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A CLASS OF NORMAL (0, 1)-MATRICES

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1. Introduction. If A is a real normal matrix A^t (the transpose of A) is a real polynomial in A. We study here those normal (0, 1)-matrices A with constant row sums which have A^t a polynomial of degree two in A. We completely determine the structure of such matrices (modulo the determination of certain block designs) by showing that if the matrix A is irreducible and *not* symmetric then either

(a) A is a skew Hadamard (ν, k, λ)-configuration [**2**], or

(b) A or $A - I$ is a Kronecker product of a skew Hadamard (ν, k, λ)-configuration with a matrix J of ones.

(These remarks must be taken modulo the natural equivalence relation of permutational similarity as the property under discussion is preserved under this relation. Indeed, in the sequel we shall say A is *equivalent* to B ($A \cong B$) if $A = PBP^t$ for some permutation matrix P.)

Since the matrix A with constant row sums (and hence also here column sums), if reducible, is a direct sum of irreducible matrices the issue rests with the irreducible case.

We remark that the problem under consideration has a graph theoretic interpretation if we think of A as the incidence matrix of a finite directed graph (loops allowed). The condition that A^t be a polynomial of degree two in A translates into a rather obvious condition on the number of directed 2-paths joining pairs of vertices of the graph. For other studies of 2-path conditions, see for example [**1**; **5**; **6**; **7**].

2. The matrix equation. Throughout, A will denote an $n \times n$, irreducible, non-symmetric (0, 1)-matrix. We suppose that

$$A^t = aA^2 + bA + cI, \qquad AJ = kJ \tag{2.1}$$

where A^t is the transpose of A, I is the identity matrix of order n and J is the $n \times n$ matrix all of whose entries are $+1$. Note then that $JA = kJ$, $a \neq 0$ and $1/a$ is a positive integer. The determination of the structure of A will depend on establishing that it has a particularly simple spectrum. Having done this we will use the following.

LEMMA 2.1. *If A is an irreducible $n \times n$ (0, 1)-matrix satisfying (2.1) and the spectrum of A consists of three numbers $k, \mu, \bar{\mu}$ for a complex number $\mu \neq \bar{\mu}$,*

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then A is skew $(A + A^t = J \pm I)$ and moreover $A^tA = (k - \lambda)I + \lambda J$ for some $\lambda \geqq 0$.

Proof. The matrix $A + A^t$ is non-negative irreducible with spectrum consisting of $2k$ and $2\mathrm{Re}(\mu)$ ($\mathrm{Re}\,\mu$ denotes the real part of μ). Since the eigenspace corresponding to $2k$ is generated by the vector $(1, 1, \ldots, 1)$ an easy argument [**3**; **4**] shows that if the factor $(x - 2k)$ is divided out of the minimal polynomial of $A + A^t$ obtaining the polynomial $g(x)$, then $g(A + A^t)$ is a multiple of J. But here $g(x) = x - \mathrm{Re}\,\mu$ so that $A + A^t = 2\mathrm{Re}\,\mu I + dJ$. Since A is not symmetric $d = 1$ and $1 + 2\mathrm{Re}\,\mu = 0$ or 2, so that $A + A^t = J \pm I$. The same argument applies to A^tA with spectrum k^2, $|\mu|^2$ giving the second conclusion in the lemma.

The Lemma says that A is the incidence matrix of a (v, k, λ)-configuration (possibly degenerate as a design) and since it is skew it must be a Hadamard design with parameters of the form

$$(4t - 1, 2t - 1, t - 1) \quad \text{or} \quad (4t - 1, 2t, t)\ t \geqq 1.$$

We now draw some preliminary conclusions about the spectrum of A from (2.1). Let $f(x) = ax^2 + bx + c$ and then note that if μ is an eigenvalue of A we have $f(\mu) = \bar{\mu}$. We thus have $f(k) = k$ and it is an easy exercise to show that f can conjugate precisely one complex pair $\mu, \bar{\mu}$. So if A is not symmetric it will have in its spectrum k, of multiplicity one, a complex pair $\mu, \bar{\mu}$ and at most one other real eigenvalue, l – the other root of $f(x) = x$. We shall show, in fact, that the only feasible values for this second real eigenvalue are 0 or 1. Before doing so we note if $l \neq k$ is a root of $f(x) = x$ we have $f(x) = ax^2 + (1 - a(k + l))x + akl$ and we discuss the case $f(0) = 0$.

3. The case $f(0) = 0$.

We consider now the case

$$A^t = aA^2 + (1 - ak)A, \qquad AJ = kJ. \tag{3.1}$$

Since if A were non-singular k would be its only real eigenvalue and Lemma 2.1 would apply, we take A singular with spectrum:

$$\Lambda(A) = \{(k, 1); (\alpha + i\beta, t); (\alpha - i\beta, t); (0, n - 2t - 1)\}, \tag{3.2}$$

where the second component denotes the multiplicity of the first as an eigenvalue of A.

We shall show that A is equivalent to a Kronecker product of a non-singular solution to an equation of the form (3.1) with a suitable size J matrix $(A \cong A_1 \otimes J)$. We note conversely that if $aA_1^2 + bA_1 = A_1^t$ the matrix $B = A_1 \otimes J_s$ satisfies $(a/s)B^2 + bB = B^t$, so we shall have a complete characterization of the singular solutions to (3.1).

A preliminary parameter relation will be useful in the sequel. With $\mu = \alpha + i\beta$ one deduces from $a\mu^2 + (1 - ak)\mu = \bar{\mu}$ that:

$$\alpha = -\frac{1}{a} + \frac{k}{2}. \tag{3.3}$$

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We now argue that A, satisfying (3.1) is reducible unless trace $A = 0$. For if $a_{ii} = 1$ we have that the (i, i) entry of A^2 is k whence row i and column i are identical. Further, if $a_{ii} = 0$ the (i, i) entry of A^2 is zero. Consider PAP^t in the form

$$PAP^t = \begin{bmatrix} I & X \\ X^t & 0 \end{bmatrix}.$$

Full trace would mean A is symmetric so there are zeros on the main diagonal and the X, X^t portions come from row i = column i if $a_{ii} = 1$. But the zeros on the diagonal force the row through the zero to have inner product zero with the column through the zero and this implies that $X = 0$ and A is reducible unless X is vaccuous, i.e., trace $A = 0$.

Next consider the possibility that for $i \neq j$, $a_{ij} = a_{ji} = 1$. From (3.1) we have $(A^2)_{ij} = k$ which says that row i is the same as column j. But $a_{ij} = 1$ and $a_{ii} = 0$ denies this. We conclude that $Z \equiv A + A^t$ is (0, 1), symmetric, irreducible with spectrum $\{(2k, 1), (2\alpha, 2t), (0, n - 2t - 1)\}$. Also trace $Z = 0$ so that $2\alpha < 0$. Considering the polynomial of Z [**3**] we have

$$Z(Z - 2\alpha I) = \mu J \tag{3.4}$$

for some μ. But the diagonal entry of $Z^2 = ZZ^t$ is $2k$ so that $\mu = 2k$. Further a line sum equality gives $2k(2k - 2\alpha) = \mu n = 2kn$ so that $n = 2k - 2\alpha$. This implies that 2α is a negative integer. We claim that (3.4) implies that for a suitable permutation matrix $P, PZP^t = (J - I)_{2t+1} \otimes J_{-2\alpha}$. (Note that trace $Z = 0$ gives

$$k + t(2\alpha) = 0 \tag{3.5}$$

so that $(2t + 1)(-2\alpha) = 2k - 2\alpha = n$.) That Z can be brought to this Kronecker product form follows from (3.4) since $z_{ij} = 0$ implies that row i of z is the same row j $((Z^2)_{ij} = 2k)$. Further Z has $2k$ ones per line and -2α zeros.

Returning to our original matrix A we have that (to within equivalence) A is "block skew" in the sense that:

$$A + A^t = \begin{bmatrix} 0_{-2\alpha} & & & * \\ & 0_{-2\alpha} & & \\ & & \cdot & \\ * & & & \cdot \\ & & & 0_{-2\alpha} \end{bmatrix}$$

where $0_{-2\alpha}$ is the zero matrix of size $(-2\alpha) \times (-2\alpha)$ and the $*$ indicates all positions contain $+1$'s. We then have A in the following partitioned form:

$$A = \begin{bmatrix} 0_{-2\alpha} & A_{12} & A_{13} & \dots & A_{1,2t} \\ A_{21} & 0_{-2\alpha} & & & \cdot \\ A_{31} & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ A_{2t,1} & \cdot & \cdot & \dots & 0_{-2\alpha} \end{bmatrix}$$

where the A_{ij} are of size $(-2\alpha) \times (-2\alpha)$ and $A_{ij} = J_{-2\alpha} - A_{ji}^t$. Now apart from the portion in the diagonal zero blocks row i and column i are complementary and from our basic equation $a_{ij} = a_{ji} = 0$ implies that row i and column j have zero inner product. Further, apart from the diagonal block zeros each line has k ones and k zeros. (Recall that $n = 2k - 2\alpha$.) Consider for example row one. Its ones locate the zeros of column one precisely and the remaining k positions in the last $n - 2\alpha$ rows of column one are then zeros. But row one must miss column two as well and hence column two is identical with column one. This argument shows, in fact, that the rows through any fixed diagonal zero block are all the same as are the columns through any fixed zero block. But then no A_{ij} can contain both a zero and a one and hence $A_{ij} = J_{-2\alpha}$ or $A_{ij} = 0_{-2\alpha}$. Since the line sum of A is $k = t(-2\alpha)$ there are $t\,J$ blocks per "block line" and we have that $A \cong A_1 \otimes J_{-2\alpha}$ where A_1 has line sums t. But $aA^2 + (1 - ak)A = A^t$ implies that $(-2\alpha a)A_1^2 + (1 - ak)A_1 = A_1^t$. Moreover, A_1 is non-singular for to achieve the spectrum of A under the Kronecker product with $J_{-2\alpha}$ we deduce the spectrum of A_1 to be

$$\left\{(t, 1); \left(-\frac{1}{2} + \frac{\beta}{2\alpha} i, t\right); \left(-\frac{1}{2} - \frac{\beta}{2\alpha} i, t\right)\right\}.$$

Since the size of A_1 is $n/-2\alpha = (2k - 2\alpha)/-2\alpha = 2t + 1$ there is no "room" for a zero eigenvalue. We thus have proven, in view of Lemma 2.1:

LEMMA 3.1. *If A is an irreducible, nonsymmetric matrix satisfying* (3.1) *then $A \cong A_1 \times J_s$ where A_1 is a skew-Hadamard incidence matrix with parameters of the form* $(4\lambda - 1, 2\lambda - 1, \lambda - 1)$.

The (v, k, λ) parameter form comes from the fact that A_1 has trace zero. Note also that any singular solution to (2.1) is covered by the Lemma 3.1 for if 0 is an eigenvalue we must have $f(0) = 0$. This means, in particular, that a singular solution to (2.1) must have trace zero.

4. The general case. We now return to the general equation (2.1), writing it in the form

$$A^t = aA^2 + (1 - a(k + l))A + aklI \tag{4.1}$$

where $l \neq k$ is a real root of $f(x) = x$. (Note that k is the spectral radius of A, and therefore is the only eigenvalue of modulus k.)

LEMMA 4.1. *l is either* 0 *or* 1.

Proof. Suppose some $a_{ii} = 0$; then from (4.1) $(A^2)_{ii} = -kl < k$ so that $-1 \leqq l \leqq 0$. Now since A is not symmetric we have some $a_{ij} = 1$, $a_{ji} = 0$. Thus $(A^2)_{ij} = 1/a + k + l$ so that l is an integer and $l = 0$ or $l = -1$. If all $a_{ii} = 1$ the matrix $A - I$ is $(0, 1)$ and satisfies an equation of the form (4.1). Since $l - 1$ is an eigenvalue it follows from the above that $l - 1$ is

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0 or -1, whence l is either 0 or 1. We now show that $l = -1$ is not feasible. If $l = -1$ and some $a_{ii} = 1$ we compute from (4.1) that $(A^2)_{ii} = 2k - 1 \leqq k$ whence $k = 1$ and $(A^2)_{ii} = 1$ for all i, so that A is a symmetric permutation matrix. Thus if $l = -1$, trace $A = 0$. But now if -1 is an eigenvalue of A the matrix $A + I$ is a singular solution to (2.1) contradicting the final remark of the last section.

Now if A satisfying (2.1) has k as its only real eigenvalue its structure is given by Lemma 2.1. If it has $l = 0$ as an eigenvalue it is covered by Lemma 3.1. The remaining case in view of Lemma 4.1 is that A has full trace and 1 is an eigenvalue. But then $A - I$ is a singular solution to (2.1) and its structure is covered by Lemma 3.1. We have proven:

THEOREM. *Let A be an irreducible, nonsymmetric* (0, 1)*-matrix with constant row sums whose transpose is a polynomial of degree two in A. Then either A is a skew Hadamard incidence matrix, or $A \cong A_1 \otimes J_s + I$, or $A \cong A_1 \otimes J_s$, where A_1 is a skew Hadamard incidence matrix with parameters of the form* $(4\lambda - 1, 2\lambda - 1, \lambda - 1)$. (*The last case holds with $s > 1$ if and only if A is singular.*)

Conversely every skew Hadamard incidence matrix A is an irreducible matrix with A^t a polynomial of degree two in A, as are $A \otimes J_s$ and $A \otimes J_s + I$ if A has parameters of the form $(4\lambda - 1, 2\lambda - 1, \lambda - 1)$.

The converse statement in the theorem is an easy consequence of the relations $A + A^t = J \pm I$, $AA^t = (k - \lambda)I + \lambda J$ and the properties of the Kronecker product.

We note that if A is a normal (0, 1) irreducible matrix with minimal polynomial of degree 3, then if A is not symmetric A^t will be a polynomial of degree two in A and moreover A will necessarily have constant line sums. Thus the structure of A is covered by the first option in the theorem.

5. A concluding example. Much of the above analysis depends solely on the fact that A is a non-negative irreducible integral matrix. We present an example to show that the (0, 1) hypothesis is critical to obtaining the spectral properties of A and hence the form of A^tA.

Let $\mathscr{C}[a_1, a_2, \ldots, a_n]$ denote the circulant matrix with first row $a_1, a_2, \ldots, a_n$. If then $X = \mathscr{C}[1, 1, 5, 7]$ we have

$$X^2 = \mathscr{C}[40, 24, 60, 72] \quad \text{and} \quad X^t = \mathscr{C}[1, 7, 5, 1].$$

One can check that

$$X^t = \frac{1}{10}X^2 - \frac{1}{5}X - \frac{14}{5}I,$$

but the spectrum of X is $\{14, -2, -4 \pm 6i\}$ and $X^tX = \mathscr{C}[76, 48, 12, 48]$.

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