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COMPLETIONS OF BOOLEAN ALGEBRAS OF PROJECTIONS AND WEAK-STAR CLOSURES OF C^*-ALGEBRAS ON DUAL BANACH SPACES

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Abstract Palmer has shown that those hermitians in the weak-star operator closure of a commutative C^*-algebra represented on a dual Banach space X *that are known to commute with the initial C^*-algebra* form the real part of a weakly closed C^*-algebra on X. Relying on a result of Murphy, it is shown in this paper that this last proviso may be dropped, and that the weak-star closure is even a W^*-algebra.

When the dual Banach space X is separable, one can prove a similar result for C^*-equivalent algebras, via a 'separable patch' completion theorem for Boolean algebras of projections on such spaces.

Keywords: numerical range; Boolean algebras of projections; dual Banach spaces; C^*-algebras

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1. Introduction

The closure in the weak operator topology of a C^*-algebra on X (see § 3 for the definition) is again a C^*-algebra, both in the commutative case and when the unit ball is relatively compact in the weak operator topology (see, for example, [**18, 20**]).

If the underlying space is a dual space, one may wonder about the closure in the weak-star, the $\sigma(X, {}'X)$, operator topology. This is a more delicate problem: multiplication is not right continuous for this topology. Nevertheless, for commutative algebras, Palmer [**18**] established that those hermitians in the weak operator closure *that are known to commute with the initial C^*-algebra* also form the real part of a C^*-algebra on X.

Reconciling the two order relations on hermitian projections (see Theorem 2.8) and using a result of Murphy [**17**] (see Theorem 2.5) allows one to drop this last proviso and then to show that the weak-star closure is even a W^*-algebra [**22**].

Further, we develop results on the closures of *C^*-equivalent-algebras* (C^*-algebras that are represented, though not isometrically, in an $L(X)$) on duals of separable Banach spaces. To do this we prove a *'separable patch' completion theorem* for Boolean algebras of projections on such spaces (Theorem 8.1).

2. Terminology and prerequisites

We follow the standard notation and sketch only a few salient details, referring the reader to [**5–7**], for example, for a full exposition and other references.

In particular, A_1 will denote the unit ball of a subset A of a normed space, and $\langle x, x' \rangle$ will denote the value of the functional x' in X' at x in X. Then $i_X : x \mapsto \hat{x}$ will denote the natural injection of a space X into its second dual X''.

Associated with the *weak* topology $\sigma(X, X')$ and the *weak-star* topology $\sigma(X, {}'X)$ (when X is the dual of ${}'X$) are the operator topologies they induce on $L(X)$, namely the weak operator topologies of pointwise convergence in $\sigma(X, X')$ and $\sigma(X, {}'X)$.

Thus, when $\mathcal{D}$ is a subset of $L(X)$, the *weak (operator) closure* $\bar{\mathcal{D}}^{\sigma(X,X')}$ is the closure of $\mathcal{D}$ in the $\sigma(X, X')$-operator topology, while when X is the dual of ${}'X$, the *weak-star (operator) closure* $\bar{\mathcal{D}}^{\sigma(X,{}'X)}$ is the closure of $\mathcal{D}$ in the $\sigma(X, {}'X)$-operator topology.

The *bounded weak-star (operator) topology* is the strongest topology agreeing with the weak-star (operator) topology on every weak-star compact set.

A set is *relatively weakly compact* if its weak operator closure is compact in the weak operator topology.

2.1. Numerical range in Banach algebras

Throughout this section, $\mathcal{A}$ will denote a complex unital Banach algebra. The *state space* of $\mathcal{A}$ is

$$S(\mathcal{A}) = \{\varphi \in \mathcal{A}' : \langle 1, \varphi \rangle = 1 = \|\varphi\|\}$$

and the *algebra numerical range* of an element a is

$$V(a) = \{\langle a, \varphi \rangle : \varphi \in S(\mathcal{A})\}.$$

The *numerical radius* $|a|_v$ $(= |V(a)|)$ of an element a determines a norm on $\mathcal{A}$ equivalent to the given norm: $\mathrm{e}^{-1}\|a\| \leqslant |a|_v \leqslant \|a\|$ for any a.

For any $a \in \mathcal{A}$ the *spectrum of* a lies inside the numerical range: $\sigma(a) \subseteq V(\mathcal{A}, a)$.

2.1.1. *Hermitian elements of Banach algebras*

Definition 2.1. An element $h \in \mathcal{A}$ is *hermitian* if its algebra numerical range is real: equivalently, if $\|\mathrm{e}^{\mathrm{i}rh}\| = 1$ $(r \in \mathbb{R})$: equivalently, if $\|1 + \mathrm{i}ra\| = 1 + o(r)$ as $r \to 0$.

Remark 2.2. If h is hermitian, then $\operatorname{co} \sigma(h) = V(\mathcal{A}, h)$, where 'co' denotes *convex hull*.

Theorem 2.3 (Sinclair's Theorem). *$\|h\| = \rho(h)$ (the spectral radius of h) for any hermitian $h \in \mathcal{A}$.*

Although the *Jordan product* of two hermitians need not be hermitian (indeed the square of an hermitian need not be hermitian [**8**]), nevertheless, we have the following theorem.

Theorem 2.4. *Let h and k be hermitian in $\mathcal{A}$. Then $\mathrm{i}(hk - kh)$ is hermitian.*

The next result is the key to finessing the absence of right weak-star operator continuity for multiplication.

Theorem 2.5 (Murphy [17]). *Let $\mathcal{A}$ be a complex unital Banach algebra. Let h, k be hermitians in $\mathcal{A}$ such that hk and h^2 [or k^2] are also hermitian. Then $hk = kh$.*

This was proved by deftly using Theorem 2.4 repeatedly to show that (when h^2 is hermitian)

$$h(hk - kh) = (hk - kh)h.$$

Kaplansky conjectured (prompted, apparently, by Jacobson [**15**, Lemma 2]) that in a Banach algebra a commutator of two elements which commutes with either of these elements must be quasi-nilpotent. This conjecture was established, independently, by Kleinecke [**16**] and Shirokov [**19**]. For an accessible exposition see [**13**, Problem 232].

Since $i(hk-kh)$ is hermitian and quasi-nilpotent it must vanish, by Sinclair's Theorem; that is, $hk = kh$.

2.2. Order

The *numerical range order relation* on the set of hermitian operators in a Banach algebra is defined as follows.

Definition 2.6. Given a hermitian element h in the Banach algebra $\mathcal{A}$, we say that $h \geqslant 0$ if and only if $V(\mathcal{A}, h) \subseteq \mathbb{R}^+$, or, equivalently, if and only if $\sigma(h) \subseteq \mathbb{R}^+$ (see Remark 2.2).

If h is hermitian, then $0 \leqslant h \leqslant 1$ if and only if $\sigma(h) \subseteq [0, 1]$; that is, if and only if h is positive and $\|h\| \leqslant 1$.

2.3. Projections

The *natural ordering on projections (idempotents)* is that $e \subseteq f$ if and only if $ef = fe = e$.

2.3.1. *Ordering of hermitian projections*

If e is a (non-trivial) *hermitian* projection, then $\|e\| = 1$ (by Sinclair's Theorem). Also, $\sigma(e) \subseteq \{0, 1\}$, so $0 \leqslant e \leqslant 1$ for any hermitian projection e.

Suppose that e and f are two hermitian projections in $\mathcal{A}$ and that $e \subseteq f$; that is, $e = ef = fe$. If χ is a character on the commutative unital subalgebra of $\mathcal{A}$ generated by e, f, then $\chi(e) = \chi(e)\chi(f)$, and both $\chi(e)$ and $\chi(f)$ are either 0 or 1. Thus, $\chi(e) \leqslant \chi(f)$ and therefore $\sigma(f - e) \geqslant 0$, so $e \leqslant f$ in the numerical range sense.

In fact, the two orders coincide; see Theorem 2.8. (This was shown for projections on a strictly convex X in [**3**, Theorem 2.17].)

Lemma 2.7. *Let e be a hermitian projection in $\mathcal{A}$ and let h be a positive hermitian in $\mathcal{A}$ with $\|h\| \leqslant 1$. If $e \leqslant h$, then $e = ehe$.*

Proof. Let $\mathcal{B} = e\mathcal{A}e$. If $\phi \in S(\mathcal{B})$, then $a \mapsto \langle eae, \phi\rangle \in S(\mathcal{A})$. Since $0 \leqslant e \leqslant h \leqslant 1$, we have

$$\langle e, \phi\rangle = \langle eee, \phi\rangle \leqslant \langle ehe, \phi\rangle \leqslant \langle e1e, \phi\rangle = \langle e, \phi\rangle,$$

from which we see that $V(\mathcal{B}, e - ehe) = \{0\}$, and thus, since $|a|_v = 0$ implies that $a = 0$, we have $e - ehe = 0$ (in $\mathcal{B}$ and therefore in $\mathcal{A}$). □

Theorem 2.8. *Suppose that e and f are hermitian projections in a Banach algebra $\mathcal{A}$ and that $e \leqslant f$ in the numerical range sense. Then $e \subseteq f$; that is, $ef = fe = e$.*

Proof. By the preceding lemma we have $e = efe$. Multiplying out shows that $(ef - fe)^3 = 0$ from which, by Sinclair's Theorem, we have $ef = fe$. □

2.4. Numerical range on a Banach space

When X is a complex Banach space we define

$$\varPi_X = \{(x, x') \in X \times X' \colon \langle x, x'\rangle = \|x\| = \|x'\| = 1\}$$

and the *spatial numerical range* $V(T)$ of the operator T to be

$$V(T) = \{\langle Tx, x'\rangle \colon (x, x') \in \varPi_X\}.$$

It is convenient to write $\omega_{x,x'} \colon T \to \langle Tx, x'\rangle$ to denote the state specified by the element (x, x') of $\varPi_X$ and $\boldsymbol{\omega}_{\varPi} = \{\omega_{x,x'} \colon (x, x') \in \varPi_X\}$ for the set of such states.

2.4.1. *Hermitian elements on a Banach space*

Definition 2.9. An operator H on X is *hermitian* if its spatial numerical range is real: equivalently, if $\|e^{irH}\| = 1$ $(r \in \mathbb{R})$: equivalently, if $\|1 + irH\| = 1 + o(r)$ as $r \to 0$.

Remark 2.10. Since, for any $T \in L(X)$,

$$V(T) \subseteq V(L(X), T) = \overline{\mathrm{co}}\, V(T)$$

(see [**5**, § 9]), we see that H (in $L(X)$) is hermitian on the space X precisely when H is hermitian in the Banach algebra $L(X)$. This is also clear from the exponential characterizations of hermiticity. The set of hermitian operators on X is an $\mathbb{R}$-linear subspace, closed in the weak and strong operator topologies, and contains I_X.

2.4.2. *Hermitians on a dual Banach space*

Let X be the dual of some other Banach space ${}'X$. (Such a predual of X need not be unique.)

An operator T on X is defined to be *lower hermitian* if its *lower numerical range*

$$V_L(T) = \{\langle Tx, \psi\rangle \colon (x, \psi) \in \varPi_L\} \subseteq \mathbb{R}$$

is real: here

$$\varPi_L = \varPi_{'X} = \{(x, \psi) \in X \times {}'X \colon \langle x, \psi\rangle = \|x\| = \|\psi\| = 1\}.$$

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Now $V_L(T) \subseteq V(T) \subseteq \overline{V_L(T)}$ for any T, as follows from the Bishop–Phelps Theorem [**4**]. Hence the following result.

Theorem 2.11. *H is hermitian if and only if H is lower hermitian. Hence, the set of hermitians on a dual space X is closed in the $\sigma(X, {}'X)$ operator topology.*

3. C^*-algebras on a Banach space

3.1. Vidav–Palmer Theorem

The now classical numerical range characterization of C^*-algebras is as follows.

Theorem 3.1 (Vidav–Palmer Theorem). *Let $\mathcal{H}$ be the set of hermitian elements of a complex unital Banach algebra $\mathcal{A}$. If $\mathcal{A} = \mathcal{H} + \mathrm{i}\mathcal{H}$, then $\mathcal{A}$ is a C^*-algebra under the given norm and the natural involution.*

By a *C^*-algebra on X* we mean a unital Banach subalgebra of $L(X)$ which satisfies the hypotheses of the Vidav–Palmer Theorem, the unit being the identity operator on X.

Any bounded linear functional on an (abstract) C^*-algebra $\mathcal{A}$, $(= \mathcal{H} + \mathrm{i}\mathcal{H})$, can be written as a linear combination of states. Hence, if (h_α) is a bounded monotone net in $\mathcal{H}$, then the scalar net $(\omega(h_\alpha))$ converges for each bounded linear functional ω on $\mathcal{A}$. In consequence we can make the following remark.

Remark 3.2. Suppose that $\mathcal{A}$ is a C^*-algebra, that $\Theta\colon \mathcal{A} \to L(X)$ is a bounded algebra homomorphism, and that (h_α) is a bounded monotone net in $\mathcal{H}$. Then the net $(\langle \Theta(h_\alpha), \psi\rangle)$ converges for each $\psi \in X'$. Consequently, if the net $(\Theta(h_\alpha))$ has a cluster point for the operator topology generated by a subfamily Γ of X', then there is only one such cluster point, and it is therefore the Γ-limit of the net.

3.2. Weak closures of C^*-algebras on X

First we recall the main results for C^*-algebras on a Banach space, before moving on, via renorming and monotone limits for hermitians, to the more subtle problems of weak-star closures that arise when the underlying space is itself a dual space (see § 7).

Some compactness or commutativity hypothesis seems to be necessary for the weak operator closure of a C^*-algebra on X again to be a C^*-algebra. Even then it is not clear whether the weak closure has to be a W^*-algebra in either the abstract or the spatial sense. We shall say that $\mathcal{A}$ is *a spatial W^*-algebra (on X)* if

(i) $\mathcal{A}$ is a C^*-algebra on X,

(ii) $\mathcal{A}$ is an abstract W^*-algebra (is a dual Banach space) and

(iii) in addition the states ω_Π are *normal* (i.e. respect suprema and infima of monotone nets).

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3.2.1. *When $\mathcal{A}$ is commutative*

The next result is originally due to Palmer [**18**, Lemma 2.7]; a short proof can be found in [**9**].

Lemma 3.3. *Suppose that $\mathcal{A}$ is a commutative C^*-algebra on X. Then*

$$\|Bx\| = \|B^*x\|$$

for all $B \in \mathcal{A}$ and $x \in X$.

This lemma allows one to extend the C^*-structure from $\mathcal{A}$ to the closure of $\mathcal{A}$ in the *strong operator topology*, which, by convexity, coincides with $\bar{\mathcal{A}}^{\sigma(X,X')}$; if a net $H_\alpha + \mathrm{i}K_\alpha$, with H_α, K_α hermitian in $\mathcal{A}$, converges in the *strong* operator topology, then its real and imaginary parts do so too [**21**].

Theorem 3.4. *Let $\mathcal{A}$ be a commutative C^*-algebra on X and let $\mathcal{H}$ be the set of hermitian elements of $\mathcal{A}$. Let $\bar{\mathcal{H}}^{\sigma(X,X')}$ be the weak operator topology closure of $\mathcal{H}$ and $\bar{\mathcal{A}}^{\sigma(X,X')}$ be the weak operator topology closure of $\mathcal{A}$. Then*

$$\bar{\mathcal{A}}^{\sigma(X,X')} = \bar{\mathcal{H}}^{\sigma(X,X')} + \mathrm{i}\bar{\mathcal{H}}^{\sigma(X,X')}$$

so $\bar{\mathcal{A}}^{\sigma(X,X')}$ is a C^*-algebra on X. Moreover, $(\bar{\mathcal{A}}^{\sigma(X,X')})_1 = \bar{\mathcal{A}}_1^{\sigma(X,X')}$ (Kaplansky Density Theorem).

3.2.2. *When $\mathcal{A}_1$ is relatively weakly compact*

Only bounded sets can be compact, so it is perhaps not surprising that here our result extends only to the *bounded weak operator* closure of $\mathcal{A}$, equal to

$$\mathcal{A}^{\sim} = \bigcup_{n=1}^{\infty} n\bar{\mathcal{A}}_1^{\sigma(X,X')}.$$

Theorem 3.5. *Suppose that $\mathcal{A}$ is a C^*-algebra on X and that its unit ball $\mathcal{A}_1$ is relatively weakly compact. Then $\mathcal{A}^{\sim}$, the closure of $\mathcal{A}$ in the bounded weak operator topology, is a spatial W^*-algebra on X and $(\mathcal{A}^{\sim})_1 = \bar{\mathcal{A}}_1^{\sigma(X,X')}$ (Kaplansky Density Theorem). Moreover, any faithful representation of $\mathcal{A}^{\sim}$ as a concrete von Neumann algebra is weak operator bicontinuous on bounded sets.*

The proof of the theorem [**22**] rests on the fact that, by the identity of comparable compact Hausdorff topologies, the weak topology on $\overline{\mathcal{A}}_1^{\sigma(X,X')}$ is the weak topology induced by the states $\boldsymbol{\omega}_\Pi$.

It is not clear whether $\mathcal{A}^{\sim}$ is $\sigma(X,X')$-closed.

Every bounded linear operator from a C^*-algebra into a Banach space not containing c_0 is automatically weakly compact. This theorem goes back to Grothendieck, Bartle–Dunford–Schwartz and others. See [**10**, Chapter VI, Notes] for an interesting discussion of its genesis and development and [**23**, Theorem 2] for a further generalization to *compressible approximate order unit spaces*. As a result of this we have the following corollary.

Corollary 3.6. *Any C^*-algebra on a space not containing c_0 has a bounded weak closure which is a spatial W^*-algebra.*

This theme has also been treated in [**12**].

3.2.3. *Commutative and compact*

When $\mathcal{A}$ is commutative and $\mathcal{A}_1$ is relatively weakly compact the weak closure $\bar{\mathcal{A}}^{\sigma(X,X')}$ is a W^*-algebra and any faithful representation as a von Neumann algebra is weakly and strongly continuous on bounded sets. This happens precisely when $\mathcal{A}$ is representable by a spectral measure (see also § 5.3).

4. Renorming of $L(X)$ and X

Any (equivalent) renorming of X induces a renorming of $L(X)$. Conversely, renormings of certain subfamilies of $L(X)$ can be used to induce renormings of X.

4.1. Renorming by a semigroup

Any bounded semigroup in $L(X)$ effects a renorming of X under which the elements of the semigroup become contractions.

Indeed, given a bounded semigroup $\mathcal{S}$ in $L(X)$ ($\|S\| \leqslant K_{\mathcal{S}}$ for some $K_{\mathcal{S}}$), with $I_X \in \mathcal{S}$, one can define a new norm $|\cdot|_{\mathcal{S}}$ on X, and then on $L(X)$, by

$$|x|_{\mathcal{S}} = \sup\{\|Sx\| \colon S \in \mathcal{S}\}, \quad x \in X.$$

Then

$$\|x\| \leqslant |x|_{\mathcal{S}} \leqslant K_{\mathcal{S}}\|x\|, \quad x \in X$$

and

$$|S|_{\mathcal{S}} \leqslant 1, \quad S \in \mathcal{S}.$$

If $\mathcal{S}$ is a group, then $|S|_{\mathcal{S}} = 1$ for all S.

4.2. Unitary renorming

Suppose we have a bounded unital isomorphism $\Theta\colon \mathcal{A} \to L(X)$ from a C^*-algebra $\mathcal{A}$ into $L(X)$, not necessarily an isometry. To distinguish, we shall term such a Θ a *representation of* $\mathcal{A}$ *in* $L(X)$, and call $\mathcal{A}$ a C^**-equivalent algebra*.

Then the group of unitaries $\mathcal{U}$ in $\mathcal{A}$ maps to a bounded group in $L(X)$ containing I_X and induces a norm

$$|x|_{\Theta} = \sup\{\|\Theta(u)x\| \colon u \in \mathcal{U}\}, \quad x \in X,$$

on X equivalent to the original norm. In turn this induces an operator norm $|\cdot|_{\Theta}$ on $L(X)$; and then $|\mathrm{e}^{\mathrm{i}t\Theta(h)}|_{\Theta} = 1$ for each hermitian $h \in \mathcal{H}$ and $t \in \mathbb{R}$. Thus, each $\Theta(h)$ is $|\cdot|_{\Theta}$-hermitian, and with this norm $\Theta(\mathcal{A})$ is a C^*-algebra on X. Moreover, Θ is then a $*$-isomorphism. The results of § 3.2 then extend to C^*-algebras represented in X.

5. Boolean algebras of projections and renorming

Boolean algebras of projections need not be bounded but when they are they become hermitian after renorming the underlying space.

5.1. Boolean algebras of projections

Consider a *Boolean algebra of projections* $\mathcal{E}$ on a complex Banach space X: a family of projections such that $I \in \mathcal{E}$ and

$$\begin{aligned} E \in \mathcal{E} &\implies E^2 = E, \\ E \in \mathcal{E} &\implies I - E \in \mathcal{E}, \\ E, F \in \mathcal{E} &\implies EF = FE \in \mathcal{E}. \end{aligned}$$

And $\mathcal{E}$ is *bounded* if every $\|E\| \leqslant K_{\mathcal{E}}$ for some constant $K_{\mathcal{E}}$. The fundamental result here is that of [**2**]; see [**11**, Proposition 5.3] for the current author's brief proof.

Lemma 5.1. *If $\mathcal{E}$ is a Boolean algebra of projections on X bounded by $K_{\mathcal{E}}$, then*

$$\mathcal{S}_{\mathcal{E}} = \left\{ \bigoplus_{\text{finite}} \lambda_j E_j \colon |\lambda_j| \leqslant 1,\ E_j \in \mathcal{E} \right\}$$

is a bounded multiplicative semigroup of operators on X; here $\bigoplus$ denotes a disjoint sum. Also

$$\|\mathcal{S}_{\mathcal{E}}\| \leqslant 4K_{\mathcal{E}}.$$

5.2. Renorming (by) a Boolean algebra of projections

Given a bounded Boolean algebra of projections $\mathcal{E}$ on X, use the recipe of § 4.1, applied to the semigroup $\mathcal{S}_{\mathcal{E}}$ of Lemma 5.1, to obtain a new norm $|\cdot|_{\mathcal{S}_{\mathcal{E}}}$ on X (abbreviated to $|\cdot|_{\mathcal{E}}$), equivalent to the original norm on X, inducing in its turn an operator norm $|\cdot|_{\mathcal{E}}$ on $L(X)$. Then $|S|_{\mathcal{E}} \leqslant 1$ for each $S \in \mathcal{S}_{\mathcal{E}}$.

Since $\mathrm{e}^{\mathrm{i}tE} = I - E + \mathrm{e}^{\mathrm{i}t}E \in \mathcal{S}_{\mathcal{E}}$ for any $E \in \mathcal{E}$ and $t \in \mathbb{R}$ we see that $|\mathrm{e}^{\mathrm{i}tE}|_{\mathcal{E}} \leqslant 1$ and therefore each $\mathrm{e}^{\mathrm{i}tE}$ is an $|\cdot|_{\mathcal{E}}$*-isometry*. Hence, we have the following result.

Theorem 5.2. *Any bounded Boolean algebra of projections $\mathcal{E}$ is (simultaneously) hermitian equivalent. Indeed, each $E \in \mathcal{E}$ is $|\cdot|_{\mathcal{E}}$-hermitian.*

Then $\mathcal{A}$ ($= \overline{\mathrm{lin}}_{\mathbb{C}}[\mathcal{E}]$), the norm closed algebra generated by $\mathcal{E}$, is a C^*-algebra under the operator norm induced by $|\cdot|_{\mathcal{E}}$, which on $\mathcal{A}$ coincides with the spectral norm: that is, $|A|_{\mathcal{E}} = \rho(A^*A)^{1/2}$.

5.3. Boolean algebras of projections and spectral measures

We write $\mathrm{Bo}(\Lambda)$ for the σ-algebra of Borel subsets of a compact space Λ, the smallest σ-algebra containing all the open subsets of Λ.

Any Boolean algebra $\mathcal{E}$ can be represented as the range of a finitely additive function on the family of Borel subsets of its Stone space:

$$\mathcal{E}\colon \mathrm{Bo}(\Lambda) \to L(X) \colon \tau \mapsto E(\tau).$$

For each $x \in X$, we obtain a finitely additive vector measure

$$\mathcal{E}x\colon \mathrm{Bo}(\Lambda) \to X \colon \tau \mapsto E(\tau)x.$$

The vector measures $\mathcal{E}x$ all are $\sigma(X, X')$-*countably additive* if and only if the range of $\mathcal{E}$ is relatively compact in $L(X)$ in the $\sigma(X, X')$-operator topology; equivalently, if and only $\mathcal{A}_1$ is relatively $\sigma(X, X')$ compact, where $\mathcal{A} = \overline{\mathrm{lin}}_{\mathbb{C}}[\mathcal{E}]$. If so, $\mathcal{E}$ is effectively the range of a spectral measure of class $(\mathrm{Bo}(\Lambda), X')$: recall that, given a σ-algebra Σ of subsets of a set Ω and a total subset Γ of X', a *spectral measure of class* (Σ, Γ) is a Boolean algebra homomorphism $\sigma \mapsto E(\sigma)$ from Σ into $L(X)$ such that $\langle E(\sigma)x, x'\rangle$ is countably additive for each $x \in X$ and $x' \in \Gamma$.

Consider a bounded Boolean algebra $\mathcal{E}$ of projections on a complex Banach space X; and let $\mathcal{A}$ be its closed linear (= algebra) span. By Theorem 5.2 we may renorm to make all the projections in $\mathcal{E}$ hermitian, and then $\mathcal{A}$ is spectrally normed. Hence, as in [**21**, Theorem 2] we have the following.

Theorem 5.3. *Let $\mathcal{E}$ be a relatively weakly compact Boolean algebra of projections on a complex Banach space X, and let $\mathcal{A}$ be the C^*-algebra generated by $\mathcal{E}$. Then $\mathcal{A}$ is representable by a spectral measure of class* $(\mathrm{Bo}(\Lambda), X')$*, the weak closure $\bar{\mathcal{A}}^{\sigma(X,X')}$ is a spatial W^*-algebra on X and any faithful representation of $\bar{\mathcal{A}}^{\sigma(X,X')}$ as a concrete von Neumann algebra on a Hilbert space is weakly bicontinuous on bounded sets.*

6. Monotone limits on a dual space

Theorem 6.1 is valid for families of hermitians on a dual Banach space, whether they belong to a C^*-algebra or not.

A net $(H_\alpha)_{\alpha \in A}$ is *upward directed*, or *increasing*, if $H_\alpha \leqslant H_\beta$ when $\alpha \leqslant \beta$ in the indexing set A; similarly, $(H_\alpha)_{\alpha \in A}$ is *downward directed*, or *decreasing*, if $H_\alpha \geqslant H_\beta$ when $\alpha \leqslant \beta$. A *monotone net* is a net that is either upward or downward directed.

Theorem 6.1. *Let X be the dual of $'X$ and let (H_α) be a bounded monotone net of hermitians in $L(X)$. Then (H_α) has a hermitian weak-star limit, say H.*

Suppose further that K and K^2 are also hermitian and that each $H_\alpha K$ is hermitian. Then HK is hermitian and $HK = KH$.

In particular, if (E_α) is a monotone net of hermitian projections on X then its weak-star limit is again a hermitian projection.

Proof. Without loss of generality assume that the net (H_α) is increasing.

For each x, by Alaoglu's Theorem $(H_\alpha x)$ has a $\sigma(X, 'X)$-cluster point, say H_x.

Since the scalar net $(\langle H_\alpha x, \psi\rangle)$ is increasing (for each $(x, \psi) \in \Pi_L$; see § 2.4.2) the cluster point is unique (see Remark 3.2), and is therefore the $\sigma(X, 'X)$-limit of the net $(H_\alpha x)$.

It follows that $H_x = Hx$ for some $H \in L(X)$ and that H is the $\sigma(X, 'X)$-operator limit of the net (H_α). So H is hermitian, by Theorem 2.11.

The second assertion follows from Murphy's Theorem (Theorem 2.5).

As for an increasing net (E_α) of hermitian projections, let E be its hermitian weak-star limit. Ultimately, that is, for large β, we have $E_\beta E_\alpha = E_\alpha$ for any given α (using Theorem 2.8) so that $EE_\alpha = E_\alpha$. Now, again by Theorem 2.5, $E_\alpha E = E_\alpha$ and taking the limit shows that $E^2 = E$. □

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For projections, this result is a lineal descendant of [**1**]: if a net (E_α) of projections (not necessarily hermitian) on a Banach space X is monotone (in the natural order) and has weak x-cluster points, that is, if

$$\bigcap_{\alpha} \overline{\{E_\beta x \colon \beta \geqslant \alpha\}}^{\sigma(X,X')} \neq \emptyset$$

for each $x \in X$, then the net converges in the *strong* operator topology.

One cannot hope for such a strengthening of Theorem 6.1. If ${}'X = l^1(\mathbb{N})$, so that $X = l^\infty(\mathbb{N})$, and if E_n is the projection onto the span of the first n coordinate projections, then $E_n \nearrow I$ in the weak-star operator topology. If, however, θ is a Banach limit on l^∞ and y is the vector in l^∞ such that $y_n = 1$ for all n, then $\langle E_n y, \theta\rangle = 0$ for all n, while $\langle Iy, \theta\rangle = 1$.

7. C^*-algebras on a dual space

Suppose now that X is a dual Banach space on which a C^*-algebra $\mathcal{A}$ acts. This hypothesis automatically provides a modicum of compactness, but in a very weak topology. Nevertheless, something can be accomplished.

7.1. W^*-closures on a dual space: commutative case

The following theorem, a principal result of the paper, is the promised advance on [**18**, Theorem 2.11].

To lighten the notation write

$$\bar{\mathcal{H}}^\sigma = \bar{\mathcal{H}}^{\sigma(X,{}'X)} \quad \text{and} \quad \bar{\mathcal{A}}^\sigma = \bar{\mathcal{H}}^\sigma + \mathrm{i}\bar{\mathcal{H}}^\sigma.$$

Theorem 7.1. *Let $\mathcal{A}$ be a commutative C^*-algebra on a Banach space X, the dual of ${}'X$, and let $\mathcal{H}$ be the set of hermitian elements of $\mathcal{A}$. Then $\bar{\mathcal{H}}^\sigma$ is weak-star and monotone closed, and is the real part of the commutative W^*-algebra $\bar{\mathcal{A}}^\sigma$ on X. Moreover, $\bar{\mathcal{A}}^\sigma$ is bounded weak-star closed and the involution is bounded weak-star continuous.*

Proof. The proof is an elaboration of that of Theorem 6.1.

First, consider a net (H_α) in $\mathcal{H}$ converging to H $(\in \bar{\mathcal{H}}^\sigma)$ in the $\sigma(X,{}'X)$ operator topology. Then H is hermitian, by Theorem 2.11.

Next, $H_\alpha K (\in \mathcal{H})$ is hermitian for each α and each $K \in \mathcal{H}$. The limit HK must be hermitian, and therefore $HK = KH$, by Theorem 2.5.

In particular, HH_α is hermitian (and equal to $H_\alpha H$) for each α, so the limit of $H_\alpha H$, which is H^2, is also hermitian and is in $\bar{\mathcal{H}}^\sigma$.

If now K is the $\sigma(X,{}'X)$-limit of a net (K_β) from $\mathcal{H}$, then $KH = \lim K_\beta H$ and so is hermitian. Thus, $HK = KH$, again by Theorem 2.5. Hence, $\bar{\mathcal{H}}^\sigma$ is a norm-closed $\mathbb{R}$-linear subspace of commuting hermitians on X, and is closed under multiplication. The Vidav–Palmer Theorem guarantees that $\bar{\mathcal{A}}^\sigma$ is a C^*-algebra.

By Theorem 6.1 $\bar{\mathcal{H}}^\sigma$ is monotone closed, and Π_L (see § 2.4.2) is a separating family of normal functionals for $\bar{\mathcal{H}}^\sigma$. A routine variation on the standard proof (see, for example, [**24**, Chapter III.3]) shows that there is a separating family of normal *states*, and this suffices to show that $\bar{\mathcal{A}}^\sigma$ is W^*.

For each lower supporting pair $(x,\psi) \in \Pi_{'X}$ the $\sigma(X,{}'X)$-continuous functional $\omega = \omega_{x,\psi}\colon A \mapsto \langle Ax, \psi\rangle$ is a state on $\bar{\mathcal{A}}^\sigma$; thus, $\omega(A^*) = \overline{\omega(A)}$ for each $A \in \bar{\mathcal{A}}^\sigma$.

Suppose that (A_α) is a bounded $\sigma(X,{}'X)$-convergent net in $\bar{\mathcal{A}}^\sigma$, with limit $A \in L(X)$, and that H_α and J_α are the real and imaginary parts of A_α. Without loss of generality we may assume that $\|A_\alpha\| \leqslant 1$ for all α. Now, by the Kaplansky Density Theorem, we have $\bar{\mathcal{A}}_1^{\sigma(X,'X)} = (\bar{\mathcal{A}}^\sigma)_1$ and $\bar{\mathcal{H}}_1^{\sigma(X,'X)} = (\bar{\mathcal{H}}^\sigma)_1$. Hence, $H_\alpha, J_\alpha \in \bar{\mathcal{H}}_1^{\sigma(X,'X)}$ for each α. Now, for each ω $(= \omega_{x,\psi})$ the bounded real nets $\omega(H_\alpha)$ and $\omega(J_\alpha)$ must each converge: for $\omega(H_\alpha) = \operatorname{Re}\omega(A_\alpha)$ and $\omega(J_\alpha) = \operatorname{Im}\omega(A_\alpha)$. Thus, (H_α) and (J_α) each have a unique cluster point (call them H and J), which must be the $\sigma(X,{}'X)$-limits of these nets, respectively. Hence, $A \in \bar{\mathcal{A}}^\sigma$ and $(A_\alpha^*) \to A^*$ weak-star. $\square$

Question 7.2. Is $\bar{\mathcal{A}}^\sigma$ also $\sigma(X,{}'X)$ closed?

7.2. Commutative C^*-equivalent-algebras on $L(('X)')$

Suppose we have a representation $\Theta\colon \mathcal{A} \to L(X)$ of a C^*-algebra $\mathcal{A}$ in $L(X)$. We may use the recipe of § 4.2 to renorm X, but have no guarantee of a *dual renorming*, one corresponding to a renorming of the predual $'X$. A warning not to be too ambitious: if Y is a Banach space for which every equivalent norm on Y' is a dual norm, then Y is reflexive [**14**, § 18F].

Nevertheless, when $\mathcal{A}$ is commutative we have a consequence of the Riesz Representation Theorem at our disposal (see the proof of [**18**, Theorem 2.5]).

Lemma 7.3. *If $\mathcal{A} = C(\Lambda)$ is a commutative C^*-algebra and if $\Theta\colon \mathcal{A} \to L(X)$, where $X[= ('X)']$, is a bounded unital algebra isomorphism, then there is a spectral measure $E(\cdot)$ on $\mathrm{Bo}(\Lambda)$, where Λ is the maximal ideal space of $\mathcal{A}$, with values in X, and of class $'X$, such that*

$$\Theta(f) = \int f(\lambda) E(\mathrm{d}\lambda).$$

From this we can derive a σ-completion for any bounded Boolean algebra of projections on X, in anticipation of the results of § 8.

Theorem 7.4. *Given a bounded Boolean algebra $\mathcal{E}$ on X, the dual space of $'X$, it can be enlarged to a $'X$-σ-complete Boolean algebra on X, namely $\bar{\mathcal{E}} = \{E(\kappa)\colon \kappa \in \mathrm{Bo}(\Lambda)\}$.*

Proof. In view of Theorem 5.2 we can apply Lemma 7.3 to the C^*-(equivalent) algebra $\overline{\mathrm{lin}}_{\mathbb{C}}[\mathcal{E}]$. $\square$

8. Boolean algebras: separable predual

If $X = ('X)'$ and $'X$ is *separable*, then X_1 is not only weak-star compact but also *weak-star metrizable*. Subject to this extra hypothesis we can extend Theorem 7.1 to the algebra generated by a Boolean algebra that is not initially hermitian.

8.1. Separable patch completion

An important step towards Theorem 8.3 is the following.

Theorem 8.1 (separable patch completion). *Suppose that X is the dual of a separable space and that $\mathcal{E}$ is a bounded Boolean algebra of projections on X.*

Let $\bar{\mathcal{E}} = \{E(\tau) : \tau \in \mathrm{Bo}(\Lambda)\}$ be the representing spectral measure for the C^*-algebra $\mathcal{A}$ generated by $\mathcal{E}$.

Let $\tilde{\mathcal{E}}$ be the family of operators on X, each of which, on each norm separable subspace of X, agrees with some member of $\bar{\mathcal{E}}$.

Then $\tilde{\mathcal{E}}$ is a monotone complete Boolean algebra of $\bar{\mathcal{E}}$-hermitian projections containing $\mathcal{E}$.

Moreover, the norms defined by $\bar{\mathcal{E}}$ and $\tilde{\mathcal{E}}$ are identical, i.e. $|x|_{\bar{\mathcal{E}}} = |x|_{\tilde{\mathcal{E}}}$ $(x \in X)$.

Proof. The proof proceeds in several stages.

Claim. *The elements of $\tilde{\mathcal{E}}$ are $\bar{\mathcal{E}}$-hermitian projections and form a Boolean algebra.*

Consider an $E \in \tilde{\mathcal{E}}$ and $x \in X$. Let $\mathcal{M} = \overline{\mathrm{lin}}[x, Ex]$ (which is certainly norm separable); choose τ $(= \tau_{\mathcal{M}})$ so that $E(\tau)|_{\mathcal{M}} = E|_{\mathcal{M}}$. Then

$$E^2 x = E(Ex) = EE(\tau)x = E(\tau)^2 x = E(\tau)x = Ex,$$

which shows that E is a projection. Thus,

$$\mathrm{e}^{\mathrm{i}tE}|_{\mathcal{M}} = \mathrm{e}^{\mathrm{i}tE(\tau)}|_{\mathcal{M}},$$

whence

$$S\mathrm{e}^{\mathrm{i}tE}|_{\mathcal{M}} \in \mathcal{S}_{\bar{\mathcal{E}}}|_{\mathcal{M}}$$

for $S \in \mathcal{S}_{\bar{\mathcal{E}}}$, and therefore

$$|\mathrm{e}^{\mathrm{i}tE}x|_{\bar{\mathcal{E}}} = \sup\{\|S\mathrm{e}^{\mathrm{i}tE}x\| : S \in \mathcal{S}_{\bar{\mathcal{E}}}\} \leqslant |x|_{\bar{\mathcal{E}}}.$$

This shows that E is $\bar{\mathcal{E}}$-hermitian.

Suppose that $E, F \in \tilde{\mathcal{E}}$. Given any norm separable subspace $\mathcal{M}$ there are τ, υ such that $E = E(\tau)$ and $F = E(\upsilon)$ when restricted to $\mathcal{M}$. Then, for $x \in \mathcal{M}$,

$$\begin{aligned} EFx &= E(\tau)Fx \\ &= E(\tau)E(\upsilon)x \\ &= E(\tau \cap \upsilon)x \\ &= E(\upsilon)E(\tau)x \\ &= E(\upsilon)Ex \\ &= FEx, \end{aligned}$$

which shows that $EF = FE \in \tilde{\mathcal{E}}$. It is immediate that $\tilde{\mathcal{E}}$ is a Boolean algebra, which establishes the claim.

Claim. *For each $x \in X$ we have $\mathcal{S}_{\tilde{\mathcal{E}}}x = \mathcal{S}_{\bar{\mathcal{E}}}x$, by construction, and therefore $|x|_{\bar{\mathcal{E}}} = |x|_{\tilde{\mathcal{E}}}$ $(x \in X)$.*

Claim. *$\tilde{\mathcal{E}}$ is monotone complete.*

Let ρ be a metric inducing the weak-star topology on X_1, which contains the closed unit $\bar{\mathcal{E}}$-ball $\{x \in X \colon |x|_{\bar{\mathcal{E}}} \leqslant 1\}$. (The latter is $\sigma(X, {}'X)$-compact if and only if the $\bar{\mathcal{E}}$ norm is a dual norm.)

Consider an increasing net (E_α) from $\tilde{\mathcal{E}}$. Then (see Remark 3.2) (E_α) has a (unique) weak-star operator limit, say E.

Let $\mathcal{M} = \overline{\text{lin}}[x_1, x_2, \dots]$ be a (norm) separable subspace of X.

For each α there is a $\tau_\alpha [= \tau_{\alpha,\mathcal{M}}] \in \text{Bo}(\Lambda)$ such that $E_\alpha|_{\mathcal{M}} = E(\tau_\alpha)|_{\mathcal{M}}$.

Now there exists α_1 such that

$$\rho(E_\alpha x_1, E x_1) < 2^{-1}, \quad \alpha \geqslant \alpha_1.$$

Next, there exists $\alpha_2 \geqslant \alpha_1$ such that

$$\left.\begin{aligned} \rho(E_\alpha x_1, E x_1) < 2^{-2}, \\ \rho(E_\alpha x_2, E x_2) < 2^{-2}, \end{aligned}\right\} \quad \alpha \geqslant \alpha_2.$$

Continuing in this manner, there is an α_k $(\geqslant \alpha_{k-1})$ such that

$$\rho(E_\alpha x_j, E x_j) < 2^{-k}, \quad 1 \leqslant j \leqslant k, \ \alpha \geqslant \alpha_k,$$

and so on.

Put $\tau_{\mathcal{M}} = \bigcup_k \tau_{\alpha_k,\mathcal{M}} \in \text{Bo}(\Lambda)$. Then

$$E x_j = E(\tau_{\mathcal{M}}) x_j, \quad j \in \mathbb{N},$$

and therefore

$$E z = E(\tau_{\mathcal{M}}) z, \quad z \in \mathcal{M}.$$

Thus, $E \in \tilde{\mathcal{E}}$. This establishes the third claim and completes the proof of the theorem. □

Corollary 8.2. *If $(E_\alpha) \nearrow E$ in $\tilde{\mathcal{E}}$ and if $F \in \tilde{\mathcal{E}}$, then $(E_\alpha F) \nearrow EF$.*

Proof. Given $\mathcal{M}$ there are τ_α and υ such that

$$E_\alpha|_{\mathcal{M}} = E(\tau_\alpha)|_{\mathcal{M}}, \qquad F|_{\mathcal{M}} = E(\upsilon)|_{\mathcal{M}},$$

and then $E_\alpha F|_{\mathcal{M}} \to E(\tau_{\cup\alpha} \cap \upsilon)|_{\mathcal{M}}$. □

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8.2. W^*-closures on the dual of a separable space

Consider a commutative C^*-algebra $\mathcal{A}$ represented in $L(X)$, where X is the dual of a separable predual $'X$. Then $\mathcal{A}$ is represented by a Boolean algebra of projections $\mathcal{E}$ as described in Lemma 7.3. As we have just seen, $\mathcal{E}$ can be extended to a monotone complete Boolean algebra of projections $\tilde{\mathcal{E}}$, and this becomes hermitian after renorming X (though perhaps not dually) according to Theorem 5.2.

Then $\mathcal{B}$ $(= \overline{\mathrm{lin}}_{\mathbb{C}}[\tilde{\mathcal{E}}])$ is a C^*-algebra containing $\mathcal{A}$, and $\mathcal{B} = \mathcal{K} + \mathrm{i}\mathcal{K}$, where $\mathcal{K} = \overline{\mathrm{lin}}_{\mathbb{R}}[\tilde{\mathcal{E}}]$, the set of hermitians in $\mathcal{B}$, is monotone complete. Thus, $\mathcal{B}$ is an AW^*-algebra, and since it has separating family of normal functionals (usually one stipulates for states, but a separating family of normal functionals is adequate; see the proof of Theorem 7.1) it must be a W^*-algebra. Since $\mathcal{A} \subseteq \mathcal{B}$ we have $\tilde{\mathcal{A}} \subseteq \mathcal{B}$.

Theorem 8.3. *Let $\mathcal{A}$ be a commutative C^*-algebra represented on X, the dual of a separable predual $'X$. Then the bounded weak-star closed algebra $\tilde{\mathcal{A}}$ generated by $\mathcal{E}$ is, under the spectral norm, a commutative (abstract) W^*-algebra.*

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