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CRITICAL ASSOCIATED METRICS ON CONTACT MANIFOLDS III

DAVID E. BLAIR

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Abstract

In the first paper of this series we studied on a compact regular contact manifold the integral of the Ricci curvature in the direction of the characteristic vector field considered as a functional on the set of all associated metrics. We showed that the critical points of this functional are the metrics for which the characteristic vector field generates a 1-parameter group of isometries and conjectured that the result might be true without the regularity of the contact structure. In the present paper we show that this conjecture is false by studying this problem on the tangent sphere bundle of a Riemannian manifold. In particular the standard associated metric is a critical point if and only if the base manifold is of constant curvature $+1$ or -1; in the latter case the characteristic vector field does not generate a 1-parameter group of isometries.

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1. Introduction

In the first paper in this series [3] we proved the following result.

THEOREM. *Let* M *be a compact regular contact manifold and* $\mathscr{A}$ *the set of associated metrics. Let* ξ *denote the characteristic vector field of the contact structure. Then* $g \in \mathscr{A}$ *is a critical point of the functional* $L(g) = \int_M \operatorname{Ric}(\xi)\, dV$ *on* $\mathscr{A}$ *if and only if* ξ *generates a* 1*-parameter group of isometries of* g.

Recall that a contact manifold is said to be *regular* if every point has a neighborhood such that any integral curve of ξ passing through the neigh-

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borhood passes through only once. The celebrated theorem of Boothby and Wang [5] states that a compact regular contact manifold is a principal circle bundle over a symplectic manifold of integral class. At first the author conjectured that the above theorem might still be true without the assumption on the regularity of the contact structure. This is not the case, however, as we shall see. For brevity a contact metric structure for which ξ is a Killing vector field is called a *K-contact* structure. First recall the result of [12] that the standard contact metric structure on the tangent sphere bundle T_1M of a Riemannian manifold (M, G) is K-contact if and only if (M, G) has constant curvature $+1$. Also recall the result of [2] that the standard contact structure on the tangent sphere bundle of a compact Riemannian manifold of nonpositive constant curvature is not regular. In this paper we prove the following theorem.

THEOREM. *Let T_1M be the tangent sphere bundle of a compact Riemannian manifold (M, G) and $\mathscr{A}$ the set of all Riemannian metrics associated to its standard contact structure. Then the standard associated metric is a critical point of the functional $L(g) = \int_{T_1M} \operatorname{Ric}(\xi)\, dV$ on $\mathscr{A}$ if and only if the base manifold is of constant curvature $+1$ or -1.*

For a contact metric structure with structure tensors (φ, ξ, η, g) (see Section 2 for definitions) let $h = \frac{1}{2}\mathscr{L}_\xi\varphi$ and $\tau = \mathscr{L}_\xi g$ where $\mathscr{L}$ denotes Lie differentiation. It is well known and easy to check that ξ is Killing if and only if $h = 0$ and that $\operatorname{Ric}(\xi) = 2n - \operatorname{tr} h^2$ [1, page 67]. Also $\tau(X, Y) = 2g(X, h\varphi Y)$. Thus the critical point question for $L(g)$ is the same as that for $\int_M |h|^2\, dV$ or $\int_M |\tau|^2\, dV$. This last integral was studied by Chern and Hamilton [7] for 3-dimensional contact manifolds as a functional on $\mathscr{A}$ regarded as the set of "CR-structures" on M and was studied again by Tanno [11] in general dimension.

2. Contact manifolds

By a *contact manifold* we mean a C^∞ manifold M^{2n+1} together with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. Given a contact form η it is well known that there exists a unique vector field ξ on M satisfying $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$; ξ is called the *characteristic vector field* of the contact structure. A Riemannian metric g is said to be an *associated metric* if there exists a tensor field φ of type $(1, 1)$ such that $d\eta(X, Y) = g(X, \varphi Y)$, $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(X) = g(X, \xi)$. These metrics can be constructed by the polarization of $d\eta$ evaluated on a local orthonormal basis of an arbitrary metric on the contact subbundle $\{\eta = 0\}$; see [1] as a general reference. We refer to (φ, ξ, η, g) as a *contact metric structure*. We also note that all associated metrics have the same volume element, namely

$$dV = \frac{1}{2^n n!}\eta \wedge (d\eta)^n.$$

For a contact metric structure (φ, ξ, η, g), let $h = \frac{1}{2}\mathscr{L}_\xi\varphi$ as above and let ∇ denote the Riemannian (Levi-Civita) connection of g. Then

$$\nabla_X\xi = -\varphi X - \varphi h X; \tag{1}$$

see [1, page 66].

3. Geometry of TM and T_1M

Let M be an $(n+1)$-dimensional C^∞ manifold and $\overline{\pi}: TM \to M$ its tangent bundle. If $(x^1, \dots, x^{n+1})$ are local coordinates on M, set $q^i = x^i \circ \overline{\pi}$; then $(q^1, \dots, q^{n+1})$ together with the fibre coordinates $(v^1, \dots, v^{n+1})$ form local coordinates on TM. If X is a vector field on M, its *vertical lift* X^V on TM is the vector field defined by $X^V\omega = \omega(X) \circ \overline{\pi}$ where ω is a 1-form on M, which on the left side of this equation is regarded as a function on TM. For an affine connection D on M, the *horizontal lift* X^H of X is defined by $X^H\omega = D_X\omega$. The *connection map* $K: TTM \to TM$ is defined by

$$KX^H = 0, \quad K(X_t^V) = X_{\overline{\pi}(t)}, \qquad t \in TM.$$

TM admits an almost complex structure J defined by

$$JX^H = X^V, \qquad JX^V = -X^H.$$

Dombrowski [8] showed that J is integrable if and only if D has vanishing curvature and torsion.

If now G is a Riemannian metric on M and D its Riemannian connection, we define a Riemannian metric $\overline{g}$ on TM, called the *Sasaki metric*, by

$$\overline{g}(X, Y) = G(\overline{\pi}_*X, \overline{\pi}_*Y) + G(KX, KY)$$

where X and Y are vector fields on TM. Since $\overline{\pi}_* \circ J = -K$ and $K \circ J = \overline{\pi}_*$, $\overline{g}$ is Hermitian for the almost complex structure J.

On TM define a 1-form β by $\beta(X)_t = G(t, \overline{\pi}_*X)$, $t \in TM$, or equivalently by the local expression $\beta = \sum G_{ij}v^i dq^j$. Then $d\beta$ is a symplectic structure on TM and in particular $2d\beta$ is the fundamental 2-form of the almost Hermitian structure $(J, \overline{g})$. Thus TM has an almost Kaehler structure which is Kaehlerian if and only if (M, G) is flat (Dombrowski [8], Tachibana and Okumura [10]).

Let $\underline{R}$ denote the curvature tensor of G, $\overline{\nabla}$ the Riemannian connection of $\overline{g}$ and $\overline{R}$ the curvature tensor of $\overline{\nabla}$. Complete formulas for $\overline{\nabla}$ and $\overline{R}$ at $t \in TM$ can be found in [9]; here we give just the ones we need.

$$(\overline{\nabla}_{X^V} Y^H)_t = -\tfrac{1}{2}(\underline{R}_{Xt} Y)^H, \tag{2}$$

$$(\overline{\nabla}_{X^H} Y^H)_t = (D_X Y)^H_t - \tfrac{1}{2}(\underline{R}_{XY} t)^V, \tag{3}$$

$$(\overline{R}_{X^H Y^V} Z^H)_t = (\tfrac{1}{4}\underline{R}_{\underline{R}_{tY} Z, Xt} + \tfrac{1}{2}\underline{R}_{XZ} Y)^V_t + \tfrac{1}{2}((D_X \underline{R})_{tY} Z)^H_t, \tag{4}$$

$$\begin{aligned}(\overline{R}_{X^H Y^H} Z^H)_t = {} & \tfrac{1}{2}((D_Z \underline{R})_{XY} t)^V_t \\ & + (\underline{R}_{XY} Z + \tfrac{1}{4}\underline{R}_{t\underline{R}_{ZYt}} X + \tfrac{1}{4}\underline{R}_{t\underline{R}_{XZt}} Y + \tfrac{1}{2}\underline{R}_{t\underline{R}_{XYt}} Z)^H_t.\end{aligned} \tag{5}$$

The tangent sphere bundle $\pi \colon T_1 M \to M$ is the hypersurface of TM defined by $\sum G_{ij} v^i v^j = 1$. The vector field $N = v^i \partial/\partial v^i$ is a unit normal, as well as the position vector for a point t. The Weingarten map A of $T_1 M$ with respect to the normal N is given by $AU = -U$ for any vertical vector U and $AX = 0$ for any horizontal vector X (see, for example, [1, page 132]). Thus many computations on $T_1 M$ involving horizontal vector fields can be done directly on TM.

Let g' denote the metric on $T_1 M$ induced from $\overline{g}$ on TM. Define φ', ξ' and η' on $T_1 M$ by

$$\xi' = -JN, \qquad JX = \varphi' X + \eta'(X) N.$$

Then η' is the contact form on $T_1 M$ induced from the 1-form β on TM as one can easily check. However $g'(X, \varphi' Y) = 2d\eta'(X, Y)$, so strictly speaking $(\varphi', \xi', \eta', g')$ is not a contact metric structure. Of course the difficulty is easily rectified and we shall take $\eta = \frac{1}{2}\eta'$ as the standard contact structure and

$$\eta = \tfrac{1}{2}\eta', \quad \xi = 2\xi', \quad \varphi = \varphi', \quad g = \tfrac{1}{4} g'$$

as the standard contact metric structure on $T_1 M$. In local coordinates $\xi = 2v^i(\partial/\partial x^i)^H$; on TM the vector field $v^i(\partial/\partial x^i)^H$ is the so-called geodesic flow.

4. Proof of the theorem

In the course of the proof we will use the following lemma of Cartan [6, pages 257–258].

LEMMA. *Let (M, G) be a Riemannian manifold, D the Riemannian connection of G and $\underline{R}$ its curvature tensor. Then (M, G) is locally symmetric if and only if $(D_X \underline{R})(Y, X, Y, X) = 0$ for all orthonormal pairs $\{X, Y\}$.*

Now turning to the proof of the theorem, let ∇ denote the Riemannian connection of g on T_1M and compute $\nabla_U\xi$ by both (1) and (2) for a vertical vector U at $t \in T_1M$; we have

$$\begin{aligned}
-\varphi U_t - \varphi h U_t = (\nabla_U \xi)_t &= \left(2\overline{\nabla}_U v^i \left(\frac{\partial}{\partial x^i}\right)^H\right)_t \\
&= 2(Uv^i)\left(\frac{\partial}{\partial x^i}\right)^H_t + 2v^i\left(-\frac{1}{2}\left(\underline{R}_{KU,t}\frac{\partial}{\partial x^i}\right)^H\right) \\
&= 2(KU)^H_t - (\underline{R}_{KU,t}t)^H.
\end{aligned}$$

Applying φ we obtain

$$hU_t = U_t - (\underline{R}_{KU,t}t)^V. \tag{6}$$

Similarly using (3) we have for a horizontal vector X orthogonal to ξ

$$hX_t = -X_t + (\underline{R}_{\pi_* X,t}t)^H. \tag{7}$$

In [3] we showed that an associated metric g was a critical point of $L(g)$ if and only if

$$R_{X\xi}\xi = -\varphi^2 X - h^2 X + 2hX. \tag{8}$$

Now for a vertical vector U we compute the left side using (4) and the Gauss equation and the right side using (6); we then have

$$(R_{U\xi}\xi)_t = (\overline{R}_{U\xi}\xi)_t = (\underline{R}_{\underline{R}_{KU,t}t,t}t)^V + 2((D_t\underline{R})_{KU,t}t)^H$$

and

$$\begin{aligned}
&-\varphi^2 U - h^2 U + 2hU \\
&\quad = U - (U - (\underline{R}_{KU,t}t)^V) + (\underline{R}_{(KU-\underline{R}_{KU,t}t),t}t)^V + 2U - 2(\underline{R}_{KU,t}t)^V \\
&\quad = 2U - (\underline{R}_{\underline{R}_{KU,t}t,t}t)^V.
\end{aligned}$$

Comparing horizontal and vertical parts, we have for any orthonormal pair $\{X, t\}$ on the base manifold (M, G)

$$(D_t\underline{R})_{Xt}t = 0 \tag{9}$$

and

$$\underline{R}_{\underline{R}_{Xt}t,t}t = X. \tag{10}$$

From (9) and the lemma of Cartan stated above we see that (M, G) is

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locally symmetric. Now working on (M, G), for each unit tangent vector $t \in T_mM$, we let $[t]^\perp$ denote the subspace of T_mM orthogonal to t and define a symmetric linear transformation $L_t\colon [t]^\perp \to [t]^\perp$ by $L_tX = \underline{R}_{Xt}t$. Then from (10) we have the $(L_t)^2 = I$ and hence that the eigenvalues of L_t are ± 1. Now M is irreducible; for if M had a locally Riemannian product structure, then, choosing t tangent to one factor and X tangent to the other, we would have $\underline{R}_{Xt}t = 0$, contradicting the fact that the only eigenvalues of L_t are ± 1. However the sectional curvature of an irreducible locally symmetric space does not change sign. Thus if for some t, L_t had both $+1$ and -1 occurring as eigenvalues, there would be sectional curvatures equal to $+1$ and -1. Consequently only one eigenvalue can occur and hence (M, G) is a space of constant curvature $+1$ or -1.

Conversely if (M, G) has constant curvature c, equations (6) and (7) become, respectively, $hU_t = (1-c)U_t$ and $hX_t = (c-1)X_t$, where X is orthogonal to ξ. Similarly equations (4) and (5) yield $(R_{\xi U}\xi)_t = -c^2U_t$ and $(R_{\xi X}\xi)_t = (3c^2 - 4c)X_t$. Now substituting these into the critical point condition (8), we see that it is satisfied if and only if $c = \pm 1$.

Appendix

There is an error in the last step of the proof of the theorem of the first paper [3] in this series, and the last paragraph should be replaced by the following.

Now suppose that the critical point g is not a K-contact metric. Since φ and h anti-commute, we may assume that all the λ_{2j-1}, $j = 1, \ldots, n$, are non-negative. Also from equation (3.3) it is easy to see that if some of the λ_{2j-1} vanish, the zero eigenspace of h is parallel along ξ and hence we may choose the corresponding X_{2j-1} and X_{2j} parallel along a fibre. Again since M is regular we may choose a vector field Y on $U \times S^1$ such that at least some $\alpha_{2i-1} \not\equiv 0$ for some $\lambda_{2i-1} \neq 0$ and Y is horizontal and projectable, that is, $[\xi, Y] = 0$. Writing $Y = \sum_i(\alpha_{2i-1}X_{2i-1} + \beta_{2i}X_{2i})$ along a fibre we have

$$\begin{aligned}
0 = [\xi, Y] &= \nabla_\xi Y - \nabla_Y \xi \\
&= \sum_i \{(\xi\alpha_{2i-1})X_{2i-1} + \alpha_{2i-1}\nabla_\xi X_{2i-1} + (\xi\beta_{2i})X_{2i} + \beta_{2i}\nabla_\xi X_{2i} \\
&\qquad + \alpha_{2i-1}X_{2i} + \lambda_{2i-1}\alpha_{2i-1}X_{2i} - \beta_{2i}X_{2i-1} + \lambda_{2i-1}\beta_{2i}X_{2i-1}\},
\end{aligned}$$

where $\nabla_{X_{2i-1}}\xi = -(1+\lambda_{2i-1})X_{2i}$ and $\nabla_{X_{2i}}\xi = (1-\lambda_{2i-1})X_{2i-1}$ follow from

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equation (2.1). Again from (3.3) we see that $g(\nabla_\xi X_{2i-1}, X_{2j}) = 0$ for $i \neq j$ and we have (3.4) for the case $i = j$. Note also that since $\nabla_\xi \varphi = 0$, $g(\nabla_\xi X_{2i-1}, X_{2j-1}) = g(\nabla_\xi X_{2i}, X_{2j})$. Now taking the inner product with X_{2j-1} and X_{2j} respectively, in our formula for $0 = [\xi, Y]$, we have

$$0 = \xi\alpha_{2j-1} + \sum_i \alpha_{2i-1} g(\nabla_\xi X_{2i}, X_{2j}) + \lambda_{2j-1}\beta_{2j}$$

and

$$0 = \xi\beta_{2j} + \sum_i \beta_{2i} g(\nabla_\xi X_{2i}, X_{2j}) + \lambda_{2j-1}\alpha_{2j-1}.$$

Multiplying the first of these by β_{2j}, the second by α_{2j-1}, and summing on j, we have

$$\xi\left(\sum_j \alpha_{2j-1}\beta_{2j}\right) = -\sum_j \lambda_{2j-1}(\alpha_{2j-1}^2 + \beta_{2j}^2) \leq 0.$$

Thus $\sum \alpha_{2j-1}\beta_{2j}$ is a non-increasing, non-constant function along the integral curve, contradicting its periodicity.

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Michigan State University
East Lansing, Michigan 48824-1027
U.S.A.

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