arrive at

$$S^{(1+1)} = 2\beta \int dt\, dz \left\{(\partial_k\, n^* \partial_k\, n) + (n^* \partial_k\, n)^2\right\}, \tag{4.4.9}$$

where the coupling constant β is given by

$$\beta = \frac{2\pi}{g_2^2}\, I, \tag{4.4.10}$$

and I is a basic normalizing integral

$$I = \int_0^\infty r dr \left\{ \left(\frac{d}{dr}\rho(r)\right)^2 + \frac{1}{r^2}\, f_{NA}^2 \left(1-\rho\right)^2 + g_2^2 \left[\frac{\rho^2}{2}(\phi_1^2 + \phi_2^2) + (1-\rho)(\phi_2 - \phi_1)^2\right]\right\}. \tag{4.4.11}$$

The theory in Eq. (4.4.9) is nothing but the two-dimensional CP($N-1$) model. To see that this is indeed the case we can eliminate the second term in (4.4.9) introducing a non-propagating U(1) gauge field. We review this in Section 4.4.3 (see also Appendix B), and then discuss the underlying physics of the model.

Thus, we obtain the CP($N-1$) model as an effective low-energy theory on the world sheet of the non-Abelian string. Its coupling constant β is related to the four-dimensional coupling g_2^2 via the basic normalizing integral (4.4.11). This integral must be viewed as an "action" for the profile function ρ.

Varying (4.4.11) with respect to ρ one obtains the second-order equation which the function ρ must satisfy, namely,

$$-\frac{d^2}{dr^2}\,\rho - \frac{1}{r}\,\frac{d}{dr}\,\rho - \frac{1}{r^2}\, f_{NA}^2(1-\rho) + \frac{g_2^2}{2}(\phi_1^2 + \phi_2^2)\rho - \frac{g_2^2}{2}\,(\phi_1 - \phi_2)^2 = 0. \tag{4.4.12}$$

After some algebra and extensive use of the first-order equations (4.2.11) one can show that the solution of (4.4.12) is

$$\rho = 1 - \frac{\phi_1}{\phi_2}. \tag{4.4.13}$$

This solution satisfies the boundary conditions (4.4.6) and (4.4.8). Substituting this solution back in the expression for the normalizing integral (4.4.11) one can check that this integral reduces to a total derivative and is given by the flux of the string determined by $f_{NA}(0) = 1$. In this way we arrive at

$$I = 1. \tag{4.4.14}$$

This result can be traced back to the fact that our theory (4.2.2) is $\mathcal{N} = 2$ supersymmetric theory, and the string is BPS saturated. In Section 4.5 we will see that this fact is crucial for the interpretation of confined monopoles as sigma-model kinks. Generally speaking, for non-BPS strings, I could be a certain function of N (see Ref. [155] for a particular example).

Equation (4.4.14) implies

$$\beta = \frac{2\pi}{g_2^2}. \tag{4.4.15}$$

The two-dimensional coupling is determined by the four-dimensional non-Abelian coupling. This relation is obtained at the classical level. In quantum theory both couplings run. Therefore, we have to specify a scale at which the relation (4.4.15) takes place. The two-dimensional CP($N-1$) model (4.4.9) is an effective low-energy theory appropriate for the description of internal string dynamics at low energies, lower than the inverse thickness of the string which is given by the masses of the gauge/quark multiplets (4.1.16) and (4.1.17) in the bulk SU(N)×U(1) theory. Thus, the parameter $g\sqrt{\xi}$ plays the role of a physical ultraviolet (UV) cut off in the action (4.4.9). This is the scale at which Eq. (4.4.15) holds. Below this scale, the coupling β runs according to its two-dimensional renormalization-group flow, see Section 4.4.3. It is worth noting that if the bulk theory were not Higgsed the running law of the bulk theory would exactly match that of the CP($N-1$) model. Indeed, Eq. (4.4.15) implies that

$$\left(4\pi\beta_{\mathrm{CP}(N-1)}\right)_0 = \left(\frac{8\pi^2}{g_2^2}\right)_0. \tag{4.4.16}$$

To get the running couplings we must add $-b_{\mathrm{CP}(N-1)}\ln(M_0/\mu)$ on the left-hand side of Eq. (4.4.16) and $-b_{\mathrm{SU}(N)\times\mathrm{U}(1)}\ln(M_0/\mu)$ on the right-hand side. The coefficients of the two- and four-dimensional Gell-Mann–Low functions coincide, $b_{\mathrm{CP}(N-1)} = b_{\mathrm{SU}(N)\times\mathrm{U}(1)} = N$.

Thus – we repeat again – the model (4.4.9) describes the low-energy limit: all higher-order terms in derivatives are neglected. Quartic in derivatives, sextic, and so on, terms certainly exist. In fact, the derivative expansion runs in powers of

$$\left(g_2\sqrt{\xi}\right)^{-1}\partial_\alpha, \tag{4.4.17}$$

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where $g_2\sqrt{\xi}$ gives the order of magnitude of masses in the bulk theory. The sigma model (4.4.9) is adequate at scales below $g_2\sqrt{\xi}$ where the higher-derivative corrections are negligibly small.

To conclude this section let us narrow down the model (4.4.9) setting $N = 2$. In this case we deal with the CP(1) model equivalent to the O(3) sigma model. The action (4.4.9) can be represented as (see Appendix B)

$$S^{(1+1)} = \frac{\beta}{2} \int dt\, dz\, (\partial_k S^a)^2, \tag{4.4.18}$$

where S^a $(a = 1, 2, 3)$ is a real unit vector, $(S^a)^2 = 1$, sweeping the two-dimensional sphere S_2. It is defined as

$$S^a = -n^* \tau^a n. \tag{4.4.19}$$

The model (4.4.18), as an effective theory on the world sheet of the non-Abelian string in SU(2)×U(1) SQCD with $\mathcal{N} = 2$ supersymmetry, was first derived in [131] in a field-theoretical framework. This derivation was generalized for arbitrary N in [154]. A brane construction of (4.4.9) was presented in [130].

4.4.2 Fermion zero modes

In Section 4.4.1 we derived the bosonic part of the effective $\mathcal{N} = 2$ supersymmetric CP$(N-1)$ model. Now we will find fermion zero modes for the non-Abelian string. Inclusion of these modes into consideration will demonstrate that the internal world sheet dynamics is given by $\mathcal{N} = 2$ supersymmetric CP$(N-1)$ model. This program was carried out in [132] for $N = 2$. Here we will focus on this construction.

The string solution (4.4.4) in the SU(2)×U(1) theory reduces to

$$\begin{aligned} \varphi &= U \begin{pmatrix} \phi_2(r) & 0 \\ 0 & \phi_1(r) \end{pmatrix} U^{-1}, \\ A_i^a(x) &= -S^a \, \varepsilon_{ij} \, \frac{x_j}{r^2} \, f_{NA}(r), \\ A_i(x) &= \varepsilon_{ij} \, \frac{x_j}{r^2} \, f(r), \end{aligned} \tag{4.4.20}$$

while the parametrization (4.4.2) reduces to

$$S^a \tau^a = U \tau^3 U^{-1}, \quad a = 1, 2, 3, \tag{4.4.21}$$

by virtue of Eq. (4.4.19).

Our string solution is 1/2 BPS-saturated. This means that four supercharges, out of eight of the four-dimensional theory (4.1), act trivially on the string solution

(4.4.20). The remaining four supercharges generate four fermion zero modes which were termed supertranslational modes because they are superpartners to two translational zero modes. The corresponding four fermionic moduli are superpartners to the coordinates x_0 and y_0 of the string center. The supertranslational fermion zero modes were found in Ref. [35] for the U(1) ANO string in $\mathcal{N} = 2$ theory. This is discussed in detail in Appendix C, see Section C.3. Transition to the non-Abelian model at hand is absolutely straightforward. We will not dwell on this procedure here.

Instead, we will focus on four *additional* fermion zero modes which arise only for the non-Abelian string, to be referred to as superorientational. They are superpartners of the bosonic orientational moduli S^a.

Let us see how one can explicitly construct these four zero modes (in CP(1)) and study their impact on the string world sheet.

At $N = 2$ the fermionic part of the action of the model (4.1.7) is

$$
\begin{aligned}
S_{\rm ferm} = \int d^4x \Bigg\{ & \frac{i}{g_2^2}\bar{\lambda}_f^a \bar{\not{D}}\lambda^{af} + \frac{i}{g_1^2}\bar{\lambda}_f \bar{\not{\partial}}\lambda^f + \mathrm{Tr}\left[\bar{\psi}\, i\bar{\not{\nabla}}\psi\right] + \mathrm{Tr}\left[\tilde{\psi}\, i\not{\nabla}\bar{\tilde{\psi}}\right] \\
& + \frac{1}{\sqrt{2}}\varepsilon^{abc}\bar{a}^a(\lambda_f^b\lambda^{cf}) + \frac{1}{\sqrt{2}}\varepsilon^{abc}(\bar{\lambda}^{bf}\bar{\lambda}_f^c)a^c \\
& + \frac{i}{\sqrt{2}}\,\mathrm{Tr}\left[\bar{q}_f(\lambda^f\psi) + (\tilde{\psi}\lambda_f)q^f + (\bar{\psi}\bar{\lambda}_f)q^f + \bar{q}^f(\bar{\lambda}_f\bar{\tilde{\psi}})\right] \\
& + \frac{i}{\sqrt{2}}\,\mathrm{Tr}\left[\bar{q}_f\tau^a(\lambda^{af}\psi) + (\tilde{\psi}\lambda_f^a)\tau^a q^f + (\bar{\psi}\bar{\lambda}_f^a)\tau^a q^f + \bar{q}^f\tau^a(\bar{\lambda}_f^a\bar{\tilde{\psi}})\right] \\
& + \frac{i}{\sqrt{2}}\,\mathrm{Tr}\left[\tilde{\psi}\left(a + a^a\tau^a\right)\psi\right] + \frac{i}{\sqrt{2}}\,\mathrm{Tr}\left[\bar{\psi}\left(a + a^a\tau^a\right)\bar{\tilde{\psi}}\right]\Bigg\},
\end{aligned}
\tag{4.4.22}
$$

where we use the matrix color-flavor notation for the matter fermions $(\psi^\alpha)^{kA}$ and $(\tilde{\psi}^\alpha)_{Ak}$. The traces in Eq. (4.4.22) are performed over the color-flavor indices. Contraction of spinor indices is assumed inside the parentheses, say, $(\lambda\psi) \equiv \lambda_\alpha\psi^\alpha$.

As was mentioned in Section 4.2, the four supercharges selected by the conditions (4.2.17) act trivially on the BPS string in the theory with the FI term of the F type. To generate superorientational fermion zero modes the following method was used in [132]. Assume the orientational moduli S^a in the string solution (4.4.20) to have a slow dependence on the world-sheet coordinates x_0 and x_3 (or t and z). Then the four (real) supercharges selected by the conditions (4.2.17) no longer act trivially. Instead, their action now generates fermion fields proportional to x_0 and x_3 derivatives of S^a.

This is exactly what one expects from the residual $\mathcal{N} = 2$ supersymmetry in the world sheet theory. The above four supercharges generate the world-sheet supersymmetry in the $\mathcal{N}=2$ two-dimensional CP(1) model,

$$\begin{aligned}
\delta\chi_1^a &= i\sqrt{2}\Big[(\partial_0 + i\partial_3)\,S^a\,\varepsilon_2 + \varepsilon^{abc}S^b\,(\partial_0 + i\partial_3)\,S^c\,\eta_2\Big],\\
\delta\chi_2^a &= i\sqrt{2}\Big[(\partial_0 - i\partial_3)\,S^a\,\varepsilon_1 + \varepsilon^{abc}S^b\,(\partial_0 - i\partial_3)\,S^c\,\eta_1\Big].
\end{aligned} \tag{4.4.23}$$

Here χ_α^a ($\alpha = 1, 2$ is the spinor index) are real two-dimensional fermions of the CP(1) model. They are superpartners of S^a and are subject to the orthogonality condition (see Appendix B)

$$S^a\chi_\alpha^a = 0. \tag{4.4.24}$$

The real parameters of the $\mathcal{N} = 2$ two-dimensional SUSY transformations ε_α and η_α are identified with the parameters of the four-dimensional SUSY transformations (with the constraint (4.2.17)) as follows:

$$\begin{aligned}
\varepsilon_1 - i\eta_1 &= \frac{1}{\sqrt{2}}(\epsilon^{21} + \epsilon^{22}) = \sqrt{2}\epsilon^{22},\\
\varepsilon_2 + i\eta_2 &= \frac{1}{\sqrt{2}}(\epsilon^{11} - \epsilon^{12}) = \sqrt{2}\epsilon^{11}.
\end{aligned} \tag{4.4.25}$$

In this way the world-sheet supersymmetry was used to re-express the fermion fields obtained upon the action of these four supercharges in terms of the $(1+1)$-dimensional fermions. This procedure gives us the superorientational fermion zero modes [132],

$$\begin{aligned}
\bar{\psi}_{Ak\dot{2}} &= \left(\frac{\tau^a}{2}\right)_{Ak}\frac{1}{2\phi_2}(\phi_1^2 - \phi_2^2)\Big[\chi_2^a + i\varepsilon^{abc}\,S^b\,\chi_2^c\Big],\\
\bar{\tilde{\psi}}_{\dot{1}}^{kA} &= \left(\frac{\tau^a}{2}\right)^{kA}\frac{1}{2\phi_2}(\phi_1^2 - \phi_2^2)\Big[\chi_1^a - i\varepsilon^{abc}\,S^b\,\chi_1^c\Big],\\
\bar{\psi}_{Ak\dot{1}} &= 0, \quad \bar{\tilde{\psi}}_{\dot{2}}^{kA} = 0,\\
\lambda^{a22} &= \frac{i}{2}\frac{x_1 + ix_2}{r^2}f_{NA}\frac{\phi_1}{\phi_2}\Big[\chi_1^a - i\varepsilon^{abc}\,S^b\,\chi_1^c\Big],\\
\lambda^{a11} &= \frac{i}{2}\frac{x_1 - ix_2}{r^2}f_{NA}\frac{\phi_1}{\phi_2}\Big[\chi_2^a + i\varepsilon^{abc}\,S^b\,\chi_2^c\Big],\\
\lambda^{a12} &= \lambda^{a11}, \quad \lambda^{a21} = \lambda^{a22},
\end{aligned} \tag{4.4.26}$$

where the dependence on x_i is encoded in the string profile functions, see Eq. (4.4.20).

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Now let us directly check that the zero modes (4.4.26) satisfy the Dirac equations of motion. From the fermion action of the model (4.4.22) we get the Dirac equations for λ^a,

$$\frac{i}{g_2^2}\bar{\not{D}}\lambda^{af} + \frac{i}{\sqrt{2}}\,\mathrm{Tr}\left(\bar{\psi}\tau^a q^f + \bar{q}^f\tau^a\bar{\tilde{\psi}}\right) = 0. \tag{4.4.27}$$

At the same time, for the matter fermions,

$$\begin{aligned} &i\not{\nabla}\bar{\psi} + \frac{i}{\sqrt{2}}\left[\bar{q}_f\lambda^f - (\tau^a\bar{q}_f)\lambda^{af} + (a - a^a\tau^a)\tilde{\psi}\right] = 0,\\ &i\not{\nabla}\bar{\tilde{\psi}} + \frac{i}{\sqrt{2}}\left[\lambda_f q^f + \lambda_f^a(\tau^a q^f) + (a + a^a\tau^a)\psi\right] = 0. \end{aligned} \tag{4.4.28}$$

Next, we substitute the orientational fermion zero modes (4.4.26) into these equations. After some algebra one can check that (4.4.26) do satisfy the Dirac equations (4.4.27) and (4.4.28) provided the first-order equations for the string profile functions (4.2.11) are satisfied.

Furthermore, it is instructive to check that the zero modes (4.4.26) do produce the fermion part of the $\mathcal{N}=2$ two-dimensional CP(1) model. To this end we return to the usual assumption that the fermion collective coordinates χ_α^a in Eq. (4.4.26) have an adiabatic dependence on the world-sheet coordinates x_k ($k = 0, 3$). This is quite similar to the procedure of Section 4.4.1 for bosonic moduli. Substituting Eq. (4.4.26) in the fermion kinetic terms in the bulk theory (4.4.22), and taking into account the derivatives of χ_α^a with respect to the world-sheet coordinates, we arrive at

$$\beta\int dt\,dz\left\{\frac{1}{2}\,\chi_1^a\,(\partial_0 - i\partial_3)\,\chi_1^a + \frac{1}{2}\,\chi_2^a\,(\partial_0 + i\partial_3)\,\chi_2^a\right\}, \tag{4.4.29}$$

where β is given by the same integral (4.4.15) as for the bosonic kinetic term, see Eq. (4.4.18).

Finally we must discuss the four-fermion interaction term in the CP(1) model. We can use the world sheet $\mathcal{N}=2$ supersymmetry to reconstruct this term. The SUSY transformations in the CP(1) model have the form (see e.g. [156] for a review)

$$\begin{aligned} \delta\chi_1^a &= i\sqrt{2}\,(\partial_1 + i\partial_3)\,S^a\,\varepsilon_2 - \sqrt{2}\varepsilon_1\,S^a(\chi_1^b\chi_2^b),\\ \delta\chi_2^a &= i\sqrt{2}\,(\partial_1 - i\partial_3)\,S^a\,\varepsilon_1 + \sqrt{2}\varepsilon_2\,S^a(\chi_1^b\chi_2^b),\\ \delta S^a &= \sqrt{2}(\varepsilon_1\chi_2^a + \varepsilon_2\chi_1^a), \end{aligned} \tag{4.4.30}$$

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where we put $\eta_\alpha = 0$ for simplicity. Imposing this supersymmetry leads to the following effective theory on the string world sheet

$$S_{\mathrm{CP}(1)} = \beta \int dt dz \left\{ \frac{1}{2}(\partial_k S^a)^2 + \frac{1}{2}\, \chi_1^a\, i(\partial_0 - i\partial_3)\, \chi_1^a \right. \\ \left. + \frac{1}{2}\, \chi_2^a\, i(\partial_0 + i\partial_3)\, \chi_2^a - \frac{1}{2}(\chi_1^a \chi_2^a)^2 \right\}, \tag{4.4.31}$$

This is indeed the action of the $\mathcal{N} = 2$ CP(1) sigma model in its entirety.

4.4.3 *Physics of the* CP($N - 1$) *model with* $\mathcal{N} = 2$

As is quite common in two dimensions, the Lagrangian of our effective theory on the string world sheet can be cast in many different (but equivalent) forms. In particular, the $\mathcal{N} = 2$ supersymmetric CP($N - 1$) model (4.4.9) can be understood as a strong-coupling limit of a U(1) gauge theory [157]. Then the bosonic part of the action takes the form

$$S = \int d^2x \left\{ 2\beta\, |\nabla_k n^\ell|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 \right. \\ \left. + 4\beta\, |\sigma|^2 |n^\ell|^2 + 2e^2 \beta^2 (|n^\ell|^2 - 1)^2 \right\}, \tag{4.4.32}$$

where $\nabla_k = \partial_k - i A_k$ while σ is a complex scalar field. The condition (4.4.3) is implemented in the limit $e^2 \to \infty$. Moreover, in this limit the gauge field A_k and its $\mathcal{N} = 2$ bosonic superpartner σ become auxiliary and can be eliminated by virtue of the equations of motion,

$$A_k = -\frac{i}{2}\, n_\ell^* \overset{\leftrightarrow}{\partial_k}\, n^\ell, \quad \sigma = 0. \tag{4.4.33}$$

Substituting Eq. (4.4.33) in the Lagrangian, we can readily rewrite the action in the form (4.4.9).

The coupling constant β is asymptotically free [158]. The running coupling, as a function of energy E, is given by the formula

$$4\pi\beta = N \ln \frac{E}{\Lambda_\sigma}, \tag{4.4.34}$$

where Λ_σ is a dynamical scale of the sigma model. The ultraviolet cut off of the sigma model on the string world sheet is determined by $g_2\sqrt{\xi}$. Equation (4.4.15) relating the two- and four-dimensional couplings is valid at this scale. Hence,

$$\Lambda_\sigma^N = g_2^N \xi^{\frac{N}{2}} e^{-\frac{8\pi^2}{g_2^2}} = \Lambda_{\mathrm{SU}(N)}^N. \tag{4.4.35}$$

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Here we take into account Eq. (4.1.20) for the dynamical scale $\Lambda_{SU(N)}$ of the SU(N) factor of the bulk theory. Note that in the bulk theory *per se*, because of the VEV's of the squark fields, the coupling constant is frozen at $g_2\sqrt{\xi}$; there are no logarithms below this scale. The logarithms of the string world-sheet theory take over. Moreover, the dynamical scales of the bulk and world-sheet theories turn out to be the same! We will explain the reason why the dynamical scale of the $(1+1)$-dimensional effective theory on the string world sheet is identical to that of the SU(N) factor of the $(3+1)$-dimensional gauge theory later, in Section 4.6.

The CP$(N-1)$ model was solved by Witten in the large-N limit [159]. We will briefly summarize Witten's results and translate them in terms of strings in four dimensions [132].

Classically the field n^ℓ can have arbitrary direction; therefore, one might naively expect a spontaneous breaking of SU(N) and the occurrence of massless Goldstone modes. Well, the Coleman theorem [160] teaches us that this cannot happen in two dimensions. Quantum effects restore the symmetry. Moreover, the condition (4.4.3) gets in effect relaxed. Due to strong coupling we have more degrees of freedom than in the original Lagrangian, namely all N fields n become dynamical and acquire masses Λ_σ.

As was shown by Witten [159], the model has N vacua. These N vacua differ from each other by the expectation value of the chiral bifermion operator, see e.g. [156]. At strong coupling the chiral condensate is the order parameter. The U(1) chiral symmetry of the CP$(N-1)$ model is explicitly broken to a discrete Z_{2N} symmetry by the chiral anomaly. The fermion condensate breaks Z_{2N} down to Z_2. That's the origin of the N-fold degeneracy of the vacuum state.

The physics of the model becomes even more transparent in the mirror representation which was established [120] for arbitrary N. In this representation one describes the CP$(N-1)$ model in terms of the Coulomb gas of instantons (see [161] where this was done for non-supersymmetric CP(1) model) to prove its equivalence to an affine Toda theory. The CP$(N-1)$ model (4.4.32) is dual to the following $\mathcal{N}=2$ affine Toda model [120, 162, 61, 163],

$$S_{\text{mirror}} = \int d^2x d^2\theta\, d^2\bar{\theta}\, \beta^{-1} \sum_{i=1}^{N-1} \bar{Y}_i\, Y_i + \left\{\Lambda_\sigma \int d^2x d^2\theta \left(\sum_{i=1}^{N-1} \exp(Y_i) + \prod_{i=1}^{N-1} \exp(-Y_i)\right) + \text{H.c.}\right\}. \tag{4.4.36}$$

Here the last term is a dual instanton-induced superpotential. In fact, the exact form of the kinetic term in the mirror representation is not known because it is not protected from quantum correction in β. However the superpotential in (4.4.36)

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is *exact*. Since the vacuum structure is entirely determined by the superpotential (4.4.36), one immediately confirms Witten's statement of N vacua.

Indeed, the scalar potential of this affine Toda theory has N minima. For example, for $N = 2$ this theory becomes $\mathcal{N} = 2$ supersymmetric sine-Gordon theory with scalar potential

$$V_{\rm SG} = \frac{\beta}{4\pi^2}\,\Lambda^2_{\rm CP(1)}\,|\sinh y|^2, \tag{4.4.37}$$

which obviously has two minima, at $y = 0$ and $y = \pm i\pi$ (warning: the points $y = i\pi$ and $y = -i\pi$ must be identified; they present one and the same vacuum).

This mirror model explicitly exhibits a mass gap of the order of Λ_σ. It shows that there are no Goldstone bosons (corresponding to the absence of the spontaneous breaking of the $\mathrm{SU}(N)_{C+F}$ symmetry). In terms of strings in the four-dimensional bulk theory, this means, in turn, that the magnetic flux orientation in the target space has no particular direction, it is smeared all over. The N vacua of the world-sheet theory (4.4.32) are heirs of the N "elementary" non-Abelian strings of the bulk theory. Note that these strings are in a highly quantum regime. They are *not* the Z_N strings of the quasiclassical $\mathrm{U}(1)^{N-1}$ theory since n^ℓ is not aligned in the vacuum.

Hori and Vafa originally derived [120] the mirror representation for the CP($N-1$) model in the form of the Toda model. Since then other useful equivalent representations were obtained, and they were expanded to include the so-called twisted masses of which we will speak in Section 4.4.4 and subsequent sections. A particularly useful mirror representation of the twisted-mass-deformed CP($N-1$) model was exploited by Dorey [30].

4.4.4 Unequal quark masses

The fact that we have N distinct vacua in the world sheet theory – N distinct elementary strings – is not quite intuitive in the above consideration. This is understandable. At the classical level the $\mathcal{N} = 2$ two-dimensional CP($N-1$) sigma model is characterized by a continuous vacuum manifold. This is in one-to-one correspondence with continuously many strings parametrized by the moduli n^ℓ. The continuous degeneracy is lifted only after quantum effects are taken into account. These quantum effects become crucial at strong coupling. Gone with this lifting is the moduli nature of the fields n^ℓ. They become massive. This is difficult to grasp.

To make the task easier and to facilitate contact between the bulk and world sheet theories, it is instructive to start from a deformed bulk theory, so that the string moduli are lifted already at the classical level. Then the origin of the N-fold degeneracy of the non-Abelian strings becomes transparent. This will help us understand, in an intuitive manner, other features listed above. After this understanding is achieved, nothing prevents us from returning to our problem – strings with non-Abelian moduli at the classical level – by smoothly suppressing the moduli-breaking deformation. The N-fold degeneracy will remain intact as it follows from the Witten index [123].

Thus, let us drop the assumption (4.1.13) of equal mass terms and introduce small mass differences. With unequal quark masses, the $\mathrm{U}(N)$ gauge group is broken by the condensation of the adjoint scalars down to $\mathrm{U}(1)^N$, see (4.1.11). Off-diagonal gauge bosons, as well as the off-diagonal fields of the quark matrix q^{kA}, (together with their fermion superpartners) acquire masses proportional to various mass differences $(m_A - m_B)$. The effective low-energy theory contains now only diagonal gauge and quark fields. The reduced action suitable for the search of the string solution takes the form

$$S = \int d^4x \left\{ \frac{1}{4g_2^2}\left(F^h_{\mu\nu}\right)^2 + \frac{1}{4g_1^2}\left(F_{\mu\nu}\right)^2 \right. \\ \left. + |\nabla_\mu \varphi^A|^2 + \frac{g_2^2}{2}\left(\bar{\varphi}_A T^h \varphi^A\right)^2 + \frac{g_1^2}{8}\left(|\varphi^A|^2 - N\xi\right)^2, \right. \tag{4.4.38}$$

where the index $h = 1, \ldots, (N-1)$ runs over the Cartan generators of the gauge group $\mathrm{SU}(N)$, while the matrix φ^{kA} is reduced to its diagonal components.

The same steps which previously lead us to Eqs. (4.2.10) now give the first-order string equations in the Abelian model (4.4.38),

$$\begin{aligned} &F_3^* + \frac{g_1^2}{2}\left(\left|\varphi^A\right|^2 - N\xi\right) = 0, \\ &F_3^{*h} + g_2^2\left(\bar{\varphi}_A T^h \varphi^A\right) = 0, \\ &(\nabla_1 + i\nabla_2)\varphi^A = 0. \end{aligned} \tag{4.4.39}$$

As soon as the Z_N-string solutions (4.2.6) have a diagonal form, they automatically satisfy the above first-order equations.

However, the Abelian Z_N strings (4.2.6) are now *the only* solutions to these equations. The family of solutions is discrete. The global $\mathrm{SU}(N)_{C+F}$ group is broken down to $\mathrm{U}(1)^{N-1}$ by the mass differences, and the continuous $\mathrm{CP}(N-1)$ moduli space of the non-Abelian string is lifted. In fact, the vector n^ℓ gets fixed in N possible positions,

$$n^\ell = \delta^{\ell\ell_0}, \quad \ell_0 = 1, \ldots, N. \tag{4.4.40}$$

These N solutions correspond to the Abelian Z_N strings, see (4.2.6) and (4.4.4). If the mass differences are much smaller than $\sqrt{\xi}$ the set of parameters n^ℓ becomes *quasi*moduli.

Now, our aim is to derive an effective two-dimensional theory on the string world sheet for unequal quark mass terms. With small mass differences we will still be able to introduce orientational quasimoduli n^ℓ. In terms of the effective two-dimensional theory on the string world sheet, unequal masses lead to a shallow potential for the quasimoduli n^ℓ. Let us derive this potential.

Below we will review the derivation carried out in [132] in the SU(2)×U(1) model. The case of general N is considered in [133]. In the $N = 2$ case two minima of the potential at $S = \{0, 0, \pm 1\}$ correspond to two distinct Z_2 strings.

We start from the expression for the non-Abelian string in the singular gauge (4.4.20) parametrized by the moduli S^a, and substitute it in the action (4.1.7). The only modification we actually have to make is to supplement our *ansatz* (4.4.20) by that for the adjoint scalar field a^a; the neutral scalar field a will stay fixed at its vacuum expectation value $a = -\sqrt{2}m$.

At large r the field a^a tends to its VEV aligned along the third axis in the color space,

$$\langle a^3 \rangle = -\frac{\Delta m}{\sqrt{2}}, \quad \Delta m = m_1 - m_2, \tag{4.4.41}$$

see Eq. (4.1.11). At the same time, at $r = 0$ it must be directed along the vector S^a. The reason for this behavior is easy to understand. The kinetic term for a^a in Eq. (4.1.7) contains the commutator term of the adjoint scalar and the gauge potential. The gauge potential is singular at the origin, as is seen from Eq. (4.4.20). This implies that a^a must be aligned along S^a at $r = 0$. Otherwise, the string tension would become divergent. The following *ansatz* for a^a ensures this behavior:

$$a^a = -\frac{\Delta m}{\sqrt{2}} \left[\delta^{a3}\, b + S^a\, S^3\, (1 - b) \right]. \tag{4.4.42}$$

Here we introduced a new profile function $b(r)$ which, as usual, will be determined from a minimization procedure. Note that at $S^a = (0, 0, \pm 1)$ the field a^a is given by its VEV, as expected. The boundary conditions for the function $b(r)$ are

$$b(\infty) = 1, \quad b(0) = 0. \tag{4.4.43}$$

Substituting Eq. (4.4.42) in conjunction with (4.4.20) in the action (4.1.7) we get the potential

$$V_{\mathrm{CP}(1)} = \gamma \int d^2x\, \frac{\Delta m^2}{2} \left(1 - S_3^2\right), \tag{4.4.44}$$

where γ is given by the integral

$$\gamma = \frac{2\pi}{g_2^2} \int_0^\infty r\,dr \left\{ \left(\frac{d}{dr} b(r) \right)^2 + \frac{1}{r^2} f_{NA}^2\, b^2 \right.$$
$$\left. + \, g_2^2 \left[\frac{1}{2} (1-b)^2 \left(\phi_1^2 + \phi_2^2 \right) + b\,(\phi_1 - \phi_2)^2 \right] \right\}. \qquad (4.4.45)$$

Here two first terms in the integrand come from the kinetic term of the adjoint scalar field a^a while the term in the square brackets comes from the potential in the action (4.1.7).

Minimization with respect to $b(r)$, with the constraint (4.4.43), yields

$$b(r) = 1 - \rho(r) = \frac{\phi_1}{\phi_2}(r), \qquad (4.4.46)$$

cf. Eqs. (4.4.11) and (4.4.13). Thus,

$$\gamma = I \times \frac{2\pi}{g_2^2} = \frac{2\pi}{g_2^2}. \qquad (4.4.47)$$

We see that the normalization integrals are the same for both the kinetic and the potential terms in the world-sheet sigma model, $\gamma = \beta$. As a result we arrive at the following effective theory on the string world sheet:

$$S_{\mathrm{CP}(1)} = \beta \int d^2x \left\{ \frac{1}{2} (\partial_k S^a)^2 + \frac{|\Delta m|^2}{2} \left(1 - S_3^2 \right) \right\}. \qquad (4.4.48)$$

This is the only functional form that allows $\mathcal{N} = 2$ completion.[9] See also Section 3.5.

The fact that we obtain this form shows that our *ansatz* is fully adequate. The informative aspect of the procedure is (i) confirmation of the *ansatz* (4.4.42) and (ii) constructive calculation of the constant in front of $(1 - S_3^2)$ in terms of the bulk parameters. The mass-splitting parameter Δm of the bulk theory exactly coincides with the twisted mass of the world-sheet model.

The CP(1) model (4.4.48) has two vacua located at $S^a = (0, 0, \pm 1)$, see Fig. 3.11. Clearly these two vacua correspond to two elementary Z_2 strings.

For generic N the potential in the CP($N-1$) model was obtained in [133]. It has the form

$$V_{\mathrm{CP}(N-1)} = 2\beta \left\{ \sum_\ell |\tilde{m}_\ell|^2 |n^\ell|^2 - \left| \sum_\ell \tilde{m}_\ell |n^\ell|^2 \right|^2 \right\}, \qquad (4.4.49)$$

[9] Note, that although the global $\mathrm{SU}(2)_{C+F}$ is broken by Δm, the extended $\mathcal{N} = 2$ supersymmetry is not.

where

$$\tilde{m}_\ell = m_\ell - m, \quad m \equiv \frac{1}{N}\sum_\ell m_\ell, \quad \ell = 1, \ldots, N. \tag{4.4.50}$$

From the perspective of the bulk theory the index ℓ of the CP($N-1$) model coincides with the flavor index, $\ell \equiv A$. The above potential has N vacua (4.4.40) which correspond to N distinct Z_N strings in the bulk theory.

The CP($N-1$) model with the potential (4.4.49) is nothing but a bosonic truncation of the $\mathcal{N}=2$ two-dimensional sigma model which was termed the twisted-mass-deformed CP($N-1$) model. This is a generalization of the massless CP($N-1$) model which preserves four supercharges. Twisted chiral superfields in two dimensions were introduced in [32] while the twisted mass as an expectation value of the twisted chiral multiplet was suggested in [31]. CP($N-1$) models with twisted mass were further studied in [30] and, in particular, the BPS spectra in these theories were determined exactly.

From the bulk theory standpoint the two-dimensional CP($N-1$) model is an effective world sheet theory for the non-Abelian string, and the emergence of $\mathcal{N}=2$ supersymmetry should be expected. As we know, the BPS nature of the strings under consideration does require the world sheet theory to have four supercharges.

The twisted-mass-deformed CP($N-1$) model can be nicely rewritten as a strong coupling limit of a U(1) gauge theory [30]. With twisted masses of the n^ℓ fields taken into account, the bosonic part of the action (4.4.32) becomes

$$S = \int d^2x \left\{ 2\beta\, |\nabla_k n^\ell|^2 + \frac{1}{4e^2}F_{kl}^2 + \frac{1}{e^2}|\partial_k\sigma|^2 \right.$$
$$\left. + \, 4\beta \left|\sigma - \frac{\tilde{m}_\ell}{\sqrt{2}}\right|^2 |n^\ell|^2 + 2e^2\beta^2(|n^\ell|^2 - 1)^2 \right\}. \tag{4.4.51}$$

In the limit $e^2 \to \infty$ the σ field can be excluded by virtue of an algebraic equation of motion which leads to the potential (4.4.49).

As was already mentioned, this sigma model gives an effective description of our non-Abelian string at low energies, i.e. at energies much lower than the inverse string thickness. Typical momenta in the theory (4.4.51) are of the order of $\tilde{m}$. Therefore, for the action (4.4.51) to be applicable we must impose the condition

$$\left|\tilde{m}_\ell\right| \ll g_2\sqrt{\xi}. \tag{4.4.52}$$

The description in terms of the twisted-mass-deformed CP($N-1$) model gives us a much better understanding of dynamics of the non-Abelian strings. If masses $\tilde{m}_\ell$ are much larger than the scale of the CP($N-1$) model Λ_σ, the coupling constant β

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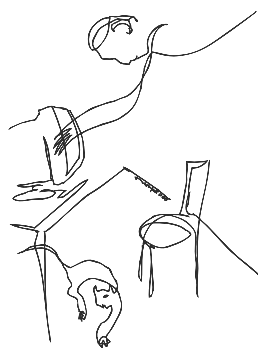
is frozen at a large scale (of the order of $\tilde{m}_\ell$) and the theory is at weak coupling. Semiclassical analysis is applicable. The theory (4.4.51) has N vacua located at

$$n^\ell = \delta^{\ell\ell_0}, \quad \sigma = \frac{\tilde{m}_{\ell_0}}{\sqrt{2}}, \quad \ell_0 = 1, \ldots, N. \tag{4.4.53}$$

They correspond to the Abelian Z_N strings of the bulk theory, see (4.4.4). As we reduce the mass differences $\tilde{m}_\ell$ and hit the value Λ_σ, the CP($N-1$) model under consideration enters the strong coupling regime. At $\tilde{m}_\ell = 0$ the global SU$(N)_{C+F}$ symmetry of the bulk theory is restored. Now n^ℓ has no particular direction. The condition (4.4.3) is relaxed. Still we have N vacua in the world sheet theory (Witten's index!). They are seen in the mirror description, see Section 4.4.3. These vacua correspond to N elementary non-Abelian strings in the strong coupling quantum regime. Thus, we see that for the BPS strings the transition from the Abelian to non-Abelian regimes is smooth. As we will discuss in Chapter 5, this is not the case for non-BPS strings. In the latter case the two regimes are separated by a phase transition [154, 164].



4.5 Confined monopoles as kinks of the CP($N-1$) model

Our bulk theory (4.1.7) is in the Higgs phase and therefore the magnetic monopoles of this theory must be in the confinement phase. If we start from a theory with the SU($N+1$) gauge group broken to SU(N)×U(1) by condensation of the adjoint scalar a from which the theory (4.1.7) emerges, the monopoles of the SU($N+1$)/SU(N) × U(1) sector can be attached to the endpoints of the Z_N strings under consideration. In the bulk theory (4.1.7) these monopoles are infinitely heavy at $m \to \infty$, and hence the Z_N strings are stable. However, the monopoles residing in the SU(N) gauge group are still present in the theory (4.1.7). As we switch on the FI parameter ξ, the squarks condense triggering confinement of these monopoles.

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In this section we will show that these monopoles manifest themselves as string junctions of the non-Abelian strings and are seen as kinks in the world sheet theory interpolating between distinct vacua of the CP$(N-1)$ model [165, 132, 133].

Our task in this section is to trace the evolution of the confined monopoles starting from the quasiclassical regime, and deep into the quantum regime. For illustrative purposes it will be even more instructive if we started from the limit of weakly confined monopoles, when in fact they present just slightly distorted 't Hooft–Polyakov monopoles (Fig. 4.3).

Let us start from the limit $|\Delta m_{AB}| \gg \sqrt{\xi}$ and assume all mass differences to be of the same order. In this limit the scalar quark expectation values can be neglected, and the vacuum structure is determined by VEV's of the adjoint field a^a, see (4.1.11). In the non-degenerate case the gauge symmetry SU(N) of our bulk model is broken down to U$(1)^{N-1}$ modulo possible discrete subgroups. This is the textbook situation for occurrence of the SU(N) 't Hooft–Polyakov monopoles. The monopole core size is of the order of $|\Delta m_{AB}|^{-1}$. The 't Hooft–Polyakov solution remains valid up to much larger distances, of the order of $\xi^{-1/2}$. At distances larger than $\sim \xi^{-1/2}$ the quark VEV's become important. As usual, the U(1) charge condensation leads to the formation of the U(1) magnetic flux tubes, with the transverse size of the order of $\xi^{-1/2}$ (see the upper picture in Fig. 4.3). The flux is quantized; the flux tube tension is tiny in the scale of the square of the monopole mass. Therefore, what we deal with in this limit is basically a very weakly confined 't Hooft–Polyakov monopole.

Let us verify that the confined monopole is a junction of two strings. Consider the junction of two Z_N strings corresponding to two "neighboring" vacua of the CP$(N-1)$ model. For the ℓ_0-th vacuum n^ℓ is given by (4.4.53) while for the ℓ_0+1-th vacuum it is given by the same equations with $\ell_0 \to \ell_0+1$. The flux of this junction is given by the difference of the fluxes of these two strings. Using (4.4.4) we get that the flux of the junction is

$$4\pi \times \text{diag}\,\frac{1}{2}\{\ldots\, 0,\, 1,\, -1,\, 0,\, \ldots\}, \tag{4.5.1}$$

with the nonvanishing entries located at positions ℓ_0 and ℓ_0+1. These are exactly the fluxes of $N-1$ distinct 't Hooft–Polyakov monopoles occurring in the SU(N) gauge theory provided that SU(N) is spontaneously broken down to U$(1)^{N-1}$. For instance, in U(2) theory the junction of two Z_2 strings is shown in Fig. 4.2.

We see that in the quasiclassical limit of large $|\Delta m_{AB}|$ the Abelian monopoles play the role of junctions of the Abelian Z_N strings. Note that in various models the monopole fluxes and those of strings were shown to match each other [166, 167, 168, 169, 140, 170, 171] so that the monopoles can be confined by strings in the Higgs phase.

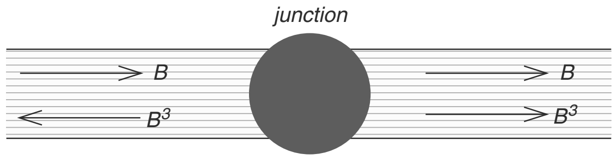

Figure 4.2. The junction of two distinct Z_2 strings in the U(2) theory.

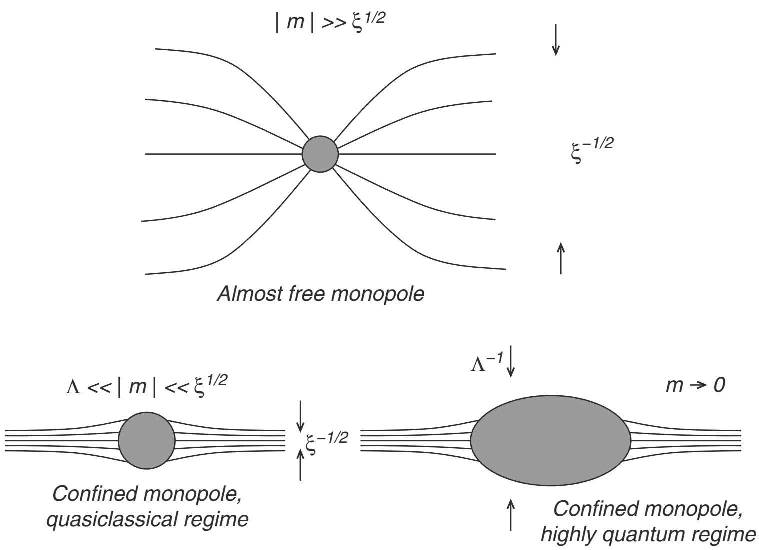

Figure 4.3. Evolution of the confined monopoles.

Now, let us reduce $|\Delta m_{AB}|$. If this parameter is limited inside the interval

$$\Lambda \ll |\Delta m_{AB}| \ll \sqrt{\xi}, \tag{4.5.2}$$

the size of the monopole ($\sim |\Delta m_{AB}|^{-1}$) becomes larger than the transverse size of the attached strings. The monopole gets squeezed in earnest by the strings – it becomes a *bona fide* confined monopole (the lower left corner of Fig. 4.3). A natural question is how this confined monopole is seen in the effective two-dimensional CP($N-1$) model (4.4.51) on the string world sheet. Since the Z_N strings of the bulk theory correspond to N vacua of the CP($N-1$) model the string junction (confined monopole) is a "domain wall" – kink – interpolating between these vacua, see Fig. 3.11.

Below we will explicitly demonstrate that in the semiclassical regime (4.5.2) the solution for the string junction in the bulk theory is in one-to-one correspondence with the kink in the world sheet theory. Then we will show that the masses of

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the monopole and kink perfectly match. This was demonstrated in [132] in the $N = 2$ case.

4.5.1 The first-order master equations

In this section we derive the first-order equations for the 1/4-BPS junction of the Z_N strings in the SU(N)×U(1) theory in the quasiclassical limit (4.5.2). In this limit Δm_{AB} is sufficiently small so that we can use our effective low-energy description in terms of the twisted-mass-deformed CP($N-1$) model (4.4.51). On the other hand, Δm_{AB} is much larger than the dynamical scale of the CP($N-1$) model; hence, the latter is in the weak coupling regime which allows one to apply quasiclassical treatment.

The geometry of our junction is shown in the left corner of Fig. 4.3. Both strings are stretched along the z axis. We assume that the monopole sits near the origin, the $n^\ell = \delta^{\ell\ell_0}$-string is at negative z while the $n^\ell = \delta^{\ell(\ell_0+1)}$-string is at positive z. The perpendicular plane is parametrized by x_1 and x_2. What is sought for is a static solution of the BPS equations, with all relevant fields depending only on x_1, x_2 and z.

Ignoring the time variable we can represent the energy functional of our theory (4.2.2) as follows (the Bogomol'nyi representation [5]):

$$
\begin{aligned}
E = \int d^3x \Bigg\{ & \left[\frac{1}{\sqrt{2}g_2} F_3^{*a} + \frac{g_2}{2\sqrt{2}} \left(\bar{\varphi}_A \tau^a \varphi^A \right) + \frac{1}{g_2} D_3 a^a \right]^2 \\
& + \left[\frac{1}{\sqrt{2}g_1} F_3^{*} + \frac{g_1}{2\sqrt{2}} \left(|\varphi^A|^2 - 2\xi \right) + \frac{1}{g_1} \partial_3 a \right]^2 \\
& + \frac{1}{g_2^2} \left| \frac{1}{\sqrt{2}} \left(F_1^{*a} + i F_2^{*a} \right) + \left(D_1 + i D_2 \right) a^a \right|^2 \\
& + \frac{1}{g_1^2} \left| \frac{1}{\sqrt{2}} \left(F_1^{*} + i F_2^{*} \right) + \left(\partial_1 + i \partial_2 \right) a \right|^2 \\
& + \left| \nabla_1 \varphi^A + i \nabla_2 \varphi^A \right|^2 \\
& + \left| \nabla_3 \varphi^A + \frac{1}{\sqrt{2}} \left(a^a \tau^a + a + \sqrt{2} m_A \right) \varphi^A \right|^2 \Bigg\}
\end{aligned}
\tag{4.5.3}
$$

plus surface terms. As compared to the Bogomol'nyi representation (4.2.9) for strings we keep here also terms involving the adjoint fields. Following our conventions we assume the quark mass terms to be real implying that the vacuum

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expectation values of the adjoint scalar fields are real too. The surface terms mentioned above are

$$E_{\text{surface}} = \xi \int d^3x\, F_3^* + \sqrt{2}\,\xi \int d^2x\, \langle a \rangle \Big|_{z=-\infty}^{z=\infty} - \sqrt{2}\, \frac{\langle a^a \rangle}{g_2^2} \int dS_n\, F_n^{*a}, \quad (4.5.4)$$

where the integral in the last term runs over a large two-dimensional sphere at $\vec{x}^{\,2} \to \infty$. The first term on the right-hand side is related to strings, the second to domain walls, while the third to monopoles (or the string junctions).

The Bogomol'nyi representation (4.5.3) leads us to the following first-order equations:

$$\begin{aligned}
&F_1^* + i F_2^* + \sqrt{2}(\partial_1 + i\partial_2)a = 0,\\
&F_1^{*a} + i F_2^{*a} + \sqrt{2}(D_1 + i D_2)a^a = 0,\\
&F_3^* + \frac{g_1^2}{2}\Big(|\varphi^A|^2 - 2\xi\Big) + \sqrt{2}\,\partial_3 a = 0,\\
&F_3^{*a} + \frac{g_2^2}{2}\Big(\bar{\varphi}_A \tau^a \varphi^A\Big) + \sqrt{2}\, D_3 a^a = 0,\\
&\nabla_3 \varphi^A = -\frac{1}{\sqrt{2}}(a^a\tau^a + a + \sqrt{2} m_A)\varphi^A,\\
&(\nabla_1 + i\nabla_2)\varphi^A = 0.
\end{aligned} \quad (4.5.5)$$

These are our master equations. Once these equations are satisfied the energy of the BPS object is given by Eq. (4.5.4).

Let us discuss the central charges (the surface terms) of the string, domain wall and monopole in more detail. Say, in the string case, the three-dimensional integral in the first term in Eq. (4.5.4) gives the length of the string times its flux. In the wall case, the two-dimensional integral in the second term in (4.5.4) gives the area of the wall times its tension. Finally, in the monopole case the integral in the last term in Eq. (4.5.4) gives the magnetic-field flux. This means that the first-order master equations (4.5.5) can be used to study *strings, domain walls, monopoles and all their possible junctions*.

It is instructive to check that the wall, the string and the monopole solutions, separately, satisfy these equations. For the domain wall this check was done in [37] where we used these equations to study the string-wall junctions (we review this in Chapter 9). Let us consider the string solution. Then the scalar fields a and a^a are given by their VEV's. The gauge flux is directed along the z axis, so that $F_1^* = F_2^* = F_1^{*a} = F_2^{*a} = 0$. All fields depend only on the perpendicular coordinates x_1 and x_2. As a result, the first two equations and the fifth one in (4.5.5) are trivially satisfied. The third and the fourth equations reduce to the first two

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equations in Eq. (4.2.10). The last equation in (4.5.5) reduces to the last equation in (4.2.10).

Now, turn to the monopole solution. The 't Hooft–Polyakov monopole equations [105, 106] arise from those in Eq. (4.5.5) in the limit $\xi = 0$. Then all quark fields vanish, and Eq. (4.5.5) reduces to the standard first-order equations for the BPS 't Hooft–Polyakov monopole (see Section 3.4),

$$F_k^{*a} + \sqrt{2}\, D_k\, a^a = 0. \tag{4.5.6}$$

The U(1) scalar field a is given by its VEV while the U(1) gauge field vanishes.

Now, Eq. (4.5.4) shows that the central charge of the SU(2) monopole is determined by $\langle a^a \rangle$ which is proportional to the quark mass difference, see (4.1.11). Thus, for the monopole on the Coulomb branch (i.e. at $\xi = 0$) Eq. (4.5.4) yields

$$M_M = \frac{4\pi\,(m_{\ell_0+1} - m_{\ell_0})}{g_2^2}. \tag{4.5.7}$$

This coincides, of course, with the Seiberg–Witten result [2] in the weak coupling limit. As we will see shortly, the same expression continues to hold even if $\Delta m_{AB} \ll \sqrt{\xi}$ (provided that Δm_{AB} is still much larger than $\Lambda_{\mathrm{SU}(N)}$). An explanation will be given in Section 4.6.

The Abelian version of the first-order equations (4.5.5) were derived in Ref. [142] where they were exploited to find the 1/4 BPS-saturated solution for the wall-string junction. The non-Abelian equations (4.5.5) in the SU(2)× U(1) theory were derived in [165] where the confined monopoles as string junctions were considered at $\Delta m \neq 0$. Then the non-Abelian equations (4.5.5) were extensively used in the analysis [37] of the wall-string junctions in the problem of non-Abelian strings ending on a stack of domain walls. Next, Eqs. (4.5.5) for the confined monopoles as string junctions were solved in [132] in the SU(2) × U(1) theory. Below we will review this solution. Later all 1/4 BPS solutions for junctions (in particular, semilocal string junctions) were found in [172].

4.5.2 The string junction solution in the quasiclassical regime

Now we will apply our master equations at $N = 2$ in order to find the junction of the $S^a = (0, 0, 1)$ and $S^a = (0, 0, -1)$-strings via an SU(2) monopole in the quasiclassical limit. We assume that the $S^a = (0, 0, 1)$-string is at negative z, while the $S^a = (0, 0, -1)$-string is at positive z. We will show that the solution of the BPS equations (4.5.5) of the four-dimensional bulk theory is determined by the kink solution in the two-dimensional sigma model (4.4.48).

To this end we will look for the solution of equations (4.5.5) in the following *ansatz*. Assume that the solution for the string junction is given, to the leading order in $\Delta m/\sqrt{\xi}$, by the same string configuration (4.4.20), (4.4.5) and (4.4.42) which we dealt with previously (in the case $\Delta m \neq 0$) with S^a slowly varying functions of z, to be determined below, replacing the constant moduli vector S^a.

Now the functions $S^a(z)$ satisfy the boundary condition

$$S^a(-\infty) = (0, 0, 1), \tag{4.5.8}$$

while

$$S^a(\infty) = (0, 0, -1). \tag{4.5.9}$$

This *ansatz* corresponds to the non-Abelian string in which the vector S^a slowly rotates from (4.5.8) at $z \to -\infty$ to (4.5.9) at $z \to \infty$. We will show that the representation (4.4.20), (4.4.5) and (4.4.42) solves the master equations (4.5.5) provided the functions $S^a(z)$ are chosen in a special way.

Note that the first equation in (4.5.5) is trivially satisfied because the field a is constant and $F_1^* = F_2^* = 0$. The last equation reduces to the first two equations in (4.2.11) because it does not contain derivatives with respect to z and, therefore, is satisfied for arbitrary functions $S^a(z)$. The same remark applies also to the third equation in Eq. (4.5.5), which reduces to the third equation in (4.2.11).

Let us inspect the fifth equation in Eq. (4.5.5). Substituting our *ansatz* in this equation and using the formula (4.4.13) for ρ we find that this equation is satisfied provided $S^a(z)$ are chosen to be the solutions of the equation

$$\partial_3 S^a = \Delta m \left(\delta^{a3} - S^a S^3\right). \tag{4.5.10}$$

Below we will show that these equations are nothing but the first-order kink equations in the massive CP(1) model.

By the same token, we can consider the second equation in (4.5.5). Upon substituting there our *ansatz*, it reduces to Eq. (4.5.10) too. Finally, consider the fourth equation in (4.5.5). One can see that in fact it contains an expansion in the parameter $\Delta m^2/\xi$. This means that the solution we have just built is not exact; it has corrections of the order of $O(\Delta m^2/\xi)$. To the leading order in this parameter the fourth equation in (4.5.5) reduces to the last equation in (4.2.11). In principle, one could go beyond the leading order. Solving the fourth equation in (4.5.5) in the next-to-leading order would allow one to determine $O(\Delta m^2/\xi)$ corrections to our solution.

Let us dwell on the meaning of Eq. (4.5.10). This equation is nothing but the equation for the kink in the CP(1) model (4.4.48). To see this let us write the

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Bogomol'nyi representation for kinks in the model (4.4.48). The energy functional can be rewritten as

$$E = \frac{\beta}{2}\int dz \left\{ \left| \partial_z S^a - \Delta m \left(\delta^{a3} - S^a S^3 \right) \right|^2 + 2\Delta m\, \partial_z S^3 \right\}. \tag{4.5.11}$$

The above representation implies the first-order equation (4.5.10) for the BPS-saturated kink. It also yields $2\beta\Delta m$ for the kink mass.

Thus, we have demonstrated that the solution describing the junction of the $S^a = (0,0,1)$ and $S^a = (0,0,-1)$ Z_2 strings is given by the non-Abelian string with a slowly varying orientation vector S^a. The variation of S^a is described in terms of the kink solution of the $(1+1)$-dimensional CP(1) model with the twisted mass.

In conclusion, we would like to match the masses of the four-dimensional monopole and two-dimensional kink. The string mass and that of the string junction is given by the first and the last terms in the surface energy (4.5.4) (the second term vanishes). The first term obviously reduces to

$$M_{\text{string}} = 2\pi\,\xi\;L, \tag{4.5.12}$$

i.e. proportional to the total string length L. Note that both the $S^a = (0,0,1)$ and $S^a = (0,0,-1)$ strings have the same tension (4.2.12). The third term should give the mass of the the monopole. The surface integral in this term reduces to the flux of the $S^a = (0,0,-1)$-string at $z \to \infty$ minus the flux of the $S^a = (0,0,1)$-string at $z \to -\infty$. The F^{*3} flux of the $S^a = (0,0,-1)$-string is 2π while the F^{*3} flux of the $S^a = (0,0,1)$-string is -2π. Thus, taking into account Eq. (4.1.11), we get

$$M_M = \frac{4\pi}{g_2^2}\,\Delta m. \tag{4.5.13}$$

Note, that although we discuss the monopole in the confinement phase at $|\Delta m| \ll \sqrt{\xi}$ (in this phase it is a junction of two strings), nevertheless the Δm and g_2^2 dependence of its mass coincides with the result (4.5.7) for the unconfined monopole on the Coulomb branch (i.e. at $\xi = 0$). This is no accident – there is a deep theoretical reason explaining the validity of this unified formula. A change occurs only in passing to a highly quantum regime depicted in the right lower corner of Fig. 4.3. We will discuss this regime shortly in Section 4.5.3.

It is instructive to compare Eq. (4.5.13) with the kink mass in the effective CP(1) model on the string world sheet. As was mentioned, the surface term in Eq. (4.5.11) gives

$$M_{\text{kink}} = 2\,\beta\,\Delta m. \tag{4.5.14}$$

Now, expressing the two-dimensional coupling constant β in terms of the coupling constant of the microscopic theory, see Eq. (4.4.15), we obtain

$$M_{\text{kink}} = \frac{4\pi}{g_2^2}\Delta m, \tag{4.5.15}$$

thus verifying that the four-dimensional calculation of M_M and the two-dimensional calculation of M_{kink} yield the same,

$$M_M = M_{\text{kink}}. \tag{4.5.16}$$

Needless to say, this is in full accordance with the physical picture that emerged from our analysis, that the two-dimensional CP(1) model is nothing but the macroscopic description of the confined monopoles occurring in the four-dimensional microscopic Yang–Mills theory. Technically the coincidence of the monopole and kink masses is based on the fact that the integral in the definition (4.4.10) of the sigma-model coupling β reduces to unity.

4.5.3 The strong coupling limit

Here we will consider the limit of small Δm_{AB}, when the effective world sheet theory develops a strong coupling regime. For illustrative purposes we will consider the simplest case, $N = 2$. Generalization to generic N is straightforward.

As we further diminish $|\Delta m|$ approaching Λ_σ and then send Δm to zero we restore the global SU(2)$_{C+F}$ symmetry. In particular, on the Coulomb branch, the SU(2) $\times$ U(1) gauge symmetry is restored. In this limit the monopole size grows, and, classically, it would explode. Moreover, the classical formula (4.5.13) interpreted literally shows that the monopole mass vanishes (see the discussion of the so-called "monopole clouds" in [112] for a review of the long-standing issue of understanding what becomes of the monopoles upon restoration of the non-Abelian gauge symmetry). Thus, classically one would say that the monopoles disappear.

That's where quantum effects on the confining string take over. As we will explain below, they materialize the confined non-Abelian monopole as a well-defined stable object [132].

While the string thickness (in the transverse direction) is $\sim \xi^{-1/2}$, the z-direction size of the kink representing the confined monopole in the highly quantum regime is much larger, $\sim \Lambda_\sigma^{-1}$, see the lower right corner in Fig. 4.3. Still, it remains finite in the limit $\Delta m \to 0$, stabilized by non-perturbative effects in the world sheet CP(1) model. This is due to the fact that the CP($N-1$) models develop a mass gap, and no massless states are present in the spectrum, see Section 4.4.3. Moreover, the mass of the confined monopole (the CP(1) model kink) is also determined by the scale Λ_σ. This sets the notion of what the confined non-Abelian monopole is. It is a kink in the massless two-dimensional CP(1) model [132].

We can get a more quantitative insight into the physics of the world-sheet theory at strong coupling if we invoke the exact BPS spectrum of the twisted-mass-deformed CP($N-1$) model obtained in [30]. For a detailed discussion in CP(1) see Section 3.5.

The exact expression for the central charge in CP($N-1$) with twisted mass was derived [30] by generalizing Witten's analysis [157] that had been carried out previously for the massless case. The BPS states saturate the central charge Z defined in Eq. (2.3.4). The exact formula for this central charge is

$$Z_{2d} = i\,\Delta m\, q + m_D\, T, \tag{4.5.17}$$

where the subscript $2d$ reminds us that the model in question is two-dimensional. The subscript D in m_D appears for historical reasons, in parallel with the Seiberg–Witten solution (it stands for dual). Furthermore, T is the topological charge of the kink under consideration, $T = \pm 1$, while the parameter q

$$q = 0,\ \pm 1,\ \pm 2, \ldots \tag{4.5.18}$$

This global U(1) charge of the "dyonic" states arises due to the presence of a U(1) group unbroken in (4.4.48) by the twisted mass (the $SU(2)_{C+F}$ symmetry is broken down to U(1) by $\Delta m \neq 0$).

The quantity m_D was introduced [30] in analogy with a_D of Ref. [2]. In the case $N = 2$ it has the form presented in Eq. (3.5.27) with the substitutions

$$m \to \Delta m, \quad \Lambda \to \Lambda_\sigma, \tag{4.5.19}$$

where Δm is now assumed to be complex. The two-dimensional central charge is normalized in such a way that $M_{\rm kink} = |Z_{2d}|$.

As we discussed in Section 3.5, there are no massless states in the CP(1) model at $\Delta m = 0$. In particular, the kink (confined monopole) mass is

$$M_M = \frac{2}{\pi}\Lambda_\sigma, \tag{4.5.20}$$

as it is clear from (3.5.27). On the other hand, in this limit both the last term in (4.5.4) and the surface term in (4.5.11) vanish for the monopole and kink masses, respectively. What's wrong?

This puzzle was solved by the following observation: anomalous terms in the central charges of both four-dimensional and two-dimensional SUSY algebras are present in these theories. In two dimensions the anomalous terms were obtained

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in [33, 34]. In four dimensions the bifermion anomalous term was discovered in [132]. We refer the reader to Section 3.4.2 for a more detailed discussion.

In the bulk theory the central charge associated with the monopole is defined through the anticommutator

$$\{\bar{Q}^{f}_{\dot\alpha}\,\bar{Q}^{g}_{\dot\beta}\} = 2\,\varepsilon_{\dot\alpha\dot\beta}\,\varepsilon^{fg}\,\bar{Z}_{4d}, \tag{4.5.21}$$

where $\bar{Z}_{4d}$ is an $\mathrm{SU}(2)_R$ singlet; the subscript $4d$ will remind us of four dimensions. It is most convenient to write $\bar{Z}_{4d}$ as a topological charge (i.e. the integral over a topological density),

$$\bar{Z}_{4d} = \int d^3x\,\bar{\zeta}^0(x). \tag{4.5.22}$$

In the model at hand[10]

$$\bar{\zeta}^{\mu} = \frac{1}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu\left(\frac{i}{g_2^2}\,a^a F^a_{\rho\sigma} + \frac{i}{g_1^2}\,a F_{\rho\sigma} - \frac{i}{2\pi^2}\,a^a F^a_{\rho\sigma}\right.$$
$$\left. + \frac{i}{8\sqrt{2}\pi^2}\left[\lambda^a_{f\alpha}(\sigma_\rho)^{\alpha\dot\alpha}(\bar\sigma_\sigma)_{\dot\alpha\beta}\lambda^{af\beta} + 2g_2^2\tilde\psi_{A\alpha}(\sigma_\rho)^{\alpha\dot\alpha}(\bar\sigma_\sigma)_{\dot\alpha\beta}\psi^{A\beta}\right]\right). \tag{4.5.23}$$

Note that the general structure of the operator in the square brackets is unambiguously fixed by dimensional arguments, the Lorentz symmetry and other symmetries of the bulk theory. The numerical coefficient was first found in [132] by matching the monopole and kink masses at $\Delta m = 0$.

The above expression is an operator equality. In the low-energy limit, the Seiberg–Witten exact solution allows one to obtain the full matrix element of the operator on the right-hand side (which includes all perturbative and non-perturbative corrections) by replacing a by a_D.

The fermion part of the anomalous term plays a crucial role in the Higgs phase for the confined monopole. On the Coulomb branch it does not contribute to the mass of the monopole due to a fast fall off of the fermion fields at infinity. On the Coulomb branch the bosonic anomalous terms become important. The relationship between the 't Hooft–Polyakov monopole mass and the $\mathcal{N}=2$ central charge is analyzed in [38], which identifies an anomaly in the central charge explaining a constant (i.e. non-logarithmic) term in the monopole mass on the Coulomb branch. The result of Ref. [38] is in agreement with the Seiberg–Witten formula for the monopole mass. In Section 3.4.2 we presented the operator form of the central charge anomaly.

[10] In Eq. (4.5.23) in the bosonic part we keep only terms containing the magnetic field $\vec{B}$ and drop those with the electric field $\vec{E}$ which are relevant for dyons. For more details see Section 3.4.2.

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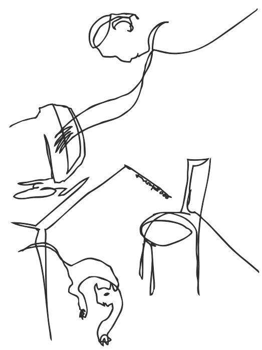
Note, that the coefficient in front of the fermionic term involving λ-fermions in (4.5.23) coincides with the one in (3.4.20) obtained by supersymmetrization of the bosonic anomalous term.



4.6 Two-dimensional kink and four-dimensional Seiberg–Witten solution

Why is the 't Hooft–Polyakov monopole mass (i.e. that on the Coulomb branch at $\xi = 0$) given by the same formula (4.5.7) as the mass (4.5.13) of the strongly confined large-ξ monopole (subject to condition $\sqrt{\xi} \gg \Delta m$)?

This fact was noted in Section 4.5.2. Now we will explain the reason lying behind this observation [132, 133]. En route, we will explain another striking observation made in Ref. [30]. A remarkably close parallel between four-dimensional SU(2) Yang–Mills theory with $N_f = 2$ and the two-dimensional CP(1) model was noted, at an observational level, by virtue of comparison of the corresponding central charges. The observation was made on the Coulomb branch of the Seiberg–Witten theory, with unconfined monopoles/dyons of the 't Hooft–Polyakov type. Valuable as it is, the parallel was quite puzzling since the solution of the CP(1) model seemed to have no physics connection to the Seiberg–Witten solution. The latter gives the mass of the unconfined monopole in the Coulomb regime at $\xi = 0$ while the CP(1) model emerges only in the Higgs regime of the bulk theory.

We want to show that the reason for the correspondence mentioned above is that in the BPS sector (and *only* in this sector) the parameter ξ, in fact, does not appear in relevant formulae. Therefore, one can vary ξ at will, in particular, making it less than $|\Delta m|$ or even tending to zero, where CP(1) is no more the string world sheet theory for our bulk model. Nevertheless, the parallel expressions for the central charges and other BPS data in four dimensions and two dimensions, trivially established at $|\Delta m| \ll \xi$, will continue to hold even on the Coulomb branch. The

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"strange coincidence" we observed in Section 4.5.2 is no accident. We deal with an exact relation which stays valid including both perturbative and non-perturbative corrections.

Physically the monopole in the Coulomb phase is very different from the one in the confinement phase, see Fig. 4.3. In the Coulomb phase it is a 't Hooft–Polyakov monopole, while in the confinement phase it becomes related to a junction of two non-Abelian strings. Still let us show that the masses of these two objects are given by the same expression,

$$M_M^{\text{Coulomb}} = M_M^{\text{confinement}} \tag{4.6.1}$$

provided that Δm and the gauge couplings are kept fixed. The superscripts refer to the Coulomb and monopole-confining phases, respectively.

The crucial observation here is that the mass of the monopole cannot depend on the FI parameter ξ. Start from the monopole in the Coulomb phase at $\xi = 0$. Its mass is given by the exact Seiberg–Witten formula [3]

$$\begin{aligned} M_M^{\text{Coulomb}} &= \sqrt{2}\left| a_D^3\left(a^3 = -\frac{\Delta m}{\sqrt{2}}\right)\right| \\ &= \left| \frac{\Delta m}{\pi} \ln \frac{\Delta m}{\Lambda_{\text{SU}(2)}} + \Delta m \sum_{k=0}^{\infty} c_k \left(\frac{\Lambda}{\Delta m}\right)^{2k} \right|, \end{aligned} \tag{4.6.2}$$

where a_D^3 is the dual Seiberg–Witten potential for the SU(2) gauge group. We take into account the fact that for $N_f = 2$ the first coefficient of the β function is 2.

In Eq. (4.6.2) $a^3 = -\Delta m/\sqrt{2}$ is the argument of a_D^3, the logarithmic term takes into account the one-loop result (4.1.19) for the SU(2) gauge coupling at the scale Δm, while the power series represents instanton-induced terms – small corrections at large a.

Now, if we switch on a small FI parameter $\xi \neq 0$ in the theory, on dimensional grounds we could expect corrections to the monopole mass in powers of $\sqrt{\xi}/\Lambda_{\text{SU}(2)}$ and/or $\sqrt{\xi}/\Delta m$ in Eq. (4.6.2).

But … these corrections are *forbidden* by the $\text{U}(1)_R$ charges. Namely, the $\text{U}(1)_R$ charges of $\Lambda_{\text{SU}(2)}$ and Δm are equal to 2 (and so is the $\text{U}(1)_R$ charge of the central charge under consideration) while ξ has a vanishing $\text{U}(1)_R$ charge. For convenience, the $\text{U}(1)_R$ charges of different fields and parameters of the microscopic theory are collected in Table 4.1. Thus, neither $(\sqrt{\xi}/\Lambda_{\text{SU}(2)})^k$ nor $(\sqrt{\xi}/\Delta m)^k$ can appear.

By the same token, we could start from the confined monopole at large ξ, and study the dependence of the monopole (string junction) mass as a function of ξ as we reduce ξ. Again, the above arguments based on the $\text{U}(1)_R$ charges tell us that

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Table 4.1. *The* $U(1)_R$ *charges of fields and parameters of the bulk theory.*

Field/parameter	a	a^a	λ^α	q	ψ^α	m_A	$\Lambda_{SU(N)}$	ξ
$U(1)_R$ charge	2	2	1	0	-1	2	2	0

no corrections in powers of $\Lambda_{SU(2)}/\sqrt{\xi}$ and $\Delta m/\sqrt{\xi}$ can appear. This leads us to Eq. (4.6.1).

Another way leading to the same conclusion is to observe that the monopole mass depends on a (anti)holomorphically, cf. Seiberg–Witten's formula (4.6.2). Thus, it cannot depend on the FI parameter ξ which is not holomorphic (it is a component of the $SU(2)_R$ triplet [127, 35]).

Now let us turn to the fact that the mass of the monopole in the confinement phase is given by the kink mass in the CP(1) model, see (4.5.16). In this way we obtain

$$M_M^{\text{Coulomb}} \leftrightarrow M_M^{\text{confinement}} \leftrightarrow M_{\text{kink}}. \tag{4.6.3}$$

In particular, at one loop, the kink mass is determined by renormalization of the CP(1)-model coupling constant β, while the monopole mass on the Coulomb branch is determined by the renormalization of g^2. This leads to the relation

$$\Lambda_\sigma = \Lambda_{SU(2)}$$

between the two- and four-dimensional dynamical scales. It was noted earlier as a "strange coincidence," see Eq. (4.4.35). The first coefficient of the β functions is two (N for generic N) for both theories. Now we know the physical reason behind this coincidence.

Clearly, the above relation can be generalized (cf. [30, 133]) to cover the $SU(N)\times U(1)$ case with $N_f = N$ flavors on the four-dimensional side, and $CP(N-1)$ sigma models on the two-dimensional side.

This correspondence can be seen in more quantitative terms [30, 133]. Four-dimensional $U(N)$ SQCD with $\mathcal{N} = 2$ and $N_f = N$ flavors is described by the degenerate Seiberg–Witten curve

$$y^2 = \frac{1}{4}\left[\prod_{i=1}^{N}(x+\tilde{m}_i) - \Lambda_{SU(N)}^N\right]^2 \tag{4.6.4}$$

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in the special point (4.1.11) on the Coulomb branch which becomes a quark vacuum upon the ξ deformation. The periods of this curve give the BPS spectrum of the two-dimensional CP($N-1$) model [30]. We quoted this spectrum for CP(1) in Eqs. (4.5.17) and (3.5.27).

In fact, Dorey demonstrated [30] that the BPS spectra of the two-dimensional CP($N-1$) model and four-dimensional SU(N) SQCD coincide with each other if one chooses a point on the Coulomb branch corresponding to the baryonic Higgs branch defined by the condition $\sum m_A = 0$ (in the SU(2) case the gauge equivalent choice is to set $m_1 = m_2$).

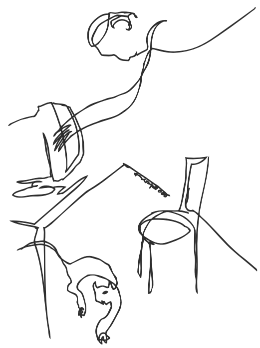
At the same time, we observe that the BPS spectra of the *massive* states in the SU(2) and U(2) theories, respectively, coincide in the corresponding quark vacua upon identification of m_A of the SU(N) theory with $\tilde{m}_A$ of the U(N) theory. In particular, in the $N = 2$ case one must identify $m_1 = m_2$ of the SU(2) theory with $\Delta m/2$ of the U(2) theory. Note that the vacuum (4.1.11) and (4.1.14) of the U(N) theory is an isolated vacuum rather than a root of a Higgs branch. There are no massless states in the U(N) bulk theory in this vacuum, see Section 4.1.2 for more details.

Note also that the BPS spectra of both theories include not only the monopole/kink and "dyonic" states but elementary excitations with $T = 0$ as well. On the two-dimensional side they correspond to elementary fields n^ℓ in the large Δm_{AB} limit. On the four-dimensional side they correspond to non-topological (i.e. $T = 0$ and $q = \pm 1$) BPS excitations of the string with masses proportional to Δm_{AB} confined on the string.

The latter can be interpreted as follows. Inside the string the squark profiles vanish, effectively bringing us into the Coulomb branch ($\xi = 0$) where the W bosons and quarks would become BPS-saturated states in the bulk. Say, for $N = 2$ on the Coulomb branch, the W boson and off-diagonal quark mass would reduce to Δm. Hence, the $T = 0$ BPS excitation of the string is a wave of such W bosons/quarks propagating along the string. One could term it a "confined W boson/quark." It is localized in the perpendicular but not in the longitudinal direction. What is important, it has no connection with the bulk Higgs phase W bosons, which are non-BPS and are much heavier than Δm. Nor do these non-topological excitations have connection to the bulk quarks of our bulk model, which are not BPS-saturated too.

To conclude, let us mention that Tong compared [173] a conformal theory with massless quarks and monopoles arising on the Coulomb branch of the four-dimensional $\mathcal{N} = 2$ SQCD (upon a special choice of the mass parameters Δm_{AB}), at the so-called Argyres–Douglas point [174], with the twisted-mass-deformed two-dimensional CP($N-1$) model.

The coincidence of the monopole and kink masses explained above ensures that the CP($N-1$) model flows to a non-trivial conformal point at these values of Δm_{AB}. The scaling dimensions of the chiral primary operators in four- and two-dimensional conformal theories were shown to agree [173]; a very nice result, indeed.



4.7 More quark flavors

In this section we will abandon the assumption $N_F = N$ and consider the theory (4.1.7) with more fundamental flavors, $N_F > N$. In this case we have a number of isolated vacua such as (4.1.11) and (4.1.14), in which N squarks out of N_f develop VEV's, while the adjoint VEV's are determined by the mass terms of these quarks, as in Eq. (4.1.11).

Now, let us focus on the equal mass case. Then the isolated vacua coalesce, and a Higgs branch develops from the common root whose location on the Coulomb branch is given by Eq. (4.1.11) (with all masses set equal). The dimension of this branch is $4N(N_f - N)$, see [143, 140]. The Higgs branch is noncompact and has a hyper-Kähler geometry [3, 143]. It has a compact base manifold defined by the condition

$$\bar{\tilde{q}}^{kA} = q^{kA}. \tag{4.7.1}$$

The dimension of this manifold is twice less than the total dimension of the Higgs branch, $2N(N_f - N)$, which implies 4 for $N_f = 3$ and 8 for $N_f = 4$ in the simplest $N = 2$ case. The BPS string solutions exist only on the base manifold of the Higgs branch. The flux tubes become non-BPS-saturated if we move away from the base along noncompact directions [175]. Therefore, we will limit ourselves to the vacua which belong to the base manifold.

Strings that emerge in multiflavor theories, i.e. $N_f > N$ (typically on the Higgs branches), as a rule are not conventional ANO strings. Rather, they become the so-called semilocal strings (for a comprehensive review see [147]). The simplest model where the semilocal strings appear is the Abelian Higgs model with two complex flavors

$$S_{\rm AH} = \int d^4x \left\{ \frac{1}{4g^2} F_{\mu\nu}^2 + \left|\nabla_\mu q^A\right|^2 + \frac{g^2}{8}\left(|q^A|^2 - \xi\right)^2 \right\}, \qquad (4.7.2)$$

where $A = 1, 2$ is the flavor index.

If $\xi \neq 0$ the scalar fields develop VEV's breaking the U(1) gauge group. The photon field becomes massive, together with one real scalar field.

In fact, for the particular choice of the quartic coupling made in Eq. (4.7.2) this scalar field has the same mass as the photon. In fact, the model (4.7.2) is the bosonic part of a supersymmetric theory; the flux tubes are classically BPS-saturated. The topological reason for the existence of the ANO flux tubes is that

$$\pi_1[\mathrm{U}(1)] = Z$$

for the U(1) gauge group. On the other hand, in Eq. (4.7.2) we can pass to the low-energy limit integrating out the massive photon and its scalar counterpart. This will lead us to a four-dimensional sigma model on the manifold

$$|q^A|^2 = \xi. \qquad (4.7.3)$$

The vacuum manifold (4.7.3) has dimension $4 - 1 - 1 = 2$, where we subtract one real condition mentioned above, as well as one phase that can be gauged away. Thus, the manifold (4.7.3) represents a two-dimensional sphere S_2. The low-energy limit of the theory (4.7.2) is the O(3) sigma model.

We should remember that

$$\pi_2[S_2] = \pi_1[\mathrm{U}(1)] = Z,$$

and this is the topological reason for the existence of instantons in the two-dimensional O(3) sigma model [176]. Uplifted in four dimensions, these instantons become string-like objects (lumps).

Just as the O(3) sigma-model instantons, the semilocal strings possess two additional zero modes associated with its complexified size modulus ρ in the model (4.7.2). Hence, the semilocal strings interpolate between the ANO strings and two-dimensional sigma-model instantons uplifted in four dimensions. At $\rho = 0$ we have the ANO string while at $\rho \to \infty$ the string becomes nothing but the two-dimensional instanton elevated in four dimensions. At generic $\rho \neq 0$ the semilocal string is characterized by a power fall-off of the profile functions at infinity, to be contrasted with the exponential fall-off characteristic of the ANO string.

Now, if we return to our non-Abelian theory (4.1.7), we will see that the semilocal strings in this theory have size moduli, in addition to the $2(N-1)$ orientational moduli n^ℓ. The total dimension of the moduli space of the semilocal string was shown [130] to be

$$2N_f = 2 + 2(N-1) + 2(N_f - N), \tag{4.7.4}$$

where the first, the second and the third terms above correspond to the translational, orientational and the size moduli.

No studies of geometry of the moduli space of the semilocal strings were carried out for quite some time due to infrared problems. It was known [177, 178] that the size-zero modes are logarithmically non-normalizable in the infrared, as is the case for the sigma-model instantons in two dimensions. This problem was addressed in [179] where non-Abelian strings in the U(2) gauge theory were treated. The effective theory on the string world sheet was shown to have the form

$$S^{(1+1)} = \beta\, M_W \int dt\, dz \left\{ \frac{\rho^2}{4}\, (\partial_k\, S^a)^2 + |\partial_k\, \rho_i|^2 \right\} \ln \frac{1}{|\rho|\; \delta\, m}, \tag{4.7.5}$$

where M_W is the W-boson mass, see Eq. (4.1.16). The subscript $i = 3, \ldots, N_f$, while ρ_i stand for $(N_f - 2)$ complex fields associated with the size moduli. The parameter δm here measures small quark mass differences, acting as an infrared regulator. It is necessary to introduce this infrared parameter, slightly lifting the size moduli ρ_i, in order to regularize the infrared logarithmic divergence.

The metric (4.7.5) is derived in [179] for large – but not too large – values of $|\rho|^2 \equiv |\rho_i^2|$ lying inside the window

$$\frac{1}{M_W} \ll |\rho| \ll \frac{1}{\delta m}. \tag{4.7.6}$$

The inequality on the left-hand side refers to the limit in which the semilocal string becomes an O(3) sigma-model lump. The inequality on the right-hand side ensures the validity of the logarithmic approximation. The action (4.7.5) was obtained in the logarithmic approximation.

For ρ_i's lying inside the window (4.7.6), with a logarithmic accuracy, one can introduce new variables

$$z_i = \rho_i \left[M_W^2\, \ln \frac{1}{|\rho|\, \delta m} \right]^{1/2}. \tag{4.7.7}$$

In terms of these new variables the metric of the world sheet theory (4.7.5) was shown[11] to become flat [179]. Corrections to this flat metric run in powers of

$$\frac{1}{M_W\,|\rho|} \quad \text{and} \quad \left(\ln\,\frac{1}{|\rho|\,\delta m}\right)^{-1}.$$

These corrections have not yet been calculated within the field-theory approach.

On the other hand, the very same problem was analyzed from the D-brane theory side. Using brane-based arguments Hanany and Tong conjectured [130, 133] (see also Ref. [152]) that the effective theory on the world sheet of the non-Abelian semilocal string is given by the strong-coupling limit ($e^2 \to \infty$) of the following two-dimensional gauge theory:

$$\begin{aligned} S = \int d^2x \Big\{ & 2\beta\,|\nabla_k n^\ell|^2 + 2\beta\,|\nabla_k z_i|^2 + \frac{1}{4e^2}F_{kl}^2 + \frac{1}{e^2}|\partial_k\sigma|^2 \\ & + 4\beta\left|\sigma - \frac{\tilde{m}_\ell}{\sqrt{2}}\right|^2 |n^\ell|^2 + 4\beta\left|\sigma - \frac{\tilde{m}_i}{\sqrt{2}}\right|^2 |z_i|^2 \\ & + 2e^2\,\beta^2\big[\,|n^\ell|^2 - |z_i|^2 - 1\big]^2\Big\}, \end{aligned} \tag{4.7.8}$$

where $\ell = 1, \ldots, N$ and $i = N+1, \ldots, N_f$. Furthermore, z_i denote $(N_f - N)$ complex fields associated with the size moduli. The fields n^ℓ and z_i have the charges $+1$ and -1 with respect to the U(1) gauge field in Eq. (4.7.8). This theory is similar to the model (4.4.51) describing the $N_f = N$ non-Abelian strings.

The Hanany–Tong conjecture is supported by yet another argument. As was discussed in Section 4.6, the BPS spectrum of dyons on the Coulomb branch of the four-dimensional theory must coincide with the BPS spectrum in the two-dimensional theory on the string world sheet. We expect that this correspondence extends to theories with $N_f > N$. The two-dimensional theory (4.7.8) was studied in [181] where it was shown that its BPS spectrum agrees with the spectrum of four-dimensional U(N) SQCD with N_f flavors. In particular, the one-loop coefficient of the β function is $2N - N_f$ in both theories. This leads to the identification of their scales, see Eq. (4.4.35). As a matter of fact, Ref. [181] deals with the SU(N) theory at the root of the baryonic Higgs branch, much in the same vein as [30]. However, as was explained in Section 4.6, the BPS spectra of the massive states in these four-dimensional theories are the same.

[11] Warning: a different metric on the moduli space of the non-Abelian semilocal string was suggested in [180]. It has a kinetic cross-term for the orientational and size moduli fields. However, at large ρ_i, inside the allowed window (4.7.6), this metric is also flat.

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The above argument shows that the two-dimensional theory (4.7.8) is a promising candidate for an effective theory on the semilocal string world sheet. In particular, the metric in (4.7.8) is asymptotically flat. The variables z_i in (4.7.8) should be identified with the ones in Eq. (4.7.7) introduced within the field-theory framework in Ref. [179]. It is quite plausible that corrections to the flat metric in powers $1/(M_W\,|\rho|)$ are properly reproduced by the world sheet theory (4.7.8). Nevertheless, the results of [179] clearly demonstrate the approximate nature of the world sheet theory (4.7.8). Namely, corrections at large ρ_i suppressed by large infrared logarithms $(\ln\,(1/|\rho|\,\delta m))^{-1}$ are certainly not captured in Eq. (4.7.8).

The implication of the "semilocal nature" of the semilocal strings which is most important from the physical standpoint is the loss of the monopole confinement [175, 179] i.e. the loss of the Meissner effect. To study the monopole confinement as a result of the squark condensation we must consider a string of a finite length L stretched between a heavy probe monopole and antimonopole from the $\mathrm{SU}(N+1)/\mathrm{SU}(N)\times\mathrm{U}(1)$ sector. The ANO string has a typical transverse size $(g\sqrt{\xi})^{-1}$. If L is much larger than this size the energy of this probe configuration is

$$V(L) = T\,L, \tag{4.7.9}$$

where T is the string tension. The linear potential in Eq. (4.7.9) ensures confinement of monopoles.

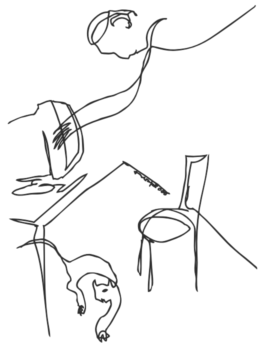
For semilocal strings this conventional picture drastically changes. Now the transverse string size can be arbitrarily large. Imagine a configuration in which the string transverse size becomes much larger than L. Then we will clearly deal with the three-dimensional rather than two-dimensional problem. The monopole flux is no longer trapped in a narrow flux tube. Instead, it freely spreads over a large three-dimensional volume, of the size of order of L in all directions. Obviously, this will give rise to a Coulomb-type potential between the probe monopoles,

$$V(L) \sim 1/L, \tag{4.7.10}$$

possibly augmented by logarithms of L. At large L the energy of this configuration is lower than the one of the flux-tube configuration (4.7.9); therefore, it is energetically favorable.

To summarize, semilocal strings can indefinitely increase their transverse size and effectively disintegrate, so that the linear potential (4.7.9) gives place to the Coulomb potential (4.7.10). In fact, lattice studies unambiguously show that the semilocal string thickness always increases upon small perturbations [182].

Formation of semilocal strings on the Higgs branches leads to a dramatic physical effect – deconfinement.



4.8 Non-Abelian k-strings

In this section we will briefly review how multi-strings, with the winding number $k > 1$, can be constructed. One can consider them as bound states of k BPS elementary strings. The Bogomol'nyi representation (4.2.9) implies that the tension of the BPS-saturated k-string is determined by its total U(1) flux, $2\pi k$. This entails, in turn, that in $\mathcal{N} = 2$ SQCD, see Eq. (4.1.7), the k-string tension has the form (4.1.7)

$$T_k = 2\pi k\, \xi. \tag{4.8.1}$$

Equation (4.8.1) implies that the elementary strings that form composite k-strings do not interact.

If one considers k elementary strings, forming the given k-string, at large separations the corresponding moduli space obviously factorizes into k copies of the moduli spaces of the elementary strings. This suggests that the dimension of the total moduli space is

$$2kN_f = 2k + 2k(N-2) + 2k(N_f - N), \tag{4.8.2}$$

see (4.7.4). The total dimension is written as a sum of dimensions of the translational, orientational and size moduli spaces. This result was confirmed by the Hanany–Tong index theorem [130] which implies (4.8.2) at any separations. The moduli space of well-separated elementary strings forming the given k-string, say, at $N_f = N$ is

$$\frac{\left[C \times \mathrm{CP}(N-1)\right]^k}{S_k}, \tag{4.8.3}$$

where S_k stands for permutations of the elementary string positions.

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An explicit solution for a non-Abelian 2-string at zero separation in the simplest bulk theory with $N = N_f = 2$ was constructed in [183]. It has a peculiar feature. If the orientation vectors of the two strings S_1^a and S_2^a are opposite, the composite 2-string becomes an Abelian ANO string. It carries no non-Abelian flux. Therefore, $\mathrm{SU}(2)_{C+F}$ rotations act trivially on this particular string. This means that the internal moduli space of this string is singular [184, 183]. The section of the orientational moduli space corresponding to $S_1^a = -S_2^a$ degenerates into a point. In [183] it was argued that the internal moduli of the 2-string at zero separation is equivalent to $\mathrm{CP}(2)/Z_2$. This differs by a discrete quotient from the result CP(2) obtained in [184]. Later results obtained in [185, 186] confirm the $\mathrm{CP}(2)/Z_2$ metric.

The metric on the k-string moduli space for generic k is not known. For Abelian k-strings exponential corrections to the flat metric were calculated in [187]. Exponentially small corrections are natural since in this case the vortices are characterized by an exponential fall off of their profile functions at large distances.

Hanany and Tong exploited [130, 133] a D-brane construction to obtain the k-string metric in terms of the Higgs branch of a two-dimensional gauge theory, see (4.4.51) and (4.7.8). What they came up with is an $\mathcal{N} = 2$ supersymmetric $\mathrm{U}(k)$ gauge theory with N fundamental and $(N_f - N)$ anti-fundamental flavors n^ℓ and ρ_i, respectively, ($\ell = 1, \ldots, N$ and $i = N, \ldots, N_f$), plus an adjoint chiral multiplet Z. The D-term condition for this theory is

$$\frac{1}{2\beta}[\bar{Z}, Z] + n^\ell \bar{n}_\ell - \rho_i \bar{\rho}^i = 1. \tag{4.8.4}$$

The metric defined by this Higgs branch has corrections to the factorized metric which run in (inverse) powers of separations between the elementary strings. Thus, it exhibits a dramatic disagreement with the field-theory expectations. Still the metric is believed [130, 133, 13, 14] to correctly reproduce some data protected by supersymmetry, such as the BPS spectrum.

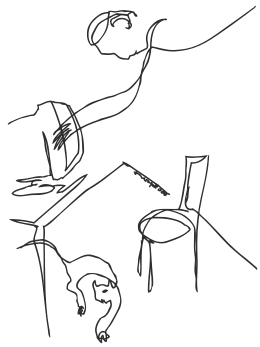
To derive all moduli of the general k-string solution the so-called moduli matrix method was developed in [188]. It was observed that the substitution

$$\varphi = S(z, \bar{z})\, H_0(z), \quad A_1 + i A_2 = S^{-1} \bar{\partial}_z S, \tag{4.8.5}$$

solves the last of the first-order equations (4.2.10). Here $z = x_1 + i x_2$ and H_0 is an $N \times N_f$ matrix with a holomorphic dependence on z.

Then the equations for the gauge field strength in (4.2.10) yield an equation on $S(z, \bar{z})$ which is rather hard to solve in the general case. It was argued, however, that the factor S involves no new moduli parameters [188]. Therefore, all moduli parameters reside in the moduli matrix $H_0(z)$.

Determining $H_0(z)$ gives one a moduli space which agrees with the moduli space corresponding to the Higgs branch (4.8.4).



4.9 A physical picture of the monopole confinement

In this section we will return to our basic $\mathcal{N} = 2$ SQCD with the U(N) gauge group and $N_f = N$ flavors (4.1.7) and discuss an emerging physical picture of the monopole confinement. As was reviewed in detail in Section 4.5, elementary confined monopoles can be viewed as junctions of two elementary strings. Therefore, the physical spectrum of the theory includes monopole-antimonopole "mesons" formed by two elementary strings in a loop configuration shown in Fig. 4.4.

If spins of such "mesons" are of order one, their mass is of the order of the square root of the string tension $\sqrt{\xi}$. Deep in the quantum non-Abelian regime ($\tilde{m}^l = 0$), the CP($N-1$)-model strings carry no average SU(N) magnetic flux [159],

$$\langle n^l \rangle = 0, \tag{4.9.1}$$

see Eq. (4.4.4). What they do carry is the U(1) magnetic flux which determines their tension.

Monopoles are seen in the world sheet theory as CP($N-1$) kinks. At $\tilde{m}^l = 0$ they become non-Abelian too, much in the same way as strings. They carry no average SU(N) magnetic flux. (Unlike strings, even in the classical regime they do not carry the U(1) magnetic flux, see (4.5.1).)

Moreover, the monopoles acquire global flavor quantum numbers. We know that the CP($N-1$) model kinks at strong coupling are described by the n^l fields [159, 120] and, therefore, in fact, they belong to the fundamental representation of the global SU(N)$_{C+F}$ group. This means that the monopole-antimonopole "mesons" formed by the string configuration shown in Fig. 4.4 can belong either to singlet or to

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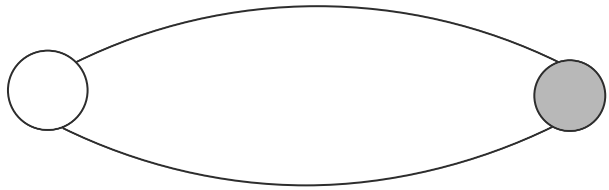

Figure 4.4. Monopole and antimonopole bound into a "meson." The binding is due to strings. Open and closed circles denote the monopole and antimonopole, respectively.

adjoint representations of the global "flavor" group $\mathrm{SU}(N)_{C+F}$, in full accordance with our expectations.

Singlets resemble glueballs. In weakly coupled bulk theory ($g_{1,2}^2 \ll 1$) the singlet mesons can decay into massive vector multiplets formed by gauge and quark fields, with mass (4.1.17), see Section 4.1.2. The monopole-antimonopole mesons with the adjoint flavor quantum numbers are also metastable in weakly coupled bulk theory, they decay into massive gauge/quark multiplets which carry the adjoint quantum numbers with respect to the global unbroken $\mathrm{SU}(N)_{C+F}$ group and have masses determined by Eq. (4.1.16).

Two elementary strings of the monopole-antimonopole meson shown in Fig. 4.4 can form a non-BPS bound state. Hence, in practice the composite meson looks as if the monopole was connected to the antimonopole by a single string. In fact, there are indications that this is what happens in the theory at hand. Interactions of elementary Z_2 strings were studied in [140] in the simplest case $N = 2$. An interaction potential for the elementary Z_2 strings with $S^a = (0, 0, +1)$ and $S^a = (0, 0, -1)$ was found to be attractive at large distances,

$$U \sim -\left\{M_{\mathrm{SU}(2)}\, R\right\}^{-1/2} e^{-M_{\mathrm{SU}(2)} R}, \tag{4.9.2}$$

where R stands for the distance between two parallel strings. The gauge boson mass is given in Eq. (4.1.16). This attractive potential leads to formation of a bound state, a composite string.

Note that we have N distinct elementary strings. As was discussed in Section 4.4.3, in the quantum regime N elementary strings differ from each other by the value of the bifermion condensate of the $\mathrm{CP}(N-1)$ model fermions [156]. Therefore, the physical picture of the monopole confinement is not absolutely similar to what we expect in QCD, see the discussion in the beginning of this section. Namely, we have N different degenerate "mesons" (at $N > 2$) of the type discussed above, associated with N different elementary strings.

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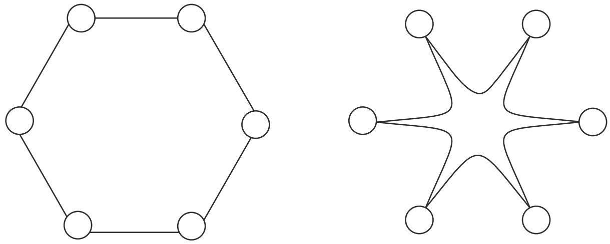

Figure 4.5. (a) A schematic picture of the "baryon" formed by monopoles and strings for $N = 6$; (b) The "baryon" acquires the shape of a star once the neighboring strings form non-BPS bound states.

In QCD (and in nature) we have instead a single meson with the given quantum numbers, plus its radial excitations which have higher masses. This is typical for BPS strings in supersymmetric gauge theories. We will see in Chapter 6 that in non-supersymmetric theories the situation is different: elementary strings are split and, therefore, different "mesons" become split too.

In addition to the "mesons" and gauge/quark multiplets, the physical spectrum contains also "baryons" built of N elementary monopoles connected to each other by elementary strings forming a closed "necklace configuration," see Fig. 4.5a. In the classical limit $\tilde{m}^l \gg \Lambda_\sigma$ all strings carry the SU(N) magnetic fluxes given by

$$\int d^2x\, F^*_{\mathrm{SU}(N)} = 2\pi \left(n \cdot n^* - \frac{1}{N} \right), \tag{4.9.3}$$

with $n^l = \delta^{ll_0}$, $l_0 = 1, \ldots, N$ for N elementary strings forming the "baryon." The monopoles carry the SU(N) magnetic fluxes given in Eq. (4.5.1) and, therefore, can be located at the corners of the polygon in Fig. 4.5a.

In the highly quantum regime, at $\tilde{m}^l = 0$, both strings and monopoles carry no average SU(N) magnetic flux, see (4.9.1). The confined monopoles are seen as kinks interpolating between the "neighboring" quantum vacua of the CP($N-1$) model (a.k.a. strings) in the closed necklace configuration in Fig. 4.5a.

As was mentioned, the monopoles/kinks acquire flavor global quantum numbers. They become fundamentals in SU$(N)_{C+F}$. Thus, the "baryon" is in the

$$\prod_1^N (N)$$

representation of SU$(N)_{C+F}$. Note that both quarks and monopoles do not carry baryon numbers. Therefore, our "baryon" has no baryon number too. The reason

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for this is that the U(1) baryon current is coupled to a gauge boson in the U(N) gauge theory that we consider here. This means, in particular, that the "baryons" can decay into the monopole "mesons" or gauge/quark multiplets.

We mentioned that the "neighboring" elementary strings can form a non-BPS bound state, a composite string. It is plausible then that in practice the monopole "baryon" actually resembles a configuration shown in Fig. 4.5b.

Let us emphasize that all states seen in the physical spectrum of the theory are gauge singlets. This goes without saying. While color charges of the gauge/quark multiplets are screened by the Higgs mechanism, the monopoles are confined by non-Abelian strings.

Let us also stress in conclusion that in the limit $\tilde{m}^l = 0$ the global group $\mathrm{SU}(N)_{C+F}$ is restored in the bulk and both strings and confined monopoles become non-Abelian. One might argue that this restoration could happen only at the classical level. One could suspect that in quantum theory a "dynamical Abelization" (i.e. a cascade breaking of the gauge symmetry $\mathrm{U}(N)\to\mathrm{U}(1)^N \to$ discrete subgroup) might occur. This could have happened if the adjoint VEV's that classically vanish at $\tilde{m}^l = 0$ (see (4.1.11)) could develop dynamically in quantum theory.

At $\tilde{m}^l \neq 0$ the global $\mathrm{SU}(N)_{C+F}$ group is explicitly broken down to $\mathrm{U}(1)^{N-1}$ by the quark masses. At $\tilde{m}^l = 0$ this group is classically restored. If it could be proven to be dynamically broken at $\tilde{m}^l = 0$, this would mean a spontaneous symmetry breaking, with obvious consequences, such as the corresponding Goldstone modes.

We want to explain why this cannot and does not happen in the theory at hand. First of all, if a global symmetry is not spontaneously broken at the tree level then it cannot be broken by quantum effects at weak coupling in "isolated" vacua. Second, if the global group $\mathrm{SU}(N)_{C+F}$ were broken spontaneously at $\tilde{m}^l = 0$ this would imply massless Goldstone bosons. However, we know that there are no massless states in the spectrum of the bulk theory, see Section 4.1.

Finally, the breaking of $\mathrm{SU}(N)_{C+F}$ in the $\tilde{m}^l = 0$ limit would mean that the twisted masses of the world sheet CP($N-1$) model would not be given by $\tilde{m}^l$; instead they would be shifted, say,

$$\tilde{m}^l_{(\mathrm{tw})} = \tilde{m}^l + c^l \Lambda_{\mathrm{CP}(N-1)},$$

where c^l are some coefficients. In Section 4.6 it was shown [132, 133] that the BPS spectrum of the CP($N-1$) model on the string should coincide with the BPS spectrum of the four-dimensional bulk theory on the Coulomb branch. The BPS spectrum of the CP($N-1$) model is determined by $\tilde{m}^l_{(\mathrm{tw})}$ while the BPS spectrum of the bulk theory on the Coulomb branch is determined by $\tilde{m}^l$. In [30] it was shown that

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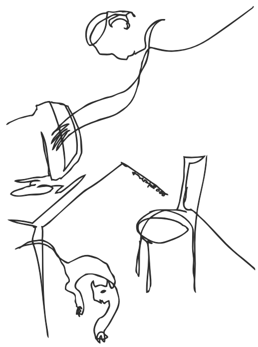
the BPS spectra of both theories coincide at $\tilde{m}^l_{(\mathrm{tw})} = \tilde{m}^l$. Thus, we conclude that $c^l = 0$, and the twisted masses vanish in the $\tilde{m}^l = 0$ limit.

Hence, the global $\mathrm{SU}(N)_{C+F}$ group is not broken in the bulk and both strings and confined monopoles become non-Abelian at $\tilde{m}^l = 0$.



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