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ON THE SEMILATTICE OF IDEMPOTENTS OF A FREE INVERSE MONOID

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Some new concepts are introduced, in particular that of a unique factorization semilattice. Necessary and sufficient conditions are given for two principal ideals of the semilattice of idempotents of a free inverse monoid $FIM(X)$ to be isomorphic and some properties of the Munn semigroup of $E[FIM(X)]$ are obtained. Some results on the embedding of semilattices in $E[FIM(X)]$ are also obtained.

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1. Preliminaries

The general terminology and notation are those of Petrich **[6]**.

Let S be a semigroup. We say that S is a *semilattice* if the following conditions hold:

$$\forall e \in S, e^2 = e;$$

$$\forall e, f \in S, ef = fe.$$

Let E be a semilattice. The *natural partial order* on E is defined by

$$e \leqq f \Leftrightarrow e = ef.$$

Suppose that $e, f \in E$ are such that $e < f$ and the condition

$$e \leqq g < f \Rightarrow e = g$$

holds for every $g \in E$. Then we say that f covers e and we denote this fact by $e \prec f$. For every $f \in E$, we define $\mathrm{Cov}(f) = \{e \in E : e \prec f\}$.

For every $e \in E$, we say that $Ee = \{fe : f \in E\}$ is the *principal ideal* of E generated by e.

Now let S be an inverse semigroup. The subset of all idempotents of S is a semilattice, usually denoted by $E(S)$, and so $E(S)$ is said to be the semilattice of idempotents of S.

Let X be a nonempty set. We define $X^{-1} = \{x^{-1} : x \in X\}$ to be a set such that

$$X \cap X^{-1} = \emptyset;$$

$$\forall x_1, x_2 \in X, x_1^{-1} = x_2^{-1} \Rightarrow x_1 = x_2.$$

Moreover, we define $(x^{-1})^{-1}=x$ for every $x\in X$. Now let $(X\cup X^{-1})^*$ denote the free monoid on $X\cup X^{-1}$ [2, §9.1], and let

$$R_X=(X\cup X^{-1})^*\Bigg\backslash\left[\bigcup_{x\in X\cup X^{-1}}(X\cup X^{-1})^*xx^{-1}(X\cup X^{-1})^*\right].$$

We define a partial order $\leqq_l$ on R_X by

$$u\leqq_l v\Leftrightarrow v\in uR_X.$$

A subset A of R_X is said to be *left closed* if

$$\forall v\in A\,\forall u\in R_X,\ u\leqq_l v\Rightarrow u\in A.$$

Finally, let E_X denote the set of all finite nonempty left closed subsets of R_X, with the operation described by

$$AB=A\cup B,$$

and let $FIM(X)$ denote the free inverse monoid on X [6, §VIII.1.].

Lemma 1.1 [5;8]. *Let X be a nonempty set. Then*

$$E_X\cong E[FIM(X)].$$

For the remainder of this paper, we assume that $E_X=E[FIM(X)]$. Let $v\in R_X$. We define

$$\bar{v}=\{u\in R_X: u\leqq_l v\}.$$

It is immediate that $\bar{v}\in E_X$ for every $v\in R_X$. It follows easily that, for every $A,B\in E_X$,

$$A\geqq B\Leftrightarrow A\subseteq B, \tag{1.1}$$

and for every $A\in E_X$,

$$A=\prod_{u\in A}\bar{u}. \tag{1.2}$$

2. Unique factorization semilattices

In this section we introduce some concepts in semilattice theory and we relate them to E_X.

Let E be a semilattice and let $e \in E$. We say that e is *irreducible* if, for every $f, g \in E$,

$$e = fg \Rightarrow e = f \text{ or } e = g.$$

The set of all irreducibles of E is denoted by $\mathrm{Irr}(E)$.
We say that e is *prime* if, for every $f, g \in E$,

$$e \geqq fg \Rightarrow e \geqq f \text{ or } e \geqq g.$$

Lemma 2.1. *Let E be a semilattice and let $e \in E$. Then*

$$e \textit{ prime} \Rightarrow e \textit{ irreducible}.$$

Proof. Suppose that e is prime and suppose that $e = fg$ for some $f, g \in E$. Then $e \leqq f$ and $e \leqq g$. Further, $e \geqq fg$ and so, since e is prime, we have $e \geqq f$ or $e \geqq g$. Hence $e = f$ or $e = g$. Thus e is irreducible.

The semilattice E is said to be a *unique factorization semilattice* (UFS) *if*

(i) E is generated by $\mathrm{Irr}(E)$;

(ii) every irreducible is prime.

All these concepts are inspired by well-known concepts for integral domains [**1**, §5.3]. We need some results on UFSs.

Lemma 2.2. *Let E denote a UFS. Let $e_1, \ldots, e_n, f_1, \ldots, f_m \in \mathrm{Irr}(E)$ be such that $e_1 \ldots e_n = f_1 \ldots f_m$. Then, for every $i \in \{1, \ldots, n\}$, there exists $j \in \{1, \ldots, m\}$ such that $e_i \geqq f_j$.*

Proof. Let $i \in \{1, \ldots, n\}$. Clearly, $e_i \geqq f_1 \ldots f_m$. Since E is a UFS, e_i is prime and an elementary induction yields $e_i \geqq f_j$ for some $j \in \{1, \ldots, m\}$.

Lemma 2.3. *Let E denote a UFS and let $e \in E$. Then*

(i) $\mathrm{Irr}(Ee) = e\,.\,\mathrm{Irr}(E)$;

(ii) *Ee is a UFS.*

Proof. (i) Let $f \in \mathrm{Irr}(Ee)$. Since E is a UFS, there exist $g_1, \ldots, g_n \in \mathrm{Irr}(E)$ such that $f = g_1 \ldots g_n$. Let I be minimal among the nonempty subsets of $\{1, \ldots, n\}$ with respect to $f = e \prod_{i \in I} g_i$. Suppose that $|I| > 1$. Since $eg_i > f$ for every $i \in I$ and $f = \prod_{i \in I} eg_i$, we obtain $f \notin \mathrm{Irr}(Ee)$, a contradiction. Hence $|I| = 1$ and so $f \in e\,.\,\mathrm{Irr}(E)$.

Conversely, let $g \in \mathrm{Irr}(E)$ and suppose that $eg = ff'$ for some $f, f' \in Ee$. We have $e \geqq f \geqq eg$ and $e \geqq f' \geqq eg$. But $g \geqq ff'$ and since E is a UFS, g is prime, so $g \geqq f$ or $g \geqq f'$. Without loss of generality, we can assume that $g \geqq f$. Hence $eg \geqq f$ and so $eg = f$. Thus $eg \in \mathrm{Irr}(Ee)$.

(ii) Let $f \in Ee$. Since E is a UFS, there exist $g_1, \ldots, g_n \in \mathrm{Irr}(E)$ such that $f = g_1 \ldots g_n$.

Therefore $f=ef=eg_1\dots g_n=(eg_1)\dots(eg_n)$. By (i), $eg_i\in\mathrm{Irr}(Ee)$ for every $i\in\{1,\dots,n\}$. Thus Ee is generated by $\mathrm{Irr}(Ee)$.

Now let $h\in\mathrm{Irr}(Ee)$ and let $a,b\in Ee$. Suppose that $h\geqq ab$. By (i), we have $h=eg$ for some $g\in\mathrm{Irr}(E)$. Hence $g\geqq ab$ and so, since g is prime, $g\geqq a$ or $g\geqq b$. We can assume that $g\geqq a$. Since $e\geqq a$, we have $h=eg\geqq a$. Thus h is prime and the lemma is proved.

Lemma 2.4. *Let E be a UFS. Let $e,f,g\in\mathrm{Irr}(E)$ be such that $f\succ e$ and $g\succ e$. Then $f=g$.*

Proof. Since $f\geqq e$ and $g\geqq e$, we have $fg\geqq e$. Suppose that $fg=e$. Since $e\in\mathrm{Irr}(E)$, we must have $e=f$ or $e=g$, a contradiction. Hence $fg>e$. Now $f\geqq fg>e$ and so, since $f\succ e$, we have $f=fg$. Therefore $f\leqq g$ and so $f=g$.

We say that a semilattice E is *upper finite* if the sets $\{f\in E\colon f\geqq e\}$ are finite for all $e\in E$.

The next lemmas state some properties of E_X.

Lemma 2.5. *Let X be a nonempty set. Then*

(i) $\mathrm{Irr}(E_X)=\{\bar{w}\colon w\in R_X\}$;

(ii) *E_X is a UFS;*

(iii) *E_X is upper finite.*

Proof. Let $A\in\mathrm{Irr}(E_X)$. By (1.2), we have $A=\prod_{u\in A}\bar{u}$. Since $A\in\mathrm{Irr}(E_X)$, we have $A=\bar{u}$ for some $u\in A$. Therefore $\mathrm{Irr}(E_X)\subseteq\{\bar{w}\colon w\in R_X\}$.

Now suppose that $w\in R_X$. We prove that $\bar{w}$ is prime. Suppose that $\bar{w}\geqq AB$ for some $A,B\in E_X$. By (1.1), we have $w\in\bar{w}\subseteq AB=A\cup B$. We can assume that $w\in A$. But A is left closed and so $\bar{w}\subseteq A$. Thus $\bar{w}\geqq A$, by (1.1), and so $\bar{w}$ is prime.

By Lemma 2.1, this implies $\bar{w}$ irreducible and so (i) is proved. Moreover, it follows that every irreducible of E_X is prime. By (i) and (1.2), $\mathrm{Irr}(E_X)$ generates E_X and so E_X is a UFS.

It follows easily from (1.1) that E_X is upper finite.

Lemma 2.6. *Let X be a nonempty set. Let $A\in E_X$ and let $B\in[\mathrm{Irr}(E_XA)]\backslash\{A\}$. Then there exists a unique $C\in\mathrm{Irr}(E_XA)$ such that $B\prec C$.*

Proof. By Lemma 2.5(ii), E_X is a UFS and so, by Lemmas 2.3 and 2.5(i), we have $B=A\bar{u}$ for some $u\in R_X$, Since $B\neq A$, we have $u\notin A$. In particular, $u\neq 1$ and so we can define $v\in R_X$ to be the maximal proper prefix of u. Let $C=A\bar{v}$. By Lemmas 2.3 and 2.5, $C\in\mathrm{Irr}(E_XA)$. Since $|C|=|B|-1$, we have $B\prec C$. The uniqueness of C follows from Lemma 2.4, replacing E by E_XA and e by B.

3. Principal ideals

In this section we shall obtain necessary and sufficient conditions for two principal ideals of E_X to be isomorphic.

Lemma 3.1. *Let X be a nonempty set and let $A \in E_X$. Then*

$$|Cov(A)| = \begin{cases} 2|A|(|X|-1)+2 & \text{if } X \text{ is finite} \\ |X| & \text{if } X \text{ is infinite.} \end{cases}$$

Proof. We assume that X is finite, the other case being obvious. We use induction on $|A|$.

Suppose that $|A|=1$. Then $A=\{1\}$ and so $Cov(A)=\{\{1,x\}: x \in X \cup X^{-1}\}$. Hence $|Cov(A)|=2|X|$ and the lemma holds.

Now suppose that the lemma holds for every $B \in E_X$ such that $|B| \leqq n$, with $n \in \mathbb{N}$. Let $A \in E_X$ be such that $|A|=n+1$. Let $v \in A$ have maximal length. Since $|A|>1$, we have $|v|>1$. Let $y \in X \cup X^{-1}$ denote the last letter of v. Let $A' \in E_X$ and suppose that $A' \in Cov(A)$. Since $|A' \backslash A|=1$, we can define $u(A')$ to be the single element of $A' \backslash A$. We define $\Gamma=\{A' \in Cov(A): v \leqq_l u(A')\}$ and $\Lambda=[Cov(A)] \backslash \Gamma$.

Let $A' \in \Gamma$. Since $|v|$ is maximal in A and $|A' \backslash A|=1$, it follows that $u(A')=vx$ for some $x \in X \cup X^{-1}$. Since y is the last letter of v and $u(A') \in R_X$, we have $x \neq y^{-1}$ and so $\Gamma=\{A \cup \{vx\}: x \in (X \cup X^{-1}) \backslash \{y^{-1}\}\}$. Hence $|\Gamma|=2|X|-1$.

Let $A_0=A \backslash \{v\}$. Since $|v|$ is maximal in A and $|v|>1$, we have $A_0 \in E_X$. We define a map $\varphi: \Lambda \to Cov(A_0)$ as follows. Suppose that $A' \in \Lambda$. Let $A'_0=A' \backslash \{v\}$. Since $v \nleqq_l u(A')$, it follows that v must be maximal in A' for $\leqq_l$. Hence $A'_0 \in E_X$. It is clear that $A'_0 \in Cov(A_0)$ and so we can define $A'\varphi=A'_0$. Moreover, φ is injective and $[Cov(A_0)] \backslash (\Lambda\varphi)=\{A\}$. Hence $|\Lambda|=|Cov(A_0)|-1$. Using the induction hypothesis, we obtain $|\Lambda|=2|A_0|(|X|-1)+2-1=2n(|X|-1)+1$. Thus $|A|=|\Gamma|+|\Lambda|=2|X|-1+2n(|X|-1)+1=2(n+1)(|X|-1)+2=2|A|(|X|-1)+2$ and the result follows by induction.

We must introduce some new concepts and notation.

Let $A \in E_X$ and let $m=|A|$. For all $k \in \mathbb{N}^0$, we define $\mathrm{Irr}_{m+k}(E_X A)=\{B \in \mathrm{Irr}(E_X A): |B|=m+k\}$. Surely, $\mathrm{Irr}(E_X A)=\bigcup_{k \geq 0} \mathrm{Irr}_{m+k}(E_X A)$. Moreover, $\mathrm{Irr}_m(E_X A)=\{A\}$ and $\mathrm{Irr}_{m+1}(E_X A)=Cov(A)$. For every $B \in \mathrm{Irr}(E_X A)$, we define $[B]_A=\{C \in \mathrm{Irr}(E_X A): C \prec B\}$. Suppose that $B \neq A$. By Lemmas 2.3 and 2.5, we have $B=A\bar{u}$ for some $u \in R_X \backslash A$. It follows easily that $[B]_A=\{A\bar{v}: v \in R_X \text{ and } v=ux \text{ for some } x \in X \cup X^{-1}\}$. Thus

$$|[B]_A| = \begin{cases} 2|X|-1 & \text{if } X \text{ is finite} \\ |X| & \text{if } X \text{ is infinite.} \end{cases} \tag{3.1}$$

Now we obtain a criterion for isomorphism.

Lemma 3.2. *Let X be a nonempty set and let $A, B \in E_X$. Then*

$$E_XA \cong E_XB \Leftrightarrow |Cov(A)| = |Cov(B)|.$$

Proof. Suppose that $\Phi: E_XA \to E_XB$ is an isomorphism. We certainly have $A\Phi = B$. Let $A' \in Cov(A)$. Since Φ is injective, we have $A'\Phi < B$. Suppose that $A'\Phi < B' < B$ for some $B' \in E_XB$. Let $A'' = B'\Phi^{-1}$. It follows easily that $A' < A'' < A$, in contradiction with $A' \in Cov(A)$. Hence no such B' exists and so $A'\Phi \in Cov(B)$. Thus $[Cov(A)]\Phi \subseteq Cov(B)$. Similarly, we obtain $[Cov(B)]\Phi^{-1} \subseteq Cov(A)$. Hence $[Cov(A)]\Phi = Cov(B)$ and so $|Cov(A)| = |Cov(B)|$.

Conversely, suppose that $|Cov(A)| = |Cov(B)|$. Suppose that $m = |A|$ and $n = |B|$. For every $k \in \mathbb{N}^0$, we define a bijection $\varphi_k: \mathrm{Irr}_{m+k}(E_XA) \to \mathrm{Irr}_{n+k}(E_XB)$ as follows.

Consider $k = 0$. Since $\mathrm{Irr}_m(E_XA) = \{A\}$ and $\mathrm{Irr}_n(E_XB) = \{B\}$, we define $A\varphi_0 = B$.

Now suppose that φ_k is defined for some $k \in \mathbb{N}^0$. Let $C \in \mathrm{Irr}_{m+k}(E_XA)$. Suppose first that $k = 0$. Then $C = A$ and $C\varphi_k = B$ and so $|[C]_A| = |Cov(A)| = |Cov(B)| = |[C\varphi_k]_B|$. Suppose now that $k > 0$. By (3.1), we obtain $|[C]_A| = |[C\varphi_k]_B|$ as well. Whatever the case, we can define a bijection $\psi_C: [C]_A \to [C\varphi_k]_B$ for every $C \in \mathrm{Irr}_{m+k}(E_XA)$. Since $\mathrm{Irr}_{m+k+1}(E_XA) = \bigcup_{C \in \mathrm{Irr}_{m+k}(E_XA)} [C]_A$ and $\mathrm{Irr}_{n+k+1}(E_XB) = \bigcup_{D \in \mathrm{Irr}_{n+k}(E_XB)} [D]_B$, there is a unique map $\varphi_{k+1}: \mathrm{Irr}_{m+k+1}(E_XA) \to \mathrm{Irr}_{n+k+1}(E_XB)$ such that, for every $C \in \mathrm{Irr}_{m+k}(E_XA), \varphi_{k+1}|_{[C]_A} = \psi_C$. Since φ_k is bijective and every ψ_C is bijective, it follows that φ_{k+1} is bijective as well. Next, we define $\varphi: \mathrm{Irr}(E_XA) \to \mathrm{Irr}(E_XB)$ to be the unique bijection such that $\varphi|_{\mathrm{Irr}_{m+k}(E_XA)} = \varphi_k$ for every $k \in \mathbb{N}^0$.

We prove that, for every $C, D \in \mathrm{Irr}(E_XA)$,

$$C \prec D \Leftrightarrow C\varphi \prec D\varphi. \tag{3.2}$$

Suppose that $C \prec D$. Then $C \in [D]_A$ and so $C\varphi = C\psi_D \in [D\varphi]_B$. Hence $C\varphi \prec D\varphi$.

Conversely, suppose that $C\varphi \prec D\varphi$. It is immediate that $C \neq A$. By Lemma 2.6, there exists a unique $C' \in \mathrm{Irr}(E_XA)$ such that $C \prec C'$. It follows from above that $C\varphi \prec C'\varphi$. By Lemma 2.4, with E replaced by E_XB and e replaced by $C\varphi$, we obtain $D\varphi = C'\varphi$. Since φ is bijective, it follows that $D = C'$ and so $C \prec D$. Thus (3.2) holds.

Since E_X is upper finite, it follows easily from (3.2) that, for every $C, D \in \mathrm{Irr}(E_XA)$,

$$C \leqq D \Leftrightarrow C\varphi \leqq D\varphi. \tag{3.3}$$

Suppose now that $C_1 \dots C_r = D_1 \dots D_s$, with $C_1, \dots, C_r, D_1, \dots, D_s \in \mathrm{Irr}(E_XA)$. Let $i \in \{1, \dots, r\}$. By Lemmas 2.2 and 2.3(ii), there exists $j \in \{1, \dots, s\}$ such that $C_i \geqq D_j$. By (3.3), we have $C_i\varphi \geqq D_j\varphi$ and so $C_1\varphi \dots C_r\varphi \geqq D_1\varphi \dots D_s\varphi$. Similarly, we obtain $D_1\varphi \dots D_s\varphi \geqq C_1\varphi \dots C_r\varphi$ and so $C_1\varphi \dots C_r\varphi = D_1\varphi \dots D_s\varphi$. A similar argument shows that $C_1\varphi \dots C_r\varphi = D_1\varphi \dots D_s\varphi$ implies $C_1 \dots C_r = D_1 \dots D_s$ and so we can define an injecive map $\Phi: E_XA \to E_XB$ as follows. Let $C \in E_XA$. By Lemma 2.3, we can write $C = C_1 \dots C_r$ for some $C_1, \dots, C_r \in \mathrm{Irr}(E_XA)$. Then we define $C\Phi = C_1\varphi \dots C_r\varphi$.

We show that Φ is an isomorphism.

Let $C \in E_XB$. By Lemma 2.3(ii), there exist $C_1, \dots, C_r \in \mathrm{Irr}(E_XB)$ such that $C = C_1 \dots C_r$. Since φ is bijective, there exist $D_1, \dots, D_r \in \mathrm{Irr}(E_XA)$ such that $C_i = D_i\varphi$ for every $i \in \{1, \dots, r\}$. Thus $C = C_1 \dots C_r = D_1\varphi \dots D_r\varphi = (D_1 \dots D_r)\Phi$ and so Φ is surjective.

Let $C, D \in E_X A$. Suppose that $C = C_1 \dots C_r$ and $D = D_1 \dots D_s$ for some $C_1, \dots, C_r, D_1, \dots, D_s \in \mathrm{Irr}(E_X A)$. Then $C\Phi . D\Phi = (C_1 \dots C_r)\Phi . (D_1 \dots D_s)\Phi = C_1\varphi \dots C_r\varphi D_1\varphi \dots D_s\varphi = (C_1 \dots C_r D_1 \dots D_s)\Phi = (CD)\Phi$. Thus Φ is a homomorphism and the lemma is proved.

We note that every isomorphism $\Phi: E_X A \to E_X B$ must induce bijections between $\mathrm{Irr}_{m+k}(E_X A)$ and $\mathrm{Irr}_{n+k}(E_X B)$ and satisfy (3.2).

Now Lemmas 3.1 and 3.2 yield:

Theorem 3.3. *Let X be a nonempty set and let $A, B \in E_X$.*

(i) *If X is infinite or $|X| = 1$, then $E_X A \cong E_X B$.*

(ii) *If X is finite and $|X| > 1$, then*

$$E_X A \cong E_X B \Leftrightarrow |A| = |B|.$$

A semilattice in which all the principal ideals are isomorphic is said to be *uniform*. It follows from Theorem 3.3 that, if X is infinite or $|X| = 1$, then E_X is uniform.

4. The Munn semigroup

We can use the results obtained in Section 3 to get information about the Munn semigroup [4] of the semillatice E_X.

Let E be a semilattice and let $U = \{(e,f) \in E \times E: Ee \cong Ef\}$. For every $(e,f) \in U$, let $T_{e,f}$ denote the set of all isomorphisms from Ee onto Ef. The *Munn semigroup* of E is defined to be $T_E = \bigcup_{(e,f)\in U} T_{e,f}$, with the usual composition of relations [3, §V.4]. This is an inverse semigroup and $E(T_E) = \{1_{Ee}: e \in E\}$ is isomorphic to E. It follows easily from the definition that, for every $e, f \in E$, $1_{Ee}\mathscr{D} = 1_{Ef}\mathscr{D}$ if and only if $(e,f) \in U$.

Theorem 4.1. *Let X be a nonempty set. Then T_{E_X} is E-unitary.*

Proof. Let $A, B, C \in E_X$ and let $\Phi: E_X A \to E_X B$ be an isomorphism. Suppose that $1_{E_X C} . \Phi \in E(T_{E_X})$. We want to prove that $\Phi \in E(T_{E_X})$. We have that $1_{E_X C} . \Phi$ is the restriction of Φ to the semilattice $(E_X C) \cap (E_X A)$, that is, $E_X CA$. Therefore we have $\Phi|_{E_X CA} = 1_{E_X CA}$ and we must show that $\Phi = 1_{E_X A}$.

Suppose that $\Phi \neq 1_{E_X A}$. We show that

$$\exists D \in \mathrm{Irr}(E_X) \text{ such that } D \ngeq A \text{ and } (AD)\Phi \neq BD. \tag{4.1}$$

Assume first that $A = B$. Since $\Phi \neq 1_{E_X A}$, there exists $U \in E_X A$ such that $U\Phi \neq U$. Since $A\Phi = B = A$, we have $U \neq A$ and so we can write $U = AD_1 \dots D_n$ for some $D_i \in \mathrm{Irr}(E_X)$, with $D_i \ngeq A, i \in \{1, \dots, n\}$. It follows that $D_i\Phi \neq D_i$ for some i and so (4.1) holds.

Now assume that $A \neq B$. Since $Cov(A) \subseteq \mathrm{Irr}(E_X A)$, and by Lemma 2.3(i), there exist

$\{D_i : i \in I\} \subseteq \mathrm{Irr}(E_X)$ such that $Cov(A) = \{AD_i : i \in I\}$. Suppose that $(AD_i)\Phi = BD_i$ for every $i \in I$. Since $[Cov(A)]\Phi = Cov(B)$, we have $Cov(B) = \{BD_i : i \in I\}$.

Suppose that $A \nsubseteq B$. Let $u \in A \setminus B$. Let u' denote the maximum prefix of u contained in B and suppose that $u = u'xu''$, with $x \in X \cup X^{-1}$ and $u'' \in R_X$. Then $B\overline{u'x} \in Cov(B)$ and so $B\overline{u'x} = BD_i$ for some $i \in I$. Since $\overline{u'x}, D_i \in \mathrm{Irr}(E_X)$, we show easily that $\overline{u'x} = D_i$. In fact, $D_i \geqq B\overline{u'x}$ and $D_i \ngeqq B$ together imply $D_i \geqq \overline{u'x}$. Similarly, $\overline{u'x} \geqq D_i$ and so $\overline{u'x} = D_i$. However, $\overline{u'x} \geqq A$, a contradiction. Thus $A \subseteq B$. Similarly, we obtain $B \subseteq A$ and so $A = B$, a contradiction. Therefore $(AD_i)\Phi \neq BD_i$ for some $i \in I$ and so (4.1) holds.

Now suppose that $D \in \mathrm{Irr}(E_X)$ is such that $D \ngeqq A$ and $(AD)\Phi \neq BD$. Let $D' \in \mathrm{Irr}(E_X)$ be such that $D' \prec D$. By Lemma 2.3(i), $AD \in \mathrm{Irr}(E_X A)$. Hence $(AD)\Phi \in \mathrm{Irr}(E_X B)$ and so, by Lemma 2.3(i), $(AD)\Phi = BU$ for some $U \in \mathrm{Irr}(E_X)$. Since $D \ngeqq A$, we have $U \ngeqq B$ and also $D' \ngeqq A$. Hence $AD' \prec AD$ and so $(AD')\Phi \prec (AD)\Phi$. Similarly, $(AD')\Phi = BU'$ for some $U' \in \mathrm{Irr}(E_X)$. Since U is prime, $BU' \prec BU$ and $U \ngeqq B$, we have $U' < U$. If $U' < U'' < U$ for some $U'' \in E_X$, then $U'' \in \mathrm{Irr}(E_X)$, $U'' \ngeqq B$ and it follows easily that $BU' < BU'' < BU$, a contradiction. Hence $U' \prec U$. Now suppose that $BU' = BD'$. Since $U', D' \in \mathrm{Irr}(E_X)$ and $U' \ngeqq B$, it follows easily that $U' = D'$. But $U' \prec U$ and $D' \prec D$, so, by Lemma 2.4, we have $U = D$, a contradiction. Hence $BU' \neq BD'$, that is, $(AD')\Phi \neq BD'$ and so (4.1) holds for $D \in \mathrm{Irr}(EX_X)$ with arbitrary large cardinal. In particular, we can assume that $|D| > |ABC|$. Suppose that $(AD)\Phi = BU$, with $U \in \mathrm{Irr}(E_X)$. Then $CAD = (CAD)\Phi = (CA)\Phi.(AD)\Phi = CABU$. Therefore $D \geqq CABU$. Since $|D| > |CAB|$, we have $D \ngeqq CAB$. Then, since D is prime, we get $D \geqq U$. Hence $|U| \geqq |D| > |ABC| \geqq |CA|$ and so $U \ngeqq CA$. But $U \geqq CAD$ and so, since U is prime, $U \geqq D$. Therefore $U = D$, a contradiction. Hence $\Phi = 1_{E_X A}$ and so T_{E_X} is E-unitary.

Let M be an inverse monoid. We say that M is *bisimple* if

$$\forall e, f \in E(M),\ e\mathscr{D} = f\mathscr{D}.$$

We say that M is *completely semisimple* if

$$\forall e, f \in E(M),\ e\mathscr{D} = f\mathscr{D} \Rightarrow e \nless f.$$

Theorem 4.2. *Let X be a nonempty set. Then*

(i) *T_{E_X} is bisimple if and only if X is infinite or $|X| = 1$;*

(ii) *T_{E_X} is completely semisimple if and only if X is finite and $|X| > 1$.*

Proof. (i) Let $A, B \in A_X$. Since $1_{E_X A}\mathscr{D} = 1_{E_X B}\mathscr{D}$ is equivalent to $E_X A \cong E_X B$, we have that T_{E_X} is bisimple if and only if E_X is uniform, and Theorem 3.3 yields the result.

(ii) Suppose that X is infinite or $|X| = 1$. Let $A, B \in E_X$ be such that $A > B$. We have that $1_{E_X A}\mathscr{D} = 1_{E_X B}\mathscr{D}$ and $1_{E_X A} > 1_{E_X B}$, so T_{E_X} is not completely semisimple.

Now suppose that X is finite and $|X| > 1$. Let $A, B \in E_X$ be such that $1_{E_X A}\mathscr{D} = 1_{E_X B}\mathscr{D}$ and $1_{E_X A} \leqq 1_{E_X B}$. Since $1_{E_X A}\mathscr{D} = 1_{E_X B}\mathscr{D}$, we have $E_X A \cong E_X B$, and by Theorem 3.3, $|A| = |B|$. Since $1_{E_X A} \leqq 1_{E_X B}$, we have $A \leqq B$. Clearly, $A \leqq B$ and $|A| = |B|$ together imply $A = B$, so T_{E_X} is completely semisimple and the lemma is proved.

5. Subsemilattices of E_X

The problem of finding necessary and sufficient conditions for a semilattice to be embeddable in E_X is still open. In this section, we obtain some results concerning some particular classes of semilattices.

Since the free inverse monoid of countable rank is itself embeddable in any free inverse monoid of rank greater than 1 [7], we will fix $X=\{x_n: n\in\mathbb{N}\}$ throughout this section.

Theorem 5.1. *Let L be a finite semilattice. Then L is embeddable in E_X.*

Proof. Let $\varphi: L\to X$ be an injective map. We define a map $\Phi: L\to E_X$ by $a\Phi=\{1\}\cup(L\backslash L^1a)\varphi$.

We show that Φ is a homomorphism. Let $a,b\in L$. Since $L^1ab=(L^1a)\cap(L^1b)$, we have $(ab)\Phi=\{1\}\cup(L\backslash L^1ab)\varphi=\{1\}\cup(L\backslash[(L^1a)\cap(L^1b)])\varphi=\{1\}\cup[(L\backslash L^1a)\cup(L\backslash L^1b)]\varphi=[\{1\}\cup(L\backslash L^1a)\varphi]\cup[\{1\}\cup(L\backslash L^1b)\varphi]=a\Phi\,.\,b\Phi$. Therefore Φ is a homomorphism.

Now suppose that $a\Phi=b\Phi$. Then $\{1\}\cup(L\backslash L^1a)\varphi=\{1\}\cup(L\backslash L^1b)\varphi$ and so $L^1a=L^1b$. Hence $a=cb$ for some $c\in L^1$, that is, $a\leqq b$. Similarly, $b\leqq a$, hence $a=b$. Thus Φ is injective and the theorem is proved.

Theorem 5.2. *Let L be a countable UFS. Then L is embeddable in E_X if and only if L is upper finite.*

Proof. Suppose that L is embeddable in E_X. Clearly, subsemilattices of upper finite semilattices are upper finite. Since E_X is upper finite, it follows that L is upper finite.

Conversely, suppose that L is upper finite.

We prove that the elements of L can be written as a sequence $(f_n: n\in\mathbb{N})$ such that

$$f_n\leqq f_m \Rightarrow n\geqq m. \tag{5.1}$$

Suppose that $L=\{e_n: n\in\mathbb{N}\}$. We define a sequence $(A_n: n\in\mathbb{N})$ of subsets of L as follows. Assuming that $A_0=\varnothing$, we define $A_n=\{g\in L: g\geqq e_n\}\backslash(A_0\cup\ldots\cup A_{n-1})$ for every $n\in\mathbb{N}$. Since L is upper finite, every A_n is finite, possibly empty. Moreover, $L=\bigcup_{n\geq 1}A_n$. Now we define the sequence $(f_n: n\in\mathbb{N})$.

Clearly, $A_1\neq\varnothing$. Let f_1 be maximial in A_1 for the natural partial order of L.

Suppose that $f_1,\ldots,f_k$ are defined for some $k\in\mathbb{N}$ and suppose that $f_k\in A_n$. If $A_n\backslash\{f_1,\ldots,f_k\}\neq\varnothing$, we choose f_{k+1} to be a maximal element of $A_n\backslash\{f_1,\ldots,f_k\}$. If $A_n\backslash\{f_1,\ldots,f_k\}=\varnothing$, we choose f_{k+1} to be a maximal element of A_{n+m}, where $m=\min\{l\in\mathbb{N}: A_{n+l}\neq\varnothing\}$. Note that $\{l\in\mathbb{N}: A_{n+l}\neq\varnothing\}$ is nonempty, since L is countable and $A_1\cup\ldots\cup A_n$ is finite.

It is immediate that $L=\{f_n: n\in\mathbb{N}\}$ and $(f_n: n\in\mathbb{N})$ satisfies (5.1).

We define a map $\varphi: L\to E_X$ as follows. Since (5.1) holds, we have $f_1\in\mathrm{Irr}(L)$. Let $k\in\mathbb{N}$. The set $B_k=\{i\in\mathbb{N}: f_i\in\mathrm{Irr}(L)$ and $f_i\geqq f_k\}$ is clearly finite. Since $\mathrm{Irr}(L)$ generates L, there exists some $f_i \in \mathrm{Irr}(L)$ such that $f_i \geqq f_k$ and so B_k is nonempty. Since L is a UFS, it is clear that $f_k = \prod_{i \in B_k} f_i$. We define $f_k \varphi = \{1\} \cup \{x_i : i \in B_k\}$.

We prove that φ is a homomorphism. Let $m, n \in \mathbb{N}$ and suppose that $f_m f_n = f_k$. We want to show that $f_m \varphi . f_n \varphi = f_k \varphi$, that is, $B_m \cup B_n = B_k$. Since $f_k \leqq f_m$ and $f_k \leqq f_n$, it follows that $B_m \cup B_n \subseteq B_k$. Now suppose that $i \in B_k$. Then $f_i \in \mathrm{Irr}(L)$ and $f_i \geqq f_k = f_m f_n$. Since L is a UFS, f_i is prime and so we have $f_i \geqq f_m$ or $f_i \geqq f_n$. Hence $i \in B_m \cup B_n$ and so $B_k \subseteq B_m \cup B_n$. Thus $B_m \cup B_n = B_k$ and φ is a homomorphism.

Now suppose that $f_m \varphi = f_n \varphi$ for some $m, n \in \mathbb{N}$. Then $B_m = B_n$ and so $f_m = \prod_{i \in B_m} f_i = \prod_{i \in B_n} f_i = f_n$. Therefore φ is injective and the theorem is proved.

We note that these results only yield sufficient conditions for a semilattice to be embeddable in E_X. We can provide a trivial example of a subsemilattice of E_X which is not a UFS. In fact, let $U, V, W, Z \in E_X$ be defined by $U = \{1, x_1, x_2\}$, $V = \{1, x_1, x_3\}$, $W = \{1, x_2, x_3\}$ and $Z = \{1, x_1, x_2, x_3\}$. Let $N = \{U, V, W, Z\}$. Obviously, N is a subsemilattice of E_X. However, N is not a UFS, since $U \in \mathrm{Irr}(N)$, $U \geqq VW$, $U \ngeqq V$ and $U \ngeqq W$.

Theorem 5.3. *There exists a countable upper finite semilattice which is not embeddable in E_X.*

Proof. Let $M = \{(m, n) \in \mathbb{N}^0 \times \mathbb{N}^0 : m \geqq n\}$, with multiplication described by

$$(m,n)\backslash(m',n') = \begin{cases} (m, \min\{n, n'\}) & \text{if } m = m' \\ (\max\{m, m'\}, 0) & \text{if } m \neq m'. \end{cases}$$

It follows from the definition that the groupoid M is commutative and every element of M is idempotent. We note that $M_0 = \{(m, 0) : m \in \mathbb{N}^0\}$ satisfies $(M_0 M) \cup (M M_0) \subseteq M_0$. Let $(m, n), (m', n'), (m'', n'') \in M$. if $m = m' = m''$, then $[(m, n)(m', n')](m'', n'') = (m, \min\{n, n', n''\}) = (m, n)[(m', n')(m'', n'')]$. Otherwise, it follows from the remark on M_0 that $[(m, n)(m', n')](m'', n'') = (\max\{m, m', m''\}, 0) = (m, n)[(m', n')(m'', n'')]$. Hence M is associative and so a semilattice.

Let $(m, n), (m', n') \in M$. It should be clear that $(m', n') \geqq (m, n)$ implies $m' \leqq m$. Since $n' \leqq m'$, there exist only finitely many $(m', n') \in M$ such that $(m', n') \geqq (m, n)$. Hence M is upper finite.

Now suppose that $\varphi : M \to E_X$ is an embedding. Let $k = |(0,0)\varphi|$. Since $(k, k) > (k, k-1) > \ldots > (k, 0)$, we have $(k, k)\varphi > \ldots > (k, 0)\varphi$. Hence $|(k, k)\varphi| < \ldots < |(k, 0)\varphi|$ and so $|(k, 0)\varphi| - |(k, k)\varphi| \geqq k$. Since $|AB| \leqq |A| + |B| - 1$ for every $A, B \in E_X$, we have $|(k, 0)\varphi| = |(0, 0)\varphi . (k, k)\varphi| \leqq |(0, 0)\varphi| + |(k, k)\varphi| - 1$. Hence $|(0, 0)\varphi| \geqq |(k, 0)\varphi| - |(k, k)\varphi| + 1 \geqq k + 1$, a contradiction. Therefore no such embedding exists.

6. The Hopf property

An algebra A is said to be *hopfian* if the only surjective endomorphisms of A are the automorphisms.

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It is known that $FIM(X)$ is hopfian if and only if X is finite [5]. However, E_X shows different behaviour.

Theorem 6.1. *Let X be a nonempty set. Then E_X is not hopfian.*

Proof. Let $x \in X$ and let

$$Y = \{u \in R_X : x^2 \leqq_l u\}.$$

Let $\iota: (X \cup X^{-1})^* \to R_X$ denote the map which associates to every $u \in (X \cup X^{-1})^*$ the corresponding reduced word, obtained by successively deleting all factors of the form xx^{-1}, $x \in X \cup X^{-1}$. Let $A \in E_X$. We define $A' = (A \backslash Y) \cup [x^{-1}(A \cap Y)]\iota$. Obviously, A' is finite and nonempty. We show that A' is left closed. Let $w \in A'$ and let $w' \in R_X$ with $w' <_l w$.

Suppose first that $w \in A \backslash Y$. Since A is left closed, we have $w' \in A$ and it is clear that $w \notin Y$ implies $w' \notin Y$. Hence $w' \in A'$.

Now suppose that $w \in [x^{-1}(A \cap Y)]\iota$. Since $1 \in A \backslash Y$, we can assume that $w' \neq 1$. Then there exists some $v \in R_X$ such that $x^2 v \in A$ and $w = xv$. Since $w' <_l w$ and $w' \neq 1$, there exists $v' \in R_X$ such that $v' <_l v$ and $w' = xv'$. Since A is left closed, $x^2 v' \in A$. Hence $w' = xv' = [x^{-1}(x^2 v')]\iota \in [x^{-1}(A \cap Y)]\iota \subseteq A'$. Thus A' is left closed.

We define a map $\varphi: E_X \to E_X$ by $A\varphi = A'$, $A \in E_X$, and we show that φ is a non-injective surjective homomorphism.

(i) *φ is not injective.*

It follows from the definition that $\{1, x, x^2\}, \varphi = \{1, x\} = \{1, x\}\varphi$ hence φ is not injective.

(ii) *φ is surjective.*

Let $A \in E_X$. Suppose that $A \cap Y = \varnothing$. Then it is immediate that $A\varphi = A$.

Now suppose that $A \cap Y \neq \varnothing$. Then $x, x^2 \in A$. Let $B = (A \backslash Y) \cup \{x^2\} \cup [x(A \cap Y)]$. Obviously, B is finite and nonempty. We show that B is left closed. Let $w \in B$ and let $w' \in R_X$ be such that $w' <_l w$. We have seen before that $A \backslash Y$ is left closed, so we can assume that $w \notin A \backslash Y$. Suppose that $w = x^2$. Since $A \cap Y \neq \varnothing$ and A is left closed, we have $x^2 \in A$ and so $w' \in A \backslash Y \subseteq B$. Now suppose that $w = x^3 u$ for some $u \in R_X$ such that $x^2 u \in A$. We can assume that $w' = x^3 u'$ and $u' <_l u$ for some $u' \in R_X$. Since $x^2 u' <_l x^2 u$ and A is left closed, we have $x^2 u' \in A$ and so $w' = x^3 u' \in [x(A \cap Y)] \subseteq B$. Thus B is left closed and so $B \in E_X$. It is immediate that $B\varphi = A$ and so φ is surjective.

(iii) *φ is a homomorphism.*

Let $A, B \in E_X$. Then $(AB)\varphi = [(A \cup B) \backslash Y] \cup (x^{-1}[(A \cup B) \cap Y])\iota = (A \backslash Y) \cup (B \backslash Y) \cup [x^{-1}(A \cap Y)]\iota \cup [x^{-1}(B \cap Y)]\iota = (A\varphi)(B\varphi)$. Thus φ is a homomorphism and the theorem is proved.

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