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ON SPINES OF 3-MANIFOLDS WITH BOUNDARY

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Abstract

We give a simple necessary and sufficient condition for the inclusion map of a subpolyhedron into a compact 3-manifold with non-empty boundary to be a homotopy equivalence.

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1. Introduction

In this note we prove the following theorem.

THEOREM 1. *Let Y be a compact, connected (triangulated) 3-manifold with $\partial Y \neq \emptyset$, and let X be a connected subpolyhedron of Y such that the maps $\pi_1(X) \to \pi_1(Y)$ and $H_2(X) \to H_2(Y)$, induced by inclusion, are isomorphisms. Then the inclusion map $X \subset Y$ is a homotopy equivalence.*

This has the following corollary.

COROLLARY 2. *Let M be a rational homology 3-sphere, and let $Q \subset P \subset M$ be polyhedra such that*

(1) *each component of P contains exactly one component of Q;*
(2) *each component of $M - Q$ contains exactly one component of $M - P$;*
(3) *for each $q \in Q$, inclusion induces an isomorphism $\pi_1(Q, q) \to \pi_1(P, q)$.*

Then Q is a deformation retract of P.

Taking $M = S^3$ in Corollary 2 gives the result that was announced as Proposition 3.4 in [3].

Theorem 1 is a straightforward consequence of the following result, which is implicit in [1].

THEOREM 3. *Let (K, L) be a pair of connected* CW *complexes such that $K - L$ has finitely many cells, each of dimension ≤ 2. Suppose that the maps $\pi_1(L) \to \pi_1(K)$ and $H_2(L) \to H_2(K)$, induced by inclusion, are isomorphisms. Then the inclusion map $L \subset K$ is a homotopy equivalence.*

We learned about the question answered by Corollary 2 in 1988, from T. Y. Kong, who was interested in it in the context of image thinning algorithms for 3-dimensional binary digital images in computer graphics (see [3]). Our original proof of Theorem 1, obtained in 1989, used 3-manifold topology, for example the sphere theorem. Later, on wondering whether the corresponding statement was true in the category of 2-complexes, we were led to Cohen's paper [1] and the realization that it essentially contained a proof of Theorem 3.

I would like to thank Dr. Kong for bringing the question mentioned above to my attention.

2. Proofs

PROOF OF THEOREM 3. Since this is not stated explicitly in [1], we describe the relevant parts of that paper and how they imply the theorem. We follow closely the notation of [1].

We may assume that L has a single 0-cell, e^0, and that $K - L$ consists of 1-cells and 2-cells. The homology exact sequence of the pair (K, L) shows that $H_1(K, L) = H_2(K, L) = 0$. It follows that the boundary homomorphism $\partial : C_2(K, L) \to C_1(K, L)$ is an isomorphism, and hence that $K - L$ has the same number of 1-cells as 2-cells. So $K = L \cup \bigcup_{j=1}^n e_j^1 \cup \bigcup_{i=1}^n e_i^2$, say.

Let $L^* = L \cup \bigcup_{j=1}^n e_j^1$. Taking e^0 as base point for π_1 throughout, let x_j be the element of $\pi_1(L^*)$ represented by e_j^1, $1 \leq j \leq n$, and let F be the free group on $\{x_1, \ldots, x_n\}$. Then $\pi_1(L^*) = \pi_1(L) * F$.

Write $G = \pi_1(L)$. Let $r_i \in G * F$ be the element represented by the attaching map of e_i^2, $(1 \leq i \leq n)$, and let $R \subset G * F$ be the normal closure of $\{r_1, \ldots, r_n\}$. Then $\pi_1(K) \cong (G * F)/R = H$, say, where the map $\pi_1(L) \to \pi_1(K)$ corresponds to the composition $\varphi : G \subset G * F \to H$.

By hypothesis, φ is an isomorphism. In particular, since φ is onto, there exists $w_j \in G$ such that $x_j w_j^{-1} \in R$, $(1 \le j \le n)$. Let $R_0 \subset G * F$ be the normal closure of $\{x_1 w_1^{-1}, \ldots, x_n w_n^{-1}\}$. Thus $R_0 \subset R$.

Clearly the composition $\varphi_0 : G \subset G * F \to (G * F)/R_0 = H_0$ is an isomorphism. But if $\pi : H_0 \to H$ is the quotient map, then $\varphi = \pi\varphi_0$. Hence π is an isomorphism, giving $R_0 = R$. Therefore $r_i \in R_0$, so we may write

$$r_i = \prod_{k=1}^{q_i} g_{ik}(x_{ik} w_{ik}^{-1})^{n_{ik}} g_{ik}^{-1}, \qquad 1 \le i \le n,$$

as in the hypothesis of Lemma 2.3 of [1]. (Here $g_{ik} \in G$, $n_{ik} \in \mathbb{Z}$, $x_{ik} = x_j$ for some j, and $w_{ik} = w_j$ for the same j.)

Let $p : \tilde{K} \to K$ be the universal cover. Since $\pi_1(L) \to \pi_1(K)$ is an isomorphism, $p^{-1}(L) = \tilde{L}$ is the universal cover of L. Note that $C_q(\tilde{K}, \tilde{L}) = 0$ for $q \neq 1, 2$, while $C_1(\tilde{K}, \tilde{L})$ and $C_2(\tilde{K}, \tilde{L})$ are free $\mathbb{Z}G$-modules of rank n, with bases corresponding to the 1-cells and 2-cells of $K - L$ respectively. Note also that under the maps $\pi_1(L^*) \to \pi_1(K)$ and $\pi_1(L) \to \pi_1(K)$ induced by inclusion, x_j and w_j $(1 \le j \le n)$ have the same image. Hence Lemma 2.3 of [1] applies to show that the boundary homomorphism $\partial : C_2(\tilde{K}, \tilde{L}) \to C_1(\tilde{K}, \tilde{L})$ is represented, with respect to the bases mentioned above, by the $n \times n$ matrix $A = (a_{ij})$ over $\mathbb{Z}G$ defined by

$$a_{ij} = \sum n_{ik} g_{ik},$$

where the sum is taken over those k for which $x_{ik} = x_j$.

Next, recall the expression for r_i given above and define $r_i' \in G * F$ by the corresponding expression

$$r_i' = \prod_{k=1}^{q_i} g_{ik} x_{ik}^{n_{ik}} g_{ik}^{-1}, \qquad 1 \le i \le n,$$

as in [1, §1]. Let $R' \subset G * F$ be the normal closure of $\{r_1', \ldots, r_n'\}$.

Let $\alpha : G * F \to G * F$ be the isomorphism defined by $\alpha \mid G =$ identity and $\alpha(x_i) = x_i w_i^{-1}$, $(1 \le i \le n)$. Then $\alpha(r_i') = r_i$, $(1 \le i \le n)$, so $\alpha(R') = R$ and α induces an isomorphism $\bar{\alpha} : H' = (G * F)/R' \to (G * F)/R = H$. Let φ' be the composition $G \subset G * F \to H'$. Then $\varphi = \bar{\alpha}\varphi'$. Since φ is an isomorphism, φ' is also. Hence, by Proposition 4.1 of [1], the matrix A is invertible.

Thus $\partial : C_2(\tilde{K}, \tilde{L}) \to C_1(\tilde{K}, \tilde{L})$ is an isomorphism, and we have $H_*(\tilde{K}, \tilde{L}) = 0$, hence $\pi_*(\tilde{K}, \tilde{L}) = 0$, and hence $\pi_*(K, L) = 0$, as in [1, Lemma 2.2]. The result follows.

PROOF OF THEOREM 1. Adding a collar to ∂Y and replacing X by a regular neighborhood, we may assume that X is a compact 3-manifold in the interior of Y.

We claim that each component of $\overline{Y - X}$ meets ∂Y. For if Z is a component that does not, then $[\partial Z] = 0$ in $H_2(Y; \mathbb{Z}_2)$. But ∂Z must consist of a proper subset of the components of ∂X, otherwise $Y = X \cup_\partial Z$ and hence $\partial Y = \emptyset$, contrary to hypothesis. Therefore $[\partial Z] \neq 0$ in $H_2(X; \mathbb{Z}_2)$. But the universal coefficient theorem shows that the map $H_2(X; \mathbb{Z}_2) \to H_2(Y; \mathbb{Z}_2)$ is an isomorphism.

Hence, starting at ∂Y, we may collapse away all the 3-simplexes of $\overline{Y - X}$, thereby collapsing Y onto $X \cup K$ where K is a finite 2-complex. The result now follows from Theorem 3.

PROOF OF COROLLARY 2. Since M is a rational homology sphere, $H^1(M) = 0$, and the cohomology exact sequence of the pair $(M, M - P)$ gives an exact sequence

$$H^0(M) \to H^0(M - P) \to H^1(M, M - P) \to 0\,,$$

and similarly for Q.

Condition (2) implies that the map $H^0(M - Q) \to H^0(M - P)$ induced by inclusion is an isomorphism. Hence so is the map $H^1(M, M - Q) \to H^1(M, M - P)$. It follows, by Alexander Duality, that $H_2(Q) \to H_2(P)$ is an isomorphism.

Now replace P by a regular neighborhood Y in M, and apply Theorem 1 to each component of Y (with X the corresponding component of Q).

3. Concluding Remarks

Here are two questions related to the above discussion. Let X and Y be either finite connected 2-complexes or compact connected 3-manifolds with non-empty boundary.

(1) If $f : X \to Y$ is a map inducing isomorphisms on π_1 and H_2, is f a homotopy equivalence?

(2) If $\pi_1(X) \cong \pi_1(Y)$ and $H_2(X) \cong H_2(Y)$, are X and Y homotopy equivalent?

Theorems 3 and 1 show that the answer to (1) is 'yes' in both cases if f is an inclusion map. On the other hand, it is easy to construct counterexamples in general. (For example, take $X = Y = S^1 \times S^2$–open 3-cell $\simeq S^1 \vee S^2$. Then

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$\pi_1(X) \cong \mathbb{Z}$, generated by z, say, and $\pi_2(X) \cong \mathbb{Z}\pi_1(X)$, generated by x, say. Define $f : X \to X$ so that $f_*(z) = z$ and $f_*(x) = (1 - z + z^2)x$.)

Question (2) for finite 2-complexes has been extensively investigated. The answer is 'no' in general; counterexamples were first given by Dunwoody [2] and Metzler [4]. In fact, in the example given in [2], X is homotopy equivalent to the exterior of the trefoil knot minus an open 3-cell. One can show, however, that the answer to (2) is 'yes' in the case of 3-manifolds with boundary.

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